Positivity of Dunkl's Intertwining Operator via the Trigonometric Setting

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1 Introduction

In [15], it was proven that Dunkl's intertwining operator between the rational Dunkl operators for a fixed finite reflection group and nonnegative multiplicity function is positive. As a consequence, we obtained an abstract Harish-Chandra-type integral representation for the Dunkl kernel, the image of the usual exponential kernel under the intertwiner. The proof was based on methods from the theory of operator semigroups and a rank-one reduction.

In the present paper, we give a new, completely different proof of these results under the only additional assumption that the underlying reflection group has to be crystallographic. In contrast to the proof of [15], where precise information on the supports of the representing measures could only be obtained by going back to estimates of the kernel from [5], this information is now directly obtained. Our new approach relies first on an asymptotic relationship between the Opdam-Cherednik kernel and the Dunkl kernel as recently observed by de Jeu [6], and second on positivity results of Sahi [17] for the Heckman-Opdam polynomials and their nonsymmetric counterparts.

2 Preliminaries

2.1 Basic notation

Let \( a \) be a finite-dimensional Euclidean vector space with inner product \( \langle \cdot, \cdot \rangle \). We use the same notation for the bilinear extension of \( \langle \cdot, \cdot \rangle \) to the complexification \( a_C \) of \( a \), and we...
identify a with its dual $a^* = \text{Hom}(a, \mathbb{R})$ via the given inner product. For $\alpha \in a \setminus \{0\}$, we write 
$\alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle$ and $\sigma_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$ for the orthogonal reflection in the hyperplane perpendicular to $\alpha$. We consider a crystallographic root system $R$ in $a$, that is, $R$ is a finite subset of $a \setminus \{0\}$ which spans $a$ and satisfies $\sigma_\alpha(R) = R$ and $\langle \alpha, \beta \rangle \in \mathbb{Z}$ for all $\alpha, \beta \in R$. We also assume that $R$ is indecomposable and reduced, that is, $R \cap R\alpha = \{\pm \alpha\}$ for all $\alpha \in R$. Let $W$ be the finite reflection group generated by the $\sigma_\alpha$, $\alpha \in R$. We will fix a positive subsystem $R_+$ of $R$ as well as a nonnegative multiplicity function $k = (k_\alpha)_{\alpha \in R}$, satisfying $k_\alpha = k_\beta$ if $\alpha$ and $\beta$ are in the same $W$-orbit.

2.2 Rational Dunkl operators and Dunkl's intertwiner

References for this section are [7, 8, 12, 15]. Let $P = \mathbb{C}[a]$ denote the vector space of complex polynomial functions on $a$, and $P_n \subset P$ the subspace of polynomials which are homogeneous of degree $n \in \mathbb{Z}_+$. The rational Dunkl operators on $a$ associated with $R$ and fixed multiplicity $k \geq 0$ are given by

$$T_\xi = T_\xi(k) = \partial_\xi + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{\langle \alpha, \alpha \rangle} (1 - \sigma_\alpha), \quad \xi \in a. \quad (2.1)$$

These operators commute and map $P$ onto itself. Moreover, there exists a unique linear isomorphism $V = V_k$ on $P$ with $V(1) = 1$, $V(P_n) = P_n$, and $T_\xi V = V\partial_\xi$ for all $\xi \in a$. According to [8], the intertwining operator $V$ can be extended to larger classes of functions as follows: for $r > 0$, let $K_r := \{x \in a : |x| \leq r\}$ denote the ball of radius $r$ and define

$$A_r := \left\{ f : K_r \to \mathbb{C}, \quad f = \sum_{n=0}^{\infty} f_n \text{ with } f_n \in P_n, \quad \|f\|_{A_r} := \sum_{n=0}^{\infty} \|f_n\|_{K_r} < \infty \right\}, \quad (2.2)$$

where $\|f\|_{K_r} := \sup_{x \in K_r} |f_n(x)|$. The space $A_r$ is a Banach space with norm $\|\cdot\|_{A_r}$ (in fact, a commutative Banach algebra). $V$ extends to a continuous linear operator on $A_r$ by $V(\sum_{n=0}^{\infty} f_n) := \sum_{n=0}^{\infty} Vf_n$. The Dunkl kernel $\text{Exp}_W$ is defined by

$$\text{Exp}_W(\cdot, z) := V(e^{\langle \cdot, z \rangle}), \quad z \in a_C. \quad (2.3)$$

It extends to a holomorphic function on $a_C \times a_C$ which is symmetric in its arguments. For $\lambda \in a_C$, $\text{Exp}_W(\lambda, \cdot)$ is the unique holomorphic solution of the joint eigenvalue problem

$$T_\xi f = \langle \lambda, \xi \rangle f \quad \forall \xi \in a, \quad f(0) = 1. \quad (2.4)$$

For $x \in a$, we denote by $C(x)$ the closure of the convex hull of the $W$-orbit $Wx$ of $x$ in $a$. Moreover, for a locally compact Hausdorff space $X$, we write $M^1(X)$ for the set of probability measures on the Borel $\sigma$-algebra of $X$. In [15], the following is proven.
Theorem 2.1. For each $x \in a$, there exists a unique probability measure $\mu_x \in M^1(a)$ such that

$$Vf(x) = \int_a f(\xi) d\mu_x(\xi) \quad \forall f \in A_{|x|}. \quad (2.5)$$

The support of $\mu_x$ is contained in $C(x)$. □

As a consequence,

$$\text{Exp}_W(x, z) = \int_a e^{\langle \xi, z \rangle} d\mu_x(\xi) \quad \forall z \in a_C. \quad (2.6)$$

In [15], the proof of the inclusion $\text{supp} \mu_x \subseteq C(x)$ requires the exponential bounds on $\text{Exp}_W$ from [5], which are by far not straightforward. As is well known, the $W$-invariant parts of the rational and trigonometric Dunkl theories are, for certain discrete sets of multiplicities, realized within the classical Harish-Chandra theory for semisimple symmetric spaces. In particular, for such $k$, the generalized Bessel functions

$$J_W(\cdot, z) = \frac{1}{|W|} \sum_{w \in W} \text{Exp}_W(\cdot, w^{-1} z), \quad z \in a_C, \quad (2.7)$$

can be identified with the spherical functions of an underlying Cartan motion group; for details see, for example, [6]. In this case, their integral representation according to (2.6) is a special case of a (Euclidean type) Harish-Chandra integral, and the inclusion $\text{supp} \mu_x \subseteq C(x)$ follows from Kostant’s convexity theorem [11, Proposition IV.4.8 and Theorem IV.10.2].

2.3 Cherednik operators and the Opdam-Cherednik kernel

The basic concepts of this as well as the following section are due to Opdam [13] (see also [14, part I]), Heckman (see [10, part I]), and Cherednik [4]. Let

$$P := \{ \lambda \in a : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \forall \alpha \in R \} \quad (2.8)$$

denote the weight lattice associated with the root system $R$. For $\lambda \in a_C$, we define the exponential $e^\lambda$ on $a_C$ by $e^\lambda(z) := e^{\langle \lambda, z \rangle}$ and denote by $\mathcal{T}$ the $C$-span of $\{ e^\lambda, \lambda \in P \}$. This is the algebra of trigonometric polynomials on $a_C$ with respect to $R$. The Cherednik operator in direction $\xi \in a$ is defined by

$$D_\xi = D_\xi(k) = \partial_{\xi} + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1}{1 - e^{-\alpha}(1 - \sigma_\alpha)} - \langle \rho(k), \xi \rangle, \quad (2.9)$$
where \( p(k) = (1/2) \sum_{\alpha \in R_+} k_\alpha \alpha \). Each \( D_\xi \) maps \( T \) onto itself and (for fixed \( k \)) the operators \( D_\xi \) commute. Notice that in contrast to the rational \( T_\xi \), they depend on the particular choice of \( R_+ \). For each \( \lambda \in a_C \), there exists a unique holomorphic function \( G(\lambda, \cdot) \) in a tubular neighborhood of \( a \) which satisfies

\[
D_\xi G(\lambda, \cdot) = (\lambda, \xi) G(\lambda, \cdot) \quad \forall \xi \in a, \quad G(\lambda, 0) = 1
\]

(see [14, Corollary I.7.6]). \( G \) is called the Opdam-Cherednik kernel. It is in fact (as a function of both arguments) holomorphic in a suitable tubular neighborhood of \( a_C \times a \) [13, Theorem 3.15]. The rational Dunkl operators can be considered a scaling limit of the Cherednik operators, and this implies limit relations between the kernels \( \text{Exp}_W \) and \( G \).

We will need the following variant of [6, Theorem 4.12].

**Proposition 2.2.** Let \( \delta > 0 \) be a constant, \( K, L \subset a_C \) compact sets, and \( h : (0, \delta) \times L \to a_C \) a continuous mapping such that \( \lim_{\epsilon \to 0} \epsilon h(\epsilon, \lambda) = \lambda \) uniformly on \( L \). Then

\[
\lim_{\epsilon \to 0} G(h(\epsilon, \lambda), \epsilon z) = \text{Exp}_W(\lambda, z)
\]

uniformly for \( (\lambda, z) \in L \times K \).

The proof is the same as for [6, Theorem 4.12], with \( \lambda/\epsilon \) replaced by \( h(\epsilon, \lambda) \). We mention that for integral \( k \), such a limit transition has first been carried out in [2] by use of shift operator methods.

### 2.4 A scaling limit for nonsymmetric Heckman-Opdam polynomials

The definition of these polynomials involves a suitable partial order on \( P \); we refer to the one used in [14]. Let

\[
P_+ := \{ \lambda \in P : (\lambda, \alpha^\vee) \geq 0 \forall \alpha \in R_+ \}
\]

(2.12)
denote the set of dominant weights associated with \( R_+ \), and \( \lambda_+ \) the unique dominant weight in the orbit \( W\lambda \). One defines \( \lambda \prec \nu \) if either \( \lambda_+ \prec \nu_+ \) in dominance ordering (i.e., \( \nu_+ - \lambda_+ \in Q_+ \), the \( \mathbb{Z}_+ \)-span of \( R_+ \)), or if \( \lambda_+ = \nu_+ \) and \( \nu < \lambda \) (in dominance ordering). Further, \( \lambda \preceq \nu \) means \( \lambda = \nu \) or \( \lambda < \nu \). The nonsymmetric Heckman-Opdam polynomials \( \{ E_\lambda : \lambda \in P \} \subset T \) associated with \( R_+ \) and \( k \) are uniquely characterized by the conditions

\[
E_\lambda = \sum_{\nu \preceq \lambda} a_{\lambda, \nu} e^\nu \quad \text{with} \quad a_{\lambda, \lambda} = 1,
\]

(2.13)

\[
D_\xi E_\lambda = (\widetilde{\lambda}, \xi) E_\lambda \quad \forall \xi \in a,
\]

(2.14)
with the shifted spectral variable \( \tilde{\lambda} = \lambda + \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} k_\alpha \epsilon(\langle \lambda, \alpha^\vee \rangle) \alpha \). Here \( \epsilon : \mathbb{R} \to \{ \pm 1 \} \) is defined by \( \epsilon(x) = 1 \) for \( x > 0 \) and \( \epsilon(x) = -1 \) for \( x \leq 0 \). For details, see [13] and [14, Section I.2.3].

On the other hand, we know (cf. (2.10)) that \( G(\tilde{\lambda}, \cdot) \) is the up-to-a-constant-factor unique holomorphic solution of (2.14). Hence

\[
E_\lambda = c_\lambda \cdot G(\tilde{\lambda}, \cdot),
\]

(2.15)

with a constant \( c_\lambda = E_\lambda(0) > 0 \). The precise value of \( c_\lambda \) is given in [14, Theorem 4.7].

**Corollary 2.3.** For \( \lambda \in \mathbb{P} \) and \( z \in a_C \),

\[
\text{Exp}_W(\lambda, z) = \lim_{n \to \infty} \frac{1}{c_{n\lambda}} E_{n\lambda} \left( \frac{z}{n} \right).
\]

(2.16)

The convergence is locally uniform with respect to \( z \).

**Proof.** Fix \( \lambda \in \mathbb{P} \) and observe that \( \tilde{n}\lambda = n\lambda + \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} k_\alpha \epsilon(\langle \lambda, \alpha^\vee \rangle) \alpha \) for all \( n \in \mathbb{N} \). Thus by Proposition 2.2 and identity (2.15), we have, locally uniformly for \( z \in a_C \),

\[
\text{Exp}_W(\lambda, z) = \lim_{n \to \infty} G(\tilde{n}\lambda, \frac{z}{n}) = \lim_{n \to \infty} \frac{1}{c_{n\lambda}} E_{n\lambda} \left( \frac{z}{n} \right).
\]

(2.17)

**Remark 2.4.** Similar results hold for the symmetric Heckman-Opdam polynomials

\[
P_\lambda(z) = \frac{|W\lambda|}{|W|} \sum_{w \in W} E_\lambda \left( w^{-1}z \right), \quad \lambda \in \mathbb{P}^+.
\]

(2.18)

They are \( W \)-invariant and related with the multivariable hypergeometric function

\[
F(\lambda, z) = \frac{1}{|W|} \sum_{w \in W} G(\lambda, w^{-1}z)
\]

(2.19)

via

\[
P_\lambda = c_\lambda^* \cdot F(\lambda + \rho, \cdot) \quad \forall \lambda \in \mathbb{P}^+
\]

(2.20)

with \( c_\lambda^* = |W\lambda| \cdot c_\lambda \) and \( \rho = \rho(k) = \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} k_\alpha \alpha \), (cf. [10, equation (4.4.10)]). This also follows from (2.15) because \( F \) is in fact \( W \)-invariant in both arguments and for \( \lambda \in \mathbb{P}^+ \), the shifted weight \( \tilde{\lambda} \) is contained in the \( W \)-orbit of \( \lambda + \rho \) [13, Proposition 2.10]. Further, **Corollary 2.3** implies that for \( \lambda \in \mathbb{P}^+ \) and \( z \in a_C \),

\[
J_W(\lambda, z) = \lim_{n \to \infty} F(n\lambda + \rho, \frac{z}{n}) = \lim_{n \to \infty} \frac{1}{c_{n\lambda}^*} P_{n\lambda} \left( \frac{z}{n} \right).
\]

(2.21)
For illustration, consider the rank-one case (type $A_1$) with $a = \mathbb{R}$ and $R_+ = \{2\alpha\}$, $\alpha = 1$. Fix $k = k_{2\alpha} \geq 0$. Then according to the example in [13, page 89f],

$$F(\lambda, z) = _2F_1 \left( a, b, c; \frac{1}{2} (1 - \cosh z) \right),$$

$$G(\lambda, z) = _2F_1 \left( a, b, c; \frac{1}{2} (1 - \cosh z) \right)$$

$$+ \frac{a}{2c} \sinh z \cdot _2F_1 \left( a + 1, b + 1, c + 1; \frac{1}{2} (1 - \cosh z) \right),$$

with $a = \lambda + k$, $b = -\lambda + k$, and $c = k + 1/2$. The weight lattice is $P = \mathbb{Z}$, and the associated Heckman-Opdam polynomials are given by

$$P_n(z) = c_n^* F(n + k, z) = c_n^* \cdot Q_n^k(\cosh z), \quad n = 0, 1, \ldots,$$

$$E_n(z) = c_n G(n, z) = c_n \left[ Q_{n-1}^k(\cosh z) + \frac{\bar{n} + k}{2k + 1} \cdot \sinh z Q_{n-1}^{k+1}(\cosh z) \right], \quad n \in \mathbb{Z},$$

with $\bar{n} = n + k$ for $n > 0$, $\bar{n} = n - k$ for $n \leq 0$, and the renormalized Gegenbauer polynomials

$$Q_n^k(x) = _2F_1 \left( n + 2k, -n, k + \frac{1}{2} \frac{1}{2} (1 - x) \right).$$

Relation (2.21) reduces to the classical limit

$$\lim_{n \to \infty} Q_n^k \left( \cos \frac{z}{n} \right) = j_{k-1/2}(z) \quad (z \in \mathbb{C})$$

for the modified Bessel functions

$$j_\alpha(z) = 2^\alpha \Gamma(\alpha + 1) \cdot \frac{I_\alpha(z)}{z^\alpha} = \Gamma(\alpha + 1) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \left( \frac{z}{2} \right)^{2n}}{n! \Gamma(n + \alpha + 1)},$$

see [1, Theorem 4.11.6]. It is clear from the explicit representation of the Gegenbauer polynomials in terms of Tchebycheff polynomials [1, equation (6.4.11)] that for $k \geq 0$, the expansion coefficients of $P_n$ with respect to the exponentials $z \mapsto e^{mz}$, $m \in \mathbb{Z}$, are all nonnegative. A closer inspection shows that the same holds for the nonsymmetric $E_n$. This is in fact a special case of a deep result for general Heckman-Opdam polynomials due to Sahi [17]: if the multiplicity function $k$ is nonnegative, then it follows from [17, Corollary 5.2 and Proposition 6.1] that the coefficients $a_{\lambda, \gamma}$ of $E_\lambda$ in (2.13) are all real and nonnegative. More precisely, if $\Pi_k := \mathbb{Z}_+[k_\alpha]$ denotes the set of polynomials in the parameters $k_\alpha$
with nonnegative integral coefficients, then for suitable \( d_\lambda \in \Pi_k \), all coefficients of \( d_\lambda E_\lambda \) are contained in \( \Pi_k \) as well. This positivity result is the key for our subsequent proof of Theorem 2.1.

3 New proof of Theorem 2.1

In contrast to our approach in [15], we first derive a positive integral representation for the Dunkl kernel. As before, \( R \) and \( k \geq 0 \) are fixed.

**Proposition 3.1.** For each \( x \in a \), there exists a unique probability measure \( \mu_x \in M^1(a) \) such that (2.6) holds. The support of \( \mu_x \) is contained in \( C(x) \).

Proof. It suffices to prove the existence of the representing measures as stated; their uniqueness is immediate from the injectivity of the (usual) Fourier-Stieltjes transform on \( M^1(a) \). Let \( \lambda \in P \). Then by Sahi’s positivity result mentioned above,

\[
G(\tilde{\lambda}, \cdot) = \frac{1}{c_\lambda} E_\lambda = \sum_{\nu \triangleleft \lambda} b_{\lambda, \nu} e^\nu,
\]

with coefficients \( b_{\lambda, \nu} \) satisfying

\[
0 \leq b_{\lambda, \nu} \leq 1, \quad \sum_{\nu \triangleleft \lambda} b_{\lambda, \nu} = 1.
\]

Now fix \( \lambda \in P \) and \( z \in a_C \). Then by Corollary 2.3,

\[
\text{Exp}_W(\lambda, z) = \lim_{n \to \infty} \frac{1}{c_{n\lambda}} E_{n\lambda} \left( \frac{z}{n} \right) = \lim_{n \to \infty} \sum_{\nu \triangleleft n\lambda} b_{n\lambda, \nu} e^{(\nu, z/n)}.
\]

Introducing the discrete probability measures

\[
\mu^n_\lambda := \sum_{\nu \triangleleft n\lambda} b_{n\lambda, \nu} \delta_{\nu/n} \in M^1(a),
\]

(where \( \delta_x \) denotes the point measure in \( x \in a \)), we may write the above relation in the form

\[
\text{Exp}_W(\lambda, z) = \lim_{n \to \infty} \int_a e^{(\xi, z)} d\mu^n_\lambda(\xi).
\]

The following lemma shows that the support of \( \mu^n_\lambda \) is contained in \( C(\lambda) \).

**Lemma 3.2.** Let \( \lambda, \nu \in P \) with \( \nu \triangleleft \lambda \). Then \( \nu \in C(\lambda) \).
Proof. Let $C := \{x \in a : \langle \alpha, x \rangle \geq 0 \forall \alpha \in R_+ \}$ be the closed Weyl chamber associated with $R_+$ and

$$C^* := \{y \in a : \langle y, x \rangle \geq 0 \forall x \in C \}$$

its closed dual cone. Notice that $Q_+ \subset C^*$. Therefore, $\nu \preceq \lambda$ implies that $\lambda_+ - \nu_+ \in C^*$. We employ the following characterization of $C(x)$ for $x \in C$ [11, Lemma IV.8.3]:

$$C(x) = \bigcup_{w \in W} w(C \cap (x - C^*))$$

This shows that $\nu \in C(\lambda)$ if and only if $\nu_+ \in \lambda_+ - C^*$, which yields the statement. ■

We now continue with the proof of Proposition 3.1. Fix $\lambda \in P$. By the preceding result, we may consider the $\mu_n^\lambda$ as probability measures on the compact set $C(\lambda)$. According to Prohorov’s theorem (see, e.g., [3]), the set $\{\mu_n^\lambda, n \in \mathbb{Z}_+ \}$ is relatively compact. Passing to a subsequence if necessary, we may therefore assume that there exists a measure $\mu_\lambda \in M_1(a)$ which is supported in $C(\lambda)$ and such that $\mu_n^\lambda \to \mu_\lambda$ weakly as $n \to \infty$. Thus in view of (3.5),

$$\text{Exp}_W(\lambda, z) = \int_a e^{\langle \xi, z \rangle} d\mu_\lambda(\xi) \quad \forall z \in a_C.$$  

(3.8)

In order to extend this representation to arbitrary arguments $x \in a$ instead of $\lambda \in P$, observe first that for $r \in \mathbb{Q}$,

$$\text{Exp}_W(r\lambda, z) = \text{Exp}_W(\lambda, rz) = \int_a e^{r\langle \xi, z \rangle} d\mu_\lambda(\xi).$$  

(3.9)

Defining $\mu_{r\lambda} \in M_1(a)$ as the image measure of $\mu_\lambda$ under the dilation $\xi \mapsto r\xi$ on $a$, we therefore obtain (2.6) for all $x \in Q.P = \{r\lambda : r \in \mathbb{Q}, \lambda \in P \}$. The set $Q.P$ is obviously dense in $a$. For arbitrary $x \in a$, choose an approximating sequence $\{x_n, n \in \mathbb{Z}_+ \} \subset Q.P$ with $\lim_{n \to \infty} x_n = x$. Using Prohorov’s theorem once more, we obtain, after passing to a subsequence, that $\mu_{x_n} \to \mu_x$ weakly for some $\mu_x \in M_1(a)$. The support of $\mu_x$ can be confined to an arbitrarily small neighbourhood of $C(x)$, and must therefore coincide with $C(x)$. We thus have

$$\text{Exp}_W(x, z) = \lim_{n \to \infty} \text{Exp}_W(x_n, z) = \int_a e^{\langle \xi, z \rangle} d\mu_x(\xi) \quad \forall z \in a_C,$$  

(3.10)

which finishes the proof of the proposition. ■
Proof of Theorem 2.1. By Proposition 3.1 and the definition of $V,$

$$\sum_{n=0}^{\infty} \frac{1}{n!} V_x(\langle x, z \rangle^n) = V_x(e^{\langle x, z \rangle}) = \int_a e^{\langle \xi, z \rangle} d\mu_x(\xi)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int_a (\langle \xi, z \rangle)^n d\mu_x(\xi) \quad (z \in a_C);$$

(3.11)

here the subscript $x$ means that $V$ is taken with respect to $x.$ Comparison of the homogeneous parts in $z$ of degree $n$ yields that

$$V_x(\langle x, z \rangle^n) = \int_a \langle \xi, z \rangle^n d\mu_x(\xi) \quad \forall n \in \mathbb{Z}^+.$$  

(3.12)

As the $C$-span of $\{ x \mapsto \langle x, z \rangle^n, z \in a_C \}$ is $\mathcal{P}_n,$ it follows by linearity that

$$V_p(x) = \int_a p(\xi) d\mu_x(\xi) \quad \forall p \in \mathcal{P}, x \in a.$$  

(3.13)

Finally, as $\mathcal{P}$ is dense in each $(A_r, \| \cdot \|_{A_r})$ and $\| \cdot \|_\infty, K_r \leq \| \cdot \|_{A_r},$ an easy approximation argument implies that this integral representation remains valid for all $f \in A_r,$ with $r \geq |x|.$ This finishes the proof. ■

We conclude this paper with a remark concerning positive product formulas. It is conjectured that (again in case $k \geq 0$) the multivariable hypergeometric function $F$ has a positive product formula. More precisely, we conjecture that for all $x, y \in a,$ there exists a probability measure $\sigma_{x,y} \in M_1(a)$ whose support is contained in the ball $K_{|x|+|y|}(0)$ and which satisfies

$$F(\lambda, x)F(\lambda, y) = \int_a F(\lambda, \xi) d\sigma_{x,y}(\xi) \quad \forall \lambda \in a_C.$$  

(3.14)

In the rank-one case, that is, for Jacobi functions, this is well known and goes back to [9]. Equation (3.14) would immediately imply a positive product formula for the generalized Bessel function $J_W$ (associated with the same multiplicity $k$). In fact, suppose there exist measures $\sigma_{x,y}$ as conjectured above, and denote for $r > 0$ the image measure of $\sigma_{x,y}$ under the dilation $\xi \mapsto r\xi$ on $a$ by $\sigma_{x,y}^r.$ Then by relation (2.21),

$$J_W(\lambda, x)J_W(\lambda, y) = \lim_{n \to \infty} F\left( n\lambda + \rho, \frac{x}{n} \right) F\left( n\lambda + \rho, \frac{y}{n} \right)$$

$$= \lim_{n \to \infty} \int_a F\left( n\lambda + \rho, \frac{\xi}{n} \right) d\sigma_{x,y}^n(\xi)$$

(3.15)
for all \( \lambda \in a_C \). As \( \text{supp } \sigma_{n,x/y} \subseteq K_{|x|+|y|}(0) \) for all \( n \in \mathbb{N} \), we may assume that there exists a probability measure \( \tau_{x,y} \in M^1(\mathbb{R}) \) with \( \text{supp } \tau_{x,y} \subseteq K_{|x|+|y|}(0) \) such that \( \sigma_{n,x/y} \to \tau_{x,y} \) weakly as \( n \to \infty \). As further \( \lim_{n \to \infty} F(n\lambda + \rho, \xi/n) = J_W(\lambda, \xi) \) locally uniformly with respect to \( \xi \), (3.15) implies the product formula

\[
J_W(\lambda, x)J_W(\lambda, y) = \int_a J_W(\lambda, \xi)d\tau_{x,y}(\xi) \quad \forall \lambda \in a_C.
\]

The uniqueness of \( \tau_{x,y} \) is immediate from the injectivity of the Dunkl transform on \( M^1(\mathbb{R}) \) (cf. [16, Theorem 2.6])

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