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On the Green Function and Poisson Integrals of the Dunkl Laplacian

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Abstract We prove the existence and study properties of the Green function of the unit ball for the Dunkl Laplacian Δ_k in $\mathbb{R}^d$. As applications we derive the Poisson-Jensen formula for Δ_k-subharmonic functions and Hardy-Stein identities for the Poisson integrals of Δ_k. We also obtain sharp estimates of the Newton potential kernel, Green function and Poisson kernel in the rank one case in $\mathbb{R}^d$. These estimates contrast sharply with the well-known results in the potential theory of the classical Laplacian.

Keywords Dunkl Laplacian · Green function · Newton kernel · Poisson kernel · Hardy-Stein identity

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1 Introduction

Dunkl operators are differential reflection operators associated with finite reflection groups which generalize the usual partial derivatives as well as the invariant differential operators of Riemannian symmetric spaces. They play an important role in harmonic analysis and the

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study of special functions of several variables. Among other applications, Dunkl operators are employed in the description of quantum integrable models of Calogero-Moser type, see e.g. [9]. Also, there are stochastic processes associated with Dunkl Laplacians which generalize Dyson's Brownian motion model, see e.g. [17, 31]. Recently, the potential theory of the Dunkl Laplacian Δ_k has found increasing attention in view of many interesting open problems and the need of developing new techniques, as many standard methods known from the case of diffusion operators do not apply, see, e.g., [7, 15, 16, 20, 26, 27]. In the present paper we study the properties of one of the fundamental objects in the potential theory of Δ_k: the Green function $G_k(x, y)$ of the unit ball $\mathbb{B}$ in $\mathbb{R}^d$. The behavior and estimates of this function and its generalizations for bounded smooth domains were intensively studied in the case of the classical Laplacian [3, 34–36], more general diffusion operators [1, 2, 8, 18, 23, 25], as well as nonlocal operators [5, 6, 19, 22, 24].

Our first result, Theorem 3.1, establishes the existence and an integral formula for $G_k(x, y)$. A more convenient two-sided bound of $G_k(x, y)$ is given in Theorem 3.2. We also prove a standard relation between $G_k(x, y)$ and the Poisson kernel $P_k(x, y)$ of $\mathbb{B}$ for Δ_k, see Proposition 3.5. As applications of Theorem 3.1 we obtain the Poisson-Jensen formula for Δ_k-subharmonic functions and Hardy-Stein identities for Δ_k-harmonic functions on $\mathbb{B}$, see Theorem 4.2 and Theorem 4.5. This leads to an equivalent characterization of the Hardy spaces of Δ_k on $\mathbb{B}$ in the spirit of [4]. We remark that the general integral representation (3.5) of $G_k(x, y)$ and the estimate of Theorem 3.2 involve the representing measure for the intertwining operator whose structure depends strongly on the underlying root system. Note that explicit formulas for the representing measure are known only in a few particular cases, and the question whether it always admits a Lebesgue density is a challenging open problem. However, the available results together with Theorem 3.2 allow us to derive explicit two-sided bounds of the Newton kernel $N_k(x, y)$, the Green function $G_k(x, y)$ and the Poisson kernel $P_k(x, y)$ for Δ_k in the rank one case in $\mathbb{R}^d$, see Theorem 5.1, Theorem 5.4, and Corollary 5.7. The obtained estimates contrast sharply with the classical results in the potential theory of the Laplacian Δ or more general diffusion operators. The main novelties in the present setting are additional singularities of $N_k(x, y)$ and $G_k(x, y)$ in $x = gy$ in dimensions higher than 3 (g is in the associated reflection group W) and the dependence of the estimate of $N_k(x, y)$, $G_k(x, y)$ and $P_k(x, y)$ on the distance to the boundary of the Weyl chamber. This makes the obtained asymptotics more complex than in the case of diffusion operators, in particular these for the Green function $G_k(x, y)$. Deriving analogous two-sided bounds in the setting of any other root system is an interesting open problem, and available informations about the representing measure for the intertwining operator are in this case essential. We should note that the existence of singularities of the Newton kernel $N_k(\cdot, y)$ on the orbit $W.y$ has recently been discussed in the case of an orthogonal root system, see [27, Proposition 2.59].

The paper is organized as follows. In Section 2 we give basic definitions and list some useful facts in the theory of Dunkl operators. In Section 3 we prove the existence and study properties of $G_k(x, y)$. In Section 4 we prove the Poisson-Jensen formula and Hardy-Stein identites. In Section 5 we derive sharp estimates of $N_k(x, y)$, $G_k(x, y)$ and $P_k(x, y)$ in the rank one case in $\mathbb{R}^d$.

2 Preliminaries

For details on the following, see [11, 12, 28] and, for a general overview, [13] or [30]. Let R be a root system in $\mathbb{R}^d$ (equipped with the usual scalar product and Euclidean norm

$|\cdot|$), and let W be the associated finite reflection group. The root system R needs not be crystallographic and W is not required to be effective, i.e. $\text{span}_{\mathbb{R}} R$ may be a proper subspace of $\mathbb{R}^d$. The dimension of $\text{span}_{\mathbb{R}} R$ is called the rank of R. An important example is $R = A_{d-1} = \{\pm(e_i - e_j) : 1 \le i < j \le d\} \subset \mathbb{R}^d$ with $W = S_d$, the symmetric group in d elements. We fix a nonnegative multiplicity function k on R, i.e. $k : R \to [0, \infty)$ is W-invariant. The (rational) Dunkl operators associated with R and k are given by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad \xi \in \mathbb{R}^d,$$

where R_+ denotes an (arbitrary) positive subsystem of R. For fixed R and k, these operators commute. Moreover, there is a unique linear isomorphism V_k on the space of polynomial functions in d variables, called the intertwining operator, which preserves the degree of homogeneity, is normalized by $V_k(1) = 1$ and intertwines the Dunkl operators with the usual partial derivatives:

$$T_\xi V_k = V_k \partial_\xi \quad \text{for all } \xi \in \mathbb{R}^d.$$

The Dunkl Laplacian is defined by

$$\Delta_k := \sum_{i=1}^d T_{\xi_i}^2$$

with an (arbitrary) orthonormal basis $(\xi_i)_{1 \le i \le d}$ of $\mathbb{R}^d$. We shall renormalize the roots according to $|\alpha| = \sqrt{2}$ for all $\alpha \in R$. Then

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),$$

where Δ is the usual Laplacian on $\mathbb{R}^d$. For $x \in \mathbb{R}^d$ denote by $C(x)$ the convex hull of the Weyl group orbit $W.x$ of x in $\mathbb{R}^d$. The intertwining operator V_k has the integral representation

$$V_k f(x) = \int_{C(x)} f(z) d\mu_x^k(z), \tag{2.1}$$

where μ_x^k is a probability measure on $C(x)$. The measures μ_x^k satisfy

$$\mu_{rx}^k(A) = \mu_x^k(r^{-1}A)$$

for all $r > 0$ and Borel sets $A \subseteq \mathbb{R}^d$. In [33], it was deduced from formula (2.1) that V_k establishes a homeomorphism of $C^\infty(\mathbb{R}^d)$ with its usual Fréchet space topology.

In the rank one case $R = \{\pm 1\} \subset \mathbb{R}$, the representation (2.1) is explicitly known ([12, Theorem 5.1]); it is given by

$$V_k f(x) = c_k \int_{-1}^1 f(tx)(1-t)^{k-1}(1+t)^k \, dt \quad \text{with } c_k = \frac{\Gamma(k+1/2)}{\sqrt{\pi}\,\Gamma(k)}. \tag{2.2}$$

We shall employ the Dunkl-type generalized translation on $C^\infty(\mathbb{R}^d)$ which was defined in [33] by

$$\tau_y f(x) := V_k^x V_k^y (V_k^{-1} f)(x+y), \;\; x, y \in \mathbb{R}^d.$$

Here the superscript denotes the relevant variable. This translation satisfies $\tau_y f(x) = \tau_x f(y)$, and we shall use the notation

$$f(x *_k y) := \tau_y f(x) = \tau_x f(y).$$

Lemma 2.1 (i) *The representing measures μ_x^k satisfy $\mu_{-x}^k(-A) = \mu_x^k(A)$.*
(ii) *Let $f \in C^\infty(\mathbb{R}^d)$ and write $f^-(x) := f(-x)$. Then $f(-x *_k -y) = f^-(x *_k y)$.*

Proof It is immediate that the Dunkl operators satisfy $T_\xi(f^-) = (T_{-\xi} f)^-$. By the characterization of V_k, it follows that $V_k(f^-) = (V_k f)^-$. This implies both assertions. □

Of particular importance in our context will be translates of functions f on $\mathbb{R}^d$ which are radial, that is $f(x) = \widetilde{f}(|x|)$ with $\widetilde{f} : [0, \infty) \to \mathbb{C}$. We recall from [29] that for each $x, y \in \mathbb{R}^d$ there exists a unique compactly supported radial probability measure $\rho_{x,y}^k$ on $\mathbb{R}^d$ such that

$$f(x *_k y) = \int_{\mathbb{R}^d} f \, d\rho_{x,y}^k \tag{2.3}$$

for all $f \in C^\infty(\mathbb{R}^d)$. This can be written explicitly as

$$f(x *_k y) = \int_{C(y)} \widetilde{f}\Big(\sqrt{|x|^2 + |y|^2 + 2\langle x, z\rangle}\Big) d\mu_y^k(z). \tag{2.4}$$

Notice that Dunkl translates of non-negative, smooth radial functions are again non-negative. Formula (2.3) allows to extend the generalized translation to measurable radial functions which are either complex-valued and bounded or have values in $[0, \infty]$. We maintain the notations $\tau_y f(x)$ and $f(x *_k y)$ for functions from these classes. In particular, for measurable radial f we have

$$f(-x *_k -y) = f(x *_k y). \tag{2.5}$$

We put

$$\gamma := \sum_{\alpha \in R_+} k_\alpha$$

and define the weight function ω_k on $\mathbb{R}^d$ by

$$\omega_k(x) := \prod_{\alpha \in R_+} |\langle \alpha, x\rangle|^{2k_\alpha}.$$

Let $\mathbb{B} = \{x \in \mathbb{R}^d : |x| < 1\}$ denote the open unit ball in $\mathbb{R}^d$ and let $\mathbb{S} = \partial\mathbb{B}$ denote the unit sphere. The Poisson kernel $P_k(x, y)$ of $\mathbb{B}$ for the Dunkl Laplacian Δ_k was defined in [12] as a reproducing kernel for Δ_k-harmonic polynomials. It can be written as

$$P_k(x, y) = V_k\left[\frac{1 - |x|^2}{(1 - 2\langle x, \cdot\rangle + |x|^2)^{\gamma + d/2}}\right](y), \;\; x \in \mathbb{B}, y \in \mathbb{S}. \tag{2.6}$$

In view of identity (2.5) with $f(x) = |x|^{-2\gamma - d}$, we obtain

$$\begin{aligned} P_k(x, y) &= \int_{C(y)} \frac{1 - |x|^2}{(1 - 2\langle x, z\rangle + |x|^2)^{\gamma + d/2}} d\mu_y(z) = (1 - |x|^2) \cdot f(-x *_k y) \\ &= (1 - |x|^2) \cdot \tau_{-y}(|x|^{-2\gamma - d}). \end{aligned} \tag{2.7}$$

The notation $f \asymp g$ will always mean that there is a constant $C > 0$ depending on k and d only (unless stated otherwise) such that $C^{-1} g \le f \le C g$.

3 The Green Function of the Ball

From now on, it is always assumed that $d + 2\gamma > 2$. Following [16], we introduce the Newton kernel in the Dunkl setting by

$$N_k(x, y) = \int_0^\infty \Gamma_k(t, x, y)dt \quad (x, y \in \mathbb{R}^d),$$

with the heat kernel

$$\Gamma_k(t, x, y) = \frac{M_k}{t^{\gamma+d/2}}\, e^{-(|x|^2+|y|^2)/4t} E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big),$$

where

$$M_k = 2^{-\gamma-d/2}\Big(\int_{\mathbb{R}^d} e^{-|x|^2/2}\omega_k(x)dx\Big)^{-1}$$

and $E_k(x, y) = V_k(e^{\langle\cdot,y\rangle})(x)$ denotes the Dunkl kernel, which has an analytic extension to $\mathbb{C}^d \times \mathbb{C}^d$ and satisfies $E_k(x, y) = E_k(y, x)$. This implies that $N_k(x, y) = N_k(y, x)$. According to [16, Proposition 6.1], the Newton kernel can be written as

$$N_k(x, y) = C_k \int_{C(y)} \Big(|x|^2 + |y|^2 - 2\langle x, z\rangle\Big)^{1-\gamma-d/2} d\mu_y(z) \tag{3.1}$$

where

$$C_k = \frac{1}{d_k(d + 2\gamma - 2)} \quad \text{and } d_k = \int_{\mathbb{S}} \omega_k(x)d\sigma(x). \tag{3.2}$$

Here σ denotes the surface measure on $\mathbb{S}$. Formula (3.1) is also easily obtained by translations. Recall that

$$\Gamma_k(t, x, y) = \tau_{-y} g_t(x) \quad \text{with } g_t(x) = \frac{M_k}{t^{\gamma+d/2}}\, e^{-|x|^2/4t},$$

which follows from [29, Lemma 2.2. and (3.2)] (see also [31]). As

$$\int_0^\infty g_t(\xi)dt \;=\; M_k\Gamma(\gamma + \frac{d}{2} - 1)\cdot\Big(\frac{2}{|\xi|}\Big)^{d-2+2\gamma} \;=\; \frac{C_k}{|\xi|^{d-2+2\gamma}},$$

it follows that

$$N_k(x, y) = \int_0^\infty \int_{\mathbb{R}^d} g_t(\xi)d\rho^k_{x,-y}(\xi)dt \;=\; C_k \cdot \tau_{-y}\big(|x|^{2-2\gamma-d}\big).$$

In view of identity (2.5), this equals the right-hand side of Eq. 3.1. Furthermore, the Newton kernel $N_k(\cdot, y)$ is Δ_k-harmonic on $\mathbb{R}^d \setminus W.y$ for fixed $y \in \mathbb{R}^d$ (see [16, Theorem 6.1]). It can be regarded as the global Green function for the Dunkl Laplacian Δ_k.

The goal of this section is to introduce and study the Green function of the ball $\mathbb{B}$ for Δ_k. For this, we recall from [14] the Kelvin transform associated with the Dunkl Laplacian, which is given by

$$K_k[u](x) = |x|^{2-2\gamma-d}u(x^*)$$

for functions u on $\mathbb{R}^d\setminus\{0\}$, where $x^* = x/|x|^2$ is the inversion with respect to the unit sphere in $\mathbb{R}^d$. By [14, Theorem 3.1], K_k preserves Δ_k-harmonic functions on $\mathbb{R}^d \setminus \{0\}$. Clearly, for a function u of class C^2 in the neighborhood of $W.x$, $x \neq 0$, we have

$$\Delta_k(K_k u)(x^*) = |x^*|^{-4}K_k(\Delta_k u)(x^*) = |x|^{2+2\gamma+d}\Delta_k u(x). \tag{3.3}$$

Following the classical case $k = 0$ (cf. [10, 32]), we define

$$G_k(x, y) := N_k(x, y) - K_k[N_k(\cdot, y)](x) \tag{3.4}$$

for $x, y \in \overline{\mathbb{B}} \times \overline{\mathbb{B}}$ with $x \neq 0$, where $K_k[N_k(\cdot, y)](x) = |x|^{2-2\gamma-d}N_k(x^*, y)$.

Theorem 3.1 *The kernel G_k is the Green function of $\mathbb{B}$ for Δ_k, that is, G_k extends to a $[0,\infty]$-valued function on $\overline{\mathbb{B}} \times \overline{\mathbb{B}}$ which is uniquely characterized by the following conditions:*

(i) $G_k(x, y) > 0$ *for all* $x, y \in \mathbb{B}$ *and* $G_k(x, y) = 0$ *for* $x \in \mathbb{S}$ *and* $y \in \mathbb{B}$.
(ii) $G_k(\cdot, y)$ *is continuous on* $\overline{\mathbb{B}} \setminus W.y$ *for any fixed* $y \in \mathbb{B}$.
(iii) $N_k(\cdot, y) - G_k(\cdot, y)$ *is* Δ_k*-harmonic on* $\mathbb{B}$ *for any fixed* $y \in \mathbb{B}$.

Moreover, the Green function G_k can be written as

$$G_k(x, y) = C_k \int_{C(y)} \Big[\Big(|x|^2 + |y|^2 - 2\langle x, z\rangle \Big)^{1-\gamma-d/2} - \Big(1 + |x|^2|y|^2 - 2\langle x, z\rangle \Big)^{1-\gamma-d/2} \Big] d\mu_y(z). \tag{3.5}$$

It satisfies $G_k(x, y) = G_k(y, x)$ for all $x, y \in \overline{\mathbb{B}}$, and $G_k(\cdot, y)$ is Δ_k-harmonic on $\mathbb{B} \setminus W.y$ for any fixed $y \in \mathbb{B}$.

Proof Fix $y \in \mathbb{B}$. As $N_k(\cdot, y)$ is Δ_k-harmonic on $\mathbb{R}^d \setminus W.y$ ([16, Theorem 6.1]), it follows from Eq. 3.3 that the Kelvin transform $K_k[N_k(\cdot, y)]$ is Δ_k-harmonic on $\mathbb{B} \setminus \{0\}$ and continuous on $\overline{\mathbb{B}} \setminus \{0\}$. By Eq. 3.1 we have

$$K_k[N_k(\cdot, y)](x) = C_k \int_{C(y)} \Big(1 + |x|^2|y|^2 - 2\langle x, z\rangle \Big)^{1-\gamma-d/2} d\mu_y(z), \tag{3.6}$$

and from this representation it is immediate by the dominated convergence theorem that $K_k[N_k(\cdot, y)]$ has a removable singularity at 0. Employing [15, Theorem 5.1], we conclude that $K_k[N_k(\cdot, y)]$ extends to a Δ_k-harmonic function on $\mathbb{B}$. Furthermore, $K_k[N_k(\cdot, y)]$ solves the Δ_k-Dirichlet problem on $\overline{\mathbb{B}}$ with the boundary values of $N_k(\cdot, y)$. Therefore, $G_k(\cdot, y)$ vanishes continuously at $\mathbb{S}$ and is Δ_k-harmonic on $\mathbb{B} \setminus W.y$. Formula (3.6) immediately gives the claimed identity (3.5). As $1 + |x|^2|y|^2 - 2\langle x, z\rangle > |x|^2 + |y|^2 - 2\langle x, z\rangle \geq 0$ for all $x, y \in \mathbb{B}$ and $z \in C(y)$, it follows from Eq. 3.5 that $G_k(x, y) > 0$ for all $x, y \in \mathbb{B}$. For the symmetry of G_k, it suffices to prove that $K_k[N_k(\cdot, y)](x)$ is symmetric in x and y for $x \neq 0$. Using the symmetry of N_k and the fact that for any $r > 0$, the representing measure μ_{rx} is just the image measure of μ_x under the dilation $z \mapsto rz$ of $\mathbb{R}^d$, we obtain

$$\begin{aligned} K_k[N_k(\cdot, y)](x) &= |x|^{2-2\gamma-d} N_k(y, x^*) \\ &= |x|^{2-2\gamma-d} \cdot C_k \int_{\mathbb{R}^d} (|y|^2 + |x^*|^2 - 2\langle y, z\rangle)^{1-\gamma-d/2} d\mu_{x/|x|^2}(z) \\ &= |x|^{2-2\gamma-d} \cdot C_k \int_{\mathbb{R}^d} (|y|^2 + |x^*|^2 - 2\langle y, \frac{z}{|x|^2}\rangle)^{1-\gamma-d/2} d\mu_x(z) \\ &= C_k \int_{\mathbb{R}^d} (|x|^2|y|^2 + 1 - 2\langle y, z\rangle)^{1-\gamma-d/2} d\mu_x(z) \\ &= K_k[N_k(\cdot, x)](y). \end{aligned}$$

Further, [16, Proposition 6.2] gives $G_k(x, x) = +\infty$. Finally, the uniqueness of the function G_k subject to the conditions $(i) - (iii)$ follows from the uniqueness of solutions to the Δ_k-Dirichlet problem on $\mathbb{B}$, see [26]. □

According to [16, Theorem 6.1], $-N_k(x,\, .)$ provides a fundamental solution for Δ_k on $\mathbb{R}^d$ in the sense that $\Delta_k(-N_k(x, .)\omega_k) = \delta_x$ in $\mathcal{D}'(\mathbb{R}^d)$. This implies that $-G_k(x, .)$ provides a fundamental solution for Δ_k in $\mathbb{B}$:

$$\Delta_k\left(-G_k(x, \cdot)\omega_k\right) = \delta_x \quad \text{in } \mathcal{D}'(\mathbb{B}).$$

Our next result provides sharp two-sided bounds for $G_k(x, y)$ which are more convenient to deal with rather than Eq. 3.5. For $x \in \mathbb{B}$ denote $\rho(x) := 1 - |x|$.

Theorem 3.2 *The two-sided bound of $G_k(x, y)$ on $\mathbb{B} \times \mathbb{B}$ is given by*

$$G_k(x, y) \asymp \int_{C(y)} \frac{(1-|x|^2)(1-|y|^2)d\mu_y(z)}{\left(1+|x|^2|y|^2-2\langle x, z\rangle\right)\left(|x|^2+|y|^2-2\langle x, z\rangle\right)^{\gamma+d/2-1}}$$
$$\asymp \int_{C(y)} \frac{\rho(x)\rho(y)d\mu_y(z)}{\left(\rho(x)\rho(y)+|x|^2+|y|^2-2\langle x, z\rangle\right)\left(|x|^2+|y|^2-2\langle x, z\rangle\right)^{\gamma+d/2-1}}.$$

Proof Note that for $x, y \in \mathbb{B}$ we have

$$1+|x|^2|y|^2-|x|^2-|y|^2 = (1-|x|^2)(1-|y|^2) \asymp \rho(x)\rho(y).$$

Hence, the estimate is a direct consequence of Theorem 3.1 and Lemma 3.3 below. □

Lemma 3.3 *Fix $p > 0$. There exists a constant $C_p > 0$ depending only on p such that for all $0 < a < b < \infty$ we have*

$$\frac{b-a}{C_p b a^p} \le \frac{1}{a^p} - \frac{1}{b^p} \le \frac{C_p(b-a)}{b a^p}.$$

Proof Assume first $p > 1$. Then by [4, Lemma 6, (11)] (see also Eq. 4.5) we get

$$\begin{aligned} b^p - a^p &\le C(b-a)^2 b^{p-2} + p a^{p-1}(b-a) \\ &\le C(b-a)(b^{p-1} - a b^{p-2} + a^{p-1}) \\ &= C b^{p-1}(b-a)\left(1 + (a/b)^{p-1} - a/b\right), \end{aligned}$$

and the lower bound obtains analogously. Furthermore, since $p > 1$, we have

$$\sup_{x\in[0,1]} |x^{p-1} - x| < 1.$$

Hence $b^p - a^p \asymp b^{p-1}(b-a)$ and

$$\frac{1}{a^p} - \frac{1}{b^p} = \frac{b^p - a^p}{(ab)^p} \asymp \frac{b-a}{b a^p}.$$

Here $\asymp$ means two-sided estimates with constants depending only on p. For $0 < p \le 1$ we let $q = p + 1$. We have

$$\frac{1}{a^p} - \frac{1}{b^p} = \frac{a}{a^q} - \frac{b}{b^q} = \frac{a b^q - b a^q}{(ab)^q}.$$

Let $c = b^{1/q}a$, $d = a^{1/q}b$. Then $0 < c < d < \infty$ and applying the estimate obtained previously we get

$$\begin{aligned} ab^q - ba^q = d^q - c^q &\asymp d^{q-1}(d-c) \\ &= a^{(q-1)/q} b^{q-1} \left(a^{1/q} b - b^{1/q} a\right) \\ &= ab^{q-1} \left(b - b^{1/q} a^{1-1/q}\right). \end{aligned}$$

Since $a < b$, we obtain

$$b - b^{1/q} a^{1-1/q} = b - (b/a)^{1/q} a \le b - a$$

and the upper bound follows. To get the lower bound define $f(x) = b^{1/q} x^{1-1/q}$ for $x \in [a, b]$. Then $f'(x) = (1-1/q)(b/x)^{1/q}$ and by the mean value theorem, for some $\xi \in (a, b)$ we have

$$b - b^{1/q} a^{1-1/q} = f(b) - f(a) = (1 - 1/q)(b/\xi)^{1/q}(b - a) \ge (1 - 1/q)(b - a).$$

Therefore

$$\frac{ab^q - ba^q}{(ab)^q} \asymp \frac{ab^{q-1}(b-a)}{(ab)^q} = \frac{b-a}{ba^p}.$$

□

A simple consequence of Theorem 3.2 is the following estimate.

Corollary 3.4 *Let $y_0 \in \mathbb{B}$ be fixed. There is a constant $C > 0$ depending on d, k and y_0 only, such that*

$$C^{-1}\rho(x)N_k(x, y_0) \le G_k(x, y_0) \le C\rho(x)N_k(x, y_0).$$

The following classical formula relates the Poisson kernel $P_k(x, y)$ to the Green function $G_k(x, y)$.

Proposition 3.5 *For all $x \in \mathbb{B}$ and $y \in \mathbb{S}$ we have*

$$P_k(x, y) = -d_k \langle y, \nabla_y G_k(x, y)\rangle.$$

Proof We use the symmetry $G_k(x, y) = G_k(y, x)$. By the dominated convergence, we can differentiate under the integral sign in Eq. 3.5 to see that for all $x \in \mathbb{B}$ and $y \in \mathbb{S}$,

$$-d_k \langle y, \nabla_y G_k(x, y)\rangle = (1 - |x|^2) \int_{C(x)} \left(|x|^2 + 1 - 2\langle y, z\rangle\right)^{-\gamma - d/2} d\mu_x(z).$$

With $f(x) = |x|^{-2\gamma-d}$ and in view of Eq. 2.5 and representation (2.7) for the kernel P_k we obtain

$$-d_k \langle y, \nabla_y G_k(x, y)\rangle = (1 - |x|^2) f(x *_k -y) = P_k(x, y).$$

□

4 Poisson-Jensen Formula and Hardy-Stein Identities

Our first goal in this section is to prove the so-called *Poisson-Jensen formula* for Δ_k-subharmonic functions on $\overline{\mathbb{B}}$. The corresponding result for classical subharmonic functions may be found in [21]. We will next use the formula to derive the *Hardy-Stein identites* for

Δ_k-harmonic functions on $\mathbb{B}$, which equivalently characterize the Hardy spaces of Δ_k in the spirit of [4].

All functions in this section are assumed to be real-valued. Let $\Omega \subset \mathbb{R}^d$ be a W-invariant open set. We will say that a function $u \in C^2(\Omega)$ is Δ_k-subharmonic on Ω if $\Delta_k u(x) \geq 0$ for all $x \in \Omega$. We refer to [16] for basic properties and other characterizations of Δ_k-subharmonic functions. We will further say that a function u is Δ_k-harmonic (resp. Δ_k-subharmonic) on $\overline{\mathbb{B}}$ if there exists $\varepsilon > 0$ such that u extends to a Δ_k-harmonic (resp. Δ_k-subharmonic) function on $\mathbb{B}_\varepsilon := \{x : |x| < 1+\varepsilon\}$. For $r > 0$ we define the dilation of a function u by $u_r(x) := u(rx)$.

The Riesz decomposition theorem [16, Theorem 7.1, see also Proposition 4.1 and Example 5.1] implies that for every $\varepsilon > 0$ and every function u which is Δ_k-subharmonic on $\mathbb{B}_\varepsilon := \{x : |x| < 1+\varepsilon\}$ there exists a unique Δ_k-harmonic function h_ε on $\mathbb{B}_{\varepsilon/2} \subset \overline{\mathbb{B}}_{\varepsilon/2} \subset \mathbb{B}_\varepsilon$ such that

$$u(x) = -\int_{\mathbb{B}_{\varepsilon/2}} N_k(x, y)\Delta_k u(x)\omega_k(x)dx + h_\varepsilon(x), \quad x \in \mathbb{B}_{\varepsilon/2}. \tag{4.1}$$

Note that $\Delta_k u(x)\omega_k(x)dx$ is the Δ_k-Riesz measure of u in this case (see [16, Section 5] for details). As in the previous section, we denote by σ the surface measure on $\mathbb{S}$ and let $\omega_k\sigma$ denote the measure on $\mathbb{S}$ given by $d\omega_k\sigma(x) = \omega_k(x)d\sigma(x)$. For $f \in L^1(\mathbb{S}, \omega_k\sigma)$ we define the Poisson integral of f by

$$P_k[f](x) := \frac{1}{d_k}\int_{\mathbb{S}} P_k(x, z)f(z)\omega_k(z)d\sigma(z), \quad x \in \mathbb{B}.$$

Our first result is the following property of the Newton kernel of Δ_k.

Lemma 4.1 *For all $x \in \mathbb{B}$ we have*

$$P_k[N_k(\cdot, y)](x) = \begin{cases} N_k(x, y) - G_k(x, y), & y \in \mathbb{B}, \\ N_k(x, y), & y \notin \mathbb{B}. \end{cases}$$

Proof For $y \in \mathbb{B}$ the statement follows from Eq. 3.4. Clearly, $K_k[N_k(\cdot, y)]$ is Δ_k-harmonic on $\mathbb{B}$ and continuous on $\overline{\mathbb{B}}$ with $K_k[N_k(\cdot, y)](x) = N_k(x, y)$ for all $x \in \mathbb{S}$. By the uniqueness of the solution to the Δ_k-Dirichlet problem [26] we have $K_k[N_k(\cdot, y)] = P_k[N_k(\cdot, y)]$ on $\mathbb{B}$. When $y \in (\overline{\mathbb{B}})^c$, then $N_k(\cdot, y)$ is Δ_k-harmonic on $\overline{\mathbb{B}}$, and hence $N_k(\cdot, y) = P_k[N_k(\cdot, y)]$ on $\mathbb{B}$ in this case. Finally, let $y \in \mathbb{S}$. Since $N_k(\cdot, y)$ is Δ_k-harmonic on $\mathbb{B}$, the dilation $N_k(\cdot, y)_r$ is Δ_k-harmonic on $\overline{\mathbb{B}}$ for any $0 < r < 1$. Hence

$$N_k(rx, y) = \frac{1}{d_k}\int_{\mathbb{S}} P_k(x, z)N_k(rz, y)\omega_k(z)d\sigma(z),$$

and it is enough to show that the right-hand side above tends to $P_k[N_k(\cdot, y)](x)$ as $r \to 1$. First note that Fatou's lemma gives $P_k[N_k(\cdot, y)](x) \leq N_k(x, y)$. By Eq. 3.1, for $z, y \in \mathbb{S}$ we have

$$N_k(rz, y) = C_k\int_{C(y)} \left(|rz|^2 + |y|^2 - 2\langle rz, v\rangle\right)^{1-\gamma-d/2} d\mu_y(v).$$

For $v \in C(y)$ write $v = \sum_{g\in W}\lambda_g(v)gy$, where $\lambda_g(v) \geq 0$ for all $g \in W$ and $\sum_{g\in W}\lambda_g(v) = 1$. This gives

$$|rz|^2 + |y|^2 - 2\langle rz, v\rangle = \sum_{g\in W}\lambda_g(v)|rz - gy|^2.$$

Furthermore, since $|z| = |gy| = 1$, we have $|rz - gy| \geq |rz - rgy|$ for any $0 < r < 1$. Consequently,

$$|rz|^2 + |y|^2 - 2\langle rz, v\rangle \geq r^2(|z|^2 + |y|^2 - 2\langle z, v\rangle),$$

and $N_k(rz, y) \leq r^{2-2\gamma-d} N_k(z, y)$. Therefore, $N_k(rz, y) \leq C N_k(z, y)$ for all $1/2 < r < 1$ and $P_k[N_k(\cdot, y)](x) \leq N_k(x, y) < \infty$. The dominated convergence theorem gives the result. □

As a consequence of Eq. 4.1 and Lemma 4.1 we obtain the following Poisson-Jensen formula.

Theorem 4.2 *Let u be Δ_k-subharmonic on $\overline{\mathbb{B}}$. Then for every $x \in \mathbb{B}$ we have*

$$u(x) = \frac{1}{d_k}\int_{\mathbb{S}} P_k(x, y)u(y)\omega_k(y)d\sigma(y) - \int_{\mathbb{B}} G_k(x, y)\Delta_k u(y)\omega_k(y)dy.$$

Proof Choose $\varepsilon > 0$ such that u extends to a Δ_k-subharmonic function on $\mathbb{B}_\varepsilon$. By Eq. 4.1,

$$u(x) = -\int_{\mathbb{B}_{\varepsilon/2}} N_k(x, y)\Delta_k u(x)\omega_k(x)dx + h_\varepsilon(x), \quad x \in \mathbb{B}_{\varepsilon/2},$$

where h_ε is Δ_k-harmonic on $\mathbb{B}_{\varepsilon/2}$. Evaluating the Poisson integral of both sides and applying Fubini's theorem and Lemma 4.1 we get

$$\begin{aligned} P_k[u](x) &= \frac{1}{d_k}\int_{\mathbb{S}} P_k(x, y)u(y)\omega_k(y)d\sigma(y) \\ &= \frac{1}{d_k}\int_{\mathbb{S}} P_k(x, y)\left(-\int_{\mathbb{B}_{\varepsilon/2}} N_k(y, z)\Delta_k u(z)\omega_k(z)dz + h_\varepsilon(y)\right)\omega_k(y)d\sigma(y) \\ &= -\int_{\mathbb{B}_{\varepsilon/2}}\left(\frac{1}{d_k}\int_{\mathbb{S}} P_k(x, y)N_k(y, z)\omega_k(y)d\sigma(y)\right)\Delta_k u(z)\omega_k(z)dz + h_\varepsilon(x) \\ &= \int_{\mathbb{B}} (G_k(x, z) - N_k(x, z))\,\Delta_k u(z)\omega_k(z)dz \\ &\quad - \int_{\mathbb{B}_{\varepsilon/2}\setminus\mathbb{B}} N_k(x, z)\Delta_k u(z)\omega_k(z)dz + h_\varepsilon(x) \\ &= \int_{\mathbb{B}} G_k(x, z)\Delta_k u(z)\omega_k(z)dz + u(x). \end{aligned}$$

□

Let $1 \leq p \leq \infty$. The Hardy space $H_k^p(\mathbb{B})$ is defined as the family of those Δ_k-harmonic functions on $\mathbb{B}$ which satisfy

$$\|u\|_{H^p} := \sup_{0\leq r<1} \|u_r\|_{L^p(\omega_k\sigma)} < \infty.$$

By [26, Theorem 2.2 and Theorem 2.3], $u \in H_k^p(\mathbb{B})$ for a given $1 < p \leq \infty$ if and only if $u = P_k[f]$ for some $f \in L^p(\mathbb{S}, \omega_k\sigma)$, and in this case $\|u\|_{H^p} = \|f\|_{L^p(\omega_k\sigma)}$. This implies that

$$\|u\|_{H^p} = \lim_{r\to 1} \|u_r\|_{L^p(\omega_k\sigma)} \tag{4.2}$$

for any Δ_k-harmonic function u on $\mathbb{B}$. As an application of Theorem 4.2, we will give an equivalent characterization of the spaces $H_k^p(\mathbb{B})$, $1 < p < \infty$, in terms of the Hardy-Stein identities. The approach is inspired by [4], where similar description was obtained for Hardy spaces of the classical Laplacian Δ and the fractional Laplacian $\Delta^{\alpha/2}$.

Let $1 < p < \infty$. For $a, b \in \mathbb{R}$ we set

$$F(a, b) = |b|^p - |a|^p - pa|a|^{p-2}(b - a) . \tag{4.3}$$

Here $F(a, b) = |b|^p$ if $a = 0$, and $F(a, b) = (p - 1)|a|^p$ if $b = 0$. For instance, if $p = 2$, then $F(a, b) = (b - a)^2$. Generally, $F(a, b)$ is the second-order Taylor remainder of $\mathbb{R} \ni x \mapsto |x|^p$, therefore by convexity, $F(a, b) \geq 0$. Furthermore, for $1 < p < \infty$ and $\varepsilon \in \mathbb{R}$ we define

$$F_\varepsilon(a, b) = (b^2 + \varepsilon^2)^{p/2} - (a^2 + \varepsilon^2)^{p/2} - pa(a^2 + \varepsilon^2)^{(p-2)/2}(b - a) . \tag{4.4}$$

Since $F_\varepsilon(a, b)$ is the second-order Taylor remainder of $\mathbb{R} \ni x \mapsto (x^2+\varepsilon^2)^{p/2}$, by convexity, $F_\varepsilon(a, b) \geq 0$. Of course, $F_\varepsilon(a, b) \to F_0(a, b) = F(a, b)$ as $\varepsilon \to 0$. The next result is proved in [4, Lemma 6].

Lemma 4.3 *For every $p > 1$ there is a constant $C > 0$ depending on p only such that*

$$C^{-1}(b - a)^2(|b| \vee |a|)^{p-2} \leq F(a, b) \leq C(b - a)^2(|b| \vee |a|)^{p-2}, \quad a, b \in \mathbb{R}. \tag{4.5}$$

If $p \in (1, 2)$, then

$$0 \leq F_\varepsilon(a, b) \leq \frac{1}{p-1} F(a, b), \quad \varepsilon, a, b \in \mathbb{R}. \tag{4.6}$$

The following explicit formulas shed some light on the meaning of the function F.

Lemma 4.4 *Let u be of class C^2 in the neighborhood of $x \in \mathbb{R}^d$. Then for $2 \leq p < \infty$ we have*

$$\begin{aligned}\Delta_k|u(x)|^p \ =\ & p(p-1)|u(x)|^{p-2}|\nabla u(x)|^2 + 2\sum_{\alpha\in R_+} k(\alpha)\frac{F(u(x), u(\sigma_\alpha(x)))}{\langle\alpha, x\rangle^2}\\ & + pu(x)|u(x)|^{p-2}\Delta_k u(x).\end{aligned} \tag{4.7}$$

When $1 < p < \infty$ and $\varepsilon > 0$, then

$$\begin{aligned}\Delta_k|u(x) + i\varepsilon|^p \ =\ & p|u(x) + i\varepsilon|^{p-4}\left[(p - 1)u(x)^2 + \varepsilon^2\right]|\nabla u(x)|^2\\ & + 2\sum_{\alpha\in R_+} k(\alpha)\frac{F_\varepsilon(u(x), u(\sigma_\alpha(x)))}{\langle\alpha, x\rangle^2} + pu(x)|u(x) + i\varepsilon|^{p-2}\Delta_k u(x).\end{aligned} \tag{4.8}$$

Proof When $2 \leq p < \infty$ or $u(x) \neq 0$ we write $|u(x)|^p = (u(x)^2)^{p/2}$ and a straightforward calculation gives

$$\begin{aligned}\nabla|u(x)|^p \ &=\ pu(x)|u(x)|^{p-2}\nabla u(x),\\ \Delta|u(x)|^p \ &=\ p(p-1)|u(x)|^{p-2}|\nabla u(x)|^2 + pu(x)|u(x)|^{p-2}\Delta u(x).\end{aligned}$$

Note that

$$|u(\sigma_\alpha(x))|^p - |u(x)|^p = F(u(x), u(\sigma_\alpha(x))) + pu(x)|u(x)|^{p-2}(u(\sigma_\alpha(x)) - u(x)).$$

Hence

$$\begin{aligned}\Delta_k|u(x)|^p &= \Delta|u(x)|^p + 2\sum_{\alpha\in R_+} k(\alpha)\left(\frac{\langle\nabla|u(x)|^p,\alpha\rangle}{\langle\alpha,x\rangle} + \frac{|u(\sigma_\alpha(x))|^p - |u(x)|^p}{\langle\alpha,x\rangle^2}\right)\\ &= p(p-1)|u(x)|^{p-2}|\nabla u(x)|^2 + pu(x)|u(x)|^{p-2}\Delta u(x)\\ &\quad +2pu(x)|u(x)|^{p-2}\sum_{\alpha\in R_+} k(\alpha)\left(\frac{\langle\nabla u(x),\alpha\rangle}{\langle\alpha,x\rangle} + \frac{(u(\sigma_\alpha(x)) - u(x))}{\langle\alpha,x\rangle^2}\right)\\ &\quad +2\sum_{\alpha\in R_+} k(\alpha)\frac{F(u(x),u(\sigma_\alpha(x)))}{\langle\alpha,x\rangle^2},\end{aligned}$$

and Eq. 4.7 follows. For $1 < p < \infty$ and $\varepsilon > 0$ we have

$$\begin{aligned}\nabla|u(x)+i\varepsilon|^p &= pu(x)|u(x)+i\varepsilon|^{p-2}\nabla u(x),\\ \Delta|u(x)+i\varepsilon|^p &= p|u(x)+i\varepsilon|^{p-4}\left[(p-1)u(x)^2+\varepsilon^2\right]|\nabla u(x)|^2\\ &\quad +p|u(x)+i\varepsilon|^{p-2}u(x)\Delta u(x),\end{aligned}$$

and

$$\begin{aligned}|u(\sigma_\alpha(x))+i\varepsilon|^p - |u(x)+i\varepsilon|^p &= F_\varepsilon(u(x),u(\sigma_\alpha(x)))\\ &\quad +pu(x)|u(x)+i\varepsilon|^{p-2}(u(\sigma_\alpha(x))-u(x)).\end{aligned}$$

The rest of the proof is similar to the previous case. □

We are now ready to prove the Hardy-Stein identities.

Theorem 4.5 *Let $1 < p < \infty$. Then for any $u \in H_k^p(\mathbb{B})$ we have*

$$\begin{aligned}\|u\|_{H^p}^p &= |u(0)|^p + C_k\int_{\mathbb{B}}(|y|^{2-2\gamma-d}-1)[p(p-1)|u(y)|^{p-2}|\nabla u(y)|^2\\ &\quad +2\sum_{\alpha\in R_+} k(\alpha)\frac{F(u(y),u(\sigma_\alpha(y)))}{\langle\alpha,y\rangle^2}]\omega_k(y)dy.\end{aligned}$$

In fact, a Δ_k-harmonic function u on $\mathbb{B}$ belongs to $H_k^p(\mathbb{B})$ if and only if the integral above is finite.

Proof Suppose v is Δ_k-subharmonic on $\mathbb{B}$. Then v_r is Δ_k-subharmonic on $\overline{\mathbb{B}}$ for any $0 < r < 1$. By Theorem 4.2,

$$v(0) = \frac{1}{d_k}\int_{\mathbb{S}} v_r(y)\omega_k(y)d\sigma(y) - \int_{\mathbb{B}} G_k(0,y)(\Delta_k v_r)(y)\omega_k(y)dy. \tag{4.9}$$

Since $(\Delta_k v_r)(x) = r^2(\Delta_k v)_r(x)$, by Eq. 3.5 we have

$$\begin{aligned}\int_{\mathbb{B}} G_k(0,y)(\Delta_k v_r)(y)\omega_k(y)dy &= C_k r^2\int_{\mathbb{B}}(|y|^{2-2\gamma-d}-1)(\Delta_k v)(ry)\omega_k(y)dy\\ &= C_k\int_{B(0,r)}(|z|^{2-2\gamma-d}-r^{2-2\gamma-d})\Delta_k v(z)\omega_k(z)dz, \quad (4.10)\end{aligned}$$

where $B(0,r) := \{x \in \mathbb{R}^d : |x| < r\}$. Let now u be Δ_k-harmonic on $\mathbb{B}$ and suppose first $2 \leq p < \infty$. Then $|u|^p$ is of class C^2 on $\mathbb{B}$ and by Eq. 4.7 we have

$$\Delta_k|u(x)|^p = p(p-1)|u(x)|^{p-2}|\nabla u(x)|^2 + 2\sum_{\alpha\in R_+} k(\alpha)\frac{F(u(x),u(\sigma_\alpha(x)))}{\langle \alpha, x\rangle^2}.$$

In particular, $\Delta_k|u|^p \geq 0$ on $\mathbb{B}$ so Eqs. 4.9 and 4.10 apply to $v = |u|^p$. Let $r \to 1$. By Eq. 4.2,

$$\frac{1}{d_k}\int_{\mathbb{S}} |u(ry)|^p \omega_k(y)d\sigma(y) \to \|u\|^p_{H^p},$$

and by the monotone convergence,

$$\begin{aligned}&C_k\int_{B(0,r)}(|y|^{2-2\gamma-d} - r^{2-2\gamma-d})\Delta_k|u(z)|^p\omega_k(z)dz\\ &\to \int_{\mathbb{B}} G_k(0,z)\Delta_k|u(z)|^p\omega_k(z)dz.\end{aligned}$$

This gives the result for $p \geq 2$. Assume now $1 < p < 2$ and let $\varepsilon > 0$. Then $|u + i\varepsilon|^p$ is of class C^2 on $\mathbb{B}$ and by Eq. 4.8 we have

$$\begin{aligned}\Delta_k|u(x)+i\varepsilon|^p &= p|u(x)+i\varepsilon|^{p-4}\left[(p-1)u(x)^2+\varepsilon^2\right]|\nabla u(x)|^2\\ &\quad +2\sum_{\alpha\in R_+} k(\alpha)\frac{F_\varepsilon(u(x),u(\sigma_\alpha(x)))}{\langle \alpha, x\rangle^2}.\end{aligned}$$

Since $\Delta_k|u + i\varepsilon|^p \geq 0$ on $\mathbb{B}$, we can apply Eqs. 4.9 and 4.10 to $v = |u + i\varepsilon|^p$. This gives

$$\begin{aligned}|u(0)+i\varepsilon|^p &= \frac{1}{d_k}\int_{\mathbb{S}}|u(ry)+i\varepsilon|^p\omega_k(y)d\sigma(y)\\ &\quad -C_k\int_{B(0,r)}(|y|^{2-2\gamma-d}-r^{2-2\gamma-d})\Delta_k|u(y)+i\varepsilon|^p\omega_k(y)dy.\end{aligned}$$

Let $\varepsilon \to 0$. Then

$$\Delta_k|u(x)+i\varepsilon|^p \to p(p-1)|u(x)|^{p-2}|\nabla u(x)|^2 + 2\sum_{\alpha\in R_+} k(\alpha)\frac{F(u(x),u(\sigma_\alpha(x)))}{\langle \alpha, x\rangle^2}$$

for a.e. $x \in \mathbb{B}$, and

$$\int_{\mathbb{S}}|u(ry)+i\varepsilon|^p\omega_k(y)d\sigma(y) \to \int_{\mathbb{S}}|u(ry)|^p\omega_k(y)d\sigma(y).$$

Fatou's lemma, Eq. 4.6 and dominated convergence give

$$\begin{aligned}|u(0)|^p &= \frac{1}{d_k}\int_{\mathbb{S}}|u(ry)|^p\omega_k(y)d\sigma(y) - C_k\int_{B(0,r)}(|y|^{2-2\gamma-d}-r^{2-2\gamma-d})\\ &\quad\times[p(p-1)|u(x)|^{p-2}|\nabla u(x)|^2 + 2\sum_{\alpha\in R_+}k(\alpha)\frac{F(u(x),u(\sigma_\alpha(x)))}{\langle\alpha,x\rangle^2}]\omega_k(y)dy.\end{aligned}$$

Let $r \to 1$. The final conclusion follows from Eq. 4.2 and monotone convergence. □

An immediate consequence of Theorem 4.5 and [26, Theorem 2.2 and Theorem 2.3] is the following identity.

Corollary 4.6 *Let* $1 < p < \infty$, $f \in L^p(\mathbb{S}, \omega_k\sigma)$ *and set* $u = P_k[f]$. *Then*

$$\int_{\mathbb{S}} |f(x)|^p \omega_k(x) d\sigma(x) = |u(0)|^p + C_k \int_{\mathbb{B}} (|y|^{2-2\gamma-d} - 1) \times [p(p-1)|u(y)|^{p-2}|\nabla u(y)|^2 + 2\sum_{\alpha\in R_+} k(\alpha) \frac{F(u(y), u(\sigma_\alpha(y)))}{\langle \alpha, y\rangle^2}]\omega_k(y)dy.$$

5 Sharp Estimates of the Green Function and Poisson Kernel in Rank One

In this part we consider the rank one case. The basic situation is that of the root system $A_1 = \{\pm\frac{1}{\sqrt{2}}(e_1 - e_2)\}$ in $\mathbb{R}^2$, where e_1, e_2 denote the standard basis vectors. We choose $\alpha = \frac{1}{\sqrt{2}}(e_1 - e_2)$ as positive root and let $\sigma_\alpha(x_1, x_2) = (x_2, x_1)$ denote the reflection corresponding to α. To simplify formulas, it will be convenient to switch to the orthonormal basis

$$(e_1', e_2') = (\frac{1}{\sqrt{2}}(e_1 - e_2), \frac{1}{\sqrt{2}}(e_1 + e_2))$$

and write $x \in \mathbb{R}^2$ as $x = (x_1, x_2)$ with coordinates x_1, x_2 with respect to the basis (e_1', e_2'). The reflection σ writes $\sigma(x_1, x_2) = (-x_1, x_2)$. By formula (2.2) we obtain

$$V_k f(y) = c_k \int_{-1}^{1} f(ty_1, y_2)(1-t)^{k-1}(1+t)^k dt.$$

This case has a nice motivation, namely the potential theory of a 2-dimensional k-Dyson Brownian Motion, which corresponds to the W-invariant Dunkl process in this case.

More generally, we will consider the rank one case with root system A_1 in $\mathbb{R}^d$, with the intertwining operator given by

$$V_k f(y) = c_k \int_{-1}^{1} f(ty_1, y_2, ..., y_d)(1-t)^{k-1}(1+t)^k dt. \tag{5.1}$$

Though this generalization seems elementary from the algebraic point of view, it reveals nontrival analytic phenomena which are strongly dependent on the underlying dimension. Note that in the rank one case we have $\gamma = k$, and as before we work under the assumption $d + 2k > 2$.

The Newton kernel (3.1) can be written as

$$N_k(x, y) = \widetilde{C}_k \int_{-1}^{1} \frac{(1-t)^{k-1}(1+t)^k dt}{\left(|x|^2 + |y|^2 - 2(tx_1y_1 + x_2y_2 + ... + x_dy_d)\right)^{k+d/2-1}}, \tag{5.2}$$

where $\widetilde{C}_k = c_k C_k$ and the constants c_k, C_k were defined in Eqs. 2.2 and 3.2. The reflection σ writes

$$\sigma(x_1, x_2, ..., x_d) = (-x_1, x_2, ..., x_d).$$

We then have

$$\begin{aligned} |x|^2 + |y|^2 - 2(tx_1y_1 + x_2y_2 + ... + x_dy_d) &= |x-y|^2 + 2x_1y_1(1-t) \\ &= |x - \sigma y|^2 - 2x_1y_1(1+t). \end{aligned} \tag{5.3}$$

Our first result in this section characterizes the asymptotic behaviour of the Newton kernel $N_k(x, y)$.

Theorem 5.1 *Let* $\Phi(x, y) := |x - y| \vee |x - \sigma y|$. *The two-sided bound of* $N_k(x, y)$ *on* $\mathbb{R}^d \times \mathbb{R}^d$ *is the following.*

1. *If* $d = 2$, *then*
$$N_k(x, y) \asymp \frac{1}{\Phi(x, y)^{2k}} \left[1 \vee \log\left(\frac{|x_1 y_1|}{|x - y|^2}\right)\right]. \tag{5.4}$$
2. *If* $d = 3$, *then*
$$N_k(x, y) \asymp \frac{1}{\Phi(x, y)^{2k}|x - y|}. \tag{5.5}$$
3. *If* $d = 4$, *then*
$$N_k(x, y) \asymp \frac{1}{\Phi(x, y)^{2k}|x - y|^2} \left[1 \vee \log\left(\frac{|x_1 y_1|}{|x - \sigma y|^2}\right)\right]. \tag{5.6}$$
4. *If* $d \geq 5$, *then*
$$N_k(x, y) \asymp \frac{1}{\Phi(x, y)^{2k}|x - y|^2(|x - y| \wedge |x - \sigma y|)^{d-4}}. \tag{5.7}$$

Theorem 5.1 is a direct consequence of Lemma 5.2 and Lemma 5.3 below.

Lemma 5.2 *The two-sided bound of* $N_k(x, y)$ *on* $\left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x_1 y_1 \geq 0\right\}$ *is as follows.*

1. *If* $d = 2$, *then*
$$N_k(x, y) \asymp \frac{1}{|x - \sigma y|^{2k}} \left[1 \vee \log\left(\frac{x_1 y_1}{|x - y|^2}\right)\right]. \tag{5.8}$$
2. *If* $d \geq 3$, *then*
$$N_k(x, y) \asymp \frac{1}{|x - \sigma y|^{2k}|x - y|^{d-2}}. \tag{5.9}$$

Proof Denote $\zeta = |x - y|^2$ and $\eta = x_1 y_1$. Since $x_1 y_1 \geq 0$ we have $\zeta + \eta \asymp |x - \sigma y|^2$. By Eqs. 5.2 and 5.3 we have
$$N_k(x, y) = C_k \int_{-1}^{1} \frac{(1-t)^{k-1}(1+t)^k dt}{(\zeta + 2\eta(1-t))^{k+d/2-1}} = C_k \int_0^2 \frac{s^{k-1}(2-s)^k ds}{(\zeta + 2\eta s)^{k+d/2-1}}. \tag{5.10}$$
We write $N_k(x, y) = C_k(I_1 + I_2)$, where
$$I_1 = \int_0^1 \frac{s^{k-1}(2-s)^k ds}{(\zeta + 2\eta s)^{k+d/2-1}} \asymp \int_0^1 \frac{s^{k-1} ds}{(\zeta + \eta s)^{k+d/2-1}},$$
and
$$I_2 = \int_1^2 \frac{s^{k-1}(2-s)^k ds}{(\zeta + 2\eta s)^{k+d/2-1}} \asymp \int_1^2 \frac{(2-s)^k ds}{(\zeta + \eta s)^{k+d/2-1}} \asymp (\zeta + \eta)^{1-k-d/2}.$$
For $\eta = 0$ the estimates of the lemma are obvious, so assume $\eta > 0$. Using the change of variables $s = u\zeta$ we get
$$I_1 \asymp \int_0^{1/\zeta} \frac{\zeta^k u^{k-1} du}{(\zeta + \zeta\eta u)^{k+d/2-1}} = \zeta^{1-d/2} \int_0^{1/\zeta} \frac{u^{k-1} du}{(1 + \eta u)^{k+d/2-1}}$$
$$= \zeta^{1-d/2} \int_0^{1/\zeta} \frac{du}{u^{d/2}(1/u + \eta)^{k+d/2-1}} = \zeta^{1-d/2} \int_\zeta^\infty \frac{w^{d/2-2} dw}{(w + \eta)^{k+d/2-1}}.$$

Let $d = 2$ and assume first $\eta \le \zeta$. Then

$$I_1 \asymp \int_\zeta^\infty \frac{dw}{w(w \vee \eta)^k} = \int_\zeta^\infty w^{-k-1} dw = \zeta^{-k}/k \asymp (\zeta + \eta)^{-k},$$

and note that the same two-sided estimate holds also for I_2. Assume $\eta > \zeta$. We have

$$I_1 \asymp \int_\zeta^\infty \frac{dw}{w(w \vee \eta)^k} = \int_\zeta^\eta \frac{dw}{w\eta^k} + \int_\eta^\infty \frac{dw}{w^{k+1}} = \eta^{-k} \log(\eta/\zeta) + \eta^{-k}/k$$
$$\asymp (\zeta + \eta)^{-k} \left[1 \vee \log(\eta/\zeta)\right].$$

It is clear that the estimate above holds also for $I_1 + I_2$, and combining it with the previous case we get (5.8).

Assume $d \ge 3$. For $\eta \le \zeta$ we get

$$I_1 \asymp \zeta^{1-d/2} \int_\zeta^\infty \frac{w^{d/2-2} dw}{(w \vee \eta)^{k+d/2-1}} = \zeta^{1-k-d/2}/k \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}.$$

When $\eta > \zeta$, then a similar reasoning as before gives

$$I_1 \asymp \zeta^{1-d/2} \left(\int_\zeta^\eta \frac{w^{d/2-2} dw}{\eta^{k+d/2-1}} + \int_\eta^\infty \frac{dw}{w^{k+1}} \right)$$
$$= \frac{1}{\eta^{k+d/2-1} \zeta^{d/2-1}} \left[\frac{2}{d-2} \left(\eta^{d/2-1} - \zeta^{d/2-1} \right) + \frac{\eta^{d/2-1}}{k} \right].$$

Since $0 < \eta^{d/2-1} - \zeta^{d/2-1} \le \eta^{d/2-1}$, we obtain

$$I_1 \asymp \eta^{-k} \zeta^{1-d/2} \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}.$$

Finally,

$$I_2 \asymp (\zeta + \eta)^{1-k-d/2} \le (\zeta + \eta)^{-k} \zeta^{1-d/2},$$

and hence $I_1 + I_2 \asymp I_1$. This proves (5.9). □

Lemma 5.3 *The two-sided bound of* $N_k(x, y)$ *on* $\left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x_1 y_1 < 0\right\}$ *is as follows.*

1. *If* $2 \le d \le 3$, *then*

$$N_k(x, y) \asymp \frac{1}{|x - y|^{2k+d-2}}. \tag{5.11}$$

2. *If* $d = 4$, *then*

$$N_k(x, y) \asymp \frac{1}{|x - y|^{2k+2}} \left[1 \vee \log \left(\frac{|x_1 y_1|}{|x - \sigma y|^2} \right) \right]. \tag{5.12}$$

3. *If* $d \ge 5$, *then*

$$N_k(x, y) \asymp \frac{1}{|x - y|^{2k+2} |x - \sigma y|^{d-4}}. \tag{5.13}$$

Proof Denote $\zeta = |x - \sigma y|^2$ and $\eta = |x_1 y_1|$. Since $x_1 y_1 < 0$ we have $\zeta + \eta \asymp |x - y|^2$. By Eqs. 5.2 and 5.3 we have

$$N_k(x, y) = C_k \int_{-1}^1 \frac{(1-t)^{k-1}(1+t)^k dt}{(\zeta + 2\eta(1+t))^{k+d/2-1}} = C_k \int_0^2 \frac{(2-s)^{k-1} s^k ds}{(\zeta + 2\eta s)^{k+d/2-1}}. \tag{5.14}$$

We write $N_k(x, y) = C_k(I_1 + I_2)$, where

$$I_1 = \int_0^1 \frac{(2-s)^{k-1}s^k ds}{(\zeta + 2\eta s)^{k+d/2-1}} \asymp \int_0^1 \frac{s^k ds}{(\zeta + \eta s)^{k+d/2-1}},$$

and

$$I_2 = \int_1^2 \frac{(2-s)^{k-1}s^k ds}{(\zeta + 2\eta s)^{k+d/2-1}} \asymp \int_1^2 \frac{(2-s)^{k-1} ds}{(\zeta + \eta s)^{k+d/2-1}} \asymp (\zeta + \eta)^{1-k-d/2}.$$

As in the proof of Lemma 5.2, we apply the change of variables $s = u\zeta$ and get

$$I_1 \asymp \zeta^{2-d/2} \int_0^{1/\zeta} \frac{u^k du}{(1 + \eta u)^{k+d/2-1}} = \zeta^{2-d/2} \int_\zeta^\infty \frac{w^{d/2-3} dw}{(w + \eta)^{k+d/2-1}}.$$

Assume $\eta \le \zeta$. Then

$$I_1 \asymp \zeta^{2-d/2} \int_\zeta^\infty w^{-k-2} dw \asymp \zeta^{1-k-d/2} \asymp (\zeta + \eta)^{1-k-d/2}. \tag{5.15}$$

When $\eta > \zeta$ we have

$$I_1 \asymp \zeta^{2-d/2} \int_\zeta^\infty \frac{w^{d/2-3} dw}{(w \vee \eta)^{k+d/2-1}} = \zeta^{2-d/2} \left(\int_\zeta^\eta \frac{w^{d/2-3} dw}{\eta^{k+d/2-1}} + \int_\eta^\infty \frac{dw}{w^{k+2}} \right)$$
$$= \zeta^{2-d/2} \left(\frac{1}{\eta^{k+d/2-1}} \int_\zeta^\eta w^{d/2-3} dw + \frac{1}{(k+1)\eta^{k+1}} \right).$$

At this point one needs to consider different values of d separately. We show only the case $d \ge 5$. For $d = 2, 3, 4$ the reasoning is similar. We have

$$I_1 \asymp \zeta^{2-d/2} \left[\frac{2\left(\eta^{d/2-2} - \zeta^{d/2-2}\right)}{(d-4)\eta^{k+d/2-1}} + \frac{1}{(k+1)\eta^{k+1}} \right] \asymp (\zeta + \eta)^{-k-1} \zeta^{2-d/2}.$$

For $\eta \le \zeta$ we have $\zeta + \eta \asymp \zeta$, and by Eq. 5.15 we get

$$I_1 \asymp (\zeta + \eta)^{1-k-d/2} \asymp (\zeta + \eta)^{-k-1} \zeta^{2-d/2}.$$

Since $d \ge 5$, the upper bound of the last estimate also dominates I_2. This proves (5.13). □

We will next give sharp two-sided estimates of $G_k(x, y)$ in the rank one case. Recall the notation $\rho(x) := 1 - |x|$.

Theorem 5.4 *Let $\Phi(x, y) := |x - y| \vee |x - \sigma y|$. The two-sided bound of $G_k(x, y)$ on $\mathbb{B} \times \mathbb{B}$ is the following.*

1. *If $d = 2$, then*

$$G_k(x, y) \asymp \frac{1}{\Phi(x, y)^{2k}} \left(1 \wedge \frac{\rho(x)\rho(y)}{|x - y|^2} \right) \left[1 \vee \log \left(\frac{|x_1 y_1| \wedge \rho(x)\rho(y)}{|x - y|^2} \right) \right]$$
$$\times \left[1 \vee \log \left(\frac{|x_1 y_1|}{\rho(x)\rho(y) \vee |x - \sigma y|^2} \right) \right]. \tag{5.16}$$

2. *If $d = 3$, then*

$$G_k(x, y) \asymp \frac{1}{\Phi(x, y)^{2k}|x - y|} \left(1 \wedge \frac{\sqrt{\rho(x)\rho(y)}}{|x - y|} \right) \tag{5.17}$$
$$\times \left(1 \wedge \frac{\sqrt{\rho(x)\rho(y)}}{|x - y| \wedge |x - \sigma y|} \right).$$

3. *If $d = 4$, then*

$$G_k(x,y) \asymp \frac{1}{\Phi(x,y)^{2k}|x-y|^2}\left(1 \wedge \frac{\rho(x)\rho(y)}{|x-y|^2 \wedge |x-\sigma y|^2}\right) \times \left[1 \vee \log\left(\frac{|x_1 y_1| \wedge \rho(x)\rho(y)}{|x-\sigma y|^2}\right)\right]. \tag{5.18}$$

4. *If $d \geq 5$, then*

$$G_k(x,y) \asymp \frac{1}{\Phi(x,y)^{2k}(|x-y| \wedge |x-\sigma y|)^{d-4}|x-y|^2} \times \left(1 \wedge \frac{\rho(x)\rho(y)}{|x-y|^2 \wedge |x-\sigma y|^2}\right). \tag{5.19}$$

Theorem 5.4 is a direct consequence of Lemma 5.5 and Lemma 5.6 below.

Lemma 5.5 *The two-sided bound of $G_k(x,y)$ on $\{(x,y) \in \mathbb{B} \times \mathbb{B} : x_1 y_1 \geq 0\}$ is the following.*

1. *If $d = 2$, then*

$$G_k(x,y) \asymp \frac{1}{|x-\sigma y|^{2k}}\left(1 \wedge \frac{\rho(x)\rho(y)}{|x-y|^2}\right) \times \left[1 \vee \log\left(\frac{x_1 y_1 \wedge \rho(x)\rho(y)}{|x-y|^2}\right)\right]. \tag{5.20}$$

2. *If $d \geq 3$, then*

$$G_k(x,y) \asymp \frac{1}{|x-\sigma y|^{2k}|x-y|^{d-2}}\left(1 \wedge \frac{\rho(x)\rho(y)}{|x-y|^2}\right). \tag{5.21}$$

Proof Let $\zeta = |x-y|^2$, $\eta = x_1 y_1$, and $\xi = \rho(x)\rho(y)$. By Theorem 3.2, Eqs. 5.1 and 5.3 we have

$$G_k(x,y) \asymp \int_{-1}^{1} \frac{\xi(1-t)^{k-1}(1+t)^k dt}{(\xi+\zeta+\eta(1-t))(\zeta+\eta(1-t))^{k+d/2-1}}. \tag{5.22}$$

Assume first $\xi \leq \zeta$. Then by Eq. 5.22,

$$G_k(x,y) \asymp \int_{-1}^{1} \frac{\xi(1-t)^{k-1}(1+t)^k dt}{(\zeta+\eta(1-t))^{k+d/2}},$$

and observe that the same integral appears in Eq. 5.10 with $d' = d+2$ instead of d. Hence, by Eq. 5.9 we get

$$G_k(x,y) \asymp \xi(\zeta+\eta)^{-k}\zeta^{-d/2}. \tag{5.23}$$

Assume $\xi > \zeta$. Using Eq. 5.22 and the substitution $s = 1-t$ we obtain

$$G_k(x,y) \asymp \int_0^2 \frac{\xi s^{k-1}(2-s)^k ds}{(\xi+\eta s)(\zeta+\eta s)^{k+d/2-1}} = I_1 + I_2,$$

where

$$I_1 \asymp \int_0^1 \frac{\xi s^{k-1} ds}{(\xi+\eta s)(\zeta+\eta s)^{k+d/2-1}}, \tag{5.24}$$

and

$$I_2 \asymp \int_1^2 \frac{\xi(2-s)^k ds}{(\xi+\eta s)(\zeta+\eta s)^{k+d/2-1}} \asymp \frac{\xi}{(\xi+\eta)(\zeta+\eta)^{k+d/2-1}}. \tag{5.25}$$

We will estimate I_1 assuming $d \geq 3$. The case $d = 2$ needs to be considered separately and the reasoning is similar.

(a) $\xi \geq \eta$. By Eq. 5.24 and the estimates from the proof of Lemma 5.2 we have

$$I_1 \asymp \int_0^1 \frac{s^{k-1}ds}{(\zeta + \eta s)^{k+d/2-1}} \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}.$$

Combining this with Eq. 5.25 give $I_1 + I_2 \asymp I_1$.

(b) $\zeta < \xi < \eta$. By Eq. 5.24 we have $I_1 \asymp I_1^{(1)} + I_1^{(2)}$, where

$$\begin{aligned} I_1^{(1)} &= \int_0^{\xi/\eta} \frac{\xi s^{k-1}ds}{(\xi + \eta s)(\zeta + \eta s)^{k+d/2-1}} \asymp \int_0^{\xi/\eta} \frac{s^{k-1}ds}{(\zeta + \eta s)^{k+d/2-1}} \\ &\asymp \eta^{1-k-d/2} \left((\eta/\zeta)^{k+d/2-1} \int_0^{\zeta/\eta} s^{k-1}ds + \int_{\zeta/\eta}^{\xi/\eta} s^{-d/2}ds \right) \\ &\asymp \eta^{1-k-d/2} (\eta/\zeta)^{d/2-1} \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}, \end{aligned}$$

and

$$\begin{aligned} I_1^{(2)} &= \int_{\xi/\eta}^1 \frac{\xi s^{k-1}ds}{(\xi + \eta s)(\zeta + \eta s)^{k+d/2-1}} \asymp \int_{\xi/\eta}^1 \frac{\xi ds}{\eta^{k+d/2} s^{d/2+1}} \\ &= \frac{2\xi}{d\eta^{k+d/2}} \left[(\eta/\xi)^{d/2} - 1 \right] \leq \eta^{-k} \xi^{1-d/2} \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}. \end{aligned}$$

Hence

$$I_1 = I_1^{(1)} + I_1^{(2)} \asymp I_1^{(1)} \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}.$$

This and Eq. 5.25 give $I_1 + I_2 \asymp I_1$.

Altogether, $G_k(x, y) \asymp (\zeta + \eta)^{-k} \zeta^{1-d/2}$ for $\xi > \zeta$, and Eq. 5.23 otherwise. This proves (5.21). □

Lemma 5.6 *The two-sided bound of $G_k(x, y)$ on $\{(x, y) \in \mathbb{B} \times \mathbb{B} : x_1 y_1 < 0\}$ is the following.*

1. *If $d = 2$, then*

$$G_k(x, y) \asymp \frac{1}{|x - y|^{2k}} \left(1 \wedge \frac{\rho(x)\rho(y)}{|x - y|^2} \right) \times \left[1 \vee \log \left(\frac{|x_1 y_1|}{\rho(x)\rho(y) \vee |x - \sigma y|^2} \right) \right]. \tag{5.26}$$

2. *If $d = 3$, then*

$$G_k(x, y) \asymp \frac{1}{|x - y|^{2k+1}} \left(1 \wedge \frac{\sqrt{\rho(x)\rho(y)}}{|x - \sigma y|} \right) \left(1 \wedge \frac{\sqrt{\rho(x)\rho(y)}}{|x - y|} \right). \tag{5.27}$$

3. *If $d = 4$, then*

$$G_k(x, y) \asymp \frac{1}{|x - y|^{2k+2}} \left(1 \wedge \frac{\rho(x)\rho(y)}{|x - \sigma y|^2} \right) \times \left[1 \vee \log \left(\frac{|x_1 y_1| \wedge \rho(x)\rho(y)}{|x - \sigma y|^2} \right) \right]. \tag{5.28}$$

4. *If $d \geq 5$, then*

$$G_k(x,y) \asymp \frac{1}{|x-y|^{2k+2}|x-\sigma y|^{d-4}} \left(1 \wedge \frac{\rho(x)\rho(y)}{|x-\sigma y|^2}\right). \tag{5.29}$$

Proof Denote $\zeta = |x-\sigma y|^2$, $\eta = |x_1 y_1|$, and $\xi = \rho(x)\rho(y)$. By Theorem 3.2, Eqs. 5.1 and 5.3 we have

$$G_k(x,y) \asymp \int_{-1}^{1} \frac{\xi(1-t)^{k-1}(1+t)^k dt}{(\xi+\zeta+\eta(1+t))(\zeta+\eta(1+t))^{k+d/2-1}}. \tag{5.30}$$

Assume first $\xi \leq \zeta$. Then by Eq. 5.30,

$$G_k(x,y) \asymp \int_{-1}^{1} \frac{\xi(1-t)^{k-1}(1+t)^k dt}{(\zeta+\eta(1+t))^{k+d/2}}.$$

Let $d = 2$. Using the estimate derived for Eq. 5.14 with $d' = 4$ instead of d, we get by Eq. 5.12 that

$$G_k(x,y) \asymp \xi(\zeta+\eta)^{-k-1}\left(1 \vee \log(\eta/\zeta)\right). \tag{5.31}$$

If $d \geq 3$, then Eq. 5.13 with $d' = d+2$ instead of d gives

$$G_k(x,y) \asymp \xi(\zeta+\eta)^{-k-1}\zeta^{1-d/2}. \tag{5.32}$$

Assume $\xi > \zeta$. Using (5.30) and substituting $s = t+1$ we get

$$G_k(x,y) \asymp \int_0^2 \frac{\xi(2-s)^{k-1}s^k ds}{(\xi+\eta s)(\zeta+\eta s)^{k+d/2-1}} = I_1 + I_2,$$

where

$$I_1 \asymp \int_0^1 \frac{\xi s^k ds}{(\xi+\eta s)(\zeta+\eta s)^{k+d/2-1}}, \tag{5.33}$$

and

$$I_2 \asymp \int_1^2 \frac{\xi(2-s)^{k-1} ds}{(\xi+\eta s)(\zeta+\eta s)^{k+d/2-1}} \asymp \frac{\xi}{(\xi+\eta)(\zeta+\eta)^{k+d/2-1}}. \tag{5.34}$$

In order to estimate I_1 we need to consider several cases.

(a) $\xi \geq \eta$. Then the estimate depends on the dimension as follows.

(i) $2 \leq d \leq 3$. Equation 5.33 and the estimates derived in the proof of Lemma 5.3 give

$$I_1 \asymp \int_0^1 \frac{s^k ds}{(\zeta+\eta s)^{k+d/2-1}} \asymp (\zeta+\eta)^{1-k-d/2}.$$

In view of Eq. 5.34, we also have $I_1 + I_2 \asymp I_1$.

(ii) $d = 4$. The same arguments as above give

$$I_1 \asymp \int_0^1 \frac{s^k ds}{(\zeta+\eta s)^{k+1}} \asymp (\zeta+\eta)^{-k-1}\left(1 \vee \log(\eta/\zeta)\right),$$

and $I_1 + I_2 \asymp I_1$.

(iii) $d \geq 5$. We get $I_1 + I_2 \asymp I_1 \asymp (\zeta+\eta)^{-k-1}\zeta^{2-d/2}$.

(b) $\xi < \eta$. Then $\zeta < \xi < \eta$. By Eq. 5.33, for any $d \geq 2$ we have $I_1 \asymp I_1^{(1)} + I_1^{(2)}$, where

$$I_1^{(1)} = \int_0^{\xi/\eta} \frac{\xi s^k ds}{(\xi + \eta s)(\zeta + \eta s)^{k+d/2-1}} \asymp \int_0^{\xi/\eta} \frac{s^k ds}{(\zeta + \eta s)^{k+d/2-1}}$$
$$\asymp \eta^{1-k-d/2} \left((\eta/\zeta)^{k+d/2-1} \int_0^{\zeta/\eta} s^k ds + \int_{\zeta/\eta}^{\xi/\eta} s^{1-d/2} ds \right), \tag{5.35}$$

and

$$I_1^{(2)} = \int_{\xi/\eta}^1 \frac{\xi s^k ds}{(\xi + \eta s)(\zeta + \eta s)^{k+d/2-1}} \asymp \frac{\xi}{\eta^{k+d/2}} \int_{\xi/\eta}^1 s^{-d/2} ds. \tag{5.36}$$

At this point we assume $d \geq 5$. For $d = 2, 3, 4$ the reasoning is similar and we omit the details. By Eq. 5.35,

$$I_1^{(1)} \asymp \eta^{1-k-d/2} \left[2\,(\eta/\zeta)^{d/2-2} - (\eta/\xi)^{d/2-2} \right] \asymp \eta^{-k-1} \zeta^{2-d/2},$$

and by Eq. 5.36,

$$I_1^{(2)} \asymp \xi \eta^{-k-d/2} \left[(\eta/\xi)^{d/2-1} - 1 \right] \leq \eta^{-k-1} \xi^{2-d/2} \leq \eta^{-k-1} \zeta^{2-d/2}.$$

It follows that $I_1 \asymp I_1^{(1)} + I_1^{(2)} \asymp I_1^{(1)}$. Furthermore, by Eq. 5.34,

$$I_2 \leq (\zeta + \eta)^{1-k-d/2} \leq (\zeta + \eta)^{-k-1} \zeta^{2-d/2}.$$

Hence

$$I_1 + I_2 \asymp I_1 \asymp \eta^{-k-1} \zeta^{2-d/2} \asymp (\zeta + \eta)^{-k-1} \zeta^{2-d/2}.$$

The same estimate holds also in (a)(iii). Combining this with Eq. 5.32 we obtain (5.29).

□

By Eqs. 2.7 and 5.1, the Poisson kernel in the rank one case in $\mathbb{R}^d$ can be written as

$$P_k(x, y) = c_k \int_{-1}^1 \frac{(1 - |x|^2)(1 - t)^{k-1}(1 + t)^k dt}{(|x|^2 + 1 - 2(t x_1 y_1 + x_2 y_2 + ... + x_d y_d))^{k+d/2}}. \tag{5.37}$$

As a consequence of the two-sided bounds of the Newton kernel obtained in Theorem 5.1 we get the following two-sided estimates of $P_k(x, y)$.

Corollary 5.7 *Let $\Phi(x, y) := |x - y| \vee |x - \sigma y|$. The two-sided bound of $P_k(x, y)$ on $\mathbb{B} \times \mathbb{S}$ is the following.*

1. *If $d = 2$, then*

$$P_k(x, y) \asymp \frac{\rho(x)}{\Phi(x, y)^{2k} |x - y|^2} \left[1 \vee \log \left(\frac{|x_1 y_1|}{|x - \sigma y|^2} \right) \right]. \tag{5.38}$$

2. *If $d \geq 3$, then*

$$P_k(x, y) \asymp \frac{\rho(x)}{\Phi(x, y)^{2k} |x - y|^2 (|x - y| \wedge |x - \sigma y|)^{d-2}}. \tag{5.39}$$

Proof In view of the formulas Eqs. 5.2 and 5.37, we can apply Theorem 5.1 with $d' = d+2$ instead of d. Hence, Eq. 5.38 follows from Eqs. 5.6 and 5.39 follows from Eq. 5.7. □

Remark 5.8 When $d = 1$, the condition $k > 1/2$ guarantees that $N_k(x, y)$ is well defined and finite, and hence also $G_k(x, y)$ and $P_k(x, y)$. Using the methods of this section one can derive the following two-sided estimates.

$$N_k(x, y) \asymp (|x| + |y|)^{1-2k}, \quad x, y \in \mathbb{R},$$
$$G_k(x, y) \asymp \frac{\sqrt{\rho(x)\rho(y)}}{(|x| + |y|)^{2k-1}} \left(1 \wedge \frac{\sqrt{\rho(x)\rho(y)}}{|x-y|}\right), \quad x, y \in (-1, 1),$$
$$P_k(x, y) \asymp \rho(x), \quad x \in (-1, 1), y \in \{-1, 1\}.$$

Remark 5.9 It is noteworthy that the explicit formulas for $N_k(x, y)$, $G_k(x, y)$ and $P_k(x, y)$ can be obtained in some particular cases, e.g., for $k \in \mathbb{N}$ and $d \in 2\mathbb{N}$ the integrands in formulas Eqs. 5.2 and 5.37 are rational functions of t. For instance, when $k = 1$ and $d = 2$ (i.e. for the root system A_1 in $\mathbb{R}^2$), we can derive the following explicit expressions

$$N_1(x, y) = \frac{1}{4\pi}\left[\frac{|x-\sigma y|^2}{2x_1^2 y_1^2}\log\left(\frac{|x-\sigma y|}{|x-y|}\right) - \frac{1}{x_1 y_1}\right], \tag{5.40}$$
$$P_1(x, y) = \frac{1-|x|^2}{4x_1^2 y_1^2}\left[\frac{2x_1 y_1}{|x-y|^2} + \log\left(\frac{|x-y|}{|x-\sigma y|}\right)\right],$$
$$G_1(x, y) = \frac{|x-\sigma y|^2}{8\pi x_1^2 y_1^2}\log\left(\frac{|x-\sigma y|}{|x-y|}\right) - \frac{|x|^2|x^*-\sigma y|^2}{8\pi x_1^2 y_1^2}\log\left(\frac{|x^*-\sigma y|}{|x^*-y|}\right).$$

Remark 5.10 ***W*-radial case and applications to the Dyson Brownian Motion.** The results of this paper can be applied to the W-invariant part of the Dunkl Laplacian,

$$\Delta_k^W f(x) = \Delta f(x) + \sum_{\alpha\in R_+} k(\alpha)\frac{\partial_\alpha f(x)}{\langle\alpha, x\rangle}.$$

Notice that for $k = 1$ and $W = S_{d-1}$ this is just the generator of the d-dimensional Dyson Brownian motion. In fact, for all integral kernels $K(x, y)$ for Δ_k considered in the paper, the following formula holds

$$K^W(x, y) = \sum_{g\in W} K(x, gy), \tag{5.41}$$

where K^W is the corresponding kernel for the operator Δ_k^W.

In the rank one case with $k = 1$ and $d = 2$, formulas Eqs. 5.40 and 5.41 give

$$N_1^W(x, y) = \frac{1}{2\pi x_1 y_1}\log\left(\frac{|x-\sigma y|}{|x-y|}\right),$$
$$P_1^W(x, y) = \frac{2(1-|x|^2)}{|x-y|^2|x-\sigma y|^2},$$
$$G_1^W(x, y) = \frac{1}{2\pi x_1 y_1}\log\left(\frac{|x^*-y||x-\sigma y|}{|x-y||x^*-\sigma y|}\right).$$

Furthermore, by multiplying the above formulas by $\omega_1(y) = y_1^2$ and going back to the initial form $A_1 = \{\pm\frac{1}{\sqrt{2}}(e_1 - e_2)\}$ with the standard basis vectors e_1, e_2 one obtains the Newton kernel, Poisson kernel and Green function of the unit ball in the setting of the potential theory of 2-dimensional Dyson Brownian motion:

$$N_1^{Dys}(x,y) = \frac{1}{2\pi}\frac{\pi(y)}{\pi(x)}\log\left(\frac{|x-\sigma_\alpha y|}{|x-y|}\right),$$
$$P_1^{Dys}(x,y) = \frac{2\pi(y)^2(1-|x|^2)}{|x-y|^2|x-\sigma_\alpha y|^2},$$
$$G_1^{Dys}(x,y) = \frac{1}{2\pi}\frac{\pi(y)}{\pi(x)}\log\left(\frac{|x^*-y||x-\sigma_\alpha y|}{|x-y||x^*-\sigma_\alpha y|}\right),$$

where x, y are in the positive Weyl chamber $C^+ = \{(z_1, z_2) : z_1 > z_2\}$, $\pi(z) = z_1 - z_2$, and $\sigma_\alpha(z_1, z_2) = (z_2, z_1)$.

References

1. Ancona, A.: First eigenvalues and comparison of Green's functions for elliptic operators on manifolds or domains. J. Anal, Math. **72**, 45–92 (1997)
2. Aronson, D.G.: On the Green's function for second order parabolic differential equations with discontinuous coefficients. Bull. Amer. Math. Soc. **69**(6), 841–847 (1963)
3. Bogdan, K.: Sharp estimates for the Green function in Lipschitz domains. J. Math. Anal. Appl. **243**, 326–337 (2000)
4. Bogdan, K., Dyda, B., Luks, T.: On Hardy spaces of local and nonlocal operators. Hiroshima Math. J. **44**(2), 193–215 (2014)
5. Bogdan, K., Jakubowski, T.: Estimates of the Green function for the fractional Laplacian perturbed by gradient. Potential Anal. **36**(3), 455–481 (2012)
6. Chen, Z.-Q., Song, R.: Estimates on Green functions and Poisson kernels for symmetric stable processes. Math. Ann. **312**(3), 465–501 (1998)
7. Ben Chrouda, M.: On the Dirichlet problem associated with the Dunkl Laplacian. Ann. Polon. Math. **117**(1), 79–87 (2016)
8. Cranston, M., Zhao, Z.: Conditional transformation of drift formula and potential theory for $\frac{1}{2}\Delta + b(\cdot)\cdot\nabla$. Comm. Math. Phys. **112**, 613–625 (1987)
9. van Diejen, J.F., Vinet, L.: Calogero-Sutherland-Moser Models. Springer-Verlag, CRM Series in Mathematical Physics (2000)
10. Doob, J.L.: Classical Potential Theory and Its Probabilistic Counterpart. Springer-Verlag, New York (1984)
11. Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc. **311**, 167–183 (1989)
12. Dunkl, C.F.: Integral kernels with reflection group invariance. Canad. J. Math. **43**, 1213–1227 (1991)
13. Dunkl, C.F., Xu, Y.: Orthogonal Polynomials of Several Variables Encyclopedia of Mathematics and Its Applications, vol. 81. Cambridge University Press, Cambridge (2001)
14. El Kamel, J., Yacoub, C.H.: Poisson integrals and kelvin transform associated to Dunkl-Laplacian operator. Global J. Pure Appl. Math. **3**(3), 351 (2007)
15. Gallardo, L., Rejeb, C.: A new mean value property for harmonic functions relative to the Dunkl-Laplacian operator and applications. Trans. Amer. Math. Soc. **368**, 3727–3753 (2016)
16. Gallardo, L., Rejeb, C.: Newtonian potentials and subharmonic functions associated to root systems, preprint (2016), available at https://hal.archives-ouvertes.fr/hal-01368871
17. Gallardo, L., Yor, M.: Some new examples of Markov processes which enjoy the time-inversion property. Probab. Theory Relat. Fields **132**, 150–162 (2005)
18. Grüter, M., Widman, K.-O.: The Green function for uniformly elliptic equations. Manuscripta Math. **37**, 303–342 (1982)
19. Grzywny, T., Ryznar, M.: Estimates of Green functions for some perturbations of fractional Laplacian. Ill. J. Math. **51**(4), 1409–1438 (2007)
20. Hassine, K.: Mean value property of δ_k-harmonic functions on W-invariant open sets. Afr. Mat. **27**(7), 1275–1286 (2016)
21. Hayman, W.K., Kennedy, P.B.: Subharmonic functions, vol. 1. Academic Press, London (1976)

22. Jakubowski, T.: The estimates for the Green function in Lipschitz domains for the symmetric stable processes. Probab. Math. Stat. **22**, 419–441 (2002)
23. Kim, P., Song, R.: Estimates on Green functions and schrödinger-type equations for non-symmetric diffusions with measure-valued drifts. J. Math. Anal. Appl. **332**, 57–80 (2007)
24. Kulczycki, T.: Properties of Green function of symmetric stable processes. Probab. Math. Stat. **17**, 339–364 (1997)
25. Littman, W., Stampacchia, G., Weinberger, H.F.: Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa (III) **17**, 43–77 (1963)
26. Maslouhi, M., Youssfi, E.H.: Harmonic functions associated to Dunkl Laplacian. Monatsh. Math. **152**, 337–345 (2007)
27. Rejeb, C.: Harmonic and subharmonic functions associated to root systems, Ph.D. thesis, University of Tours, University of Tunis El Manar. available at https://tel.archives-ouvertes.fr/tel-01291741/ (2015)
28. Rösler, M.: Positivity of Dunkl's intertwining operator. Duke Math. J. **98**, 445–463 (1999)
29. Rösler, M.: A positive radial product formula for the Dunkl kernel. Trans. Amer. Math. Soc. **355**, 2413–2438 (2003)
30. Rösler, M.: Dunkl operators: Theory and applications. In: Orthogonal polynomials and special functions, Leuven 2002, Springer Lecture Notes in Math, vol. 1817, pp. 93–135 (2003)
31. Rösler, M., Voit, M.: Dunkl Theory, Convolution Algebras, and Related Markov Processes. In: Graczyk, P., Rösler, M., Yor, M. (eds.) Harmonic and Stochastic Analysis of Dunkl Processes, pp. 1–112. Hermann, Paris (2008). Travaux en cours 71
32. Stein, E.M.: Boundary Behavior of Holomorphic Functions of Several Complex Variables, Princeton University Press and University of Tokyo Press, Princeton, New Jersey (1972)
33. Trimèche, K.: Paley-wiener Theorems for the Dunkl transform and Dunkl translation operators. Integral Transform. Spec. Funct. **13**, 17–38 (2002)
34. Widman, K.-O.: Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations. Math. Scand. **21**, 17–37 (1967)
35. Zhao, Z.: Uniform boundedness of conditional gauge and schrödinger equations. Comm. Math. Phys. **93**, 19–31 (1984)
36. Zhao, Z.: Green function for schrödinger operator and conditioned Feynman-Kac gauge. J. Math. Anal. Appl. **116**, 309–334 (1986)