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Olshanski spherical functions for infinite dimensional motion groups of fixed rank

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Abstract. Consider the Gelfand pairs $(G_p, K_p) := (M_{p,q} \rtimes U_p, U_p)$ associated with motion groups over the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with $p \geq q$ and fixed q as well as the inductive limit for $p \to \infty$, the Olshanski spherical pair (G_∞, K_∞). We classify all Olshanski spherical functions of (G_∞, K_∞) as functions on the cone Π_q of positive semidefinite $q \times q$-matrices and show that they appear as (locally) uniform limits of spherical functions of (G_p, K_p) as $p \to \infty$. The latter are given by Bessel functions on Π_q. Moreover, we determine all positive definite Olshanski spherical functions and discuss related positive integral representations for matrix Bessel functions.

We also extend the results to the pairs $(M_{p,q} \rtimes (U_p \times U_q), (U_p \times U_q))$ which are related to the Cartan motion groups of non-compact Grassmannians. Here Dunkl-Bessel functions of type B (for finite p) and of type A (for $p \to \infty$) appear as spherical functions.
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1. Introduction

Let $M_{p,q} := M_{p,q}(\mathbb{F})$ be the vector space of $p \times q$-matrices over one of the fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and $U_p = U_p(\mathbb{F})$ the group of unitary $p \times p$-matrices over $\mathbb{F}$. Consider the Gelfand pairs $(G_p, K_p) := (M_{p,q} \rtimes U_p, U_p)$ associated with motion groups over $\mathbb{F}$ where q is fixed and $p \geq q$ is increasing. The inductive limit $(G_\infty, K_\infty) := (\lim_\rightarrow G_p, \lim_\rightarrow K_p) = (M_{\infty,q} \rtimes U_\infty, U_\infty)$ is an Olshanski spherical pair; see [F], [Ol3] for the notion. The main purpose of this paper is to classify all Olshanski spherical functions of this pair, i.e., the continuous K_∞-biinvariant functions $\phi : G_\infty \to \mathbb{C}$ which satisfy the product formula

$$\phi(g)\phi(h) = \lim_{p\to\infty} \int_{K_p} \phi(gkh)\, dk \qquad \text{for} \quad g, h \in G_\infty \tag{1.1}$$

with respect to the normalized Haar measures dk on K_p. We shall obtain a classification in terms of certain exponential functions which relies on the fact that for integers $p \geq q$ as well for $p = \infty$, the double coset space $G_p//K_p$ may be identified with the cone Π_q of positive semidefinite $q\times q$-matrices over $\mathbb{F}$, and that for finite p, formula (1.1) can be made explicit by the results of [R1]. It is known (see e.g. [H], [FK], [R1] and references cited there) that for finite p, Bessel functions J_μ of index $\mu = pd/2$ on the cone Π_q provide spherical functions of (G_p, K_p); here $d := dim_{\mathbb{R}}\mathbb{F} = 1, 2, 4$. Moreover, it is also known that under suitable rescaling, the Bessel functions J_μ converge (locally) uniformly to exponential functions, see [Ol1] and [RV2], and references therein. From this we obtain that all Olshanski spherical functions appear as (locally) uniform limits of spherical functions on (G_p, K_p) for $p \to \infty$.

The classification of those Olshanski spherical functions which are positive definite is also easily achieved. The connection between positive definite spherical functions of (G_p, K_p) with different values of p leads to the existence of positive integral representations for the involved matrix Bessel functions $J_{pd/2}$ as already explained by Schoenberg [S] in a general setting. Following mainly [H] and [FK] we shall derive explicit formulas for these integral representations of J_ν in terms of J_μ for $\nu \geq \mu$ in two different ways. A comparison of both appoaches will imply an identity for projections of beta distributions on matrix cones.

Besides the Gelfand pairs $(M_{p,q} \rtimes U_p, U_p)$ we also consider the Cartan motion pairs $(M_{p,q} \rtimes (U_p \times U_q), U_p \times U_q)$ and the associated Olshanski spherical functions as $p \to \infty$. For finite p, the spherical functions are now Dunkl-Bessel functions of type B which converge for $p \to \infty$ to certain Dunkl-Bessel functions of type A, and these in turn are just the Olshanski spherical functions of $(M_{\infty,q} \rtimes (U_\infty \times U_q), U_\infty \times U_q)$; see [BF], [Du], [O], [R1], [RV1] for the background. The results and their proofs are very similar to the first case; only the classification of the positive definite Olshanski spherical functions without using representation theory needs more care.

The approach to Olshanski spherical functions taken in this paper is similar to the one for noncompact infinite dimensional Grassmannian manifolds in [RKV] where the finite dimensional spherical functions are Heckman-Opdam hypergeometric functions of type B which converge to hypergeometric functions of type A. In this case, the classification of the Olshanksi spherical functions is based on a product formula in [R2]. The present paper as well as [RKV] were partly motivated by related results in [DOW].

Our approach via special functions does not use explicit representation theory for Olshanski spherical pairs in contrast to [Ol1], [Ol2], [Ol3], [OV], [P] or further related references. This has the slight advantage that we can deal with general spherical functions from the beginning. We point out that most of the results on positive definite Olshanksi spherical functions are already contained in the above mentioned papers of Olshanski.

This paper is organized as follows: In Sections 2 and 3 we collect some facts about matrix Bessel functions and Dunkl-Bessel functions of type B and type A. The Gelfand pairs $(M_{p,q} \rtimes U_p, U_p)$ and the corresponding Olshanski spherical functions will then be studied in Section 4, while Section 5 is devoted to the pairs $(M_{p,q} \rtimes (U_p \times U_q), U_p \times U_q)$. In Section 6 we return to matrix Bessel functions and study their positive integral representations.

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cerning positive definite Olshanksi spherical functions as well as the proof of Lemma 5.7 and Theorem 5.9.

2. Bessel functions on matrix cones

In this section we collect some facts about Bessel functions on matrix cones. The material is mainly taken from [FK], [H], and [R1], and is in part slightly generalized.

Let $\mathbb{F}$ be one of the fields $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ with real dimension $d = 1, 2$ or 4 respectively. Denote the usual conjugation in $\mathbb{F}$ by $t \mapsto \overline{t}$, the real part of $t \in \mathbb{F}$ by $\mathfrak{R}t = \frac{1}{2}(t + \overline{t})$.

For $p, q \in \mathbb{N}$ we denote by $M_{p,q} := M_{p,q}(\mathbb{F})$ the vector space of all $p \times q$-matrices over $\mathbb{F}$ and put $M_q := M_q(\mathbb{F}) := M_{q,q}(\mathbb{F})$ for abbreviation. Let further

$$H_q = H_q(\mathbb{F}) = \{x \in M_q(\mathbb{F}) : x = x^*\}$$

be the space of Hermitian $q \times q$-matrices over $\mathbb{F}$, where $x^* = \overline{x}^t$ denotes the usual involution on M_q. The space H_q is a real Euclidean vector space with scalar product $\langle x, y \rangle := tr(xy)$ and norm $\|x\| = \langle x, x \rangle^{1/2}$, where tr is the trace. Notice that $tr(xy)$ is real for $x, y \in H_q$. H_q is a (Euclidean) Jordan algebra with the usual Jordan product $x \circ y = \frac{1}{2}(xy + yx)$. The real dimension of H_q is given by

$$dim_{\mathbb{R}} H_q = q + \frac{d}{2} q(q-1).$$

We also need the complexification $H_q^{\mathbb{C}}$ of H_q which is a Jordan algebra over $\mathbb{C}$ in the natural way. The extension of the scalar product $\langle\, .\, , .\, \rangle$ to a hermitian scalar product on $H_q^{\mathbb{C}}$ will again be denoted by $\langle x, y \rangle$. For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, the space $H_q^{\mathbb{C}}$ may be written as

$$H_q^{\mathbb{C}} = \{a + ib : \ a, b \in H_q(\mathbb{F})\} \subset M_q(\mathbb{C}).$$

More precisely, for $\mathbb{F} = \mathbb{R}$ we have $H_q^{\mathbb{C}}(\mathbb{R}) = \{a \in M_q(\mathbb{C}) : \ a^t = a\}$, while for $\mathbb{F} = \mathbb{C}$, $H_q^{\mathbb{C}}(\mathbb{C}) = M_q(\mathbb{C})$; c.f. Section VIII.5 of [FK]. The complexification of the Jordan algebra $H_q(\mathbb{H})$ can be desribed as follows, c.f. Sections V.2 and VIII.5 of [FK]: One realizes matrices $z \in H_q(\mathbb{H})$ as complex Hermitian matrices

$$z = \begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \in H_{2q}(\mathbb{C})$$

where $x \in H_q(\mathbb{C})$ is Hermitian and $y \in Skew_q(\mathbb{C})$ is skew-symmetric. Then with $J := \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix} \in M_{2q}$, the mapping

$$z \mapsto -Jz = \begin{pmatrix} \overline{y} & -\overline{x} \\ x & y \end{pmatrix} \in Skew_{2q}(\mathbb{C})$$

defines a real Jordan algebra isomorphism from $H_q(\mathbb{H})$ onto the Jordan algebra $V_q := \{u \in Skew_{2q}(\mathbb{C}) : \ u^* = JuJ\}$ with Jordan product $u \circ v := \frac{1}{2}(uJv + vJu)$. The complexification of this Jordan algebra is just $Skew_{2q}(\mathbb{C})$ (with the same

Jordan-product). We thus identify $H_q^{\mathbb{C}}(\mathbb{H})$ and $Skew_{2q}(\mathbb{C})$ as complex Jordan algebras.

Let

$$\Pi_q := \{x^2 : \ x \in H_q\} = \{x^*x : x \in H_q\}$$

denote the set of all positive semidefinite matrices in H_q, and Ω_q its interior consisting of all strictly positive definite matrices. Ω_q is a symmetric cone; see [FK] for details. On H_q we use the standard partial ordering

$$x \le y :\Longleftrightarrow \ y - x \in \Pi_q \, .$$

To define Bessel functions, we need the spherical polynomials

$$\Phi_\lambda(x) = \int_{U_q} \Delta_\lambda(uxu^{-1})du, \quad x \in H_q$$

which are indexed by partitions $\lambda = (\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_q) \in \mathbb{N}_0^q$ (for short, $\lambda \ge 0$). Here du denotes the normalized Haar measure of $U_q = U_q(\mathbb{F})$, and Δ_λ is the power function

$$\Delta_\lambda(x) := \Delta_1(x)^{\lambda_1 - \lambda_2} \Delta_2(x)^{\lambda_2 - \lambda_3} \cdot \ldots \cdot \Delta_q(x)^{\lambda_q} \quad (x \in H_q)$$

with the principal minors $\Delta_i(x)$ of x; see [FK]. There is a renormalization $Z_\lambda = c_\lambda \Phi_\lambda$ with suitable constants $c_\lambda > 0$ depending on Π_q such that

$$(tr\, x)^k \ = \sum_{|\lambda| = k} Z_\lambda(x) \qquad \text{for} \quad k \in \mathbb{N}_0\, ; \tag{2.1}$$

see Section XI.5. of [FK] where the Z_λ are called zonal polynomials. By construction, the Z_λ are invariant under conjugation by U_q and thus depend only on the eigenvalues of their argument. More precisely, for $x \in H_q$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$,

$$Z_\lambda(x) = C_\lambda^\alpha(\xi) \quad \text{with} \quad \alpha = \frac{2}{d} \tag{2.2}$$

where the C_λ^α are the Jack polynomials of index α (c.f. [FK], [M], [R1]). These are homogeneous of degree $|\lambda|$ and symmetric in their arguments. Notice that the zonal polynomials Z_λ naturally extend to $H_q^{\mathbb{C}}$.

The Bessel functions associated with the cone Ω_q are defined as ${}_0F_1$-hypergeometric series

$$J_\mu(z) = \sum_{\lambda \ge 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(z), \quad z \in H_q^{\mathbb{C}} \tag{2.3}$$

where the generalized Pochhammer symbol $(\mu)_\lambda$ is given by

$$(\mu)_\lambda = \ (\mu)_\lambda^{2/d} \quad \text{with} \ \ (\mu)_\lambda^\alpha := \prod_{j=1}^q \big(\mu - \frac{1}{\alpha}(j-1)\big)_{\lambda_j} \quad (\alpha > 0).$$

The index $\mu \in \mathbb{C}$ is required to satisfy $(\mu)_\lambda^\alpha \neq 0$ for all $\lambda \ge 0$. In this case, J_μ is holomorphic on $H_q^{\mathbb{C}}$.

If $q = 1$, then $\Pi_q = \mathbb{R}_+$, and the Bessel function J_μ is independent of d with

$$J_\mu(\frac{z^2}{4}) = j_{\mu-1}(z)$$

where $j_\kappa(z) = {}_0F_1(\kappa + 1; -z^2/4)$ is the modified Bessel function in one variable.

We need the following product formula for the J_μ from [R1]:

Proposition 2.1. *For $\mu \in \mathbb{C}$ with $\Re\mu > d(q - 1/2)$ and all $r, s \in H_q^{\mathbb{C}}$,*

$$J_\mu(r^2)J_\mu(s^2) = \frac{1}{\kappa_\mu} \int_{B_q} J_\mu(r^2 + s^2 + rws + sw^*r)\, \Delta(I_q - w^*w)^{\mu-\gamma}\, dw$$

where Δ denotes the determinant on H_q, dw the Lebesgue measure on the ball

$$B_q := \{w \in M_q : \ w^*w < I_q\},$$

and

$$\gamma := d(q - 1/2) + 1; \quad \kappa_\mu := \left(\int_{B_q} \Delta(I_q - w^*w)^{\mu-\gamma}\, dw \right)^{-1}.$$

Proof. For $r, s \in H_q$ we refer to Eq. (3.8) of [R1]. As both sides of the equation are holomorphic in $r, s \in H_q^{\mathbb{C}}$, the result is true in general. ∎

For the limit case $\mu = \gamma - 1$ there exists a degenerated product formula; see Section 3.5 of [R1].

For real $\mu \ge \gamma - 1$, product formula 2.1 leads to a positive (hypergroup) convolution structure on the Banach space $M_b(\Pi_q)$ of all bounded signed Borel measures on Π_q, as follows: define first the convolution of point measures

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{B_q} f\big(\sqrt{r^2 + s^2 + swr + rw^*s}\,\big)\, \Delta(I - ww^*)^{\mu-\gamma}\, dw \quad (r, s \in \Pi_q) \tag{2.4}$$

for $f \in C_b(\Pi_q)$, and then extend this convolution in a weakly continuous, bilinear way to $M_b(\Pi_q)$. This generates a probability preserving commutative Banach algebra $(M_b(\Pi_q), *_\mu)$; see [R1] for details. For $\mu = pd/2$ with integers $p \ge 2q$, this convolution is just the double coset convolution associated with the Gelfand pair $(M_{p,q} \rtimes U_p, U_p)$ (see [R1]) which we shall consider in more detail in Section 4.

For $s \in H_q^{\mathbb{C}}$ we define the continuous function

$$f_s^\mu(r) := J_\mu(\frac{1}{4} rsr) \qquad (r \in H_q^{\mathbb{C}}). \tag{2.5}$$

as well as the function

$$\phi_s^\mu(r) := f_{s^2}^\mu(r) = J_\mu(\frac{1}{4} rs^2r), \tag{2.6}$$

where the latter definition is in correspondence with the notion in [R1]. The following facts slightly generalize results from [R1].

Lemma 2.2. (1) *For all* $r, s \in H_q^{\mathbb{C}}$, $\phi_s^\mu(r) = \phi_r^\mu(s)$.

(2) *For all* $r, t \in \Pi_q$ *and* $s \in H_q^{\mathbb{C}}$,

$$f_s^\mu(r) f_s^\mu(t) = \int_{\Pi_q} f_s^\mu(z)\, d(\delta_r *_\mu \delta_t)(z), \tag{2.7}$$

i.e., the functions f_s^μ *with* $s \in H_q^{\mathbb{C}}$ *are multiplicative with respect to* $*_\mu$.

Proof. Both statements are known for $r, s, t \in \Pi_q$; see the results for ϕ_s in Section 3 of [R1] and notice for the second statement that each $s \in \Pi_q$ may be written as $s = \tilde{s}^2$ with some unique $\tilde{s} \in \Pi_q$. The general statements now follow easily by analytic continuation from Π_q to $H_q^{\mathbb{C}}$. For this, notice that $H_q^{\mathbb{C}} \setminus \Pi_q$ is connected, and that the identity theorem can be applied because Π_q has non-empty interior in H_q. ■

3. Dunkl-Bessel functions on Weyl chambers of type B and type A

There is a close connection between the Bessel functions on the cone Π_q and the theory of Dunkl operators associated with the root system B_q, see [R1]. We briefly review this connection. We do not go into details of Dunkl theory, but refer to [BF], [Du], [O] and [R1]. For a reduced root system $R \subset \mathbb{R}^q$ and a multiplicity function $k : R \to [0, \infty)$ (i.e. k is invariant under the action of the corresponding reflection group), we denote by J_k^R the Dunkl-Bessel function associated with R and k. It is obtained from the Dunkl kernel by symmetrization with respect to the underlying reflection group. Dunkl-Bessel functions generalize the spherical functions of Euclidean type symmetric spaces, which occur for crystallographic root systems and specific discrete values of k, see [O]. For the root system $A_{q-1} = \{\pm(e_i - e_j) : i < j\} \subset \mathbb{R}^q$, the multiplicity k is a single real parameter. If $k > 0$, then according to formulas (3.22) and (3.37) of [BF] the associated Dunkl-Bessel function is the generalized ${}_0F_0$-hypergeometric function

$$J_k^A(\xi, \eta) = {}_0F_0^\alpha(\xi, \eta) := \sum_{\lambda \geq 0} \frac{1}{|\lambda|!} \cdot \frac{C_\lambda^\alpha(\xi) C_\lambda^\alpha(\eta)}{C_\lambda^\alpha(\mathbf{1})} \quad (\xi,\, \eta \in \mathbb{C}^q),$$

with $\mathbf{1} = (1, \ldots, 1)$, $\alpha = 1/k$. For $k = \frac{d}{2}$ with $d = \dim_{\mathbb{R}} \mathbb{F}$, the Dunkl-Bessel functions $\xi \mapsto J_{d/2}^A(\xi, \eta)$ are known to be the spherical functions of the flat symmetric space $H_q(\mathbb{F}) \rtimes U_q(\mathbb{F})/U_q(\mathbb{F})$ and therefore have the Harish-Chandra type integral representation

$$J_{d/2}^A(\xi, \eta) = \int_{U_q(\mathbb{F})} e^{\operatorname{tr}(\underline{\eta} u \underline{\xi} u^{-1})} du, \tag{3.1}$$

with $\underline{\xi} = \operatorname{diag}(\xi_1, \ldots, \xi_q)$. See also formula (7) of [RV1], where an alterantive proof of (3.1) is given.

For the root system $B_q = \{\pm e_i, \pm e_i \pm e_j : i < j\}$, we have $k = (k_1, k_2)$ where k_1 and k_2 correspond to the roots $\pm e_i$ and $\pm e_i \pm e_j$ respectively. The associated Dunkl-Bessel function is

$$J_k^B(\xi, \eta) = {}_0F_1^\alpha\big(\mu; \frac{\xi^2}{2}, \frac{\eta^2}{2}\big) \quad \text{with } \alpha = 1/k_2,\ \mu = k_1 + (q-1)k_2 + 1/2$$

where $\xi^2 = (\xi_1^2, \dots, \xi_q^2)$ and

$$ {}_0F_1^\alpha(\mu;\xi,\eta) := \sum_{\lambda\ge 0} \frac{1}{(\mu)_\lambda^\alpha |\lambda|!}\cdot \frac{C_\lambda^\alpha(\xi)C_\lambda^\alpha(\eta)}{C_\lambda^\alpha(\mathbf{1})}. $$

The Dunkl-Bessel function J_k^B is invariant in both arguments under the action of the hyperoctahedral group which is generated by sign changes and permutations of the coordinates. For suitable multiplicity k, it is related the matrix Bessel function J_μ and to the functions $\phi_s^\mu(r) = J_\mu\big(\frac{1}{4}sr^2s\big)$ of Section 2, as follows: Consider the B_q-Weyl chamber

$$ C_q^B := \{\xi = (\xi_1, \dots, \xi_q) \in \mathbb{R}^q : \xi_1 \ge \dots \ge \xi_q \ge 0\} \tag{3.2} $$

For $\xi \in C_q^B$ we denote by $\underline{\xi} \in \Pi_q$ the diagonal matrix with entries $\xi_1, \dots, \xi_q$. Let

$$ k(\mu, d) := \big(\mu - (d(q-1)+1)/2,\ d/2\big). $$

Then for $\xi, \eta \in C_q^B$,

$$ J_{k(\mu,d)}^B(\xi, i\eta) = \int_{U_q} J_\mu\big(\frac{1}{4}\underline{\eta}u\underline{\xi}^2u^{-1}\underline{\eta}\big)\,du; \tag{3.3} $$

see Section 4 of [R1]. Dunkl-Bessel functions of type B will play an important role in Section 5 of this paper, in connection with the study of the Olshanski spherical pairs $(M_{\infty,q} \rtimes (U_\infty \times U_q), U_\infty \times U_q)$.

4. Olshanski spherical functions related to $M_{\infty,q} \rtimes U_\infty$

In this section, we consider Olshanski spherical pairs associated with matrix cones. For a general background on Olshanski spherical pairs and their spherical functions we refer to Faraut [F] and Olshanski [Ol1], [Ol2], [Ol3].

We fix the field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and the rank $q \in \mathbb{N}$ as in Section 2. Consider the Gelfand pairs (G_p, K_p) with $G_p = M_{p,q} \rtimes U_p$ and $K_p := U_p$ where U_p acts on $M_{p,q}$ by left multiplication. In all cases, G_p will be regarded as a closed subgroup of G_{p+1} with $K_p = G_p \cap K_{p+1}$. Consider the inductive limits $G_\infty := \lim_\to G_p$ and $K_\infty := \lim_\to K_p$. Then (G_∞, K_∞) is an Olshanski spherical pair. A continuous function $\phi : G_\infty \to \mathbb{C}$ is called an Olshanski spherical function of (G_∞, K_∞) if ϕ is K_∞-biinvariant and satisfies the product formula

$$ \phi(g)\phi(h) = \lim_{p\to\infty} \int_{K_p} \phi(gkh)\,dk \qquad \text{for all } g, h \in G_\infty \tag{4.1} $$

with respect to the normalized Haar measure dk. Further, an Olshanski spherical function ϕ of (G_∞, K_∞) is called positive definite if ϕ is positive definite on G_∞ in the usual sense, i.e. if for all $n \in \mathbb{N}$, $g_1, \dots, g_n \in G_\infty$ and $c_1, \dots, c_n \in \mathbb{C}$,

$$ \sum_{k,l=1}^n c_k \overline{c_l}\, \phi(g_k g_l^{-1}) \ge 0. $$

Positive definite continuous functions on G_∞ are always bounded. Furthermore, in our situation

$$K_\infty g^{-1} K_\infty = K_\infty g K_\infty \qquad \text{for} \qquad g \in G_\infty.$$

This implies that positive definite spherical functions on G_∞ are automatically $\mathbb{R}$-valued.

We shall now classify the Olshanski spherical functions of (G_∞, K_∞) as well as those which are positive definite in addition.

For this we recapitulate that for each p, the space of double cosets $G_p//K_p$ can be topologically identified with the cone Π_q via the homeomorphism

$$K_p(x,k)K_p \mapsto \sqrt{x^*x} \qquad \text{for } x \in M_{p,q},\ k \in K_p\,.$$

In other words, if for $a \in \Pi_q$ we consider the matrix $\begin{pmatrix} a \\ 0 \end{pmatrix} \in M_{p,q}$, then the inverse of the above homeomorphism can be written as

$$F_p : \Pi_q \to G_p//K_p, \quad r \longmapsto K_p g_a K_p \qquad \text{with} \quad g_a := (\begin{pmatrix} a \\ 0 \end{pmatrix}, I_p).$$

By definition of the inductive limit topology, the mapping

$$F : \Pi_q \to G_\infty//K_\infty, \quad a \longmapsto K_\infty g_a K_\infty \qquad \text{with} \quad g_a := (\begin{pmatrix} a \\ 0_\infty \end{pmatrix}, I_\infty)$$

also provides a homeomorphism for $p = \infty$. We will therefore use the agreement that for all integers $p \geq q$ as well as for $p = \infty$, continuous and K_p-biinvariant functions on G_p are identified with continuous functions on Π_q. When doing so, our notations immediately imply the following

Lemma 4.1. *Let $p_1 \geq p_2 \geq q$. If a continuous function on Π_q corresponds to a K_{p_1}-biinvariant, positive definite function on G_{p_1}, then it also corresponds to a K_{p_2}-biinvariant, positive definite function on G_{p_2}. Moreover, a continuous function on Π_q corresponds to a K_∞-biinvariant, positive definite function on G_∞ if and only if it corresponds to a K_p-biinvariant, positive definite function on G_p for all integers $p \geq q$.*

Proof. The first statement follows immediately from the definition of positive definite functions and the fact that for $p_1 \geq p_2 \geq q$ (where possibly $p_1 = \infty$), the canonical projections

$$P_p : G_p \to G_p//K_p \equiv \Pi_q\,, \quad P_p((x,k)) := \sqrt{x^*x}$$

satisfy $P_{p_1}|_{G_{p_2}} = P_{p_2}$. The remaining part of the second statement follows also easily by these arguments. ∎

We next turn to the (not necessarily positive definite) spherical functions of (G_p, K_p) for finite $p \geq q$. We know from Lemma 2.2 that the functions $f_s^{pd/2}$

of Section 2 with $s \in H_q^{\mathbb{C}}$ are spherical. On the other hand, using results of Wolf [W] in combination with an integral representation of the matrix Bessel functions $f_s^{pd/2}$ (see Eq. (3.4) of [R1] and Propos. XVI.2.3 of [FK]), we obtain the following classification:

Theorem 4.2. *Let $p \geq q$ be finite. Then $\{f_s^{pd/2} : s \in H_q^{\mathbb{C}}\}$ is the set of all spherical functions of (G_p, K_p). Moreover, the set of positive definite spherical functions of (G_p, K_p) is given by $\{f_s^{pd/2} : s \in \Pi_q\}$.*

Proof. According to [W], the complete set of spherical functions of (G_p, K_p) can be described as follows:

Consider $M_{p,q}$ as a real vector space of dimension dpq with Euclidean scalar product

$$(x|y) := \Re tr(x^* y)$$

and extend this form in a bilinear way to the complexification $M_{p,q}^{\mathbb{C}}$ of $M_{p,q}$. Then it is easily checked that for each $y \in M_{p,q}^{\mathbb{C}}$, the function

$$\tilde{\phi}_y(x, v) := \int_{U_p} e^{-i(ux|y)}\, du \qquad (x \in M_{p,q}, v \in U_p) \tag{4.2}$$

defines a spherical function of (G_p, K_p). Moreover, by Theorem 4.4 of [W], all spherical functions are given in this way, and by Theorem 5.4 of [W], the set of positive-definite spherical functions is made up by those $\tilde{\phi}_y$ with $y \in M_{p,q}$.

We next show that all functions of the form (4.2) are in fact Bessel functions $f_s^{pd/2}$ for suitable $s \in H_q^{\mathbb{C}}$. For this we again regard $M_{p,q}$ and H_q as vector spaces over $\mathbb{R}$, and denote the complex-bilinear extension of the $\mathbb{R}$-bilinear mapping

$$M_{p,q} \times M_{p,q} \to M_q, \quad (x, y) \longmapsto x^* y$$

(to a mapping $M_{p,q}^{\mathbb{C}} \times M_{p,q}^{\mathbb{C}} \to M_q^{\mathbb{C}}$) again by $x^* y$. For $y \in M_{p,q}^{\mathbb{C}}$ we then obtain easily that $y^* y \in H_q^{\mathbb{C}}$.

Now fix a matrix $y \in M_{p,q}^{\mathbb{C}}$. Then $y^* y \in H_q^{\mathbb{C}}$, and for all $r \in \Pi_q$,

$$f_{y^*y}^{pd/2}(r) = J_{pd/2}(\frac{1}{4} r y^* y r) = J_{pd/2}(\frac{1}{4}(yr)^* yr).$$

We conclude from Eqs. (3.3) and (3.4) of [R1] (see also Propos. XVI.2.3 of [FK]) that for all $x \in M_{p,q}$,

$$J_{pd/2}(\frac{1}{4} x^* x) = \int_{U_p} e^{-i(u\sigma_0|x)} du \quad \text{with } \sigma_0 = \begin{pmatrix} I_q \\ 0 \end{pmatrix} \in M_{p,q}. \tag{4.3}$$

Using the complex-bilinear and thus analytic extension above, we conclude that (4.3) remains correct for all $x \in M_{p,q}^{\mathbb{C}}$. Now let $r \in \Pi_q$, $y \in M_{p,q}^{\mathbb{C}}$. Then

$$J_{pd/2}(\frac{1}{4}(yr)^* yr) = \int_{U_p} e^{-i(u\sigma_0|yr)} du.$$

As

$$(u\sigma_0|yr) = \Re tr((u\sigma_0)^* yr) = \Re tr((u\sigma_0 r)^* y) = (u\sigma_0 r|y),$$

we obtain

$$f_{y^*y}^{pd/2}(r) = \int_{U_p} e^{-i(u\sigma_0 r|y)} du = \tilde{\phi}_y(\sigma_0 r, v) \tag{4.4}$$

with arbitrary $v \in U_p$. As the functions on both sides are biinvariant, and the $r \in \Pi_q$ form a set of representatives of all double cosets as descibed in the beginning of this section, the proof of the first statement of the theorem is complete. Moreover, equation (4.4) in combination with Theorem 5.4 of [W] leads to the stated classification of the positive definite spherical functions. ■

We mention that the statement about the positive definite spherical functions above can be also obtained by hypergroup methods from Theorem 3.12 of [R1] in combination with results of [J].

We now turn to the case $p = \infty$. The Olshanski spherical functions of (G_∞, K_∞) can be characterized as follows:

Lemma 4.3. *A continuous K_∞-biinvariant function $\phi : G_\infty \to \mathbb{C}$ is Olshanski spherical if and only if the continuous function $\tilde{\phi}(b) := \phi(g_b)$ on Π_q satisfies the product formula*

$$\tilde{\phi}(a) \cdot \tilde{\phi}(b) \;=\; \tilde{\phi}(\sqrt{a^2 + b^2}), \quad a, b \in \Pi_q. \tag{4.5}$$

Proof. Let ϕ be a continuous K_∞-biinvariant function on G_∞ and $\tilde{\phi} \in C(\Pi_q)$ as described in the lemma. Then, by (4.1) and the product formula (2.4), ϕ is Olshanski spherical iff $\tilde{\phi}$ satisfies

$$\tilde{\phi}(a)\cdot\tilde{\phi}(b) \;=\; \lim_{p\to\infty} \frac{1}{\kappa_{pd/2}} \int_{B_q} \tilde{\phi}(\sqrt{a^2 + b^2 + awb + bw^*a})\cdot\Delta(I - w^*w)^{pd/2-\gamma}\, dw \tag{4.6}$$

for $a, b \in \Pi_q$ with $\gamma = d(q - 1/2) + 1$. The probability measures

$$\kappa_{pd/2}^{-1} \cdot \Delta(I - w^*w)^{pd/2-\gamma}\, dw$$

are compactly supported in B_q and tend weakly to the point measure δ_0 for $p \to \infty$. Therefore (4.6) is equivalent to

$$\tilde{\phi}(a) \cdot \tilde{\phi}(b) \;=\; \tilde{\phi}(\sqrt{a^2 + b^2})$$

as claimed. ■

We remark that precise estimates for the order of convergence of the probability measures $\kappa_{pd/2}^{-1} \cdot \Delta(I - w^*w)^{pd/2-\gamma}\, dw$ are given in [V2].

We now solve the functional equation (4.5):

Lemma 4.4. *A continuous function $\tilde{\phi} \in C(\Pi_q)$ with $\tilde{\phi}(0) = 1$ satisfies (4.5) if and only if there exists some $b \in H_q^{\mathbb{C}}$ such that*

$$\tilde{\phi}(a) = \exp\big(-\langle a^2, b\rangle\big) \;=: \psi_b(a), \quad a \in \Pi_q\,.$$

Proof. Clearly, all ψ_b satisfy (4.5). Conversely, if a function $\tilde{\phi} \in C(\Pi_q)$ satisfies (4.5), then $\psi(a) := \tilde{\phi}(\sqrt{a})$ satisfies $\psi(a)\psi(b) = \psi(a+b)$ for $a, b \in \Pi_q$ and thus, because of $\psi(0) = 1$ and continuity, $\psi(a) \neq 0$ for all $a \in \Pi_q$. Each $a \in H_q$ can be written as $a = d - cI_q$ for some $d \in \Pi_q$ and some $c \in [0, \infty[$, and it is easily checked that $\psi(a) := \psi(d)/\psi(cI_q)$ defines a well-defined function $\psi \in C(H_q)$ with $\psi(a)\psi(b) = \psi(a+b)$ for $a, b \in H_q^{\mathbb{C}}$. The assertion now follows by the well-known characterization of the exponential function on a Euclidean space by its functional equation. ∎

The preceding lemmata immediately lead to the following characterization of the Olshanski spherical functions of (G_∞, K_∞).

Theorem 4.5. *A continuous K_∞-biinvariant function $\phi : G_\infty \to \mathbb{C}$ is Olshanski spherical if and only if*

$$\phi(g_a) = \psi_b(a) = \exp\big(-\langle a^2, b\rangle\big) \qquad (a \in \Pi_q)$$

for some $b \in H_q^{\mathbb{C}}$.

We next investigate which Olshanski spherical functions of (G_∞, K_∞) appear as limits of spherical functions on (G_p, K_p) for $p \to \infty$. For this we employ the known convergence of $J_\mu(\mu y)$ to $e^{-tr(y)}$ for $\mu \to \infty$:

Lemma 4.6. *For $y \in H_q^{\mathbb{C}}$,*

$$\lim_{\mu\to\infty} J_\mu(\mu y) = e^{-tr(y)} \qquad \textit{locally uniformly.}$$

On the cone Π_q, this convergence is even uniform. More precisely, there exists a constant $C = C(q, \mathbb{F})$ such that

$$|J_\mu(\mu y) - e^{-tr(y)}| \leq C/\mu \qquad \textit{for all } y \in \Pi_q,\ \mu \geq 2q.$$

Proof. For the second statement we refer to Proposition 3.2 of [RV2]; see also Lemma 4.8 of [Ol1] or references cited there for the uniform convergence.

The first statement can be obtained either from the integral representation (3.12) of [R1] for J_μ, or by power series expansion as in the proof of Proposition 3.5 in [RV2]. We outline the second approach: Using the expansions (2.3) and (2.1), we have

$$J_\mu(\mu y) - e^{-tr(y)} = \sum_{\lambda\geq 0} \frac{1}{|\lambda|!}\Big(\frac{\mu^{|\lambda|}}{(\mu)_\lambda} - 1\Big) \cdot Z_\lambda(-y)$$

where by Lemma 3.4 of [RV2]

$$\Big|1 - \frac{\mu^{|\lambda|}}{(\mu)_\lambda}\Big| \leq dq \cdot 2^{dq(q-1)/2} \cdot \frac{|\lambda|^2}{\mu}.$$

It is easily checked that the series

$$\sum_{\lambda\geq 0} \frac{|\lambda|^2}{|\lambda|!} \cdot Z_\lambda(-y)$$

converges absolutely and locally uniformly for $y \in H_q^{\mathbb{C}}$ (c.f. Section XV.1 of [FK] and our normalization of the Z_λ instead of the Φ_λ there). This immediately leads to the locally uniform convergence of order $1/\mu$. ∎

We conclude from Lemma 4.6 that for all $y \in \Pi_q$, $b \in H_q^{\mathbb{C}}$ and for $\mu = pd/2 \to \infty$, the functions f_s^μ of Section 2 satisfy

$$f_{\mu b}^\mu(y) = J_\mu(\frac{\mu}{4} yby) \to exp(-tr(yby)/4) = \psi_{b/4}(y)$$

uniformly or locally uniformly depending on b. According to Theorem 4.2, the functions $f_{\mu b}^\mu$ with $b \in H_q^{\mathbb{C}}$ form the spherical functions of (G_p, K_p). Considering biinvariant functions on G_p and G_∞ as functions on Π_q as above, we obtain

Corollary 4.7. *All Olshanski spherical functions ψ_b of (G_∞, K_∞) with $b \in H_q^{\mathbb{C}}$ appear as locally uniform limits of spherical functions of (G_p, K_p).*

Moreover, those Olshanski spherical functions ψ_b with $b \in \Pi_q$ even appear as uniform limits of positive definite spherical functions of (G_p, K_p) as $p \to \infty$.

We finally determine the Olshanski spherical functions ψ_b which are positive definite.

Theorem 4.8. *The positive definite Olshanski spherical functions of (G_∞, K_∞) are precisely given by the functions ψ_b with $b \in \Pi_q$.*

Proof. Assume that ψ_b is a positive definite Olshanski spherical function. Then ψ_b must be $\mathbb{R}$-valued. From $\psi_b(a) = exp\,(-\langle a^2, b\rangle)$ for $a \in \Pi_q$ we infer that $b \in H_q$. Moreover, as ψ_b is in addition bounded on Π_q, it follows easily that $b \in \Pi_q$. In fact, if b would have a negative eigenvalue with eigenvector u, we could choose a matrix $a \in \Pi_q$ with the same eigenvector u associated to the eigenvalue 1, and with all other eigenvalues equal to 0. It is then clear that $\psi_b(ca) = exp\,(-c^2\langle a^2, b\rangle)$ tends to ∞ for $c \to \infty$.

For the converse statement consider $b \in \Pi_q$. Let $p \geq q$ be a fixed integer. Then for each integer $\tilde{p} \geq p$, the spherical function $f_b^{\tilde{p}d/2} \in C(\Pi_q)$ corresponds to a $K_{\tilde{p}}$-biinvariant positive definite function on $G_{\tilde{p}}$, and thus by Lemma 4.1, to a K_p-biinvariant positive definite function on G_p. As positive definiteness is preserved under limits, it follows from Corollary 4.7 that $\psi_b \in C(\Pi_q)$ is a K_p-biinvariant positive definite function on G_p. This holds for all p, and therefore Lemma 4.1 ensures that ψ_b is a positive definite function on G_∞ as claimed. ∎

In summary, the Olshanski spherical functions of (G_∞, K_∞) admit a classification which is in complete accordance with that for finite p in Theorem 4.2.

5. Olshanski spherical functions related to $M_{\infty,q} \rtimes (U_\infty \times U_q)$

In this section we consider the Gelfand pairs (G_p, K_p) with $G_p = M_{p,q} \rtimes (U_p \times U_q)$ and $K_p := U_p \times U_q$ for fixed $q \geq 1$, where the groups U_p and U_q act on $M_{p,q}$ by multiplication from the left and right, respectively. Consider the Olshanski spherical pairs (G_∞, K_∞) with $G_\infty := \lim_\to G_p$ and $K_\infty := \lim_\to K_p$.

Again, we investigate the Olshanski spherical functions of (G_∞, K_∞). We recapitulate first that for each p, the double coset space $G_p//K_p$ can be topologically identified with the Weyl chamber

$$C_q^B := \{\xi = (\xi_1, \dots, \xi_q) \in \mathbb{R}^q : \xi_1 \ge \dots \ge \xi_q \ge 0\}$$

of type B via

$$K_p(x,k)K_p \mapsto \sigma(\sqrt{x^*x}\,) \qquad \text{for} \qquad x \in M_{p,q},\ k \in K_p$$

independently of p where σ stands for the ordered spectrum of a positive semidefinite matrix. In other words, if for $\xi \in C_q^B$ we consider the diagonal matrix $\underline{\xi} = diag(\xi_1, \dots, \xi_q) \in M_q$ as well as $\begin{pmatrix} \underline{\xi} \\ 0 \end{pmatrix} \in M_{p,q}$, then the inverse of this homeomorphism can be written as

$$F_p : C_q^B \to G_p//K_p, \quad b \longmapsto K_p g_\xi K_p \qquad \text{with} \qquad g_\xi := \left(\begin{pmatrix} \underline{\xi} \\ 0 \end{pmatrix}, I_p \times I_q\right).$$

By definition of the inductive limit topology, the F_p induce a homeomorphism $F : C_q^B \to G_\infty//K_\infty$. Again we use the agreement that for all integers p and for $p = \infty$, K_p-biinvariant continuous functions on G_p will be identified with continuous functions on Π_q. When doing so, the statement of Lemma 4.1 transfers to the present setting without changes.

We now turn to the classification of spherical functions:

Proposition 5.1. *The spherical functions of (G_p, K_p), considered as functions on the chamber C_q^B, are precisely given by the Dunkl-Bessel functions*

$$\phi_\eta^{pd/2}(\xi) := J_{k(pd/2,d)}^B(\xi, i\eta), \quad \eta \in \mathbb{C}^q.$$

Moreover, the positive definite spherical functions of (G_p, K_p) are given by those $\phi_\eta^{pd/2}$ with $\eta \in \mathbb{R}^q$.

Proof. The first statement is known from Dunkl theory, see [O]. The second then again follows from Theorem 5.4 of [W]. ∎

For the case $p = \infty$, we start with the following

Lemma 5.2. *A continuous K_∞-biinvariant function $\phi : G_\infty \to \mathbb{C}$ is Olshanski spherical if and only if the continuous function $\tilde\phi(\xi) := \phi(g_\xi)$ on C_q^B satisfies*

$$\tilde\phi(\xi) \cdot \tilde\phi(\eta) = \int_{U_q} \tilde\phi\big(\sigma(\sqrt{\underline{\xi}^2 + u\underline{\eta}^2u^{-1}})\big)\, du, \qquad \xi, \eta \in C_q^B. \tag{5.1}$$

Proof. Let ϕ be a continuous K_∞-biinvariant function on G_∞ and $\tilde\phi$ defined as above. Then, by (4.1) and the product formula for the spherical functions of

(G_p, K_p) (see e.g., p. 771 of [R1]), ϕ is Olshanski spherical iff $\tilde{\phi}$ satisfies

$$\tilde{\phi}(\xi)\cdot\tilde{\phi}(\eta) = \lim_{p\to\infty}\frac{1}{\kappa_{pd/2}}\int_{U_q}\int_{B_q}\tilde{\phi}\big(\sigma(\sqrt{\underline{\xi}^2 + u\underline{\eta}^2u^* + \underline{\xi}wu\underline{\eta}u^* + u\underline{\eta}u^*w^*\underline{\xi}})\big)$$
$$\cdot\,\Delta(I-w^*w)^{pd/2-\gamma}\,dw\,du, \qquad \xi,\eta\in C_q^B. \tag{5.2}$$

As the probability measure

$$\kappa_{pd/2}^{-1}\cdot\Delta(I-w^*w)^{pd/2-\gamma}\,dw$$

on B_q tends weakly to the point measure δ_0 for $p\to\infty$, (5.2) is equivalent to the condition of the lemma. ■

We now solve the functional equation (5.1) by using the Dunkl-Bessel functions J_k^A of type A on the Weyl chamber

$$C_q^A := \{\xi\in\mathbb{R}^q:\ \xi_1\ge\ldots\ge\xi_q\}\supset C_q^B.$$

For this we identify the space of double cosets of the Gelfand pairs $(H_q\rtimes U_q, U_q)$ (where U_q acts on H_q by conjugation) with C_q^A and recall from Section 3 that the spherical functions of $(H_q\rtimes U_q, U_q)$ are given by the functions $x\mapsto J_{d/2}^A(\xi,\eta)$ with $\eta\in\mathbb{C}^q$ by [O].

Lemma 5.3. *A continuous function $\tilde{\phi}$ on C_q^B with $\tilde{\phi}(0)=1$ satisfies (5.1) if and only if there exists some $\eta\in\mathbb{C}^q$ such that $\tilde{\phi}(\xi)=J_{d/2}^A(\xi^2,\eta)$ for all $\xi\in C_q^B$. Here $\xi^2\in C_q^B$ means the vector which is obtained from ξ by taking squares in each component.*

Proof. Let $\tilde{\phi}\in C(C_q^B)$ with $\tilde{\phi}(0)=1$. Then $\tilde{\phi}$ satisfies (5.1) if and only if the function $\psi(\xi):=\tilde{\phi}(\sqrt{\xi})$ on C_q^B satisfies

$$\psi(\xi)\cdot\psi(\eta) = \int_{U_q}\psi(\sigma(\underline{\xi}+u\underline{\eta}u^{-1}))\,du, \qquad \xi,\eta\in C_q^B. \tag{5.3}$$

In particular, for $a,b\in[0,\infty[$ we have

$$\psi((a,\ldots,a))\psi((b,\ldots,b)) = \psi((a+b,\ldots,a+b)).$$

As ψ is continuous with $\psi(0)=1$, this implies that there exists some $c\in\mathbb{C}$ such that

$$\psi((a,\ldots,a)) = e^{ca}\quad\text{for all } a\in[0,\infty[.$$

Precisely as in the proof or Lemma 4.4, it is now seen that ψ can be uniquely extended from C_q^B to a continuous function on C_q^A which satisfies (5.3) for all $\xi,\eta\in C_q^A$, namely by putting

$$\psi(\xi_1,\ldots,\xi_q) := \psi(\xi_1-\xi_q,\ldots,\xi_{q-1}-\xi_q,0)\cdot e^{c\xi_q}\quad\text{for } (\xi_1,\ldots,\xi_q)\in C_q^A.$$

Thus the extension $\psi\in C(C_q^A)$ is a spherical function of the Gelfand pair $(H_q\rtimes U_q, U_q)$. On the other hand, we know that the spherical functions of this Gelfand pair are precisely the Dunkl-Bessel functions $\xi\mapsto J_{d/2}^A(\xi,\eta)$ with $\eta\in\mathbb{C}^q$. This proves the claim. ■

The preceding lemmata yield the following characterization of the Olshanski spherical functions.

Theorem 5.4. *A continuous K_∞-biinvariant function $\phi : G_\infty \to \mathbb{C}$ is Olshanski spherical if and only if for some $\eta \in \mathbb{C}^q$,*

$$\phi(g_\xi) = J^A_{d/2}(\xi^2, -\eta) \; =: \psi_\eta(\xi)$$

for all $\xi \in C_q^B$.

We next study which Olshanski spherical functions of (G_∞, K_∞) appear as limits of spherical functions on (G_p, K_p) for $p \to \infty$. For this we employ the following lemma:

Lemma 5.5. *The Dunkl Bessel functions satisfy*

$$\lim_{\mu\to\infty} J^B_{k(\mu,d)}(2\sqrt{\mu}\xi, i\eta) = J^A_{d/2}(\xi^2, -\eta^2)$$

locally uniformly in $(\xi, \eta) \in C_q^B \times \mathbb{C}^q$. On $C_q^B \times C_q^B$, the convergence is even uniform.

Proof. This follows from the integral representations (3.1) and (3.3), together with the locally uniform limit for the matrix Bessel functions according to Lemma 4.6. ∎

Notice that the squared entries $\xi^2, -\eta^2$ in the limit on the right hand side above reflect the fact that the limit admits B-symmetry while the Bessel function $J^A_{d/2}$ only admits A-symmetry.

Lemma 5.5 in combination with Proposition 5.1 and Theorem 5.4 implies the following result (again, we consider biinvariant functions on G_p and G_∞ as functions on C_q^B):

Theorem 5.6. *All Olshanski spherical functions ψ_η, $\eta \in \mathbb{C}^q$ of (G_∞, K_∞) appear as locally uniform limits of spherical functions of (G_p, K_p) for $p \to \infty$.*

Moreover, those Olshanski spherical functions ψ_η with $\eta \in C_q^B$ appear even as uniform limits of the spherical functions of (G_p, K_p) for $p \to \infty$.

We discuss the locally uniform convergence of the spherical functions of (G_p, K_p) in this theorem more explicitly:

Lemma 5.7. *Consider the spherical functions $\phi_\eta^{pd/2}$ of (G_p, K_p) as introduced in Proposition 5.1 as well as the spherical functions ψ_η of (G_∞, K_∞). Let $\eta \in \mathbb{C}^q$ and consider a sequence $(\eta_p)_{p\geq q} \subset \mathbb{C}^q$ with*

$$\lim_{p\to\infty} \eta_p = \eta.$$

Then

$$\lim_{p\to\infty} \phi^{pd/2}_{\sqrt{2pd}\cdot\eta_p} = \psi_{\eta^2} \quad \textit{locally uniformly on } C_q^B.$$

Moreover, if the sequence $(\eta_p)_{p\geq q}$ *is real-valued with* $\lim_{p\to\infty}|\eta_p| = \infty$*, then the spherical functions* $\phi^{pd/2}_{\sqrt{2pd}\cdot\eta_p}$ *do not converge locally uniformly on* C_q^B*.*

Proof. The first statement follows from Lemma 5.5. For the second statement consider a sequence $(\eta_p = (\eta_{p,1}, \ldots, \eta_{p,q}))_p \subset \mathbb{R}^q$ which tends to ∞ such that the $\phi^{pd/2}_{\sqrt{2pd}\cdot\eta_p}$ converge locally uniformly on C_q^B. Note that the limit is continuous on C_q^B. By the B_q-symmetry of the $\phi^{pd/2}_\eta$ in the parameter η, we may assume in addition that $\eta_{p,1} \geq \eta_{p,2} \geq \ldots \geq \eta_{p,q} \geq 0$ for all p, and in particular $\eta_{p,1} \to \infty$ for $p \to \infty$. Moreover, after passing to a subsequence, we may assume that $(\eta_p/\eta_{p,1})_p \subset \mathbb{R}^q$ converges to some $\tilde{\eta} = (1, \tilde{\eta}_2, \ldots, \tilde{\eta}_q) \in \mathbb{R}^q$. Then $\psi_{\tilde{\eta}}$ is not the constant function 1 i.e., we can find $\xi \in C_q^B$ with $\psi_{\tilde{\eta}}(\xi) \neq 1$. Therefore, by the first part of the lemma,

$$\phi^{pd/2}_{\sqrt{2pd}\cdot\eta_p}(\xi/\eta_{p,1}) = \phi^{pd/2}_{\frac{\sqrt{2pd}}{\eta_{p,1}}\cdot\eta_p}(\xi) \longrightarrow \psi_{\tilde{\eta}}(\xi) \neq 1.$$

As $\phi^{pd/2}_\eta(0) = 1$ for all η, this shows that $\phi^{pd/2}_{\sqrt{2pd}\cdot\eta_p}$ cannot converge locally uniformly. ∎

We finally turn to the question which of the Olshanski spherical functions ψ_η are positive definite. The same argument as in the second part of the proof of Theorem 4.8 immediately implies that the functions ψ_η with $\eta \in C_q^B$ are positive definite Olshanski spherical functions of (G_∞, K_∞). The converse statement is more complicated than in the situation of Theorem 4.8, because now there exist spherical functions ψ_η with $\eta \in \mathbb{C}^q \setminus \mathbb{R}^q$ which are $\mathbb{R}$-valued and bounded. To overcome this problem, we use Lemma 5.7 in combination with the following result from Section 22 of [Ol3]:

Proposition 5.8. *Let* $(G_\infty, K_\infty) = \lim_{p\to\infty}(G_p, K_p)$ *be an Olshanski spherical pair. Then each positive definite spherical function of* (G_∞, K_∞) *is a locally uniform limit of a sequence of positive definite spherical functions of* (G_p, K_p) *for* $p \to \infty$*.*

Theorem 5.9. *An Olshanski spherical function* ψ_η *of* (G_∞, K_∞) *is positive definite if and only if* $\eta \in \mathbb{R}^q$*.*

Proof. The if-part follows by the same arguments as in the proof of Theorem 4.8.

For the converse direction consider a positive definite Olshanski spherical function ψ_b of (G_∞, K_∞) with $b \in \mathbb{C}^q$. Then, by Proposition 5.8 and the classification of the positive definite spherical functions of (G_p, K_p) in Proposition 5.1, there exists a sequence $(\eta_p)_{p\geq q} \subset \mathbb{R}^q$ of indices such that the positive definite spherical functions $\phi^{pd/2}_{\eta_p}$ of (G_p, K_p) converge to ψ_b locally uniformly on C_q^B. Therefore, by the second part of Lemma 5.7, the sequence $(\frac{\eta_p}{\sqrt{2pd}})_{p\geq q} \subset \mathbb{R}^q$ does not converge to infinity, i.e., this sequence contains a bounded subsequence and hence a convergent subsequence with a limit $\eta \in \mathbb{R}^q$, i.e., $\lim_{p_k} \frac{\eta_{p_k}}{\sqrt{2p_k d}} = \eta$. Therefore, by the first part of Lemma 5.7, $\phi^{p_k d/2}_{\eta_{p_k}} \to \psi_\eta$. This implies that $\psi_b = \psi_\eta$ on C_q^B and, by analyticity, also on $\mathbb{C}^q$. As two Dunkl-Bessel functions of type A with parameters $b \in \mathbb{C}^q$ and $\eta \in \mathbb{R}^q$ are equal precisely if b and η are in the same Weyl group orbit, we conclude that $b \in \mathbb{R}^q$ as claimed. ∎

We finally point out that Theorem 5.9 can also be obtained from Theorem 4.10 of Olshanski [Ol1] by using admissible representations.

6. Positive integral representations of matrix Bessel functions

Consider integers $p_2 \geq p_1 \geq q$ and the associated indices $\mu_k := p_k d/2$ $(k = 1, 2)$ of the matrix Bessel functions. Then the functions $\phi_s^{\mu_2}$ with $s \in \Pi_q$ as introduced in (2.6) represent positive definite biinvariant functions on G_{p_1}, and thus by Lemma 4.1, positive definite biinvariant functions on G_{p_2}. Therefore, by Bochner's theorem for hypergroups (see [J]), which may be applied to the associated matrix Bessel hypergroup on Π_q of index μ_1, the function $\phi_s^{\mu_2}$ has a representation

$$\phi_s^{\mu_2}(x) = \int_{\Pi_q} \phi_t^{\mu_1}(x)\, d\nu_{p_1,p_2;s}(t) \qquad (x \in \Pi_q) \tag{6.1}$$

with some unique probability measure $\nu_{p_1,p_2;s} \in M^1(\Pi_q)$.

In this section we shall determine these measures explicitely in two different ways. Comparison of these results will then lead to a projection result for multivariate beta distributions. Our first approach was already carried out in [H] for $\mathbb{F} = \mathbb{R}$; it relies on the following Laplace transform identity for the Bessel functions J_μ which holds for general $\mathbb{F}$; see Proposition XV.2.1 and Corollary VII.1.3 of [FK]:

Proposition 6.1. *For all $\mu \in \mathbb{C}$ with $\Re\mu > d(q-1)/2$ and $y \in \Omega_q$,*

$$\int_{\Pi_q} J_\mu(x) e^{-\langle x,y\rangle} \Delta(x)^{\mu - n/q}\, dx = \Gamma_\Omega^q(\mu) \cdot \Delta(y)^{-\mu} \cdot e^{-tr(y^{-1})}$$

with the Gamma function

$$\Gamma_\Omega^q(\mu) = \int_{\Pi_q} e^{-tr(x)} \Delta(x)^{\mu-n/q}\, dx = (2\pi)^{(n-q)/2}\Gamma(\mu)\Gamma(\mu - d/2)\cdots\Gamma(\mu-(q-1)d/2).$$

By the transformation formula for linear maps, we have for $m \in \Pi_q$

$$\int_{\Pi_q} f(mxm)\, dx = \Delta(m)^{2n/q} \int_{\Pi_q} f(x)\, dx.$$

We thus obtain for $\mu \in \mathbb{C}$ with $\Re\mu > d(q-1)/2$, $m \in \Pi_q$ and $y \in \Omega_q$, that

$$\int_{\Pi_q} J_\mu(xm) e^{-\langle x,y\rangle} \Delta(x)^{\mu-n/q}\, dx = \Gamma_\Omega^q(\mu) \cdot \Delta(y)^{-\mu} \cdot e^{-tr(my^{-1})}; \tag{6.2}$$

c.f. equation (2.5) of [H] for $\mathbb{F} = \mathbb{R}$, where a minus sign in the exponential is missing. We notice that (6.2) may be also interpreted as a well-known formula for the Hankel transforms of Wishart distributions; see for instance [FK] or Section 5 of [V1]. Using injectivity and the convolution theorem for the Laplace transform, we deduce from (6.2) the following addition theorem:

Proposition 6.2. *For all $\mu,\nu\in\mathbb{C}$ with $\Re\mu > d(q-1)/2$ and $\Re\nu > d(q-1)/2$ and all $m_1, m_2 \in \Pi_q$,*

$$J_{\mu+\nu}(x(m_1+m_2))\cdot\Delta(x)^{\mu+\nu-n/q} = \tag{6.3}$$
$$= \frac{\Gamma_\Omega^q(\mu+\nu)}{\Gamma_\Omega^q(\mu)\Gamma_\Omega^q(\nu)}\int_{\{y\in\Pi_q:\, y\le x\}} J_\mu(ym_1)\Delta(y)^{\mu-n/q}J_\nu((x-y)m_2)\Delta(x-y)^{\nu-n/q}\,dy.$$

Taking $m_1 = m$, $m_2 = 0$, $x = I_q$ (the identity matrix), and defining

$$\Pi_q^I := \{y\in\Pi_q:\; y\le I_q\},$$

we obtain the following Sonine-type integral representation of $J_{\mu+\nu}$ in terms of J_μ, which was for $\mathbb{F}=\mathbb{R}$ already proven in [H]:

Corollary 6.3. *For all $\mu,\nu\in\mathbb{C}$ with $\Re\mu,\Re\nu > d(q-1)/2$ and $m\in\Pi_q$,*

$$J_{\mu+\nu}(m) = \frac{\Gamma_\Omega^q(\mu+\nu)}{\Gamma_\Omega^q(\mu)\Gamma_\Omega^q(\nu)}\int_{\Pi_q^I} J_\mu(ym)\Delta(y)^{\mu-n/q}\Delta(I_q-y)^{\nu-n/q}\,dy.$$

Notice that for $m=0$ this formula implies the known beta integral

$$\int_{\Pi_q^I}\Delta(y)^{\mu-n/q}\Delta(I_q-y)^{\nu-n/q}\,dy = \frac{\Gamma_\Omega^q(\mu)\Gamma_\Omega^q(\nu)}{\Gamma_\Omega^q(\mu+\nu)} =: B_\Omega^q(\mu,\nu). \tag{6.4}$$

Using these notions, we define the beta distributions $d\beta_{q;\mu,\nu}\in M^1(\Pi_q^I)$ on Π_q^I for real parameters $\mu,\nu > d(q-1)/2$ by

$$d\beta_{q;\mu,\nu}(y) := \frac{1}{B_\Omega^q(\mu,\nu)}\Delta(y)^{\mu-n/q}\Delta(I_q-y)^{\nu-n/q}\,dy$$

with $n = n(q,\mathbb{F})$ as in Section 2. Using Lemma 2.2(1), we obtain the following explicit form of relation (6.1):

Corollary 6.4. *For all $\mu,\nu\in\mathbb{C}$ with $\Re\mu,\Re\nu > d(q-1)/2$ and $s,x\in\Pi_q$,*

$$\phi_s^{\mu+\nu}(x) = \int_{\Pi_q^I}\phi^\mu_{\sqrt{sys}}(x)\,d\beta_{q;\mu,\nu}(y).$$

Remark 6.5. (1) We conjecture that for all real parameters $\nu\ge 0$, $\mu > d(q-1)/2$ and all $s\in\Pi_q$ there is a probability measure $m_{\mu,\nu,s}\in M^1(\Pi_q)$ with compact support such that

$$\phi_s^{\mu+\nu}(x) = \int_{\Pi_q}\phi_y^\mu(x)\,dm_{\mu,\nu,s}(y)\quad\text{for all } x\in\Pi_q$$

(2) One can easily combine Corollary 6.4 with Lemma 4.6 for $\nu\to\infty$. This yields an explicit integral representation of the Olshanski spherical functions ψ_b of Section 4 for $b\in\Pi_q$ in terms of the ϕ_s^μ. This formula describes ψ_b as the Hankel transform of order μ of some Wishart distribution and is well-known, see Proposition XV.2.1 of [FK] and Lemma 5.4. of [V1], which is adapted to our notation.

Next, we derive Corollaries 6.3 and 6.4 in a different, more geometric way as follows: Consider integers $1 \leq q \leq \tilde{p}$ and $p \geq 2\tilde{p}$. For $p, \tilde{p}$ define the associated parameters $\mu := pd/2$ and $\tilde{\mu} := \tilde{p}d/2$ of the Bessel functions on Π_q. From the fact that the ϕ_λ^μ are spherical functions we conclude that for $\lambda, x \in \Pi_q$

$$\phi_\lambda^\mu(x) = \int_{U_p} \exp\Big(i \cdot \Re tr\Big((x,0)\, u \begin{pmatrix} \lambda \\ 0 \end{pmatrix}\Big)\Big) du$$

with $(x,0) \in M_{q,p}$ and $\begin{pmatrix} \lambda \\ 0 \end{pmatrix} \in M_{p,q}$ (see also [R1]). Now let $w \in B_{\tilde{p}}$ be the upper left $\tilde{p} \times \tilde{p}$-block of u, i.e. $u = \begin{pmatrix} w & * \\ * & * \end{pmatrix}$. In the following, C_1, C_2, C_3 are constants depending on $q, \tilde{p}, p, d$. We first conclude from Lemma 2.1 of [R2] that for $p \geq 2\tilde{p}$,

$$\phi_\lambda^\mu(x) = C_1 \int_{B_{\tilde{p}}} \exp\Big(i \cdot \Re tr\Big((x,0) w \begin{pmatrix} \lambda \\ 0 \end{pmatrix}\Big)\Big) \cdot \Delta(I - w^* w)^{pd/2 - d(\tilde{p}-1/2)-1} dw$$

where now $(x,0) \in M_{q,\tilde{p}}$ and $\begin{pmatrix} \lambda \\ 0 \end{pmatrix} \in M_{\tilde{p},q}$. Further, integration with respect to polar coordinates on $M_{\tilde{p}}$ according to [FT] (see also p.759 of [R1]) gives

$$\int_{M_{\tilde{p}}} f(w) dw = C_2 \int_{\Omega_{\tilde{p}}} \int_{U_{\tilde{p}}} f(u\sqrt{r}) \cdot \Delta(r)^{d/2-1} du\, dr.$$

Therefore

$$\phi_\lambda^\mu(x) = C_3 \int_{\Pi_{\tilde{p}}^I} \Big(\int_{U_{\tilde{p}}} \exp\Big(i \cdot \Re tr\Big((x,0) u \sqrt{r} \begin{pmatrix} \lambda \\ 0 \end{pmatrix}\Big)\Big) du \Big) \cdot \\ \cdot \Delta(r)^{d/2-1} \Delta(I - r)^{pd/2 - d(\tilde{p}-1/2)-1} dr.$$

By the definition of $\phi_\lambda^{\tilde{\mu}}$, we obtain

$$\phi_\lambda^\mu(x) = C_3 \int_{\Pi_{\tilde{p}}^I} \phi_{\lambda(r)}^{\tilde{\mu}}(x) \Delta(r)^{d/2-1} \Delta(I - r)^{pd/2 - d(\tilde{p}-1/2)-1} dr$$

with $\lambda(r) := \sqrt{(\lambda, 0) r \begin{pmatrix} \lambda \\ 0 \end{pmatrix}} \in \Pi_q$.

Putting $x = 0$ and using (6.4) for $\tilde{p}$ instead of q, we finally obtain from the definition of n that

$$\phi_\lambda^\mu(x) = \int_{\Pi_{\tilde{p}}^I} \phi_{\lambda(r)}^{\tilde{\mu}}(x)\, d\beta_{\tilde{p}; \tilde{p}d/2, d(p-\tilde{p})/2}(r). \tag{6.5}$$

We now compare equation (6.5) with Corollary 6.4 for $\mu = \tilde{p}d/2$, $\nu = (p - \tilde{p})d/2$ and arbitrary $x \in \Pi_q$. By the injectivity of the Hankel transform of index $\tilde{p}$ on Π_q and by analytic continuation with respect to μ and ν, we shall obtain the following projection result for multivariate beta distributions:

Proposition 6.6. *For integers $\tilde p \geq q \geq 1$, consider the projection*

$$P_{\tilde p,q}:\Pi_{\tilde p}^I \to \Pi_q^I, \quad r=\begin{pmatrix} y & * \\ * & * \end{pmatrix} \longmapsto y.$$

Then for all real-valued parameters $\mu,\nu > d(\tilde p-1)/2$, the measure $\beta_{q;\mu,\nu}$ is the push-forward of the measure $\beta_{\tilde p;\mu,\nu}$ under $P_{\tilde p,q}$,

$$P_{\tilde p,q}(\beta_{\tilde p;\mu,\nu}) = \beta_{q;\mu,\nu}.$$

Proof. Assume first that $\mu = d\tilde p/2$ and $\nu = d(p-\tilde p)/2$ for some integer $p \geq 2\tilde p$. In this case, the statement follows from the preceding arguments.

Now consider integers $r \geq \tilde p \geq q$ and $p \geq 2r$. We obtain

$$\begin{aligned}\beta_{q;dr/2,d(p-r)/2} &= P_{r,q}(\beta_{r;dr/2,d(p-r)/2}) = P_{\tilde p,q}\circ P_{r,\tilde p}(\beta_{r;dr/2,d(p-r)/2}) \\ &= P_{\tilde p,q}(\beta_{\tilde p;dr/2,d(p-r)/2}).\end{aligned}$$

This equality means that for the parameters $\mu = dr/2$ and $\nu = d(p-r)/2$ with integers p,r with $r \geq \tilde p$ and $p \geq 2r$, we have for all bounded continuous functions $f \in C_b(\Pi_q^I)$ the identity

$$\int_{\Pi_{\tilde p}^I} f(P(x))\, d\beta_{\tilde p;\mu,\nu}(x) = \int_{\Pi_q^I} f(y)\, d\beta_{q;\mu,\nu}(y). \tag{6.6}$$

By definition of the beta distributions, both sides of this identity are, for fixed f, analytic in the variables $\mu,\nu \in \mathbb{C}$ with $\Re\mu, \Re\nu > d(\tilde p-1)/2$. Precisely as in [R1], it is easily checked that both sides also satisfy the exponential growth conditions of Carlson's Theorem (p. 186 of [T]). This yields that (6.6) holds for general μ and ν, which finishes the proof. ∎

We mention that Proposition 6.6 can also be derived, at least for $\mathbb{F}=\mathbb{R}$ and $\mu = p/2$ and $\nu = r/2$ with suitable integers p,r, from the construction of the multivariate beta distributions in statistics; see Section 10.2 of [Fa] and references cited there. Let us sketch this approach: we consider $M_{p,\tilde p}$- and $M_{r,\tilde p}$-valued independent random variables X and Y respectively such that all entries of X and Y are i.i.d. standard-normal distributed. Then $S := X^tX$, $T := Y^tY$, and $S+T$ are $\Pi_{\tilde p}$-valued and Wishart-distributed. Now form the unique lower triangular matrix C with nonnegative entries satisfying $CC^t = S+T$. It is well-known (Section 10.2 of [Fa] and references cited there) that $L := C^{-1}S(C^t)^{-1}$ is a $\Pi_{\tilde p}^I$-valued variable with distribution $\beta_{\tilde p;p/2,r/2}$.

Now consider some integer $1 \leq q \leq \tilde p$. For any matrix $A \in M_{\tilde p}$ we denote by $\tilde A$ its upper left $q\times q$-block. It is now easily checked from the triagular structure of C and C^{-1} that

$$\tilde L = \widetilde{C^{-1}}\tilde S\widetilde{(C^t)^{-1}} = (\tilde C)^{-1}\tilde S((\tilde C)^t)^{-1}$$

with $\tilde C\tilde C^t = \tilde S + \tilde T$. This observation readily leads to a further proof of Proposition 6.6 for $\mu = p/2$ and $\nu = r/2$. Again, application of Carlson's theorem implies the general result.

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