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INTEGRAL REPRESENTATION AND UNIFORM LIMITS FOR SOME HECKMAN-OPDAM HYPERGEOMETRIC FUNCTIONS OF TYPE *BC*

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Abstract. The Heckman-Opdam hypergeometric functions of type BC extend classical Jacobi functions in one variable and include the spherical functions of non-compact Grassmann manifolds over the real, complex or quaternionic numbers. There are various limit transitions known for such hypergeometric functions, see e.g. [dJ], [RKV]. In the present paper, we use an explicit form of the Harish-Chandra integral representation as well as an interpolated variant, in order to obtain two limit results, each of them for three continuous classes of hypergeometric functions of type BC which extend the group cases over the fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$. These limits are distinguished from the known results by explicit and uniform error bounds. The first limit realizes the approximation of the spherical functions of infinite dimensional Grassmannians of fixed rank; here hypergeometric functions of type A appear as limits. The second limit is a contraction limit towards Bessel functions of Dunkl type.

1. Introduction

The theory of hypergeometric functions associated with root systems provides a framework which generalizes the classical theory of spherical functions on Riemannian symmetric spaces; see [H], [HS] and [O2] for the general theory, as well as [Sch] and [NPP] for some more recent developments. Here we consider the non-compact Grassmannians $\mathcal{G}_{p,q}(\mathbb{F}) = G/K$ over one of the (skew-) fields $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, where G is one of the indefinite orthogonal, unitary or symplectic groups $SO_0(q,p)$, $SU(q,p)$ or $Sp(q,p)$ with $p > q$, and K is the maximal compact subgroup $K = SO(q) \times SO(p)$, $S(U(q) \times U(p))$ or $Sp(q) \times Sp(p)$, respectively. The real rank of G/K is q, and the restricted root system $\Delta(\mathfrak{g}, \mathfrak{a})$ is of type BC. Let $F_{BC}(\lambda, k; t)$ denote the Heckman-Opdam hypergeometric function associated with the root system

$$R = 2 \cdot BC_q = \{\pm 2e_i, \pm 4e_i, \pm 2e_i \pm 2e_j : 1 \leq i < j \leq q\} \subset \mathbb{R}^q,$$

with spectral variable $\lambda \in \mathbb{C}^q$ and multiplicity parameter k. The spherical functions of $G/K = \mathcal{G}_{p,q}(\mathbb{F})$, which are K-biinvariant as functions on G, are then given by

$$\phi_\lambda^p(a_t) = F_{BC}(i\lambda, k_p; t) \qquad (t \in \mathbb{R}^q)$$

with $\lambda \in \mathbb{C}^q$ and multiplicity

$$k_p = (d(p-q)/2, (d-1)/2, d/2)$$

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corresponding to the roots $\pm 2e_i$, $\pm 4e_i$ and $2(\pm e_i \pm e_j)$ respectively; here $d \in \{1, 2, 4\}$ denotes the dimension of $\mathbb{R}, \mathbb{C}, \mathbb{H}$ over $\mathbb{R}$; see [R2] and Remark 2.3 of [H]. In [R2], the product formula for spherical functions,

$$\phi(g)\phi(h) = \int_K \phi(gkh)dk \quad (g, h \in G),$$

was made explicit in such a way that it could be extended to a product formula for the hypergeometric function F_{BC} with mulitplicity k_p corresponding to arbitrary real parameters $p > 2q - 1$. This led to three continuous series of positive product formulas for F_{BC} corresponding to $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ as well as associated commutative, probability-preserving convolution algebras of measures (hypergroups in the sense of [J]) on the BC_q-Weyl chamber

$$C_q = \{t = (t_1, \ldots, t_q) \in \mathbb{R}^q : t_1 \geq \ldots \geq t_q \geq 0\}.$$

On the other hand, the spherical functions of G/K have the Harish-Chandra integral representation

$$\phi_\lambda^p(a_t) = \int_K e^{(i\lambda - \rho)(H(a_t k))} dk, \quad \lambda \in \mathfrak{a}_\mathbb{C}^* \cong \mathbb{C}^q,$$

see [Hel] or [GV] for the general theory and Section 2 for details in our particular case. The Harish-Chandra integral was made explicit by Sawyer [Sa] for the real Grassmannians $\mathcal{G}_{p,q}(\mathbb{R})$. In the present paper, we extend Sawyer's representation to general $\mathbb{F}$ and further reduce it to a form which allows an extension from the spherical case with integers $p \geq 2q$ to a positive integral representation for the three classes of hypergeometric functions F_{BC} as above, with arbitrary real parameters $p > 2q-1$, the rank q being fixed. This (in part) generalizes the well-known integral representation of Jacobi functions, which are the hypergeometric functions of type BC in rank one (see [K1]). We also give an analogous integral representation for the corresponding Heckman-Opdam polynomials.

Our integral representation (Theorem 2.4) for the spherical functions of $\mathcal{G}_{p,q}(\mathbb{F})$ is closely related to those for the spherical functions of the type A symmetric spaces $GL(q, \mathbb{F})/U(q, \mathbb{F})$. In particular, we obtain immediately that for $p \to \infty$, the spherical functions of $\mathcal{G}_{p,q}(\mathbb{F})$ tend to the spherical functions of $GL(q, \mathbb{F})/U(q, \mathbb{F})$, a result which was proven recently by completely different methods and in more generality in [RKV], see also the note [K2] for the polynomial case. As described in [RKV], this limit transition realizes the approximation of the Olshanski-spherical functions of infinite dimensional Grassmannians of fixed rank q over $\mathbb{F}$, which can be naturally identified with the spherical functions of $GL(q, \mathbb{F})/U(q, \mathbb{F})$.

As a main result of the present paper, we shall deduce from our explicit integral representation a result on the rate of convergence (Theorem 4.2): the convergence of the bounded hypergeometric functions F_{BC}, with multiplicities depending on p as above, is of order $O(p^{-1/2})$ for $p \to \infty$, uniformly on the chamber C_q and locally uniformly in the spectral variable. Moreover, a corresponding result is obtained in the unbounded case. It seems that these results cannot be obtained by the methods of [RKV]. Corresponding results for $q = 1$, i.e., for Jacobi functions, can be found in [V2]. We also mention that our convergence results are related to further limits, e.g., to limits in [D] and [SK] for multivariate polynomials as well as to the convergence of (multivariable) Bessel functions of type B to those of type A and related results for matrix Bessel functions in [RV2], [RV3]. We point out that these convergence results with error bounds may serve as a basis to derive central limit theorems for random walks on the Grassmannians $\mathcal{G}_{p,q}(\mathbb{F})$ when for fixed rank q, the time parameter of the random walks as well as the dimension parameter p tend to infinity in a coupled way. For results in this direction we refer to [RV3], [V2].

In generalization of the contraction principle for Riemannian symmetric spaces, Heckman-Opdam hypergeometric functions can be approximated for small space variables and large spectral parameters by corresponding Bessel functions of Dunkl type. This was first proven in [dJ] by an asymptotic analysis of the Cherednik system; see also [RV1]. In the present paper, we shall use the integral representation of Theorem 2.4 in order to obtain this approximation in our series of BC-cases (which include the spherical functions on Grassmannians), again with an explicit error estimate. For the case $q = 1$ and the use of the error estimate in the proof of central limit theorems we refer to [V2] and references cited there.

We finally mention that the Harish-Chandra integral in Proposition 5.4.1 of [HS] for the K-spherical functions of the symmetric spaces $U(p, q)/(U(p) \times SU(q))$ over $\mathbb{C}$ may be used to derive an explicit integral representation for Heckman-Opdam hypergeometric functions of type BC for a different class of parameters than considered here. For such cases, associated convolution structures have been derived in [V3].

The organization of this paper is as follows: In Section 2 we treat the Harish-Chandra integral representation for the spherical functions of $\mathcal{G}_{p,q}(\mathbb{F})$ as well as for the associated three continuous series of Heckman-Opdam hypergeometric functions. In Section 3 we deduce the convergence of the spherical functions of $\mathcal{G}_{p,q}(\mathbb{F})$ to those of $GL(q, \mathbb{F})/U(q, \mathbb{F})$ as $p \to \infty$. Section 4 is then devoted to explicit estimates for the rate of convergence. In particular, in order to obtain a uniform rate for $t \in C_q$, we need a technical result on the convex hull of Weyl group orbits of the weighted half-sum ρ of roots which will be proven separately in an appendix (Section 6). The quantitative contraction estimates between hypergeometric functions of type BC and Bessel functions of type B will be presented in Section 5.

2. An integral representation for spherical functions on Grassmann manifolds and hypergeometric functions of type BC

In this section, we extend Sawyer's ([Sa]) integral representation for spherical functions on real Grassmannians and deduce an explicit integral representation (Theorem 2.4) for three continuous series for hypergeometric functions of type BC.

Let $\mathbb{F}$ be one of the (skew-) fields $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and $d = \dim_{\mathbb{R}}\mathbb{F} \in \{1, 2, 4\}$. On $\mathbb{F}$, we have the standard involution $x \mapsto \overline{x}$ and norm $|x| = (\overline{x}x)^{1/2}$. By $M_{q,p}(\mathbb{F})$ we denote the set of $q \times p$ matrices over $\mathbb{F}$, also viewed as $\mathbb{F}$-linear transformations from $\mathbb{F}^p$ to $\mathbb{F}^q$, which are considered as right $\mathbb{F}$-vector spaces. We write $M_q(\mathbb{F}) = M_{q,q}(\mathbb{F})$.

We consider the Grassmannians $G/K = \mathcal{G}_{p,q}(\mathbb{F})$ where G is one of the groups $SO_0(p, q), SU(p, q)$ or $Sp(p, q)$, and K is the maximal compact subgroup $K = SO(p) \times SO(q)$, $S(U(p) \times U(q))$, $Sp(p) \times Sp(q)$, respectively. Note that G is the identity component of $SU(q, p; \mathbb{F})$, where $U(q, p; \mathbb{F})$ is the isometry group for the quadratic form

$$|x_1|^2 + \ldots + |x_q|^2 - |x_{q+1}|^2 - \ldots - |x_{p+q}|^2$$

on $\mathbb{F}^{p+q}$. In the same way, K is a subgroup of $U(q, \mathbb{F}) \times U(p, \mathbb{F})$ where

$$U(q, \mathbb{F}) = \{X \in M_q(\mathbb{F}) : X^*X = I_q\}$$

is the unitary group over $\mathbb{F}$; here $X^* = \overline{X}^t$ denotes the conjugate transpose. The Lie algebra $\mathfrak{g}$ of G consists of the matrices

$$X = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \in M_{p+q}(\mathbb{F})$$

with blocks $A = -A^* \in M_q(\mathbb{F})$ and $D = -D^* \in M_p(\mathbb{F})$ satisfying $\text{tr}A + \text{tr}D = 0$, as well as $B \in M_{q,p}(\mathbb{F})$. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the associated Cartan decomposition of $\mathfrak{g}$, with $\mathfrak{p}$ consisting of the (q,p)-block matrices

$$X = \begin{pmatrix} 0 & X \\ X^* & 0 \end{pmatrix}, \quad X \in M_{q,p}(\mathbb{F}).$$

In accordance with [Sa], we use as a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ the set of matrices

$$H_t = \begin{pmatrix} 0_{q\times q} & \underline{t} & 0_{q\times(p-q)} \\ \underline{t} & 0_{q\times q} & 0_{q\times(p-q)} \\ 0_{(p-q)\times q} & 0_{(p-q)\times q} & 0_{(p-q)\times(p-q)} \end{pmatrix}$$

where $\underline{t} = \text{diag}(t_1, \ldots, t_q)$ is the diagonal matrix corresponding to $t = (t_1, \ldots, t_q) \in \mathbb{R}^q$. We remark that our present notions are adjusted to those of [Sa] (with p and q exchanged), and are slightly different from those used in [R2].

The restricted root system $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ consists of the non-zero linear functionals $\alpha \in \mathfrak{a}^*$ such that

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \ \forall H \in \mathfrak{a}\} \neq \{0\}.$$

In our case, the root system is of type B_q if $\mathbb{F} = \mathbb{R}$ and of type BC_q if $\mathbb{F} = \mathbb{C}$ or $\mathbb{H}$. The multiplicities $m_\alpha = \dim_{\mathbb{R}} \mathfrak{g}_\alpha$ can be found e.g. in table 9 of [OV]. We shall need an explicit description of the root spaces. For this, define $f_i \in \mathfrak{a}^*$ by $f_i(H_t) = t_i$, $i = 1, \ldots, q$. We shall write matrices from $\mathfrak{g}$ in $(q, q, p-q)$-block form. By E_{ij} we denote a matrix of appropriate size which has entries 0 except in position (i,j), where the entry is 1. Notice that $E_{ij} \cdot \lambda = \lambda \cdot E_{ij}$ for $\lambda \in \mathbb{F}$. The following list of roots is easily verified by block multiplications; in the real case, it matches Theorem 5 of [Sa].

(1) $\alpha = \pm f_i$, $1 \leq i \leq q$. The root space $\mathfrak{g}_\alpha$ is given by $\mathfrak{g}_\alpha = \{X_{ir}^{\pm}(\lambda) : \lambda \in \mathbb{F}, r = 1, \ldots, p-q\}$ with

$$X_{ir}^{\pm}(\lambda) = \begin{pmatrix} 0 & 0 & \lambda E_{ir} \\ 0 & 0 & \pm\lambda E_{ir} \\ \overline{\lambda} E_{ri} & \mp\overline{\lambda} E_{ri} & 0 \end{pmatrix}.$$

The multiplicity of α is $m_\alpha = d(p-q)$.

(2) $\alpha = \pm(f_i - f_j)$, $1 \leq i < j \leq q$. In this case, $\mathfrak{g}_\alpha = \{Y_{ij}^{\pm}(\lambda) : \lambda \in \mathbb{F}\}$ with

$$Y_{ij}^{\pm}(\lambda) = \begin{pmatrix} \pm(\lambda E_{ij} - \overline{\lambda} E_{ji}) & \lambda E_{ij} + \overline{\lambda} E_{ji} & 0 \\ \lambda E_{ij} + \overline{\lambda} E_{ji} & \pm(\lambda E_{ij} - \overline{\lambda} E_{ji}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The multiplicity is $m_\alpha = d$.

(3) $\alpha = \pm(f_i + f_j)$, $1 \leq i < j \leq q$. Here $\mathfrak{g}_\alpha = \{Z_{ij}^{\pm}(\lambda) : \lambda \in \mathbb{F}\}$ with

$$Z_{ij}^{\pm}(\lambda) = \begin{pmatrix} \pm(\lambda E_{ij} - \overline{\lambda} E_{ji}) & -\lambda E_{ij} + \overline{\lambda} E_{ji} & 0 \\ -\overline{\lambda} E_{ji} + \lambda E_{ij} & \pm(\overline{\lambda} E_{ji} - \lambda E_{ij}) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Again, the multiplicity is $m_\alpha = d$.

(4) $\alpha = \pm 2f_i\,,\ 1 \le i \le q$. This family of roots occurs only for $\mathbb{F} = \mathbb{C}, \mathbb{H}$. The root spaces are given by $\mathfrak{g}_\alpha = \{\lambda \cdot W_i^\pm : \lambda \in \mathbb{F}, \overline{\lambda} = -\lambda\}$ with

$$W_i^\pm = \begin{pmatrix} E_{ii} & 0 & \mp E_{ii} \\ 0 & 0 & 0 \\ \pm E_{ii} & 0 & -E_{ii} \end{pmatrix}.$$

In order to obtain a unified notion, we consider $\alpha = \pm 2f_i$ also a root if $\mathbb{F} = \mathbb{R}$, with multiplicity zero. Then $m_\alpha = d - 1$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

In our unified notion, Σ is of type BC_q in all cases, with the understanding that 0 may occur as a multiplicity on the long roots. As usual, we choose the positive subsystem

$$\Sigma_+ = \{f_i, 2f_i, 1 \le i \le q\} \cup \{f_i \pm f_j,\, 1 \le i < j \le q\}.$$

Then the weighted half-sum of positive roots is

$$\rho^{BC} = \rho^{BC}(p) = \frac{1}{2} \sum_{\alpha \in \Sigma_+} m_\alpha \alpha \ = \sum_{i=1}^{q} \big(\frac{d}{2}(p + q + 2 - 2i) - 1\big) f_i\,. \tag{2.1}$$

Let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma_+} \mathfrak{g}_\alpha$$

and $N = \exp \mathfrak{n}$, $A = \exp \mathfrak{a}$. Then A is abelian, N is nilpotent, and $G = KAN$ is an Iwasawa decomposition of G. The spherical functions of G/K are given by the Harish-Chandra integral formula

$$\phi_\lambda^p(a_t) = \int_K e^{(i\lambda - \rho^{BC})(H(a_t k))} dk, \quad \lambda \in \mathfrak{a}_\mathbb{C}^* \tag{2.2}$$

where $H(g) \in A$ denotes the unique abelian part of $g \in G$ in the Iwasawa decomposition $G = KAN$ (see e.g. [GV]), and

$$a_t = \exp(H_t) = \begin{pmatrix} \cosh \underline{t} & \sinh \underline{t} & 0 \\ \sinh \underline{t} & \cosh \underline{t} & 0 \\ 0 & 0 & I_{p-q} \end{pmatrix} \tag{2.3}$$

with $\cosh \underline{t} = \operatorname{diag}(\cosh t_1, \dots, \cosh t_q)$, $\sinh \underline{t} = \operatorname{diag}(\cosh t_1, \dots, \cosh t_q)$.
We shall identify $\mathfrak{a}_\mathbb{C}^*$ with $\mathbb{C}^q$ via $\lambda \mapsto (\lambda_1, \dots, \lambda_q)$ for $\lambda \in \mathfrak{a}_\mathbb{C}^*$ given by $\lambda(H_t) = \sum_{r=1}^q \lambda_r t_r\,,\ \lambda_r \in \mathbb{C}$.

In order to state a more explicit form of the Harish-Chandra integral above, we need some further notation. For a Hermitian square matrix $A = (a_{ij})$ over $\mathbb{F}$ we denote by $\Delta(A)$ the determinant of A, i.e. the product of its eigenvalues (which are real) and by $\Delta_r(A) = \Delta((a_{ij})_{1 \le i,j \le r})$ its r-th principal minor, see [FK] for details.

We introduce the usual power functions on the cone

$$\Omega_q = \{x \in M_q(\mathbb{F}) : x = x^*, x \text{ strictly positive definite}\},$$

(c.f. [FK]), Chap.VII.1.): For $\lambda \in \mathbb{C}^q \cong \mathfrak{a}_\mathbb{C}^*$ and $x \in \Omega_q$, we define

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1 - \lambda_2} \cdot \ldots \cdot \Delta_{q-1}(x)^{\lambda_{q-1} - \lambda_q} \cdot \Delta_q(x)^{\lambda_q}. \tag{2.4}$$

We also define the projection matrix

$$\sigma_0 := \begin{pmatrix} I_q \\ 0_{(p-q)\times q} \end{pmatrix} \in M_{p,q}(\mathbb{F}).$$

The following result generalizes Theorem 16 of [Sa].

Theorem 2.1. *For the Grassmannian $\mathcal{G}_{p,q}(\mathbb{F})$, the spherical functions* (2.2) *are given by*

$$\phi_\lambda^p(a_t) = \int_K \Delta_{(i\lambda-\rho^{BC})/2}(x_t(k))dk, \ \lambda \in \mathbb{C}^q$$

where for $k = \begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix} \in K$ *with* $u \in U(q,\mathbb{F}), v \in U(p,\mathbb{F})$,

$$x_t(k) := (\cosh \underline{t}\, u + \sinh \underline{t}\, \sigma_0^* v \sigma_0)^*(\cosh \underline{t}\, u + \sinh \underline{t}\, \sigma_0^* v\sigma_0) \in \Omega_q\,.$$

Proof. We closely follow [Sa]. Let

$$S = \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} I_q & 0_{q\times(p-q)} & J_q \\ I_q & 0_{q\times(p-q)} & -J_q \\ 0_{(p-q)\times q} & \sqrt{2}I_{p-q} & 0_{(p-q)\times q}\end{pmatrix} \quad \text{with } J_q = (\delta_{i,q+1-j})_{i,j} \in M_q(\mathbb{F}).$$

Notice that $S^*S = I_{p+q}$. Using the explicit form of the root spaces above, one checks that S^*XS is strictly upper triangular for each $X \in \mathfrak{n}$. Thus for $n \in N$, the matrix S^*nS is upper triangular with entries 1 in the diagonal. Furthermore,

$$S^* \exp(H_t)S = \text{diag}(e^{t_1}, \dots, e^{t_q}, 1, \dots, 1, e^{-t_q}, \dots, e^{-t_1})$$

with $p-q$ entries 1. Consider $g = k\exp(H_t)n \in KAN$ and let $1 \le r \le q$. As in the proof of Proposition 14 of [Sa], we calculate the principal minors

$$\Delta_r(S^*g^*gS) = \Delta_r((S^*nS)^*(S^*\exp(2H_t)S)S^*nS) = e^{2(t_1+\ldots+t_r)}.$$

Writing $g = k\exp(H_t)n$ in (q,p)-block form as $g = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$, the upper left $q\times q$-block of S^*g^*gS becomes

$$(A+B\sigma_0)^*(A+B\sigma_0) \quad \text{with } \sigma_0 = \begin{pmatrix} I_q \\ 0_{(p-q)\times q}\end{pmatrix} \in M_{p,q}(\mathbb{F}).$$

Thus

$$t_r = \frac{1}{2}\log\frac{\Delta_r((A+B\sigma_0)^*(A+B\sigma_0))}{\Delta_{r-1}((A+B\sigma_0)^*(A+B\sigma_0))}, \tag{2.5}$$

with the agreement $\Delta_0 := 1$. Notice that this generalizes Proposition 14 of [Sa], and that the arguments of Δ_r and Δ_{r-1} belong to the cone Ω_q, because gS is non-singular.

Now consider $g = a_t k$ with $k = \begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix} \in K$. We have

$$a_t k = \begin{pmatrix} \cosh \underline{t} & \sinh \underline{t} & 0 \\ \sinh \underline{t} & \cosh \underline{t} & 0 \\ 0 & 0 & I_{p-q}\end{pmatrix} \cdot \begin{pmatrix} u & 0 \\ 0 & v\end{pmatrix} = \begin{pmatrix} \cosh \underline{t}\, u & \sinh \underline{t}\, \sigma_0^* v \\ * & * \end{pmatrix}.$$

By (2.5), this gives

$$e^{\lambda(H(a_tk))} = \prod_{r=1}^q \Big(\frac{\Delta_r(x_t(k))}{\Delta_{r-1}(x_t(k))}\Big)^{\lambda_r/2} = \Delta_{\lambda/2}(x_t(k)),$$

which proves the statement. □

For $p \ge 2q$ we may reduce the integral in Theorem 2.1 by techniques from [R1], [R2]. For this, consider the ball

$$B_q = \{w \in M_q(\mathbb{F}) : w^*w < I\},$$

where $A < B$ means that $B - A$ is (strictly) positive definite. Define also the probability measure m_p on B_q given by

$$dm_p(w) = \frac{1}{\kappa_{pd/2}} \cdot \Delta(I - w^*w)^{pd/2-\gamma} dw, \tag{2.6}$$

where

$$\gamma := d\big(q - \frac{1}{2}\big) + 1,$$

dw is the Lebesgue measure on the ball B_q, and

$$\kappa_{pd/2} = \int_{B_q} \Delta(I - w^*w)^{pd/2-\gamma}\, dw. \tag{2.7}$$

Notice that m_p is a probability measure on B_q.

By $U_0(q,\mathbb{F})$ we denote the identity component of $U(q,\mathbb{F})$. Notice that $U(q,\mathbb{F}) = U_0(q,\mathbb{F})$ for $\mathbb{F} = \mathbb{C},\mathbb{H}$, while $U_0(q,\mathbb{R}) = SO(q)$. With these notions, we obtain the following integral representation:

Corollary 2.2. *Let $p \geq 2q$ be an integer. Then the spherical functions* (2.2) *can be written as*

$$\phi_\lambda^p(a_t) = \int_{U_0(q,\mathbb{F})\times B_q} \Delta_{(i\lambda-\rho^{BC})/2}(g_t(u,w))\, dm_p(w)du \tag{2.8}$$

where du denotes the normalized Haar measure on $U_0(q)$, and

$$g_t(u,w) = u^{-1}(\cosh \underline{t} + \sinh \underline{t}\, w)^*(\cosh \underline{t} + \sinh \underline{t}\, w)u\,.$$

The same formula holds with the argument $g_t(u,w)$ replaced by

$$\widetilde{g}_t(u,w) = u^{-1}(\cosh \underline{t} + \sinh \underline{t}\, w)(\cosh \underline{t} + \sinh \underline{t}\, w)^*u.$$

Proof. In a first step, we replace the integral over K in Theorem 2.1 by an integral over $U_0(q,\mathbb{F}) \times U(p,\mathbb{F})$. This is achieved in the same way as for the integral (2.5) in [R2]; it is important in this context that the argument $x_t(k)$ depends only on the upper left $q\times q$-block of v. Lemma 2.1 of [R2] then gives the first formula with the argument $(\cosh \underline{t}\, u + \sinh \underline{t}\, w)^*(\cosh \underline{t}\, u + \sinh \underline{t}\, w)$ instead of $g_t(u,w)$, which is then obtained by a change of variables $w \mapsto wu$.

For the proof of the second equation, notice that for $a := \cosh t + \sinh t \cdot w \in M_q(\mathbb{F})$, the matrices a^*a and aa^* have the same eigenvalues with the same multiplicities. Therefore, $a^*a = vaa^*v^*$ with some $v \in U(q,\mathbb{F})$ for $\mathbb{F} = \mathbb{R},\mathbb{C}$. In fact, this also valid for $\mathbb{H}$. To check this, write $a \in M_q(\mathbb{H})$ as $a = a_1 + ja_2$ for complex matrices $a_1, a_2 \in M_q(\mathbb{C})$, and form

$$\chi_a := \begin{pmatrix} a_1 & a_2 \\ -\bar{a}_2 & \bar{a}_1 \end{pmatrix} \quad \in M_{2q}(\mathbb{C}).$$

The mapping $\chi : M_q(\mathbb{H}) \to M_{2q}(\mathbb{C})$, $a \mapsto \chi_a$, is a $*$-homomorphism of algebras, and $\chi_a^*\chi_a$ and $\chi_a\chi_a^*$ have the same eigenvalues as a^*a and aa^* respectively with the doubled multiplicities; see the survey [Zh]. Thus, a^*a and aa^* have the same eigenvalues with the same multiplicities, and hence $a^*a = vaa^*v^*$ with some $v \in U(q,\mathbb{H})$.

Using $a^*a = vaa^*v^*$ for some $v \in U(q,\mathbb{F})$, we see that for each fixed $w \in B_q$

$$\int_{U_0(q,\mathbb{F})} \Delta_{(i\lambda-\rho)/2}(\widetilde{g}(t,u,w))du = \int_{U_0(q,\mathbb{F})} \Delta_{(i\lambda-\rho)/2}(u^*vaa^*v^*u)du = \int_{U_0(q,\mathbb{F})} \Delta_{(i\lambda-\rho)/2}(g(t,u,w))du.$$

This yields the second equation. □

We now identify $t \in C_q$ with the matrices $a_t \in G$ as above and regard the spherical functions ϕ_λ^p above as functions on the Weyl chamber C_q. With this agreement we now extend the integral representation (2.8) above from integer parameters $p \geq 2q$ to arbitrary real parameters $p \geq 2q-1$. For this we fix $\mathbb{F}$ (and thus $d = 1, 2, 4$) and define the functions

$$\phi_\lambda^p(t) := F_{BC}(i\lambda, k_p; t) \qquad (t \in C_q,\ \lambda \in \mathbb{C}^q) \tag{2.9}$$

with

$$k_p = (d(p-q)/2, (d-1)/2, d/2),$$

which are analytic in p with $\operatorname{Re} p > q$. Note that for integers p, the functions ϕ_λ^p are precisely the spherical functions (2.2). For the extension of the integral representation, we shall employ Carlson's theorem on analytic continuation which we recapitulate from [Ti], p.186:

Theorem 2.3. *Let $f(z)$ be holomorphic in a neighbourhood of $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$ satisfying $f(z) = O\big(e^{c|z|}\big)$ on $\operatorname{Re} z \geq 0$ for some $c < \pi$. If $f(z) = 0$ for all nonnegative integers z, then f is identically zero for $\operatorname{Re} z > 0$.*

We shall prove:

Theorem 2.4. *Let $p \in \mathbb{R}$ with $p > 2q-1$. Then the functions (2.9) satisfy*

$$\phi_\lambda^p(t) = \int_{B_q \times U_0(q,\mathbb{F})} \Delta_{(i\lambda - \rho^{BC})/2}(g_t(u,w))\, dm_p(w) du \tag{2.10}$$

for all $\lambda \in \mathbb{C}^q$ and $t \in C_q$, where again the argument g_t may be replaced by $\widetilde{g}_t$ as in Corollary 2.2.

Proof. We first observe that both sides of (2.10) are analytic in p and λ. In order to employ Carlson's theorem to extend (2.8) to $p \in]2q-1, \infty[$, we need a suitable exponential growth bound on F_{BC} w.r.t. p in some right half plane. Such exponential estimates are available only for real, nonnegative multiplicities; see Proposition 6.1 of [O2], [Sch], and Section 3 of [RKV]. We thus proceed in two steps and closely follow the proof of Theorem 4.1 of [R2], where a product formula is obtained by analytic continuation. We first restrict our attention to a discrete set of spectral parameters λ for which F_{BC} is a (renormalized) Jacobi polynomial and where the growth condition is easily checked. Carlson's theorem then leads to (2.10) for this discrete set of parameters λ and all $p \in]2q-1, \infty[$. In a further step we fix $p \in]2q-1, \infty[$ and extend (2.10) to all $\lambda \in \mathbb{C}^q$.

Let us go into details. We need some notation and facts from [O2] and [HS]. For $R = 2 \cdot BC_q$ with the set R_+ of positive roots, consider the weighted half-sum of positive roots

$$\rho(k) := \frac{1}{2} \sum_{\alpha \in R_+} k(\alpha)\alpha = \sum_{i=1}^{q} (k_1 + 2k_2 + 2k_3(q-i)) e_i \tag{2.11}$$

as well the c-function

$$c(\lambda, k) := \prod_{\alpha \in R_+} \frac{\Gamma(\langle \lambda, \alpha^\vee \rangle + \frac{1}{2}k(\frac{\alpha}{2}))}{\Gamma(\langle \lambda, \alpha^\vee \rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha))} \cdot \prod_{\alpha \in R_+} \frac{\Gamma(\langle \rho(k), \alpha^\vee \rangle + \frac{1}{2}k(\frac{\alpha}{2}) + k(\alpha))}{\Gamma(\langle \rho(k), \alpha^\vee \rangle + \frac{1}{2}k(\frac{\alpha}{2}))} \tag{2.12}$$

with the usual inner product on $\mathbb{C}^q$ and the conventions $\alpha^\vee := 2\alpha/\langle\alpha,\alpha\rangle$ and $k(\frac{\alpha}{2}) = 0$ for $\frac{\alpha}{2} \notin R$. The c-function is meromorphic on $\mathbb{C}^q \times \mathbb{C}^3$. We consider the dual root system $R^\vee = \{\alpha^\vee : \alpha \in R\}$, the coroot lattice $Q^\vee = \mathbb{Z}.R^\vee$, and the weight lattice $P = \{\lambda \in \mathbb{R}^q : \langle\lambda,\alpha^\vee\rangle \in \mathbb{Z}\ \forall\, \alpha \in R\}$. Further, denote by $P_+ = \{\lambda \in P : \langle\lambda,\alpha^\vee\rangle \geq 0\ \forall\, \alpha \in R_+\}$ the set of dominant weights associated with R_+. In our case, $P_+ = C_q \cap 2\mathbb{Z}^q$. According to Eq. (4.4.10) of [HS], we have for $k \geq 0$ and $\lambda \in P_+$ the connection

$$F_{BC}(\lambda + \rho(k), k; t) = c(\lambda + \rho(k), k) P_\lambda(k;t) \tag{2.13}$$

where the P_λ are the Heckman-Opdam Jacobi polynomials associated with BC_q. We also consider the specific multiplicities $k_p := (d(p-q)/2, (d-1)/2, d/2)$ and the associated constants $\rho(k_p) = \rho^{BC}$ as in (2.1). With these notations we obtain from (2.13) and (2.9) that the integral representation (2.10) can be written as

$$P_\lambda(k_p;t) = \frac{1}{c(\lambda+\rho(k_p),k_p)} \cdot \frac{1}{\kappa_{pd/2}} \int_{B_q} \int_{U_0(q,\mathbb{F})} \Delta_{\lambda/2}(g_t(u,w)) \Delta(I - w^*w)^{pd/2-\gamma} dw du. \tag{2.14}$$

Exactly as in the proof of Theorem 4.1 of [R2], it is now checked that both sides of (2.14) are, as functions of p, of polynomial growth in the half-plane $\{p \in \mathbb{C} : \mathrm{Re}(pd/2) > \gamma - 1\}$; we omit the details. We may therefore apply Carlson's theorem to (2.14), and this proves (2.10) for p with $\mathrm{Re}(pd/2) > \gamma - 1$ and all spectral parameters of the form $-i(\lambda + \rho(k_p))$ with $\lambda \in P_+$.

We next fix $p \in \mathbb{R}$ with $p > 2q-1$ (in which case k_p is nonnegative) and extend (2.10) with respect to the spectral parameter λ. According to Proposition 6.1 of [O2],

$$|F_{BC}(\lambda, k_p; t)| \leq |W|^{1/2} e^{\max_{w\in W} \mathrm{Re}\, \langle w\lambda, t\rangle}$$

where W is the Weyl group of BC_q. Let C_q^0 denote the interior of C_q and $H' := \{\lambda \in \mathbb{C}^q : \mathrm{Re}\lambda \in C_q^0\}$. Then

$$\mathrm{Re}\langle w\lambda, t\rangle \leq \mathrm{Re}\langle\lambda, t\rangle \qquad \text{for} \quad \lambda \in H',\ t \in C_q,\ w \in W.$$

Now fix $t \in C_q$ and p as above, and choose a vector $a \in C_q^0$ sufficiently large. Then (2.10) for the spectral parameter $\lambda + \rho(k_p)$ is equivalent to

$$e^{-\langle\lambda,a+t\rangle} \phi^p_{-i(\lambda+\rho(k_p))}(t) = \int_{B_q \times U_0(q,\mathbb{F})} e^{-\langle\lambda,a+t\rangle} \cdot \Delta_{\lambda/2}(g_t(u,w))\, dm_p(w) du.$$

The left hand side remains bounded for $\lambda \in H'$. Moreover, for $a \in C_q^0$ sufficiently large,

$$\sup_{(u,w)\in U_0(q,\mathbb{F})\times B_q;\lambda\in H'} |e^{-\langle\lambda,a+t\rangle} \cdot \Delta_{\lambda/2}(g_t(u,w))| < \infty,$$

which proves that also the right hand side remains bounded for $\lambda \in H'$. By a q-fold application of Carlson's theorem we thus may extend the preceding equation from $\lambda \in P_+$ to $\lambda \in H'$. A classical analytic continuation now finishes the proof. □

The above proof reveals in particular the following integral representation for Heckman-Opdam polynomials of type BC:

Corollary 2.5. *Let $k_p = (d(p-q)/2, (d-1)/2, d/2)$ with $p \in \mathbb{R}$, $p > 2q-1$. Then the Heckman-Opdam polynomials of type BC_q with multiplicity k_p have the integral* *representation*

$$P_\lambda(k_p;t) = \frac{1}{c(\lambda+\rho(k_p),k_p)}\int_{B_q\times U_0(q,\mathbb{F})}\Delta_{\lambda/2}(g_t(u,w))dm_p(w)du \quad \text{for } t\in\mathbb{C}^q.$$

Here $\lambda\in P_+ = C_q\cap 2\mathbb{Z}^q$ *and*

$$g_t(u,w) = u^{-1}(\cosh\underline{t} + w^*\sinh\underline{t})(\cosh\underline{t} + \sinh\underline{t}\,w)u\,.$$

Remark 2.6. For the limit case $p = 2q-1$, a degenerate version of the integral representation (2.10) is available. For this we follow Section 3 of [R1].

We fix q and consider the matrix ball $B_q := \{w\in M_q(\mathbb{F}):\ w^*w < I_q\}$ as above as well as the ball $B := \{y\in\mathbb{F}^q:\ \|y\|_2 = (\sum_{j=1}^q \overline{y}_j y_j)^{1/2} < 1\}$ and the sphere $S := \{y\in\mathbb{F}^q:\ \|y\|_2 = 1\}$. By Lemma 3.7 and Corollary 3.8 of [R1], the mapping

$$P(y_1,\ldots,y_q) := \begin{pmatrix} y_1 \\ y_2(I_q - y_1^*y_1)^{1/2} \\ \vdots \\ y_q(I_q - y_{q-1}^*y_{q-1})^{1/2}\cdots(I_q-y_1^*y_1)^{1/2}\end{pmatrix} \tag{2.15}$$

establishes a diffeomorphism $P: B^q\to B_q$. The image of the measure $dm_p(w)$ under P^{-1} is given by

$$\frac{1}{\kappa_{pd/2}}\prod_{j=1}^q (1-\|y_j\|_2^2)^{d(p-q-j+1)/2-1}dy_1\ldots dy_q\,. \tag{2.16}$$

Thus for $p > 2q-1$, the integral representation (2.10) may be rewritten as

$$\phi_\lambda^p(t) = \frac{1}{\kappa_{pd/2}}\int_{B^q}\int_{U_0(q,\mathbb{F})}\Delta_{(i\lambda-\rho^{BC})/2}(g_t(u,P(y)))\cdot\prod_{j=1}^q(1-\|y_j\|_2^2)^{d(p-q-j+1)/2-1}dy_1\ldots dy_q dw \tag{2.17}$$

where $dy_1,\ldots,dy_q$ means integration w.r.t. the Lebesgue measure on $\mathbb{F}^q$. Moreover, for $p\downarrow 2q-1$, (2.17) and continuity lead to the following degenerated product formula:

$$\begin{aligned}\phi_\lambda^{2q-1}(t) = \frac{1}{\kappa_{(2q-1)d/2}}&\int_{B^{q-1}}\int_S\int_{U_0(q,\mathbb{F})}\Delta_{(i\lambda-\rho^{BC})/2}(g_t(u,P(y)))\,\cdot \\ &\cdot\prod_{j=1}^{q-1}(1-\|y_j\|_2^2)^{d(q-j)/2-1}dy_1\ldots dy_{q-1}\,d\sigma(y_q)\,dw\end{aligned} \tag{2.18}$$

where $\sigma\in M^1(S)$ is the uniform distribution on the sphere S and

$$\kappa_{(2q-1)d/2} = \int_{B^{q-1}}\int_S\prod_{j=1}^{q-1}(1-\|y_j\|_2^2)^{d(q-j)/2-1}dy_1\ldots dy_{q-1}\,d\sigma(y_q).$$

Notice that the ϕ_λ^{2q-1} are the spherical functions of the Grassmannian $\mathcal{G}_{2q-1,q}(\mathbb{F})$.

3. The connection with spherical functions of type A_{q-1}

We shall compare the spherical functions of the Grassmannians $\mathcal{G}_{p,q}(\mathbb{F})$ with the spherical functions of the symmetric space $\mathcal{P}_q(\mathbb{F}) = G/K$ with $G = GL(q,\mathbb{F}), K = U(q,\mathbb{F})$. It is well-known that G has the Iwasawa decomposition $G = KAN$ where $A = \exp\mathfrak{a}$, $\mathfrak{a} = \{H_t = \underline{t},\, t = (t_1,\ldots,t_q)\in\mathbb{R}^q\}$ and N is the unipotent group consisting of all upper triangular matrices with entries 1 in the diagonal. The restricted root system $\Delta(\mathfrak{g},\mathfrak{a})$ is of type A_{q-1}, with a positive subsystem given by

$$\Delta_+ = \{f_i - f_j : 1 \le i < j \le q\}.$$

Here the multiplicity is $m_\alpha = d$ for all $\alpha \in \Delta_+$ and the weighted half-sum of positive roots is

$$\rho^A = \sum_{i=1}^{q} \frac{d}{2}(q+1-2i) f_i\,.$$

Again, $\mathfrak{a}_{\mathbb{C}}^*$ may be identified with $\mathbb{C}^q$ via $\lambda \mapsto (\lambda_1,\ldots,\lambda_q)$ for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ given by $\lambda(H_t) = \sum_{r=1}^q \lambda_r t_r\,,\ \lambda_r \in \mathbb{C}$. We briefly recall the further well-known calculation, which is similar to the Grassmannian case: For $g = k\exp(H_t)n \in KAN$ one obtains $\Delta_r(g^*g) = e^{2(t_1+\ldots+t_r)}$ and thus

$$t_r = \frac{1}{2}\log\frac{\Delta_r(g^*g)}{\Delta_{r-1}(g^*g)} \qquad (r = 1,\ldots,q).$$

If $g = a_t k$ with $a_t = \exp(H_t) = e^{\underline{t}}$ and $k \in K$, then $g^*g = k^{-1}e^{2\underline{t}}k$. The spherical functions of $G/K = \mathcal{P}_q(\mathbb{F})$ are given by

$$\psi_\lambda^A(e^{\underline{t}}) = \int_K e^{(i\lambda-\rho^A)(H(a_tk))}dk, \quad \lambda \in \mathbb{C}^q. \tag{3.1}$$

The above considerations lead to the known integral representation

$$\psi_\lambda^A(e^{\underline{t}}) = \int_{U(q,\mathbb{F})} \Delta_{(i\lambda-\rho^A)/2}\big(u^{-1}e^{2\underline{t}}\,u\big)du \;=\; \int_{U_0(q,\mathbb{F})} \Delta_{(i\lambda-\rho^A)/2}\big(u^{-1}e^{2\underline{t}}\,u\big)du. \tag{3.2}$$

We also remark that the functions ψ_λ^A can be written in terms of the Heckman-Opdam hypergeometric function F_A associated with the root system $2A_{q-1} = \{\pm 2(e_i - e_j) : 1 \le i < j \le q\}$, as follows:

$$\psi_\lambda^A(e^{\underline{t}}) = e^{\langle t-\pi(t),\lambda\rangle} \cdot F_A(\pi(\lambda), d/2; \pi(t)) \qquad (\lambda \in \mathbb{C}^q,\ t \in \mathbb{R}^q). \tag{3.3}$$

Here π denotes the orthogonal projection $\mathbb{R}^q \to \mathbb{R}_0^q := \{t \in \mathbb{R}^q : t_1 + \ldots + t_q = 0\}$; see Eq. (6.7) of [RKV] and note our rescaling of the root system by the factor 2.

We compare (3.2) with the integral (2.8) for the spherical functions of $\mathcal{G}_{p,q}(\mathbb{F})$ and, more generally, with representation (2.10) for the hypergeometric functions $\phi^p_{\lambda-i\rho^{BC}}$. As for $p \to \infty$ the probability measures m_p on B_q tend weakly to the point measure at the zero matrix, we obtain:

Corollary 3.1. *The spherical functions of $\mathcal{G}_{p,q}(\mathbb{F})$, and more generally, the hypergeometric functions $\phi^p_{\lambda-i\rho^{BC}}$ with $p \in \mathbb{R}, p > 2q-1$ are related to the spherical functions of $\mathcal{P}_q(\mathbb{F})$ by*

$$\lim_{p\to\infty} \phi^p_{\lambda-i\rho^{BC}}(t) = \psi^A_{\lambda-i\rho^A}(\cosh \underline{t}) \quad (t \in \mathbb{R}^q).$$

This result was already obtained in Corollary 6.1 of [RKV] by completely different methods, namely as a special case of a general limit transition for hypergeometric functions of type BC. However, the approach in [RKV] seems not suitable to gain information on the rate of convergence. In the following section, we study the integral representations (3.2) and (2.8) (or (2.10) for continuous p) in order to derive precise estimates on the rate of convergence.

4. The rate of convergence for $p \to \infty$

The main result of this section is Theorem 4.2. It sharpens the qualitative limit of Corollary 3.1 for the Heckman-Opdam hypergeometric functions ϕ_λ^p by a precise estimate of the approximation error. Again, $p > 2q-1$ varies and the rank q as well as the dimension $d = 1, 2, 4$ of $\mathbb{F}$ are fixed. For convenience, we consider the type A spherical functions ψ_λ^A as functions on $\mathbb{R}^q$ and study

$$\psi_\lambda(t) := \psi_\lambda^A(\cosh \underline{t}) = \int_{U_0(q,\mathbb{F})} \Delta_{(i\lambda-\rho^A)/2}\big(u^{-1}\cosh^2 \underline{t} u\big)du. \tag{4.1}$$

We write

$$\|\lambda\|_1 := |\lambda_1| + \ldots + |\lambda_q| \quad \text{for } \lambda \in \mathbb{C}^q;$$
$$\tilde{t} := \min(t_1, 1) \geq 0 \;\text{ for }\; t = (t_1, \ldots, t_q) \in C_q\,.$$

The action of the Weyl group W of type BC_q extends in a natural way to $\mathbb{C}^q$. We write

$$\rho := \rho^{BC}(p)$$

for the weighted half-sum defined in (2.1). Moreover, $co(W.\rho) \subset \mathbb{R}^q$ denotes the convex hull of the W-orbit of ρ.

Let us recapitulate the following known properties of ϕ_λ^p:

Lemma 4.1. (1) *For all $t \in C_q$, $\lambda \in \mathbb{C}^q$, and $p \in \mathbb{R}$ with $p \geq q$,*

$$\left|\phi_{\lambda-i\rho}^p(t)\right| \leq e^{\max_{w\in W} \operatorname{Im}\langle w\lambda, t\rangle}.$$

(2) *ϕ_λ^p is bounded if and only if $\operatorname{Im}\lambda \in co(W.\rho)$. In this case, $\|\phi_\lambda^p\|_\infty = 1$.*
(3) *If λ is purely imaginary, then ϕ_λ^p is real-valued and strictly positive on C_q.*

Proof. (1) follows from Corollary 3.4 of [RKV]. For part (2) we refer to Theorem 5.4 of [R2] and Theorem 4.2 of [NPP] (the proof of the only-if-part in [R2] contains a gap). Part (3) follows from Lemma 3.1 of [Sch]. □

Notice that by Corollary 3.1, the same estimates as in Lemma 4.1 hold for the function $\psi_{\lambda-i\rho^A}(t)$. The following theorem is the main result of this section:

Theorem 4.2. *There exists a universal constant $C = C(\mathbb{F}, q)$ as follows:*
(1) *For all $p > 2q-1$, $t \in C_q$ and $\lambda \in \mathbb{C}^q$,*

$$\left|\phi_{\lambda-i\rho}^p(t) - \psi_{\lambda-i\rho^A}(t)\right| \leq C \cdot \frac{\|\lambda\|_1 \cdot \tilde{t}}{p^{1/2}} \cdot e^{\max_{w\in W} \operatorname{Im}\langle w\lambda, t\rangle}.$$

(2) *Let $p > 2q-1$, $t \in C_q$, and $\lambda \in \mathbb{C}^q$ such that $\operatorname{Im}\lambda - \rho$ is contained in $co(W.\rho)$, i.e., $\phi_{\lambda-i\rho}^p$ is bounded on C_q. Then*

$$\left|\phi_{\lambda-i\rho}^p(t) - \psi_{\lambda-i\rho^A}(t)\right| \leq C \cdot \frac{\|\lambda\|_1 \cdot \tilde{t}}{p^{1/2}}.$$

In particular, for these spectral parameters λ the convergence is uniform of order $O(p^{-1/2})$ in $t \in C_q$.

We briefly discuss this result in the rank-one case $q = 1$. Here the Heckman-Opdam functions ϕ_λ^p are Jacobi functions $\phi_\lambda^{(\alpha,\beta)}$ as studied in Koornwinder [K1]. More precisely,

$$\phi_\lambda^p(t) = \phi_\lambda^{(\alpha,\beta)}(t) \qquad \text{with } \alpha = dp/2,\ \beta = d/2-1,\ d = 1,2,4$$

and $\rho = \alpha + \beta + 1 = d(p+1)/2$. Furthermore,

$$\psi_\lambda(t) = e^{i\lambda \cdot \ln(\cosh t)} = (\cosh t)^{i\lambda}$$

independently of d, and $\rho^A = 0$. Thus, Theorem 4.2 implies for $q = 1$ the following

Corollary 4.3. *There exists a constant $C > 0$ as follows:*

(1) *For $\beta = -1/2, 0, 1$, all $t \in [0, \infty[$, $\alpha > 0$, and $\lambda \in \mathbb{C}$,*

$$\left|\phi^{(\alpha,\beta)}_{\lambda - i\rho}(t) - (\cosh t)^{i\lambda}\right| \le C \cdot \frac{|\lambda| \min(t,1)}{\sqrt{\alpha}} \cdot e^{|\operatorname{Im}\lambda| \cdot t}.$$

(2) *Let $\beta = -1/2, 0, 1$, $t \in [0, \infty[$, $\alpha > 0$, and $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda \in [0, 2\rho]$. Then*

$$\left|\phi^{(\alpha,\beta)}_{\lambda - i\rho}(t) - (\cosh t)^{i\lambda}\right| \le C \cdot \frac{|\lambda| \min(t,1)}{\sqrt{\alpha}}.$$

Remarks 4.4. (1) For $\operatorname{Im}\lambda = 0$ and all $\beta \ge -1/2$, Corollary 4.3(2) was proven in [V2]. The proof there relies on the well-known integral representation for the Jacobi functions for $\alpha \ge \beta \ge -1/2$ in [K1] and is similar to that given here. Corollary 4.3 (2) for $\operatorname{Im}\lambda = 0$ is used in [V2] to derive a central limit theorem for the hyperbolic distances of radial random walks on hyperbolic spaces from their starting point when the number of time steps as well as the dimensions of the hyperbolic spaces tend to infinity. Similar results can be derived from Theorem 4.2 for $q \ge 2$.

(2) Corollary 4.3 corresponds to the convergence of the known one-dimensional Jacobi convolutions $*_{(\alpha,\beta)}$ to a semigroup convolution on $[0, \infty[$ in [V1] where the multiplicative functions of the limit semigroup are precisely the functions $t \mapsto (\cosh t)^{i\lambda}$; i.e., the convergence of the convolution structures $*_{(\alpha,\beta)}$ for $\alpha \to \infty$ corresponds to the convergence of the multiplicative functions. The same picture appears for $q > 1$; see [R2] for the explicit convolution and [RKV] for the corresponding limit transition. In [K2], a corresponding result for polynomials was derived.

(3) There are similar limit results to those of Theorem 4.2 for Dunkl-type Bessel functions of types A and B, and for Bessel functions on matrix cones with applications in probability; see [RV2], [RV3].

We now turn to the proof of Theorem 4.2. In fact, our main result is essentially a consequence of Lemma 4.1 and the following technical variant of Theorem 4.2:

Theorem 4.5. *For each $n \in \mathbb{N}$ there is a constant $C = C(\mathbb{F}, q, n)$ such that for all $p > 2q - 1$, $t \in C_q$ and $\lambda \in \mathbb{C}^q$,*

$$\left|\phi^p_{\lambda - i\rho}(t) - \psi_{\lambda - i\rho^A}(t)\right| \le C \cdot \left(\phi^p_{\frac{2n}{2n-1} i \operatorname{Im}\lambda - i\rho}(t)^{\frac{2n-1}{2n}} + \psi_{\frac{2n}{2n-1} i \operatorname{Im}\lambda - i\rho^A}(t)^{\frac{2n-1}{2n}}\right) \frac{\|\lambda\|_1 \cdot \tilde{t}}{p^{1/2}}.$$

Notice that the functions ϕ, ψ on the right side take positive values by Lemma 4.1. In fact, Theorem 4.2(1) follows immediately from Lemma 4.1(1) and Theorem 4.5 with $n = 1$. For the proof of Theorem 4.2(2), consider $\lambda \in \mathbb{C}^q$ with $\operatorname{Im}\lambda - \rho \in co(W.\rho)$. As ϕ^p_λ is W-invariant in the spectral variable λ and the mapping $\lambda \mapsto -\lambda$ is an element of W, we may assume without loss of generality that $\operatorname{Im}\lambda - \rho \in -C_q$. Now choose $\epsilon_0 = \epsilon_0(q) > 0$ according to the following Lemma 4.6, and choose $n \in \mathbb{N}$ such that $\epsilon := (2n-1)^{-1} \le \epsilon_0$. Lemma 4.6 below for $y := \operatorname{Im}\lambda - \rho$ thus implies that

$$\frac{2n}{2n-1}\operatorname{Im}\lambda - \rho = (1+\epsilon)\operatorname{Im}\lambda - \rho = (1+\epsilon)y + \epsilon\rho \in co(W.\rho).$$

This fact, Lemma 4.1(2), and Theorem 4.5 then lead to Theorem 4.2(2) as claimed.

Lemma 4.6. *For each dimension q there exists a constant $\epsilon_0 = \epsilon_0(q) > 0$ such that for all $0 < \epsilon \le \epsilon_0$, all ρ in the interior of C_q, and all $y \in co(W.\rho) \cap (-C_q)$,*

$$(1+\epsilon)y + \epsilon\rho \in co(W.\rho).$$

The proof of this lemma will be postponed to an appendix at the end of this paper. We here only mention that for $q = 1, 2$ the lemma can be easily checked with $\epsilon_0 = 1$ at hand of a picture, but for $q \ge 3$ the situation is more complicated, and the lemma is then no longer true with $\epsilon_0 = 1$.

We now turn to the technical proof of Theorem 4.5. We decompose it into several steps. We first recall the integral representation (2.10),

$$\phi^p_{\lambda - i\rho}(t) = \int_{B_q} \int_{U_0(q,\mathbb{F})} \Delta_{i\lambda/2}(\widetilde{g}_t(u,w))\, dm_p(w) du \tag{4.2}$$

with the probability measure dm_p as in Section 2 and

$$\widetilde{g}_t(u,w) = u^*(\cosh \underline{t} + \sinh \underline{t}\, w)(\cosh \underline{t} + \sinh \underline{t}\, w)^* u. \tag{4.3}$$

In order to analyze the principal minors $\Delta_1, \ldots, \Delta_q$ appearing in the definition of the power function $\Delta_{i\lambda/2}$, we use the singular values $\sigma_1(a) \ge \sigma_2(a) \ge \ldots \ge \sigma_q(a)$ of a matrix $a \in M_q$ ordered by size, i.e., the square roots of the eigenvalues of a^*a. We need the following known estimates for singular values:

Lemma 4.7. *For all matrices $a_1, a_2 \in M_q(\mathbb{F})$ and $i = 1, \ldots, q$,*

$$|\sigma_i(a_1 + a_2) - \sigma_i(a_1)| \le \sigma_1(a_2) \qquad \text{and} \qquad \sigma_i(a_1 \cdot a_2) \le \sigma_i(a_1)\sigma_1(a_2).$$

Proof. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$ we refer to Theorem 3.3.16 of [HJ]. The case $\mathbb{F} = \mathbb{H}$ can be reduced to $\mathbb{F} = \mathbb{C}$ by the same arguments as in the second part of the proof of Corollary 2.2. □

Lemma 4.8. *For $t \in C_q$, $w \in B_q$, $u \in U_0(q,\mathbb{F})$ and $r = 1, \ldots, q$,*

$$\frac{\Delta_r(\widetilde{g}_t(u,w))}{\Delta_r(\widetilde{g}_t(u,0))} \in \left[(1 - \widetilde{t}\,\sigma_1(w))^{2r}, (1 + \widetilde{t}\,\sigma_1(w))^{2r}\right], \quad \text{with } \widetilde{t} := \min(t_1, 1).$$

Proof. We write the matrix $\widetilde{g}_t(u,w)$ as

$$\widetilde{g}_t(u,w) = b(I + \widetilde{w})(I + \widetilde{w}^*)b^* \tag{4.4}$$

with

$$b := u^* \cosh \underline{t}, \quad \widetilde{w} := (\cosh \underline{t})^{-1} \sinh \underline{t} \cdot w \;=\; \tanh \underline{t} \cdot w$$

The inequalities of Lemma 4.7 imply for $i = 1, \ldots, q$ that

$$\begin{aligned} |1 - \sigma_i(I + \widetilde{w})| &= |\sigma_i(I) - \sigma_i(I + \widetilde{w})| \le \sigma_1(\widetilde{w}) = \sigma_1(\tanh \underline{t} \cdot w) \\ &\le \sigma_1(\tanh \underline{t}\,) \cdot \sigma_1(w). \end{aligned}$$

As $0 \le \tanh x \le \min(x, 1)$ for $x \ge 0$ and $x \mapsto \tanh x$ is increasing, we conclude that

$$\sigma_1(\tanh \underline{t}\,) \le \min(t_1, 1) = \widetilde{t}$$

and thus

$$|1 - \sigma_i(I + \widetilde{w})| \le \widetilde{t} \cdot \sigma_1(w) \in [0, 1]. \tag{4.5}$$

This implies for $i = 1, \ldots, q$ that

$$(1 - \widetilde{t}\,\sigma_1(w))^2 \le \sigma_i(I + \widetilde{w})^2 \le (1 + \widetilde{t}\,\sigma_1(w))^2. \tag{4.6}$$

This leads to the matrix inequality

$$(1-\widetilde{t}\sigma_1(w))^2 I \le (I+\widetilde{w})(I+\widetilde{w}^*) \le (1+\widetilde{t}\sigma_1(w))^2 I,$$

and thus

$$(1-\widetilde{t}\sigma_1(w))^2\, bb^* \le b(I+\widetilde{w})(I+\widetilde{w}^*)b^* \le (1+\widetilde{t}\sigma_1(w))^2\, bb^*.$$

As for Hermitian matrices a, b with $0 \le a \le b$ the determinants satisfy $0 \le \Delta(a) \le \Delta(b)$, we finally obtain

$$\Delta_r(b(I+\tilde{w})(I+\tilde{w}^*)b^*) \in \left[(1-\tilde{t}\sigma_1(w))^{2r}\Delta_r(bb^*), (1+\tilde{t}\sigma_1(w))^{2r}\Delta_r(bb^*)\right] \quad (4.7)$$

as claimed. □

For the next step in the proof of Theorem 4.5 we use the integral representation (4.1),

$$\begin{aligned}\psi_{\lambda-i\rho^A}(t) &= \int_{U_0(q,\mathbb{F})} \Delta_{i\lambda/2}(u^{-1}(\cosh \underline{t})^2 u)du \\ &= \int_{B_q}\int_{U_0(q,\mathbb{F})} \Delta_{i\lambda/2}(\widetilde{g}_t(u,0))dm_p(w)du. \end{aligned} \quad (4.8)$$

Using Lemma 4.8, we estimate the difference of the integrands in (4.2) and (4.8). We shall obtain the following result.

Lemma 4.9. *Let $t \in \mathbb{R}^q$ and $\lambda \in \mathbb{C}^q$. Then for all $n \in \mathbb{N}$,*

$$\begin{aligned}\left|\phi^p_{\lambda-i\rho}(t) - \psi_{\lambda-i\rho^A}(t)\right| \le 8q\|\lambda\|_1\, \widetilde{t}\cdot \Big(\frac{1}{\kappa_{pd/2}}\int_{B_q} \sigma_1(w)^{2n}\Delta(I-w^*w)^{pd/2-\gamma-2n}\, dw\Big)^{1/2n} \\ \cdot\Big(\phi^p_{\frac{2n}{2n-1}i\mathrm{Im}\lambda-i\rho}(t)^{\frac{2n-1}{2n}} + \psi_{\frac{2n}{2n-1}i\mathrm{Im}\lambda-i\rho^A}(t)^{\frac{2n-1}{2n}}\Big)\end{aligned}$$

Proof. We write the difference

$$D := \left|\Delta_{i\lambda/2}(\widetilde{g}_t(u,w)) - \Delta_{i\lambda/2}(\widetilde{g}_t(u,0))\right|$$

of the integrands in (4.2), (4.8) as $D = |e^\alpha - e^\beta|$ with

$$\alpha := \alpha(t,\lambda,u,w) = \frac{i}{2}\sum_{r=1}^q(\lambda_r-\lambda_{r+1})\cdot\ln\Delta_r(\widetilde{g}_t(u,w))$$

and

$$\beta := \beta(t,\lambda,u) = \frac{i}{2}\sum_{r=1}^q(\lambda_r-\lambda_{r+1})\cdot\ln\Delta_r(\widetilde{g}_t(u,0))$$

with the agreement $\lambda_{q+1}=0$. We further write the functions α, β as $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$. By elementary calculus, we obtain

$$\begin{aligned}|e^\alpha - e^\beta| = |e^{\alpha_1+i\alpha_2} - e^{\beta_1+i\beta_2}| &\le |e^{i\alpha_2}|\cdot|e^{\alpha_1}-e^{\beta_1}| + e^{\beta_1}\cdot|e^{i\alpha_2}-e^{i\beta_2}| \\ &\le |e^{\alpha_1}-e^{\beta_1}| + \sqrt{2}\cdot e^{\beta_1}|\alpha_2-\beta_2| \\ &\le |\alpha_1-\beta_1|\cdot(e^{\alpha_1}+e^{\beta_1}) + \sqrt{2}(e^{\alpha_1}+e^{\beta_1})|\alpha_2-\beta_2| \\ &\le 2\cdot|\alpha-\beta|\cdot(e^{\alpha_1}+e^{\beta_1}).\end{aligned} \quad (4.9)$$

We have

$$|\alpha-\beta| \le \|\lambda\|_1\cdot \max_{r=1,\dots,q}\left|\ln\Delta_r(\widetilde{g}(t,u,w)) - \ln\Delta_r(\widetilde{g}(t,u,0))\right|.$$

Hence we obtain from Lemma 4.8, together with the elementary inequality

$$|\ln(1+z)| \le \frac{|z|}{1-|z|} \quad \text{for} \quad |z|<1 \tag{4.10}$$

and with $\widetilde{t} \in [0,1]$ that

$$|\alpha-\beta| \le \|\lambda\|_1 \cdot 2q \cdot \frac{\widetilde{t}\,\sigma_1(w)}{1-\tilde{t}\,\sigma_1(w)} \le \|\lambda\|_1 \cdot 2q\,\widetilde{t} \cdot \frac{\sigma_1(w)}{1-\sigma_1(w)}.$$

Furthermore, as $1 \ge \sigma_1(w) \ge \ldots \ge \sigma_q(w) \ge 0$ for $w \in B_q$, we have

$$\frac{1}{1-\sigma_1(w)} \le \frac{2}{1-\sigma_1(w)^2} \le 2\prod_{r=1}^{q} \frac{1}{1-\sigma_r(w)^2} = \frac{2}{\Delta(I-w^*w)}. \tag{4.11}$$

We thus conclude that

$$D \le 2(e^{\alpha_1}+e^{\beta_1})|\alpha-\beta| \le 8q(e^{\alpha_1}+e^{\beta_1})\|\lambda\|_1\,\widetilde{t} \cdot \frac{\sigma_1(w)}{\Delta(I-w^*w)}.$$

By this this estimate and Hölders inequality we obtain

$$\begin{aligned}
&\left|\phi^p_{\lambda-i\rho}(t)-\psi_{\lambda-i\rho^A}(t)\right| \le \qquad (4.12)\\
&\le\ 8q\|\lambda\|_1\,\widetilde{t} \cdot \int_{B_q\times U_0(q,\mathbb{F})} (e^{\alpha_1}+e^{\beta_1}) \frac{\sigma_1(w)}{\Delta(I-w^*w)} dm_p(w)du\\
&\le 8q\|\lambda\|_1\,\widetilde{t} \cdot \Big(\int_{B_q} \frac{\sigma_1(w)^{2n}}{\Delta(I-w^*w)^{2n}} dm_p(w)\Big)^{1/2n} \times\\
&\quad \times \Big[\Big(\int_{B_q\times U_0(q,\mathbb{F})} e^{\frac{2n}{2n-1}\alpha_1} dm_p(w)du\Big)^{\frac{2n-1}{2n}} + \Big(\int_{B_q\times U_0(q,\mathbb{F})} e^{\frac{2n}{2n-1}\beta_1} dm_p(w)du\Big)^{\frac{2n-1}{2n}}\Big].
\end{aligned}$$

In view of (4.2) and (4.8), the $[\ldots]$-term in the last two lines is equal to

$$\phi^p_{\frac{2n}{2n-1}i\mathrm{Im}\lambda-i\rho}(t)^{\frac{2n-1}{2n}} + \psi_{\frac{2n}{2n-1}i\mathrm{Im}\lambda-i\rho^A}(t)^{\frac{2n-1}{2n}},$$

and the lemma follows. □

The estimate of Theorem 4.5 is now a consequence of Lemma 4.9 and the following result:

Lemma 4.10. *For each $n \in \mathbb{N}$ there is a constant $C = C(\mathbb{F}, q, n) > 0$ such that for all $p \ge 2q$,*

$$R(p) := \int_{B_q} \frac{\sigma_1(w)^{2n}}{\Delta(I-w^*w)^{2n}} dm_p(w) \ \le\ \frac{C}{p^n}\,.$$

Proof. We transform the integral in the lemma. The diffeomorphism $P: B^q \to B_q$ introduced in Remark 2.6, where B is the ball $B := \{y \in \mathbb{F}^q : \|y\|_2 < 1\}$. We recall from [R1] that for $w = P(y_1,\ldots,y_q)$, one has $\Delta(I-w^*w) = \prod_{j=1}^q (1-\|y_j\|_2^2)$. With (2.16) in mind, we obtain

$$R(p) = \frac{1}{\kappa_{pd/2}} \cdot \int_{B^q} \sigma_1(P(y_1,\ldots,y_q))^{2n} \cdot \prod_{j=1}^{q} (1-\|y_j\|_2^2)^{d(p-q-j+1)/2-1-2n} d(y_1,\ldots,y_q). \tag{4.13}$$

Moreover, the j,j-element $(ww^*)_{jj}$ of ww^* satisfies

$$(ww^*)_{jj} = y_j(I-y_1^*y_1)^{1/2}\ldots(I-y_{j-1}^*y_{j-1})^{1/2}(I-y_{j-1}^*y_{j-1})^{1/2}\ldots(I-y_1^*y_1)^{1/2}y_j^*.$$

As the hermitian matrix $I - y^*y$ has eigenvalues in $[0,1]$, it follows readily that $0 \le (ww^*)_{jj} \le \|y_j\|_2^2$ and hence

$$\sigma_1(w)^2 \le \sum_{j=1}^q (ww^*)_{jj} \le \sum_{j=1}^q \|y_j\|_2^2.$$

Therefore,

$$\sigma_1(w)^{2n} \le C \cdot \sum_{j=1}^q \|y_j\|_2^{2n}$$

with some constant $C > 0$. This leads to the estimate

$$R(p) \le \frac{C}{\kappa_{pd/2}} \sum_{j=1}^q \int_{B^q} \|y_j\|_2^{2n} \cdot \prod_{r=1}^q (1 - \|y_r\|_2^2)^{d(p-q-r+1)/2\,-1-2n} d(y_1, \dots, y_q). \quad (4.14)$$

Using polar coordinates, we obtain for $y = y_r$ and arbitrary $\alpha > 0$ that

$$\int_B (1 - \|y\|_2^2)^{\alpha-1}\, dy = \omega_{dq} \int_0^1 x^{dq-1}(1-x^2)^{\alpha-1}\, dx = \omega_{dq} \cdot \frac{\Gamma(\alpha)\,\Gamma\big(\frac{dq}{2}\big)}{2 \cdot \Gamma\big(\alpha + \frac{dq}{2}\big)}$$

and

$$\int_B \|y\|_2^{2n}(1 - \|y\|_2^2)^{\alpha-1} dy = \omega_{dq} \int_0^1 x^{dq-1+2n}(1-x^2)^{\alpha-1} dx = \omega_{dq} \cdot \frac{\Gamma(\alpha)\,\Gamma\big(n + \frac{dq}{2}\big)}{2 \cdot \Gamma\big(\alpha + n + \frac{dq}{2}\big)}$$

with the surface measure $\omega_{dq} := vol(S^{dq-1})$ of the unit sphere in $\mathbb{R}^{dq}$ as normalization constant. These formulas yield that

$$\begin{aligned} \kappa_{pd/2} &= \int_{B^q} \prod_{r=1}^q (1 - \|y_r\|_2^2)^{d(p-q-r+1)/2-1} d(y_1, \dots, y_q) \\ &= \Big(\frac{\omega_{dq}}{2} \cdot \Gamma\big(\frac{dq}{2}\big)\Big)^q \cdot \prod_{r=1}^q \frac{\Gamma\big(\frac{d}{2}(p-q-r+1)\big)}{\Gamma\big(\frac{d}{2}(p-r+1)\big)} \end{aligned} \quad (4.15)$$

and

$$I_j(p) := \frac{1}{\kappa_{pd/2}} \cdot \int_{B^q} \|y_j\|_2^{2n} \cdot \prod_{r=1}^q (1 - \|y_r\|_2^2)^{d(p-q-r+1)/2-1-2n}\, d(y_1, \dots, y_q) =$$

$$= \frac{\Gamma\big(n + \frac{dq}{2}\big)}{\Gamma\big(\frac{dq}{2}\big)} \cdot \frac{\prod_{r=1}^q \Gamma\big(\frac{d}{2}(p-q-r+1) - 2n\big)}{\Gamma\big(\frac{d}{2}(p-j+1) - n\big) \cdot \prod_{r \ne j} \Gamma\big(\frac{d}{2}(p-r+1) - 2n\big)} \cdot \prod_{r=1}^q \frac{\Gamma\big(\frac{d}{2}(p-r+1)\big)}{\Gamma\big(\frac{d}{2}(p-q-r+1)\big)}\,.$$

From the asymptotics of the gamma function we obtain for $p \to \infty$ the asymptotic equality

$$I_j(p) \sim \frac{\Gamma\big(n + \frac{dq}{2}\big)}{\Gamma\big(\frac{dq}{2}\big)} \cdot \Big(\frac{dp}{2}\Big)^{-n} \quad (p \to \infty).$$

This implies that $R(p)$ is of order $O(p^{-n})$ for $p \to \infty$. □

The proof of Theorem 4.5 is now complete.

5. Convergence to Bessel functions of type B

In this section we consider the Heckman-Opdam function ϕ_λ^p for fixed $p \in \mathbb{R}$ with $p \geq 2q-1$ in a scaling limit. More precisely, we use the integral repesentation of Theorem 2.4 in order to derive convergence of the rescaled functions $\phi^p_{n\lambda - i\rho}(t/n)$ for $n \to \infty$ to Dunkl-type Bessel functions associated with root system B_q. While such limit transitions are well-known in a general context from the asymptotics of the hypergeometric system, we here obtain a precise estimate for the rate of convergence.

To explain the result, let us first recall some facts on Bessel functions from [FK],[Ka] and [R1].

Multivariate Bessel functions 5.1. Let $\mathbf{m} = (m_1, \ldots, m_q)$ be a partition of length q with integers $m_1 \geq m_2 \geq \ldots \geq m_q \geq 0$ and let $|\mathbf{m}| := m_1 + \ldots + m_q$. For $x \in \mathbb{C}$ and a parameter $\alpha > 0$, the generalized Pochhammer symbol is given by

$$(x)^\alpha_{\mathbf{m}} \;=\; \prod_{j=1}^q \big(x - \frac{1}{\alpha}(j-1)\big)_{m_j}. \tag{5.1}$$

For $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with $d = \dim_{\mathbb{R}}\mathbb{F}$ and partitions $\mathbf{m}$, the spherical polynomials $\Phi_{\mathbf{m}}$ are defined by

$$\Phi_{\mathbf{m}}(x) = \int_{U_q} \Delta_{\mathbf{m}}(uxu^{-1})du \quad \text{for } x \in H_q(\mathbb{F})$$

where $\Delta_{\mathbf{m}}$ is the power function of Eq. (2.4). We also consider the renormalized polynomials $Z_{\mathbf{m}} = c_{\mathbf{m}} \cdot \Phi_{\mathbf{m}}$ with certain normalization constants $c_{\mathbf{m}} > 0$ which are characterized by the formula

$$(\operatorname{tr} x)^k \;=\; \sum_{|\mathbf{m}|=k} Z_{\mathbf{m}}(x) \qquad \text{for } k \in \mathbb{N}_0,\, x \in H_q(\mathbb{F}). \tag{5.2}$$

By construction, the $\Phi_{\mathbf{m}}$ and $Z_{\mathbf{m}}$ are invariant under conjugation by $U(q,\mathbb{F})$ and thus depend only on the eigenvalues of their argument. More precisely, for a Hermitian matrix $x \in H_q(\mathbb{F})$ with eigenvalues $\xi = (\xi_1, \ldots, \xi_q) \in \mathbb{R}^q$, we have $Z_{\mathbf{m}}(x) = C^\alpha_{\mathbf{m}}(\xi)$ where the $C^\alpha_{\mathbf{m}}$ are the Jack polynomials of index $\alpha := 2/d$; see Section XI of [FK] and references cited there. The Jack polynomials are homogeneous of degree $|\mathbf{m}|$ and symmetric in their arguments.

Following Kaneko [Ka] (see also Section 2.2 of [R1]) we define Bessel functions in two arguments

$$J_\mu(\xi,\eta) := \sum_{\mathbf{m}} \frac{(-1)^{|\mathbf{m}|}}{(\mu)^\alpha_{\mathbf{m}} |\mathbf{m}|!} \cdot \frac{C^\alpha_{\mathbf{m}}(\xi) C^\alpha_{\mathbf{m}}(\eta)}{C^\alpha_{\mathbf{m}}(1,\ldots,1)} \tag{5.3}$$

for $\mu \in \mathbb{C}$ with $(\mu)^\alpha_{\mathbf{m}} \neq 0$ for all partitions $\mathbf{m}$ and with fixed parameter $\alpha := 2/d$. A comparison of (5.3) with the explicit form of the Dunkl-type Bessel functions J^B_k associated with root system B_q in [BF] shows that the Bessel function J_μ can be expressed in terms of J^B_k as

$$J_\mu\big(\frac{\xi^2}{2}, \frac{\eta^2}{2}\big) = J^B_k(\xi, i\eta),$$

with the multiplicity parameter $k := k(\mu,d) := (\mu - (q-1)d/2 - 1/2, d/2)$. For the details see Section 4.3 of [R1] and [O1] for the general context.

For certain indices μ, the Bessel functions J_μ appear as the spherical functions of the Euclidean-type symmetric spaces G_0/K where $K = U(p,\mathbb{F}) \times U(q,\mathbb{F})$ and $G_0 = K \ltimes M_{p,q}(\mathbb{F})$ is the Cartan motion group associated with the Grassmannian $\mathcal{G}_{p,q}(\mathbb{F})$. The double coset space $G_0//K$ is naturally identified with the Weyl chamber C_q, with $t \in C_q$ corresponding to the double coset of $(I_p, I_q, \underline{t}) \in G_0$. So we may consider biinvariant functions on G_0 as functions on C_q. It is well known (see Section 4 of [R1]) that the spherical functions of (G_0, K) are given in terms of the Bessel function J_μ as follows:

Proposition 5.2. *The spherical functions of (G_0, K) are given by the Dunkl-type Bessel functions*

$$\widetilde{\phi}^p_\lambda(t) := J^B_k(t, i\lambda) = J_\mu\Big(\frac{\lambda^2}{2}, \frac{t^2}{2}\Big), \quad \lambda \in \mathbb{C}^q$$

with $\mu := pd/2$ and k as in Section 5.1. Moreover, $\widetilde{\phi}^p_\lambda$ is bounded precisely for $\lambda \in \mathbb{R}^q$.

The spherical functions of (G_0, K) with dimension parameters $p \geq 2q$ admit a Harish-Chandra integral representation which can be extended by Carlson's theorem to all real parameters $p > 2q-1$ and thus to the corresponding indices μ. This leads to the following

Proposition 5.3. *For all real parameters $p > 2q-1$ and all $t \in C_q$ and $\lambda \in \mathbb{C}^q$,*

$$\widetilde{\phi}^p_\lambda(t) = \int_{B_q} \int_{U_0(q,\mathbb{F})} e^{-i \operatorname{Re} \operatorname{tr}(w\underline{t}u\underline{\lambda})} dm_p(w) du \tag{5.4}$$

with the probability measure $m_p \in M^1(B_q)$ of Eq. (2.6). Moreover, for $p = 2q-1$ and with the notations of Remark 2.6,

$$\widetilde{\phi}^p_\lambda(t) = \frac{1}{\kappa_{(2q-1)d/2}} \int_{B^{q-1}\times S} \int_{U_0(q,\mathbb{F})} e^{-i \operatorname{Re} \operatorname{tr}(P(y)\underline{t}u\underline{\lambda})} \cdot \prod_{j=1}^{q-1} (1 - \|y_j\|_2^2)^{q-1-j} dy_1 \ldots dy_{q-1} d\sigma(y_q) du. \tag{5.5}$$

Proof. For $p > 2q-1$ and $\lambda \in C_q$, the first formula is immediate by a combination of the integral representations (3.12) and (4.4) in [R1] (in the latter, integration over $U(q,\mathbb{F})$ may be replaced by integration over $U_0(q,\mathbb{F})$.) The general case $\lambda \in \mathbb{C}^q$ then follows by analytic continuation.

The singular limit case $p = 2q-1$ can be derived in the same way as in [R1]; see also Remark 2.6. We omit the details. □

A comparison of these integral representations for the Bessel functions $\widetilde{\phi}^p_\lambda$ with the integral representation for the Heckman-Opdam functions ϕ^p_λ of Section 2 leads to the following theorem, which is the main result of this section.

Theorem 5.4. *For each compact subset $K \subset \mathbb{R}^q$ there exists a constant $C = C(K) > 0$ such that for all $p \in \mathbb{R}$ with $p \geq 2q-1$, all $\lambda \in \mathbb{R}^q$, $t \in K$, and all $n \in \mathbb{N}$,*

$$|\phi^p_{n\lambda - i\rho}(t/n) - \widetilde{\phi}^p_\lambda(t)| \leq C \cdot \frac{\|\lambda\|_1}{n}.$$

Here again, $\|\lambda\|_1 = |\lambda_1| + \ldots + |\lambda_q|$.

Proof. We only give a proof for the non-degenerate case $p > 2q-1$. The case $p = 2q-1$ follows in the same way from (5.5) and Remark 2.6.

We substitute $w \mapsto -u^*w^*$ in the integral (5.4) and obtain

$$\widetilde{\phi}_\lambda^p(t) = \int_{B_q}\int_{U_0(q,\mathbb{F})} e^{i\cdot \mathrm{Re}\,\mathrm{tr}(u^*w^*\underline{t}\,u\underline{\lambda})} dm_p(w)du\,.$$

Moreover, denoting the trace of the upper left $r\times r$-block of a $q\times q$-matrix by tr_r , we have

$$\begin{aligned}
\mathrm{Re}\,\mathrm{tr}(u^*w^*\underline{t}u\underline{\lambda}\,) &= \frac{1}{2}\cdot \sum_{r=1}^q (u^*((\underline{t}w)^* + \underline{t}w)u)_{rr}\cdot \lambda_r \\
&= \sum_{r=1}^q \big[\,\mathrm{tr}_r(u^*((\underline{t}w)^*+\underline{t}w)u) - \mathrm{tr}_{r-1}(u^*((\underline{t}w)^*+\underline{t}w)u)\big]\cdot \lambda_r/2 \\
&= \sum_{r=1}^q \mathrm{tr}_r(u^*((\underline{t}w)^*+\underline{t}w)u)\cdot(\lambda_r - \lambda_{r+1})/2
\end{aligned}$$

with $\lambda_{q+1} := 0$. Thus,

$$\widetilde{\phi}_\lambda^p(t) = \int_{U_0(q,\mathbb{F})\times B_q} \prod_{r=1}^q exp\,(i\cdot \mathrm{tr}_r(u^*((tw)^*+tw)u)\cdot(\lambda_r-\lambda_{r+1})/2)\,dm_p(w)du.$$

Further, according to Theorem 2.4,

$$\phi^p_{n\lambda - i\rho}(t/n) = \int_{U_0(q,\mathbb{F})\times B_q} \prod_{r=1}^q \Delta_r(g_{t/n}(u,w))^{in(\lambda_r-\lambda_{r+1})/2}\,dm_p(w)du$$

with the positive definite matrix

$$g_{t/n}(u,w) = u^*(\cosh(t/n) + \sinh(t/n)\cdot w)^*(\cosh(t/n)+\sinh(t/n)\cdot w)u.$$

Using the well-known estimate

$$\left|\prod_{r=1}^q a_r - \prod_{r=1}^q b_r\right| \le \sum_{r=1}^q |a_r - b_r| \qquad \text{for } a_r, b_r \in \{z\in\mathbb{C}:\ |z| = 1\},$$

we obtain

$$\begin{aligned}
C &:= |\phi^p_{n\lambda-i\rho}(t/n) - \widetilde{\phi}_\lambda^p(t)| \\
&\le \sum_{r=1}^q \int_{U_0(q,\mathbb{F})\times B_q}\Big|\Delta_r(g_{t/n}(u,w))^{in(\lambda_r-\lambda_{r+1})/2} \\
&\qquad\qquad - exp\,(i\cdot \mathrm{tr}_r(u^*((\underline{t}w)^*+\underline{t}w)u)\cdot(\lambda_r-\lambda_{r+1})/2)\Big|\,dm_p(w)du.
\end{aligned}$$

Further, by the inequality

$$|e^{ix} - e^{iy}| \le \sqrt{2}\cdot|x-y| \qquad \text{for } x,y\in\mathbb{R},$$

we obtain

$$C \le \frac{1}{\sqrt{2}}\sum_{r=1}^q |\lambda_r - \lambda_{r+1}|\cdot C_r$$

with

$$C_r := \int_{U_0(q,\mathbb{F})\times B_q}\big|n\ln\Delta_r(g_{t/n}(u,w)) - \mathrm{tr}_r(u^*((\underline{t}w)^*+\underline{t}w)u)\big|\,dm_p(w)du.$$

We now write $g_{t/n}(u,w) = I + A/n + H/n^2$ with $A := u^*((tw)^* + tw)u$ and some Hermitian matrix $H = H(u,w,t,n)$ which stays in a compact subset of M_q for $(u,w,t,n) \in U_0(q,\mathbb{F}) \times B_q \times K \times \mathbb{N}$. Therefore,

$$n \ln \Delta_r(g_{t/n}(u,w)) = n \ln \Delta_r(I + A/n + H/n^2) = n \ln\bigl(1 + \mathrm{tr}_r(A)/n + h/n^2\bigr)$$

with some constant $h = h(u,w,t,n) \in \mathbb{C}$ which remains bounded for the arguments under consideration. Using the power series for $\ln(1+z)$, we get

$$n \ln \Delta_r(g_{t/n}(u,w)) - \mathrm{tr}_r(A) = O(1/n) \qquad \text{for } n \to \infty,$$

uniformly in u, w and $t \in K$. This yields the assertion. □

Remarks 5.5. (1) Similar to the results in Section 4, Theorem 5.4 can be extended from $\lambda \in \mathbb{R}^q$ to $\lambda \in \mathbb{C}^q$ with suitable exponential bounds on the right side of the estimate.

(2) We point out that one may also compare the integral representation for the spherical functions of the symmetric spaces $GL(q,\mathbb{F})/U(q,\mathbb{F})$ in Section 3 with the integral representation for the spherical functions $\widetilde{\psi}_\lambda$ of $(U(q,\mathbb{F}) \ltimes H_q(\mathbb{F}), U(q,\mathbb{F}))$, where $U(q,\mathbb{F})$ acts by conjugation on the space $H_q(\mathbb{F})$ of all Hermitian $q \times q$-matrices. In this case, the methods of the preceding proof lead to a result analogous to that of Theorem 5.4. Moreover, for real spectral variables λ it is possible to combine this result with Theorems 5.4 and 4.2(2), in order to obtain a convergence result for the Dunkl-type Bessel functions $\widetilde{\phi}_\lambda^p$ to the functions $\widetilde{\psi}_\lambda$ for $p \to \infty$ with explicit error bounds, similar to Theorem 4.2(2). However, these results will be weaker than those which were derived directly in [RV2].

6. Appendix: On convex hulls of Weyl group orbits

In this appendix we present a proof of Lemma 4.6. We start with some general facts, where we assume that R is a crystallographic root system of rank q in a Euclidean vector space $(V, \langle\,.\,\rangle)$ with Weyl group W. We fix a closed Weyl chamber C_q for R and denote by $\alpha_1, \ldots, \alpha_q \subset R$ the simple roots associated with C_q. We further introduce the dual cone

$$C_q^+ := \{x \in V : \langle x, y\rangle \geq 0\}.$$

It is well-known (see e.g. Lemma IV.8.3 of [Hel]) that for each $x \in C_q^+$,

$$co(W.x) \cap C_q = C_q \cap (x - C_q^+). \tag{6.1}$$

Lemma 6.1. *Suppose that R is irreducible.*

(1) *Let $x, y \in C_q \setminus \{0\}$. Then $\langle x, y\rangle > 0$.*

(2) *There exists a constant $\epsilon_0 > 0$ such that the ball $B_{\epsilon_0}(0) = \{x \in V : \|x\| < \epsilon_0\}$ is contained in $co(W.x)$ for each $x \in C_q$ with $\|x\| = 1$.*

Proof. (1) Let $\lambda_1, \ldots, \lambda_q \in V$ denote the fundamental weights associated with $\alpha_1, \ldots, \alpha_q$, defined by $\langle \lambda_j, \alpha_i^\vee\rangle = \delta_{ij}$ with $\alpha_i^\vee = 2\alpha_i/\langle \alpha_i, \alpha_i\rangle$. Then both x and y can be written as linear combinations of the λ_i with non-negative coefficients (see [Hu], Section 13.1). By our assumption on R and Section 13 of [Hu], the weights λ_i satisfy $\langle \lambda_i, \lambda_j\rangle > 0$ for all i, j. We therefore obtain that $\langle x, y\rangle > 0$.

(2) Let $C_q^1 := \{x \in C_q : \|x\| = 1\}$ and consider the continuous mapping $(x, y) \mapsto \langle x, y\rangle$ on the compact set $C_q^1 \times C_q^1$. By part (1), there exists some $\epsilon_0 > 0$ such that

$$\langle x, y\rangle > \epsilon_0 \quad \text{for all } \ x, y \in C_q^1.$$

Now fix $x \in C_q^1$. We claim that $B_{\epsilon_0}(0) \subseteq co(W.x)$. For this, let $z \in B_{\epsilon_0}(0) \cap C_q$. Then for each $y \in C_q^1$, we have

$$\langle z, y\rangle < \epsilon < \langle x, y\rangle.$$

This shows that $x - z \in C_q^+$ and $z \in x - C_q^+$. In view of (6.1), we thus obtain

$$B_{\epsilon_0}(0) \cap C_q \subseteq co(W.x) \cap C_q.$$

The claim is now immediate. □

We now fix some $\rho \in C_q$ and consider the compact convex set

$$K := co(W.\rho) \cap C_q\,.$$

We collect some simple facts on the extreme points of K.

Lemma 6.2. (1) *The topological boundary ∂C_q of C_q is contained in the union of the reflecting hyperplanes $H_{\alpha_1}, \ldots, H_{\alpha_q}$ associated with the simple reflections $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_2}$, and C_q is the intersection of q closed half-spaces.*
(2) *The closed cone $\rho - C_q^+$ is also the intersection of q closed half-spaces corresponding to hyperplanes $H_1^+, \ldots, H_q^+$.*
(3) *K is a compact convex polytope which is obtained as the intersection of $2q$ closed half-spaces. Moreover, if x is an extreme point of K, then $x = 0$, $x = \rho$, or $x \in \partial C_q \cap \partial(co(W.\rho))$.*
(4) *If $x \in K$ is an extreme point different from 0 and ρ, then there exists $k \in \{1, \ldots, q-1\}$ such that x is contained in the q-fold intersection of k hyperplanes H_{α_j} and $q - k$ hyperplanes H_l^+.*

Proof. (1) See Section 10.1 of [Hu].
(2) This follows from (1) and the definition of the dual cone.
(3) The first statement is clear by (1), (2) and (6.1). For the second statement, consider some extreme point x of $K = C_q \cap (\rho - C_q^+)$. If x is contained in the interior of C_q, then it is easily checked that x has to be an extreme point of the cone $\rho - C_q^+$ which implies $x = \rho$. Moreover, if x is contained in the interior of $\rho - C_q^+$ then by the same reasons, x has to be an extreme point of C_q and hence $x = 0$.This yields the assertion.
(4) This follows from (3).

□

Lemma 6.3. *Let W_1, W_2 be reflection groups acting on V_1 and V_2 respectively. Let $\rho_i \in V_i$ and $a_i \in co(W_i.\rho_i)$ for $i = 1, 2$. Then $(a_1, a_2) \in V_1 \times V_2$ satisfies $(a_1, a_2) \in co((W_1 \times W_2)(\rho_1, \rho_2))$.*

Proof. For $i = 1, 2$, we have $a_i = \sum_{w_i \in W_i} \lambda_{w_i}^i w_i \rho_i$ with $\lambda_{w_i}^i \geq 0$ and $\sum_{w_i \in W_i} \lambda_{w_i}^i = 1$. Therefore,

$$(a_1, a_2) = \sum_{w_1 \in W_1} \sum_{w_2 \in W_2} \lambda_{w_1}^1 \lambda_{w_2}^2 \cdot (w_1\rho_1, w_2\rho_2)$$

as claimed. □

We finally turn to the proof of Lemma 4.6. As for Weyl groups of type B the mapping $x \mapsto -x$ on $\mathbb{R}^q$ corresponds to the action of some Weyl group element, Lemma 4.6 is a consequence of part (1) of the following result.

Proposition 6.4. *Consider a root system R of rank q in a Euclidean space V with reflection group $W \subset O(V)$ and a fixed closed Weyl chamber C_q in one of the following cases:*

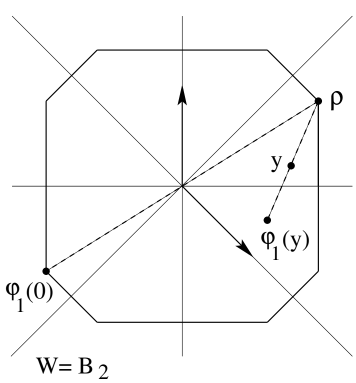
(1) *$R = B_q$ and $V = \mathbb{R}^q$, or*
(2) *$R = A_q$ and the symmetric group $W = S_{q+1}$ acts either on $V = \mathbb{R}^{q+1}$ or $V = (1, \ldots, 1)^\perp \subset \mathbb{R}^{q+1}$ in a non-effective or effective way.*

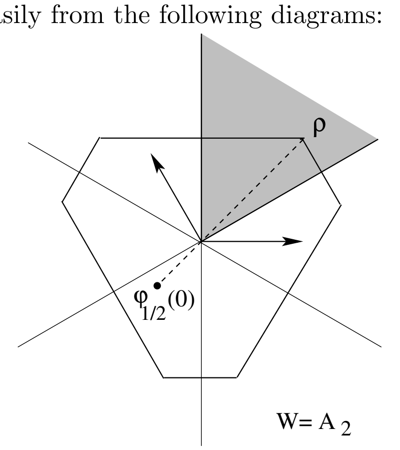
Then there exists some $\epsilon_0 > 0$ (depending on R) such that for all $0 \le \epsilon \le \epsilon_0$, $\rho \in C_q$, and $y \in co(W.\rho) \cap C_q$,

$$(1+\epsilon)y - \epsilon\rho \in co(W.\rho).$$

Notice that for fixed y, the point $(1+\epsilon)y - \epsilon\rho = y + \epsilon(y - \rho)$ is opposite to ρ with respect to y on the line through y and ρ, with distance $\epsilon\|y - \rho\|$ from y. In case $\epsilon = 1$, it is obtained from ρ by reflection in y.

For the root systems A_1, B_1 and B_2 the maximal parameter is $\epsilon_0 = 1$ while in the reduced A_2-case the maximal parameter is $\epsilon_0 = 1/2$. In fact, the cases A_1, B_1 are trivial, while the cases A_2, B_2 follow easily from the following diagrams:





Proof of Proposition 6.4. For the proof of the general case, we fix $\rho \in C_q$ and consider

$$K := co(W.\rho) \cap C_q$$

as well as for $\epsilon > 0$, its image $K_\epsilon := \phi_\epsilon(K)$ under the affine mapping

$$\phi_\epsilon : y \mapsto (1+\epsilon)y - \epsilon\rho\,.$$

Clearly, K_ϵ is again compact and convex, and ϕ_ϵ maps extreme points of K onto extreme point of K_ϵ. For the proof of Proposition 6.4 it suffices to prove that extreme points of K are mapped to points in $co(W.\rho)$ for $\epsilon \in [0, \epsilon_0]$ with $\epsilon_0 > 0$ sufficiently small. For the proof of this statement, we may assume that in addition $\|\rho\|_2 = 1$ holds, and that, by a continuity argument, ρ is contained in the interior of C_q.

We prove Proposition 6.4 by induction on q first for the A_q-cases and then for B_q, where the A-cases are used. The proposition is clear for A_1 and B_1. Let $y \in K$ be an extreme point. By Lemma 6.2(4), we have 3 cases of extreme points:

If $y = \rho$, then $\phi_\epsilon(\rho) = \rho$, and the claimed statement is trivial.

Moreover, if $y = 0$, then $\phi_\epsilon(0) = -\epsilon\rho$, and the statement follows in all cases with $\epsilon_0 > 0$ as in Lemma 6.1(2).

We now turn to the third case. Assume that S_{q+1} acts on the vector space $V_q := (1, \ldots, 1)^\perp \subset \mathbb{R}^{q+1}$ where C_q is the closed Weyl chamber associated with the simple roots

$$\alpha_1 := e_1 - e_2, \alpha_2 := e_2 - e_3, \ldots, \alpha_q := e_q - e_{q+1},$$

and $e_1, \ldots, e_{q+1}$ is the standard basis of $\mathbb{R}^{q+1}$. We first study the extreme point $x_0 \in C_q \cap co(W.\rho)$ contained in the intersection of the hyperplanes $H_{\alpha_1}, \ldots, H_{\alpha_{q-1}} \subset V_q$ and the hyperplane

$$H := \{x \in V_q : \langle x, e_{q+1}\rangle = \langle \rho, e_{q+1}\rangle\}$$

which contains the q affinely independent points $\rho, \sigma_{\alpha_1}(\rho), \ldots, \sigma_{\alpha_{q-1}}(\rho)$ (notice that ρ is in the interior of C_q). We observe that S_q as a subgroup of S_{q+1} acts on H by permutations of the first q components. We now identify H with the vector space $V_{q-1} \subset \mathbb{R}^q$ via the affine mapping

$$(x_1, \ldots, x_q, \rho_{q+1}) \mapsto (x_1 - \rho_{q+1}/q, \ldots, x_q - \rho_{q+1}/q).$$

In terms of this identification, the action of S_q on H is just the usual action of S_q on V_{q-1} with the simple reflections $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_{q-1}}$. We now regard the points $\rho, x_0, \phi_\epsilon(x_0), \sigma_{\alpha_1}(\rho), \ldots, \sigma_{\alpha_{q-1}}(\rho) \in H$ as points of V_{q-1} and may apply the assumption in the induction for A_{q-1}. This shows that $\phi_{\epsilon_0}(x_0)$ is contained in $co(S_q.\rho) \subset co(S_{q+1}.\rho)$ for $\epsilon_0 > 0$ sufficiently small. This proves the claim for this extreme point x_0.

The case of the extreme point in the intersection of $H_{\alpha_2}, \ldots, H_{\alpha_q}$ and the corresponding hyperplane H containing the q points $\rho, \sigma_{\alpha_2}(\rho), \ldots, \sigma_{\alpha_q}(\rho)$ can be handled in the same way.

For the next type of an extreme point, we fix $k = 2, \ldots, q-1$ and define

$$S := \rho_1 + \ldots + \rho_k = -(\rho_{k+1} + \ldots + \rho_{q+1}).$$

We now consider the extreme point x_0 which is contained in the intersection of the hyperplanes $H_{\alpha_1}, \ldots, H_{\alpha_{k-1}}, H_{\alpha_{k+1}}, \ldots, H_{\alpha_q}$ and the hyperplane

$$H := \{(x_1, \ldots, x_{q+1}) \in \mathbb{R}^{q+1} : x_1 + \ldots + x_k = S, x_{k+1} + \ldots + x_{q+1} = -S\} \subset V_q.$$

H contains the affinely independent q points $\rho, \sigma_{\alpha_1}(\rho), \ldots, \sigma_{\alpha_{k-1}}(\rho), \sigma_{\alpha_{k+1}}(\rho), \ldots, \sigma_{\alpha_q}(\rho)$. We write H as $H := H_1 \times H_2$ with $H_1 := \{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_1 + \ldots + x_k = S\}$ and $H_2 := \{(x_{k+1}, \ldots, x_{q+1}) \in \mathbb{R}^{q+1-k} : x_{k+1} + \ldots + x_{q+1} = -S\}$ where the group $S_k \times S_{q+1-k}$ as a subgroup of S_{q+1} acts on H. We now identify H_1 with $V_{k-1} \subset \mathbb{R}^k$ via the affine mapping

$$p_1 : (x_1, \ldots, x_k) \mapsto (x_1 - S/k, \ldots, x_k - S/k),$$

and H_2 with $V_{q-k} \subset \mathbb{R}^{q+1-k}$ via

$$p_2 : (x_{k+1}, \ldots, x_{q+1}) \mapsto (x_{k+1} + S/(q+1-k), \ldots, x_{q+1} + S/(q+1-k)).$$

In terms of this identification of H with $V_{k-1} \times V_{q-k}$, the action of $S_k \times S_{q+1-k}$ above on H is just the usual action of $S_k \times S_{q+1-k}$ on $V_{k-1} \times V_{q-k}$. We now consider the Weyl chamber $C_{k-1} \subset V_{k-1}$ associated with the reflections $\sigma_{\alpha_1}, \ldots, \sigma_{\alpha_{k-1}}$. We see that $p_1(\rho) \in C_{k-1}$, and that the points

$$p_1(\rho), p_1(x_0), p_1(\phi_\epsilon(x_0)), \sigma_{\alpha_1}(p_1(\rho)), \ldots, \sigma_{\alpha_{k-1}}(p_1(\rho)) \in V_{k-1}$$

are related in a way such that we may apply the induction assumption for A_{k-1}. We conclude that $p_1(\phi_\epsilon(x_0))$ is contained in $co(S_k.p_1(\rho))$ for sufficiently small $\epsilon > 0$. In the same way, $p_2(\phi_{\epsilon_0}(x_0)) \in co(S_{q+1-k}.p_2(\rho))$ for sufficiently small $\epsilon > 0$. In view of Lemma 6.3 we conclude that there exists some $\epsilon_0 > 0$ such that $\phi_\epsilon(x_0) \in co\big((S_k \times S_{q+1-k}).\rho\big) \subset co(S_{q+1}.\rho)$ for $0 \le \epsilon \le \epsilon_0$ as claimed.

We next study the extreme points x_0 with the property that for some $k \in \{1, \ldots, q-1\}$, the point x_0 is contained in the k reflecting hyperplanes $H_{\alpha_{j_1}}, \ldots, H_{\alpha_{j_k}}$ with $1 \le j_1 < \ldots < j_k \le q+1$ as well as in the k-dimensional affine subspace $H \subset V_q$ which is spanned by the $k+1$ affinely independent points $\rho, \sigma_{\alpha_{j_1}}(\rho), \ldots, \sigma_{\alpha_{j_k}}(\rho)$. As in the preceding case, we split the problem into several lower dimensional problems which can be handled separately by induction. Again, by Lemma 6.3 we obtain some $\epsilon_0 > 0$ such that $\phi_{\epsilon_0}(x_0) \in co(S_{q+1}.\rho)$ for $\epsilon \le \epsilon_0$. This completes the proof for the A_q-case.

We finally consider the case B_q for $q > 1$. We assume that C_q is the Weyl chamber associated with the simple roots

$$\alpha_1 := e_1 - e_2, \alpha_2 := e_2 - e_3, \ldots, \alpha_{q-1} := e_{q-1} - e_q, \alpha_q = e_q.$$

We here immediately study the general case where for some $k \in \{1, \ldots, q-1\}$, the extreme point x_0 is contained in the k reflecting hyperplanes $H_{\alpha_{j_1}}, \ldots, H_{\alpha_{j_k}}$ with $1 \le j_1 < \ldots < j_k \le q+1$ as well as in the affine subspace $H \subset \mathbb{R}^{q+1}$ of dimension k which is spanned by the $k+1$ points $\rho, \sigma_{\alpha_{j_1}}(\rho), \ldots, \sigma_{\alpha_{j_k}}(\rho)$. As in the preceding case, we split the problem into several lower dimensional problems which can be handled either as a lower-dimensional B-case or as a known A-case. The proof is again completed by induction and by use of Lemma 6.3. □

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