BESSEL-TYPE SIGNED HYPERGROUPS ON $\mathbb{R}$

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ABSTRACT

In this paper we introduce a family of convolution structures on the real line which are derived from the Bessel-Kingman hypergroups by a certain complexification of spherical Bessel functions. Though their convolution is not positivity-preserving, these "Bessel-type signed hypergroups" provide a natural extension of the usual group structure on the real line.

Introduction

While on the half-line $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$ a variety of non-isomorphic hypergroup structures is known, there do not exist any nontrivial hypergroup structures on $\mathbb{R}$ (Zeuner[13]). As Zeuner's proof shows, this is mainly due to the hypergroup axiom on support continuity of the convolution with respect to the Michael topology. So one may ask whether the situation becomes different if the hypergroup axioms are weakened. One possibility of doing so is provided by the framework of so-called signed hypergroups as introduced by the author in [8]. The concept of signed hypergroups generalizes the hypergroup axiomatics in several points, mainly in abandoning positivity and support-continuity of the convolution. It is aimed to allow the development of a commutative harmonic analysis in close analogy to the hypergroup case; see Rösler [8, 9], and for a survey, Ross [10].

In this contribution we construct a class of commutative signed hypergroups $X_\alpha, \alpha > -\frac{1}{2}$ on $\mathbb{R}$, which are intimately connected with the Bessel-Kingman hypergroups on $\mathbb{R}_+$. The characters of X_α are given as complex combinations of the normalized spherical Bessel functions j_α and $j_{\alpha+1}$, namely,

$$\Psi_\lambda^\alpha(x) = j_\alpha(\lambda x) + iC_\alpha \lambda x\, j_{\alpha+1}(\lambda x), \quad \lambda, x \in \mathbb{R},$$

with $C_\alpha = \frac{1}{2}(\alpha+1)^{-1}$. This is a natural extension of the formula

$$e^{i\lambda x} = \cos \lambda x + i \sin \lambda x = j_{-1/2}(\lambda x) + i\lambda x\, j_{1/2}(\lambda x)$$

for the usual group characters on $\mathbb{R}$.

1. Preliminaries

If not stated otherwise, our basic notation is the same as in the monograph of Bloom and Heyer [2], to which we also refer for an introduction to hypergroups. Here we recapitulate some facts on Bessel-Kingman hypergroups which will be of interest later on, and give a short summary on commutative signed hypergroups. For a background on Bessel hypergroups we refer to Kingman [7]; for details concerning signed hypergroups see Rösler [8, 9].

1.1. Bessel-Kingman hypergroups

For $\alpha > -1$, let

$$j_\alpha(z) := 2^\alpha \Gamma(\alpha+1)\,\frac{J_\alpha(z)}{z^\alpha} = \Gamma(\alpha+1)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\alpha+1)}\Big(\frac{z}{2}\Big)^{2k}, \quad z \in \mathbb{C},$$

denote the normalized spherical Bessel functions of order α. In case $\alpha > -\frac{1}{2}$, they satisfy the product formula

$$j_\alpha(x)j_\alpha(y) = M_\alpha \int_0^\pi j_\alpha(\sqrt{x^2+y^2-2xy\cos\theta})\sin^{2\alpha}\theta\, d\theta \tag{1.1}$$

for $x, y > 0$, with

$$M_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\frac{1}{2})};$$

see e.g. Watson [12], 11.4. This implies the following product formula for the functions $\varphi_\lambda^\alpha(x) := j_\alpha(\lambda x)$, with $\alpha > -\frac{1}{2}$ and parameter $\lambda \in \mathbb{C}$:

$$\varphi_\lambda^\alpha(x)\varphi_\lambda^\alpha(y) = \int_0^\infty \varphi_\lambda^\alpha(z)\, k_\alpha(x,y,z) z^{2\alpha+1} dz \quad \text{for } x, y > 0,$$

where

$$k_\alpha(x,y,z) = 2^{2\alpha-1} M_\alpha \cdot \frac{\Delta(x,y,z)^{2\alpha-1}}{(xyz)^{2\alpha}} \cdot 1_{[|x-y|, x+y]}(z). \tag{1.2}$$

Here 1_A is the indicator function of A and

$$\Delta(x,y,z) := \frac{1}{4}\sqrt{(x+y+z)(x+y-z)(x-y+z)(y+z-x)}$$

denotes the area of the triangle with sides $x, y, z > 0$.

The φ_λ^α, $\lambda \in \mathbb{C}$ are exactly the multiplicative functions of the Bessel-Kingman hypergroup H_α on $\mathbb{R}_+$, with the convolution on $M^b(\mathbb{R}_+)$ being defined by $d(\varepsilon_x * \varepsilon_y)(z) = k_\alpha(x,y,z) z^{2\alpha+1} dz$. H_α is a commutative hypergroup with 0 as neutral element and the identity mapping as involution. Its dual space, that is the space of bounded multiplicative and symmetric functions on H_α, is given by $\hat{H}_\alpha = \{\varphi_\lambda^\alpha : \lambda \in \mathbb{R}_+\}$. H_α is self-dual with Haar- and Plancherel measure $(2^\alpha\Gamma(\alpha+1))^{-1}x^{2\alpha+1}dx$. For details, we refer to Bloom and Heyer [2].

1.2. Commutative signed hypergroups

For a locally compact Hausdorff space X let $M^b_{\mathbb{R}}(X)$ denote the subspace of real measures from $M^b(X)$, and τ_* the $\sigma(M^b(X), C_0(X))$-topology on $M^b(X)$.
A *commutative signed hypergroup* is a triple (X, m, ω) consisting of a locally compact, σ-compact Hausdorff space X, a distinguished positive Radon measure $m \in M_+(X)$ with $\operatorname{supp} m = X$ and a commutative τ_*-continuous mapping $\omega : X \times X \to M^b_{\mathbb{R}}(X)$, $(x,y) \mapsto \varepsilon_x * \varepsilon_y$, satisfying the following axioms:

(A1) For each $x \in X$ and $f \in C_b(X)$, the translate $T^x f : y \mapsto \varepsilon_x * \varepsilon_y(f)$ again belongs to $C_b(X)$. Furthermore, for $f \in C_c(X)$ and any compact subset $K \subset X$, the set $\bigcup_{x\in K} \operatorname{supp}(T^x f)$ is relatively compact in X.

(A2) $\|\varepsilon_x * \varepsilon_y\| \le C$ for all $x, y \in X$, where $C > 0$ is a constant.

(A3) The canonical continuation $*$ of ω to $M^b(X)$, which is given by

$$\mu * \nu(f) := \int_{X\times X} \varepsilon_x * \varepsilon_y(f)\, d(\mu\otimes\nu)(x,y) \quad \text{for } f \in C_0(X),$$

is associative.

(A4) There exists a neutral element $e \in X$ with $\varepsilon_e * \varepsilon_x = \varepsilon_x$ for all $x \in X$.

(A5) There exists an involutive homeomorphism $^-$ on X such that

$$(\varepsilon_x * \varepsilon_y)^- = \varepsilon_{\overline{y}} * \varepsilon_{\overline{x}} \quad \text{for all } x, y \in X,$$

where $\mu^-(A) := \mu(A^-)$ for Borel measures μ on X and Borel sets $A \subset X$.

(A6) For all $f, g \in C_c(X)$ and $x \in X$ the following adjoint relation holds:

$$\int_X (T^x f) g\, dm = \int_X f(T^{\overline{x}} g)\, dm.$$

The algebra $(M^b(X), *)$ becomes a commutative Banach-*-algebra with unit ε_e, the involution $\mu \mapsto \overline{\mu^-}$, and with the norm $\|\mu\|' := \|L_\mu\|$, where $L_\mu(\nu) := \mu * \nu$ for $\mu, \nu \in M^b(X)$. $L^1(X, m)$ with the same multiplication and norm is a closed *-ideal in $(M^b(X), *, \|.\|')$. Equipped with the topology of uniform convergence on compact subsets of X, the character spaces

$$\mathcal{X}_b(X) := \{\varphi \in C_b(X) : \varphi \not\equiv 0,\ \varepsilon_x * \varepsilon_y(\varphi) = \varphi(x)\varphi(y) \text{ for all } x, y \in X\} \quad \text{and}$$
$$\widehat{X} := \{\varphi \in \mathcal{X}_b(X) : \varphi(\overline{x}) = \overline{\varphi(x)} \text{ for all } x \in X\}$$

are canonically homeomorphic to the spectrum $\Delta(L^1(X,m))$ and its symmetric part $\Delta_S(L^1(X,m))$ respectively. Fourier transforms are defined in the usual way, and there hold generalized Bochner and Plancherel theorems. In particular, there exists a unique measure $\pi \in M_+(\widehat{X})$, the Plancherel measure of (X, m, ω), with

$$\int_X |f|^2\, dm = \int_{\widehat{X}} |\widehat{f}|^2 d\pi \quad \text{for all } f \in C_c(X).$$

2. Modified Bessel functions on $\mathbb{R}$ and their product formula

For $\alpha > -1$ and $\lambda \in \mathbb{C}$ we define the modified Bessel functions Ψ_λ^α on $\mathbb{C}$ by

$$\Psi_\lambda^\alpha(z) := \varphi_\lambda^\alpha(z) + iC_\alpha\,\lambda z\,\varphi_\lambda^{\alpha+1}(z), \quad \text{with } C_\alpha = \frac{1}{2(\alpha+1)}.$$

The Ψ_λ^α are holomorphic on $\mathbb{C}$; for $\lambda z \in \mathbb{C} \setminus \{x \in \mathbb{R} : x \le 0\}$ they can also be written as

$$\Psi_\lambda^\alpha(z) = \frac{2^\alpha\Gamma(\alpha+1)}{(\lambda z)^\alpha}\big(J_\alpha(\lambda z) + iJ_{\alpha+1}(\lambda z)\big).$$

Note that in the special case $\alpha = -\frac{1}{2}$ we just have $\Psi_\lambda^{-1/2}(z) = e^{i\lambda z}$.

2.1. Lemma. *(Laplace-representation of Ψ_λ^α). Let $\alpha > -\frac{1}{2}$. Then*

$$\Psi_\lambda^\alpha(z) = M_\alpha \int_{-1}^1 e^{i\lambda zt}(1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}}dt \quad \textit{for all } \lambda, z \in \mathbb{C}.$$

Proof. Mehler's formula for Bessel functions with index $\alpha > -\frac{1}{2}$ (cf. Szegö [11], (1.71.6)) can be written as

$$\varphi_\lambda^\alpha(z) = M_\alpha \int_{-1}^1 \cos(\lambda zt)(1-t^2)^{\alpha-\frac{1}{2}}dt.$$

With this representation for $\varphi_\lambda^{\alpha+1}$, partial integration yields

$$\lambda z\varphi_\lambda^{\alpha+1}(z) = (2\alpha+1)M_{\alpha+1}\int_{-1}^1 \sin(\lambda zt)(1-t^2)^{\alpha-\frac{1}{2}}\,t\,dt,$$

and from these two the assertion follows.

2.2. Corollary. *For $\alpha \ge -\frac{1}{2}$ and $\lambda \in \mathbb{R}$ the estimate $|\Psi_\lambda^\alpha(z)| \le e^{|\lambda\cdot Im(z)|}$ holds. In particular,*

$$|\Psi_\lambda^\alpha(x)| \le 1 = \Psi_\lambda^\alpha(0) \quad \textit{for all } \lambda, x \in \mathbb{R}.$$

In case $\alpha = -\frac{1}{2}$ this is obvious; for $\alpha > -\frac{1}{2}$, we have

$$|\Psi_\lambda^\alpha(z)| \le M_\alpha \int_{-1}^{1} e^{-\lambda t \cdot Im(z)}(1-t)^{\alpha-\frac{1}{2}}(1+t)^{\alpha+\frac{1}{2}}dt \le e^{|\lambda \cdot Im(z)|}\cdot\Psi_\lambda^\alpha(0) = e^{|\lambda\cdot Im(z)|}.$$

We will now derive a product formula with a quasi-positive kernel for the modified Bessel functions Ψ_λ^α on $\mathbb{R}$. We shall use the following abbreviations:

2.3. Notation. *For $x, y, z \in \mathbb{R}$ put*

$$\sigma_{x,y,z} := \begin{cases} \frac{1}{2xy}(x^2+y^2-z^2) & \text{if } x, y \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

as well as $\varrho(x, y, z) := \frac{1}{2}(1 - \sigma_{x,y,z} + \sigma_{z,x,y} + \sigma_{z,y,x})$.

2.4. Theorem. (1) *For $\alpha > -\frac{1}{2}$ the modified Bessel functions Ψ_λ^α, $\lambda \in \mathbb{C}$, satisfy the following product formula:*

$$\Psi_\lambda^\alpha(x)\Psi_\lambda^\alpha(y) = \int_{\mathbb{R}} \Psi_\lambda^\alpha(z)\, d\mu_{x,y}^\alpha(z) \quad \text{for } x, y \in \mathbb{R}. \tag{2.1}$$

The $\mu_{x,y}^\alpha \in M^b(\mathbb{R})$ are given by

$$d\mu_{x,y}^\alpha(z) := \begin{cases} \mathcal{K}_\alpha(x,y,z)|z|^{2\alpha+1}dz & \text{if } x, y \neq 0, \\ d\varepsilon_x(z) & \text{if } y = 0, \\ d\varepsilon_y(z) & \text{if } x = 0, \end{cases}$$

with kernel

$$\mathcal{K}_\alpha(x,y,z) = k_\alpha(|x|,|y|,|z|)\varrho(x,y,z)$$

where k_α is the Bessel kernel (1.2).

(2) *The measures $\mu_{x,y}^\alpha$ have the following properties:*

(i) $\operatorname{supp}\mu_{x,y}^\alpha = \big[-|x|-|y|, -\big||x|-|y|\big|\big] \cup \big[\big||x|-|y|\big|, |x|+|y|\big]$ *for $x, y \neq 0$.*

(ii) $\mu_{x,y}^\alpha \in M^b(\mathbb{R})$ *with* $\mu_{x,y}^\alpha(\mathbb{R}) = 1$ *and* $\|\mu_{x,y}^\alpha\| \le 4$ *for all $x, y \in \mathbb{R}$.*

Note that in general the measures $\mu_{x,y}^\alpha$ are not positive: if $x, y \neq 0$, $x \neq y$, then $\varrho(x, y, y-x) = -1$, and hence there exists a neighbourhood of $y-x$ in $\operatorname{supp}\mu_{x,y}^\alpha$ where $z \mapsto \mathcal{K}_\alpha(x,y,z)$ is strictly negative.

Proof of Theorem 2.4. (1) We need two modified product formulas for spherical Bessel functions. First note that $x j_{\alpha+1}(x) = -\frac{1}{C_\alpha}\frac{d}{dx} j_\alpha(x)$ (Szegö [11], (1.71.5)). Therefore derivation of the usual product formula (1.1) yields

$$x j_{\alpha+1}(x)\, j_\alpha(y) = M_\alpha \int_0^\pi j_{\alpha+1}\big(\sqrt{x^2+y^2-2xy\cos\theta}\big)(x - y\cos\theta)\sin^{2\alpha}\theta\, d\theta.$$

Substitute $z := \sqrt{x^2-y^2-2xy\cos\theta}$ and note that $x - y\cos\theta = z\cdot\sigma_{z,x,y}$. Thus

$$x j_{\alpha+1}(x)\, j_\alpha(y) = \int_0^\infty z j_{\alpha+1}(z)\,\sigma_{z,x,y}\, k_\alpha(x,y,z)\, z^{2\alpha+1}dz \tag{2.2}$$

for $x, y > 0$. Further we claim that for $x, y > 0$,

$$C_\alpha^2\, xy\, j_{\alpha+1}(x)\, j_{\alpha+1}(y) = \int_0^\infty j_\alpha(z)\,\sigma_{x,y,z}\, k_\alpha(x,y,z)\, z^{2\alpha+1}dz. \tag{2.3}$$

For the proof of (2.3) recall the following expansions of Gegenbauer (see e.g. Askey [1], (4.36), (4.37)): For $x, y > 0$ and $\theta \in \mathbb{R}$,

$$\frac{J_\alpha\big(\sqrt{x^2+y^2-2xy\cos\theta}\big)}{\sqrt{x^2+y^2-2xy\cos\theta}^{\,\alpha}} = 2^\alpha\Gamma(\alpha)\sum_{n=0}^\infty (n+\alpha)\frac{J_{n+\alpha}(x)}{x^\alpha}\,\frac{J_{n+\alpha}(y)}{y^\alpha}\, C_n^\alpha(\cos\theta)$$

$$\text{if } \alpha > -\frac{1}{2},\ \alpha \neq 0,$$

$$J_0\big(\sqrt{x^2+y^2-2xy\cos\theta}\big) = J_0(x)J_0(y) + 2\sum_{n=0}^\infty J_n(x)J_n(y)\cos n\theta,$$

where the C_n^α are the Gegenbauer polynomials

$$C_n^\alpha(x) = \frac{(2\alpha)_n}{(\alpha+\frac{1}{2})_n} P_n^{(\alpha-1/2,\alpha-1/2)}(x).$$

Both series converge uniformly in $\theta \in \mathbb{R}$. Multiply by $C_1^\alpha(\cos\theta)$ and $\cos n\theta$ respectively. By orthogonality of the C_n^α, integration then gives

$$C_\alpha^2\, x\, j_{\alpha+1}(x)\, y\, j_{\alpha+1}(y) = M_\alpha \int_0^\pi j_\alpha\big(\sqrt{x^2+y^2-2xy\cos\theta}\big)\cos\theta\,\sin^{2\alpha}\theta\, d\theta$$

for $\alpha > -\frac{1}{2}$. Now the same substitution as before yields (2.3).
By the homogeneity of $k_\alpha(x,y,z)z^{2\alpha+1}$ and $\sigma_{x,y,z}$ it is also clear that for $\lambda, x, y > 0$ the following identities hold:

$$\lambda x\, j_{\alpha+1}(\lambda x)\, j_\alpha(\lambda y) = \int_0^\infty \lambda z j_{\alpha+1}(\lambda z)\, k_\alpha(x,y,z)\,\sigma_{z,x,y}\, z^{2\alpha+1}dz,$$

$$\lambda^2 xy\, j_{\alpha+1}(\lambda x)\, j_{\alpha+1}(\lambda y) = \int_0^\infty j_\alpha(\lambda z)\, k_\alpha(x,y,z)\,\sigma_{x,y,z}\, z^{2\alpha+1}dz.$$

Now suppose that $\lambda > 0$ and $x, y \in \mathbb{R}\setminus\{0\}$. Then of course,

$$j_\alpha(\lambda x)\, j_\alpha(\lambda y) = \frac{1}{2}\int_{\mathbb{R}} j_\alpha(\lambda z)\, k_\alpha(|x|,|y|,|z|)\,|z|^{2\alpha+1}dz.$$

Further, $z \mapsto \sigma_{x,y,z}$ is even and $z \mapsto \sigma_{z,x,y}$ as well as $z \mapsto \sigma_{z,y,x}$ are odd. Thus

$$\begin{aligned}\lambda x\, j_{\alpha+1}(\lambda x)\, j_\alpha(\lambda y) &= sgn(x)\int_0^\infty \lambda z\, j_{\alpha+1}(\lambda z)\, k_\alpha(|x|,|y|,z)\,\sigma_{z,|x|,|y|}\, z^{2\alpha+1}dz\\ &= \frac{1}{2}\int_{\mathbb{R}} \lambda z\, j_{\alpha+1}(\lambda z)\, k_\alpha(|x|,|y|,|z|)\,\sigma_{z,x,y}\,|z|^{2\alpha+1}dz,\end{aligned}$$

as well as

$$\begin{aligned}C_\alpha^2\,\lambda^2\, xy\, j_{\alpha+1}(\lambda x)\, j_{\alpha+1}(\lambda y) &= sgn(xy)\int_0^\infty j_\alpha(\lambda z)\, k_\alpha(|x|,|y|,z)\,\sigma_{|x|,|y|,z}\, z^{2\alpha+1}dz\\ &= \frac{1}{2}\int_{\mathbb{R}} j_\alpha(\lambda z)\, k_\alpha(|x|,|y|,|z|)\,\sigma_{x,y,z}\,|z|^{2\alpha+1}dz.\end{aligned}$$

Combining these gives

$$\Psi_\lambda^\alpha(x)\Psi_\lambda^\alpha(y) = \frac{1}{2}\int_{\mathbb{R}} \Psi_\lambda^\alpha(z)\, k_\alpha(|x|,|y|,|z|)(1-\sigma_{x,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x})|z|^{2\alpha+1}dz.$$

As $\Psi_\lambda^\alpha(0)=1$ for all λ, formula (2.1) is thus proved in case $\lambda>0$. For arbitrary $\lambda\in\mathbb{C}$, it is obtained by analytic continuation: The mapping $(\lambda,z)\mapsto\Psi_\lambda^\alpha(z)$ is continuous on $\mathbb{C}\times\mathbb{R}$ and bounded on $U\times\operatorname{supp}\mu_{x,y}^\alpha$ for each bounded $U\subset\mathbb{C}$ and $x,y\in\mathbb{R}$. For fixed $z\in\mathbb{R}$, the function $\lambda\mapsto\Psi_\lambda^\alpha(z)$ is holomorphic on $\mathbb{C}$, and so both sides in (2.1) are holomorphic on $\mathbb{C}$ as functions of λ.

(2) The statement on the support of $\mu_{x,y}^\alpha$ is clear, because $z\mapsto\varrho(x,y,z)$ does not vanish on any open subset of $\mathbb{R}$. Furthermore, if $x,y\neq 0$ then $z\in\operatorname{supp}\mu_{x,y}^\alpha$ holds if and only if $|x|,|y|,|z|$ are the sides of a (perhaps degenerated) triangle. So if $z\in\operatorname{supp}\mu_{x,y}^\alpha$, then $|\sigma_{x,y,z}|\le 1, |\sigma_{z,x,y}|\le 1$ and $|\sigma_{z,y,x}|\le 1$. This yields the assertion about $\|\mu_{x,y}^\alpha\|$.

3. Bessel-type signed hypergroups on $\mathbb{R}$

The product formula of the modified Bessel functions gives rise to signed hypergroup structures on $\mathbb{R}$ in a canonical way; this is stated in the following theorem. Furthermore, we shall derive some duality results. Throughout this section we assume that $\alpha>-\frac{1}{2}$.

3.1. Theorem. *Define $\omega_\alpha:\mathbb{R}\times\mathbb{R}\to M_{\mathbb{R}}^b(\mathbb{R})$ by $\omega_\alpha(x,y):=\mu_{x,y}^\alpha$ and put*

$$dm_\alpha(x) := \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}|x|^{2\alpha+1}dx,\quad x\in\mathbb{R}.$$

Then $X_\alpha:=(\mathbb{R},m_\alpha,\omega_\alpha)$ is a commutative signed hypergroup with neutral element 0 and involution $x\mapsto -x$.

For the proof we need two preparatory lemmata:

3.2. Lemma. *For $f\in C_b(\mathbb{R})$ write $f=f_e+f_o$ with $f_e\in C_b(\mathbb{R})$ even, $f_o\in C_b(\mathbb{R})$ odd. Then*

$$\begin{aligned}\int_{\mathbb{R}} f\,d\mu_{x,y}^\alpha &= M_\alpha\int_0^\pi f_e(\sqrt{x^2+y^2-2|xy|\cos\theta})\,h^e(x,y,\theta)\sin^{2\alpha}\theta\,d\theta\\ &\quad + M_\alpha\int_0^\pi f_o(\sqrt{x^2+y^2-2|xy|\cos\theta})\,h^o(x,y,\theta)\sin^{2\alpha}\theta\,d\theta\end{aligned}\tag{3.1}$$

for all $x,y\in\mathbb{R}$, where

$$h^e(x,y,\theta) = 1-sgn(xy)\cos\theta,$$

$$h^o(x,y,\theta) = \begin{cases}\dfrac{(x+y)(1-sgn(xy)\cos\theta)}{\sqrt{x^2+y^2-2|xy|\cos\theta}} & \textit{if } (x,y)\neq(0,0),\\ 0 & \textit{otherwise.}\end{cases}$$

Proof. In case $xy=0$ formula (3.1) is obvious. (Note that $f_e(|x|)=f_e(x)$, $f_o(|x|)sgn(x)=f_o(x)$, $f_o(0)=0$, and that $\int_0^\pi\sin^{2\alpha}\theta\,d\theta = M_\alpha^{-1}$.) Now suppose $xy\neq 0$. First note that

$$\begin{aligned}\int_{\mathbb{R}} f\,d\mu_{x,y}^\alpha &= \int_0^\infty f_e(z)\,k_\alpha(|x|,|y|,z)(1-\sigma_{x,y,z})\,z^{2\alpha+1}dz\\ &\quad+\int_0^\infty f_o(z)\,k_\alpha(|x|,|y|,z)(\sigma_{z,x,y}+\sigma_{z,y,x})\,z^{2\alpha+1}dz.\end{aligned}$$

For $z\in supp\,\mu_{x,y}^\alpha$ we may substitute $\cos\theta:=\dfrac{x^2+y^2-z^2}{2|xy|}$ with $\theta\in[0,\pi]$. A short calculation then yields that

$$1-\sigma_{x,y,z} = h^e(x,y,\theta),\quad \sigma_{z,x,y}+\sigma_{z,y,x} = h^o(x,y,\theta),$$

from which the assertion follows.

3.3. Lemma. *Suppose $\mu\in M^b(\mathbb{R})$ with $\int_{\mathbb{R}}\Psi_\lambda^\alpha(x)\,d\mu(x)=0$ for all $\lambda\in\mathbb{R}$. Then $\mu=0$.*

Proof. As the subspace $M_0:=\{\mu\in M^b(\mathbb{R}):\mu(\{0\})=0\}$ is $\sigma(M^b(\mathbb{R}),C_b(\mathbb{R}))$-dense in $M^b(\mathbb{R})$ we may suppose that $\mu\in M_0$. Further, we have the identities

$$\Psi_\lambda^\alpha(x)+\Psi_{-\lambda}^\alpha(x) = 2\varphi_\lambda^\alpha(x),\quad \Psi_\lambda^\alpha(x)-\Psi_{-\lambda}^\alpha(x) = 2iC_\alpha\,\lambda x\,\varphi_\lambda^{\alpha+1}(x).$$

By assumption of the lemma, they lead to the following two relations:

$$\int_0^\infty \varphi_\lambda^\alpha(x)\, d(\mu+\mu^-)(x) = \int_{\mathbb{R}} \varphi_\lambda^\alpha(x)\, d\mu(x) = 0 \text{ for all } \lambda \in \mathbb{R},$$

$$\int_0^\infty \varphi_\lambda^{\alpha+1}(x)\, x\, d(\mu-\mu^-)(x) = \int_{\mathbb{R}} \varphi_\lambda^{\alpha+1}(x)\, x\, d\mu(x) = 0 \text{ for all } \lambda \in \mathbb{R}\setminus\{0\},$$

and a continuity argument shows that the second relation also holds for $\lambda = 0$. So by injectivity of the Fourier transform on Bessel-Kingman hypergroups it follows that $\mu+\mu^- = 0$ and $x(\mu-\mu^-) = 0$, which implies $\mu = 0$.

Proof of Theorem 3.1. It is clear that the $\mu_{x,y}^\alpha$ are real with $\mu_{x,y}^\alpha = \mu_{y,x}^\alpha$. So the first thing to show is that for $f \in C_b(\mathbb{R})$ the mapping $(x,y) \mapsto \mu_{x,y}^\alpha(f)$ is continuous on $\mathbb{R}^2$. This yields τ_*-continuity of $(x,y) \mapsto \mu_{x,y}^\alpha$, and also that $T^x f \in C_b(\mathbb{R})$ for $f \in C_b(\mathbb{R})$, by norm-boundedness of the $\mu_{x,y}^\alpha$. The basis of our proof is (3.1), and we may restrict to even and odd test functions.

(a) If $f \in C_b(\mathbb{R})$ is even, then (3.1) becomes

$$\int_{\mathbb{R}} f\, d\mu_{x,y}^\alpha = M_\alpha \int_0^\pi f(\sqrt{x^2+y^2-2|xy|\cos\theta}) \sin^{2\alpha}\theta\, d\theta$$
$$- sgn(xy)\cdot M_\alpha \int_0^\pi f(\sqrt{x^2+y^2-2|xy|\cos\theta}) \cos\theta\, \sin^{2\alpha}\theta\, d\theta.$$

Both integrals are continuous functions in $(x,y) \in \mathbb{R}^2$, and for $xy = 0$, where $sgn(xy)$ is discontinuous, the latter takes the value 0. But as $sgn(xy)$ is bounded, it follows that $(x,y) \mapsto \mu_{x,y}^\alpha(f)$ is continuous on $\mathbb{R}^2$.

(b) For odd f the statement is clear other than along the lines $|x| = |y|$ and $xy = 0$. If $|x| = |y|$, then the weight h^o defined in Lemma 3.2 is continuous in (x,y,θ) unless $\sqrt{x^2+y^2-2|xy|\cos\theta} = 0$. But as $f(0) = 0$, and as $|h^o|$ is bounded by 2 according to the boundedness properties of σ, the product $f(\sqrt{x^2+y^2-2|xy|\cos\theta})\cdot h^o(x,y,\theta)$ becomes continuous in each (x,y,θ) with $|x| = |y|$. Hence $(x,y) \mapsto \mu_{x,y}^\alpha(f)$ is also continuous on the diagonals. To handle the remaining case $xy = 0, |x| \neq |y|$, consider the function

$$(x,y,\theta) \mapsto (x+y)\frac{f(\sqrt{x^2+y^2-2|xy|\cos\theta})}{\sqrt{x^2+y^2-2|xy|\cos\theta}}.$$

It is continuous in every (x,y,θ) with $|x| \neq |y|$. So by the same argument as used in (a), it is now easily seen that $(x,y) \mapsto \mu_{x,y}^\alpha(f)$ is continuous in all (x,y) with $xy = 0, |x| \neq |y|$.

Summing up, we have shown that $(x,y) \mapsto \mu_{x,y}^\alpha(f)$ is continuous on $\mathbb{R}^2$ for each $f \in C_b(\mathbb{R})$. Concerning axiom (A1), it remains to note that for $f \in C_c(\mathbb{R})$ with $\operatorname{supp} f \subseteq [-M,M]$ the support of $T^x f$ is contained in $[-|x|-M,\, M+|x|]$. To prove associativity of the canonical continuation $*$ of ω_α it suffices to consider point measures. But by definition of ω_α,

$$(\varepsilon_x * (\varepsilon_y * \varepsilon_z))(\Psi_\lambda^\alpha) = \Psi_\lambda^\alpha(x)\Psi_\lambda^\alpha(y)\Psi_\lambda^\alpha(z) = ((\varepsilon_x * \varepsilon_y) * \varepsilon_z)(\Psi_\lambda^\alpha)$$

for all $x,y,z \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. Lemma 3.3 now yields the assertion. (A4) is obvious; finally, (A5) and (A6) are immediate consequences of the symmetry relations $\mathcal{K}_\alpha(-x,-y,-z) = \mathcal{K}_\alpha(x,y,z)$ and $\mathcal{K}_\alpha(x,y,z) = \mathcal{K}_\alpha(-x,z,y)$ respectively.

3.4. Remark. The mapping $(x,y) \mapsto \operatorname{supp}\mu_{x,y}^\alpha$ is not continuous with respect to the Michael topology on the space $\mathcal{C}(\mathbb{R})$ of compact subsets of $\mathbb{R}$. This is because $\operatorname{supp}\mu_{x,y}^\alpha \cap \{\xi \in \mathbb{R} : \xi < 0\} \neq \emptyset$ for all $x,y > 0$, while $\operatorname{supp}\mu_{x,0}^\alpha = \{x\}$.

The characterization of multiplicative functions on our Bessel-type signed hypergroups can be accomplished by a reduction to the well-known corresponding results for Bessel-Kingman hypergroups. It leads to results which extend the group case in a natural way.

In addition to the character spaces $\mathcal{X}_b(X_\alpha)$ and $\widehat{X_\alpha}$, let $\mathcal{X}(X_\alpha)$ and X_α^* denote the spaces of continuous multiplicative functions and semi-characters respectively:

$$\mathcal{X}(X_\alpha) := \{\varphi \in C(X) : \varphi \not\equiv 0,\ \mu_{x,y}^\alpha(\varphi) = \varphi(x)\varphi(y) \text{ for all } x,y \in \mathbb{R}\},$$
$$X_\alpha^* := \{\varphi \in \mathcal{X}(X_\alpha) : \varphi(-x) = \overline{\varphi(x)} \text{ for all } x \in \mathbb{R}\}.$$

3.5. Theorem.

(a) $\mathcal{X}(X_\alpha) = \{\Psi_\lambda^\alpha : \lambda \in \mathbb{C}\}$.

(b) $X_\alpha^* = \mathcal{X}_b(X_\alpha) = \widehat{X_\alpha} = \{\Psi_\lambda^\alpha : \lambda \in \mathbb{R}\}$,

and the mapping $\mathbb{R} \to \widehat{X_\alpha}$, $\lambda \mapsto \Psi_\lambda^\alpha$, *is a homeomorphism.*

Proof. By definition of the convolution on X_α it is clear that each Ψ_λ^α, $\lambda \in \mathbb{C}$ belongs to $\mathcal{X}(X_\alpha)$. Now take $\varphi \in \mathcal{X}(X_\alpha)$ and define $F(x) := \frac{1}{2}(\varphi(x)+\varphi(-x))$. A short computation shows that for $x,y > 0$,

$$F(x)F(y) = \frac{1}{2}\int_{\mathbb{R}} \varphi(z)\, k_\alpha(x,y,|z|)|z|^{2\alpha+1}dz = \int_0^\infty F(z)\, k_\alpha(x,y,z)\, z^{2\alpha+1}dz.$$

Thus F is a multiplicative function on the Bessel-Kingman hypergroup H_α. According to Zeuner [13], it follows that $F = \varphi_\lambda^\alpha$ for some $\lambda \in \mathbb{C}$.

Now set $G(x) := \frac{1}{2}(\varphi(x) - \varphi(-x))$. Then for $x, y > 0$,

$$G(x)G(y) = -\frac{1}{2}\int_{\mathbb{R}} \varphi(z)\, k_\alpha(x,y,|z|)\sigma_{x,y,z}|z|^{2\alpha+1}dz$$
$$= -\int_0^\infty \varphi_\lambda^\alpha(z)\, k_\alpha(x,y,z)\sigma_{x,y,z}\, z^{2\alpha+1}dz.$$

But at the same time, this last expression is equal to $\Phi_\lambda^\alpha(x)\Phi_\lambda^\alpha(y)$ where $\Phi_\lambda^\alpha(x) := \frac{1}{2}\big(\Psi_\lambda^\alpha(x) - \Psi_\lambda^\alpha(-x)\big)$. Hence we have

$$G(x)G(y) = \Phi_\lambda^\alpha(x)\Phi_\lambda^\alpha(y) \quad \text{for all } x, y > 0,$$

from which it follows that $G = \Phi_\lambda^\alpha$ or $G = -\Phi_\lambda^\alpha = \Phi_{-\lambda}^\alpha$. We have shown that $\varphi = \Psi_\lambda^\alpha$ or $\varphi = \Psi_{-\lambda}^\alpha$, and thus the proof of (a) is complete.

For the proof of (b), note that by Corollary 2.2 it is clear that $\{\Psi_\lambda^\alpha : \lambda \in \mathbb{R}\} \subseteq \widehat{X_\alpha}$. On the other hand, if $\Psi_\lambda^\alpha \in X_\alpha^*$ then

$$\varphi_\lambda^\alpha(x) - iC_\alpha\, \lambda x\, \varphi_\lambda^{\alpha+1}(x) = \overline{\varphi_\lambda^\alpha(x)} - iC_\alpha x\, \overline{\lambda\varphi_\lambda^{\alpha+1}(x)}$$

for all $x \in \mathbb{R}$, and by comparison of even and odd parts in x we obtain that both $\varphi_\lambda^\alpha(x)$ and $\lambda\varphi_\lambda^{\alpha+1}(x)$ must be real for all $x \in \mathbb{R}$. But this is only possible if λ is real. The final statement is standard; we omit its proof.

3.6. Proposition. *With the identification of $\widehat{X_\alpha}$ and $\mathbb{R}$ as above the Plancherel measure of the signed hypergroup X_α is given by*

$$d\pi_\alpha(\lambda) = dm_\alpha(\lambda) = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)}|\lambda|^{2\alpha+1}d\lambda.$$

Proof. Again we reduce our discussion to the associated Bessel-Kingman hypergroup H_α. For this, denote the Fourier transform on H_α by $\mathcal{F}$, and let

$$d\widetilde{\pi}_\alpha(\lambda) = d\widetilde{m}_\alpha(\lambda) = \frac{1}{2^\alpha\Gamma(\alpha+1)}\lambda^{2\alpha+1}d\lambda \quad (\lambda \geq 0)$$

denote the Plancherel- and Haar meaure on H_α. Then, for all even functions $f \in C_c(\mathbb{R})$ and $\lambda \in \mathbb{R}$,

$$\widehat{f}(\lambda) = \int_{\mathbb{R}} \overline{\Psi_\lambda^\alpha(x)}\, f(x)dm_\alpha(x) = 2\int_0^\infty \varphi_\lambda^\alpha(x)f(x)dm_\alpha(x) = \mathcal{F}(f)(|\lambda|). \quad (3.2)$$

Thus the Plancherel formula for H_α yields that for even $f \in C_c(\mathbb{R})$,

$$\int_{\mathbb{R}} |\widehat{f}|^2\, dm_\alpha = \int_0^\infty |\mathcal{F}(f)|^2 d\widetilde{\pi}_\alpha = \int_0^\infty |f|^2 d\widetilde{m}_\alpha = \int_{\mathbb{R}} |f|^2 dm_\alpha.$$

Hence, by the Plancherel formula for the Bessel-type signed hypergroup X_α, we find that

$$\int_{\mathbb{R}} |\widehat{f}|^2 d\pi_\alpha = \int_{\mathbb{R}} |\widehat{f}|^2 dm_\alpha \quad \text{for all even } f \in C_c(\mathbb{R}). \quad (3.3)$$

To complete the proof we have to show that (3.3) implies $\pi_\alpha = m_\alpha$. For this first notice that just as for usual hypergroups, the Plancherel measure of X_α must be symmetric, i.e. $d\pi_\alpha(-\lambda) = d\pi_\alpha(\lambda)$. Therefore it is sufficient to prove $\int_{\mathbb{R}} g\, d\pi_\alpha = \int_{\mathbb{R}} g\, dm_\alpha$ for all $g \in C_0^{e,+}(\mathbb{R}) := \{h \in C_0(\mathbb{R}) : h \geq 0 \text{ and even}\}$. In view of (3.3) this follows if we have shown that $\{|\widehat{f}|^2 : f \in C_c(\mathbb{R}) \text{ even}\}$ is $\|.\|_\infty$-dense in $C_0^{e,+}(\mathbb{R})$. So take $g \in C_0^{e,+}(\mathbb{R})$ and $\varepsilon > 0$. Then there exists $f \in C_c(\mathbb{R})$ with $\|\widehat{f} - \sqrt{g}\|_\infty < \varepsilon$. Now write $f = f_e + f_o$ with f_e even and f_o odd. Then by (3.2), $\widehat{f_e}$ is even, and a similar calculation shows that $\widehat{f_o}$ is odd. It follows that

$$\|\widehat{f_e} - \sqrt{g}\|_\infty \leq \|(\widehat{f_e} - \sqrt{g}) + \widehat{f_o}\|_\infty = \|\widehat{f} - \sqrt{g}\|_\infty < \varepsilon,$$

which immediately takes care of the claim.

3.7. Remarks.

1. Let (X, m, ω) be a commutative signed hypergroup with dual $\widehat{X}$ and Plancherel measure π. Assume that for all $\alpha, \beta \in \widehat{X}$, there exists a measure $\varrho_{\alpha,\beta} \in M_{\mathbb{R}}^b(\widehat{X})$ such that

$$\alpha(x)\beta(x) = \int_{\widehat{X}} \gamma(x)\, d\varrho_{\alpha,\beta}(\gamma) \quad \text{for all } x \in X.$$

Then we say that $\widehat{X}$ carries a dual signed hypergroup structure if $(\widehat{X}, \pi, \Omega)$, with $\Omega(\alpha,\beta) := \varrho_{\alpha,\beta}$, is a commutative signed hypergroup. In our case,

$$\Psi_\lambda^\alpha(x) = \Psi_x^\alpha(\lambda) \quad \text{for all } \lambda, x \in \mathbb{R}.$$

Together with Proposition 3.6 this shows that $\widehat{X}_\alpha$ carries a commutative signed hypergroup structure which is isomorphic to X_α in the obvious way.

2. The Fourier transform on the Bessel-type signed hypergroup X_α is exactly the Dunkl transform with parameter $\alpha + \frac{1}{2}$, associated with the reflection group $\mathbb{Z}_2$ on $\mathbb{R}$. This is a special case of a generalized Hankel transform introduced by Dunkl [3, 4] in connection with finite reflection groups on $\mathbb{R}^N$, $N \geq 1$. The Dunkl transform involves an integral kernel, which in the special case of $\mathbb{Z}_2$ and $\mathbb{R}$ is just given by

$$K_\alpha(x, -iy) = \Gamma(\alpha+\tfrac{1}{2})\Big(\frac{|xy|}{2}\Big)^{-\alpha+\frac{1}{2}}\big(J_{\alpha-\frac{1}{2}}(|xy|) - i\, sgn(xy) J_{\alpha+\frac{1}{2}}(|xy|)\big) = \Psi_{-y}^{\alpha-\frac{1}{2}}(x),$$

with parameter $\alpha > 0$ (see Dunkl [4], Section 4). For a further study of the Dunkl transform we also refer to de Jeu [5].

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