# Dunkl theory, convolution algebras, and related Markov processes

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**Abstract**

These lecture notes are intended as an introduction to the theory of rational Dunkl operators, the associated special functions and related Markov processes with an emphasis on examples which are related to Riemannian symmetric spaces of Euclidean type and Bessel hypergroups on the matrix cones of positive semidefinite matrices.

We start with a comprehensive introduction into Dunkl theory: Dunkl operators, the intertwining operator and its positivity, the Dunkl kernel and the Dunkl transform, the Dunkl Laplacian and the associate heat semigroup. We further give an outline of the connection with Calogero-Moser-Sutherland models and generalized Hermite polynomials. Moreover, of central interest will be product formulas, generalized translations and associated commutative hypergroup structures on closed Weyl chambers. In particular, we explain how Dunkl theory for particular multiplicities is related to Riemannian symmetric spaces of Euclidean type and Bessel hypergroups on the matrix cones, and how this leads to a bunch of multiplicities for which the Weyl-group invariant Dunkl theory admits a probability preserving translation and an associated commutative hypergroup structure on the closed Weyl chamber. We finally discuss Markov processes on $\mathbb{R}^N$ which are related with Dunkl theory with an emphasis on connections to random walk on groups and hypergroups. In particular associated martingales, martingale characterizations, moment functions and Appell characters are studied. Of particular interest are Dunkl processes, i.e., diffusion-reflection processes with the Dunkl Laplacians as generators.

# Contents

# 1 Introduction

Since their invention exactly twenty years ago, Dunkl operators have initiated an intense development within the area of harmonic analysis and special functions associated with root systems. A basic motivation for this subject comes from the theory of Riemannian symmetric spaces whose spherical functions can be considered as multi-variable special functions depending on certain discrete sets of parameters. For spaces of rank one, they can be imbedded into classes of one-variable hypergeometric functions.

Dunkl operators provide a tool to extend the theory of spherical functions to a theory of multivariable hypergeometric functions. Roughly speaking, Dunkl operators are commuting differential-reflection operators on a Euclidean space which are associated with a finite reflection group and have a continuous set of parameters, called the multiplicities. For fixed root system and multiplicities, the associated Dunkl operators commute. They lead to commutative algebras which generalize the algebras of invariant differerential operators on Riemannian symmetric spaces. Actually, there are two levels of Dunkl operator theory: first the full theory, involving operators with reflection parts. Second, the Weyl group invariant theory (in probability, such as in [De] in this volume, sometimes called "radial" ), where the associated special functions are Weyl group invariant and include the spherical functions of Riemannian symmetric spaces for particular multiplicity values.

The first class of Dunkl operators, nowadays often called "rational" Dunkl operators, were introduced by C.F. Dunkl in a series of papers ([D1-5]), where he built up the framework for a theory of special functions and integral transforms in several variables related with reflection groups. The rational operators generalize the theory of symmetric spaces of Euclidean type. Besides them, there are now various further classes of Dunkl-type operators, in particular the trigonometric Dunkl operators of Heckman, Opdam and Cherednik, which generalize the theory of Riemannian symmetric spaces of the compact and non-compact type (see [He2] or [O3]), and the important $q$-analogues of Macdonald and Cherednik. There are various kinds of limit transitions between these theories and their special functions. Apart from the context of symmetric spaces, Dunkl operators are also relevant in mathematical physics, namely for the analysis of quantum many body systems of Calogero-Moser-Sutherland type. These describe algebraically integrable systems in one dimension; a good bibliography is contained in [vDV].

In this article, we shall restrict ourselves to the rational case with multiplicity function $k \geq 0$ in which case the most satisfying results and applications in analysis and probability are available. We give a general introduction to rational Dunkl theory, discuss examples, and sketch the beginning of applications in probability.

To be more precise, these lecture notes are organized as follows: We start with a general introduction to rational Dunkl theory: Dunkl operators, the intertwining operator and its positivity, the Dunkl kernel and the Dunkl transform, as well as the Dunkl Laplacian and the associated heat semigroup. We further give an outline of the connection with Calogero-Moser-Sutherland models and generalized Hermite polynomials, and we derive product formulas and generalized translations which are positive and probability preserving in some important cases. We also explain how Dunkl theory for particular multiplicities is related to Riemannian symmetric spaces of Euclidean type. Furthermore, we present hypergroup convolution algebras on cones of positive semidefinite matrices and show how these lead to a bunch of multiplicities for which the Weyl-group invariant Dunkl theory admits probability preserving translations and commutative hypergroup structures on the closed Weyl chambers of type $B_N$. In the final chapter, we discuss Markov processes related to Dunkl operators with an emphasis on connections to random walks on groups and hypergroups. We study associated martingales, martingale characterizations, moment functions and Appell characters. Of particular interest are the Dunkl processes, i.e., diffusion-reflection processes with the Dunkl Laplacians as generators, as these processes share many well-known features of Brownian motions on $\mathbb{R}^N$. For a more detailed discussion of these processes and their projections to Weyl chambers in view of stochastic analysis we refer to the contributions [De] and [CGY] in this volume.

# 2 Dunkl theory

This section gives an introduction to the theory of rational Dunkl operators, which we call Dunkl operators for short, and to the Dunkl transform. Main references are [D2], [D4],[D5], [DX],[dJ1], [He1], [R2], [R3], [R6] and [R7]. For a background on reflection groups and root systems the reader is referred to [Hu] and [GB]. We can by far not be complete here. In particular, we do not touch the field of rational Cherednik algebras, but rather focus on aspects which are of interest in connection with stochastic analysis, such as Fourier analysis and positivity results.

## 2.1 Root systems and reflection groups

The basic ingredient in the theory of Dunkl operators are root systems and finite reflection groups, acting on some Euclidean space $(E, \langle \,.\,,.\, \rangle)$ of finite dimension $N$. It will be no restriction to assume that that $E = \mathbb{R}^N$ with the standard Euclidean inner product $\langle x, y\rangle = \sum_{j=1}^N x_j y_j$. For $\alpha \in \mathbb{R}^N \setminus \{0\}$, we denote by $\sigma_\alpha$ the orthogonal reflection in the hyperplane $\langle\alpha\rangle^\perp$ perpendicular to $\alpha$, i.e.

$$\sigma_\alpha(x) = x - 2\,\frac{\langle\alpha, x\rangle}{|\alpha|^2}\,\alpha\,,$$

where $|x| := \sqrt{\langle x, x\rangle}$. Each reflection $\sigma_\alpha$ is orthogonal with respect to the standard inner product.

**2.1 Definition.** A finite subset $R \subset \mathbb{R}^N \setminus \{0\}$ is called a *root system*, if

$$\sigma_\alpha(R) = R \text{ for all } \alpha \in R.$$

The dimension of $span_{\mathbb{R}} R$ is called the rank of $R$. There are two possible additional requirements: $R$ is called

- *reduced*, if $\alpha \in R$ implies $2\alpha \notin R$.
- *crystallographic* if $R$ has full rank $N$ and

$$\frac{2\langle\alpha, \beta\rangle}{\langle\beta, \beta\rangle} \in \mathbb{Z} \quad \text{for all } \alpha, \beta \in R.$$

The group $W = W(R) \subseteq O(N, \mathbb{R})$ which is generated by the reflections $\{\sigma_\alpha,\, \alpha \in R\}$ is called the *reflection group* (or *Coxeter group*) associated with $R$. The dimension of $span_{\mathbb{R}} R$ is called the rank of $R$.

If $R$ is crystallographic, then $span_{\mathbb{Z}} R$ forms a lattice in $\mathbb{R}^N$ (called the *root-lattice*) which is stabilized by the action of the associated reflection group.

In rational Dunkl theory, one usually works with reduced root systems which are not necessarily crystallographic. On the other hand, the root systems occuring in Lie theory and in geometric contexts associated with Riemannian symmetric spaces are always crystallographic, and this requirement is also fundamental in the theory of trigonometric Dunkl operators.

**2.2 Lemma.** (1) *If $\alpha$ is in $R$, then also $-\alpha$ is in $R$.*

(2) *For any root system $R$ in $\mathbb{R}^N$, the reflection group $W = W(R)$ is finite.*

(3) *The set of reflections contained in* $W$ *is exactly* $\{\sigma_\alpha\,,\,\alpha\in R\}$.

(4) $w\sigma_\alpha w^{-1}=\sigma_{w\alpha}$ *for all* $w\in W$ *and* $\alpha\in R$.

*Proof.* (1) This follows since $\sigma_\alpha(\alpha)=-\alpha$. (2) As $R$ is fixed under the action of $W$, the assignment $\varphi(w)(\alpha):=w\alpha$ defines a homomorphism $\varphi:W\to S(R)$ of $W$ into the symmetric group $S(R)$ of $R$. This homomorphism is easily checked to be injective. Thus $W$ is naturally identified with a subgroup of $S(R)$, which is finite. Part (3) is slightly more involved. An elegant proof can be found in Section 4.2 of [DX]. Part (4) is straight forward. □

Properties (3) and (4) imply in particular that there is a bijective correspondence between the conjugacy classes of reflections in $W$ and the orbits in $R$ under the natural action of $W$. We shall need some more concepts: Each root system can be written as a disjoint union $R=R_+\cup(-R_+)$, where $R_+$ and $-R_+$ are separated by a hyperplane $\langle\{x\in\mathbb{R}^N:\langle\beta,x\rangle=0\}$ with $\beta\notin R$. Such a set $R_+$ is called a *positive subsystem.* The set of reflecting hyperplanes $\{\langle\alpha\rangle^\perp,\,\alpha\in R\}$ divides $\mathbb{R}^N$ into connected open components, called the *Weyl chambers* of $R$. It can be shown that the topological closure $\overline{C}$ of any chamber $C$ is a fundamental domain for $W$, i.e. $\overline{C}$ is naturally homeomorphic with the space $(\mathbb{R}^N)^W$ of all $W$-orbits in $\mathbb{R}^N$, endowed with the quotient topology (see Section 1.12 of [Hu]). $W$ permutes the reflecting hyperplanes as well as the chambers.

**2.3 Examples.** (1) $I_2(n)$, $n\geq 3$: Root systems of the *dihedral groups.* Define $\mathcal{D}_n$ to be the dihedral group of order $2n$, consisting of the orthogonal transformations in the Euclidean plane $\mathbb{R}^2$ which preserve a regular $n$-sided polygon centered at the origin. It is generated by the reflection at the $x$-axis and the reflection at the line through the origin which meets the $x$-axis at the angle $\pi/n$. Root system $I_2(n)$ is crystallographic only for $n=2,3,4,6$.

(2) $A_{N-1}$. Let $S_N$ denote the symmetric group in $N$ elements. It acts faithfully on $\mathbb{R}^N$ by permuting the standard basis vectors $e_1,\ldots,e_N$. Each transposition $(ij)$ acts as a reflection $\sigma_{ij}$ sending $e_i-e_j$ to its negative. Since $S_N$ is generated by transpositions, it is a finite reflection group. The root system of $S_N$ is called $A_{N-1}$ and is given by

$$A_{N-1}=\{\pm(e_i-e_j),\,1\leq i<j\leq N\}.$$

This root system is crystallographic. Its span is the orthogonal complement of the vector $e_1+\ldots+e_N$, and thus the rank is $N-1$.

(3) $B_N$. Here $W$ is the reflection group in $\mathbb{R}^N$ generated by the transpositions $\sigma_{ij}$ as above, as well as the sign changes $\sigma_i:e_i\mapsto -e_i$, $i=1,\ldots,N$. The group of sign changes is isomorphic to $\mathbb{Z}_2^N$, intersects $S_N$ trivially and is normalized by $S_N$, so $W$ is isomorphic with the semidirect product $S_N\ltimes\mathbb{Z}_2^N$. The corresponding root system is called $B_N$; it is given by

$$B_N=\{\pm e_i,\,1\leq i\leq N\}\cup\{\pm(e_i\pm e_j),1\leq i<j\leq N\}.$$

$B_N$ is crystallographic and has rank $N$.

(4) $BC_N$. This is the root system in $\mathbb{R}^N$ given by

$$BC_N = \{\pm e_i,\, \pm 2e_i,\, 1 \leq i \leq N\} \cup \{\pm(e_i \pm e_j), 1 \leq i < j \leq N\}.$$

It is crystallographic, but not reduced.

A root system $R$ is called *irreducible*, if it cannot be written as the orthogonal disjoint union $R = R_1 \cup R_2$ of two root systems $R_1$, $R_2$. Any root system can be uniquely written as an orthogonal disjoint union of irreducible root systems. There exists a classification of all irreducible, reduced root systems in terms of Coxeter graphs. There are 5 infinite series: $A_N, B_N, C_N, D_N$ (which are crystallographic), as well as the rank 2 root systems $I_2(n)$ corresponding to the dihedral groups. Apart from those, there is a finite number of exceptional root systems. The root systems $BC_N$ are the only irreducible crystallographic root systems which are not reduced. We mention that the root system of a complex semisimple Lie algebra is always crystallographic and reduced, and it is irreducible exactly if the Lie algebra is simple. For further details on root systems, the reader is referred to [Hu] and [Kn].

## 2.2 Dunkl operators

Let $R$ be a reduced (not necessarily crystallographic) root system in $\mathbb{R}^N$ and $W$ the associated reflection group. The Dunkl operators attached with $R$ are modifications of the usual partial derivatives by reflection parts, which are coupled by parameters. The parameters are given in terms of a so-called multiplicity function:

**2.4 Definition.** A function $k : R \to \mathbb{C}$ on the root system $R$ is called a *multiplicity function*, if it is invariant under the natural action of $W$ on $R$.

The set of multiplicity functions forms a $\mathbb{C}$-vector space whose dimension is equal to the number of $W$-orbits in $R$.

**2.5 Definition.** Let $k : R \to \mathbb{C}$ be a multiplicity function on $R$. Then for $\xi \in \mathbb{R}^N$, the *Dunkl operator* $T_\xi := T_\xi(k)$ is defined on $C^1(\mathbb{R}^N)$ by

$$T_\xi f(x) := \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha)\, \langle \alpha, \xi \rangle\, \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Here $\partial_\xi$ denotes the directional derivative corresponding to $\xi$, and $R_+$ is a fixed positive subsystem of $R$. For the $i$-th standard basis vector $\xi = e_i \in \mathbb{R}^N$ we use the abbreviation $T_i = T_{e_i}$.

The operators $T_\xi$ were introduced and first studied by C.F. Dunkl ([D1-5]). By the $W$-invariance of $k$, their definition does not depend on the special choice of $R_+$. Also, the length of the roots is irrelevant in the formula for $T_\xi$. This is the basic reason for the convention which requires reduced root systems: a Dunkl operator with summation over a non-reduced root system can be replaced by a counterpart with summation about an associated reduced counterpart, the multiplicities being modified accordingly. Note further that the dependence of $T_\xi$ on $\xi$ is linear. In case $k = 0$, the $T_\xi(k)$ reduce to the corresponding directional derivatives.

Dunkl operators enjoy regularity properties similar to usual partial derivatives on various spaces of functions. In order to formulate them, we introduce some further standard notation. We denote by $\Pi := \mathbb{C}[\mathbb{R}^N]$ the $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$. It has a natural grading

$$\Pi = \bigoplus_{n\geq 0} \mathcal{P}_n\,,$$

where $\mathcal{P}_n$ is the subspace of homogeneous polynomials of (total) degree $n$. Further, $\mathcal{S}(\mathbb{R}^N)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$ with the usual locally convex topology.

**2.6 Lemma.** (1) *If $f \in C^m(\mathbb{R}^N)$ with $m \geq 1$, then $T_\xi f \in C^{m-1}(\mathbb{R}^N)$.*

(2) *$T_\xi$ leaves $C_c^\infty(\mathbb{R}^N)$ and $\mathcal{S}(\mathbb{R}^N)$ invariant.*

(3) *$T_\xi$ is homogeneous of degree $-1$ on $\mathcal{P}$, that is, $T_\xi\, p \in \mathcal{P}_{n-1}$ for $p \in \mathcal{P}_n$.*

*Proof.* By the fundamental theorem of calculus, one obtains for $\alpha \in R$ the representation

$$\frac{f(\sigma_\alpha x) - f(x)}{\langle \alpha, x\rangle} \;=\; \int_0^1 \partial_\alpha f\big(x - 2t\frac{\langle \alpha, x\rangle}{\langle \alpha, \alpha\rangle}\alpha\big)dt.$$

¿From this, (1) (3) and the first part of (2) are immediate; the proof of (2) for $\mathcal{S}(\mathbb{R}^N)$ is also straightforward but more technical; it can be found in [dJ1]. □

The Dunkl operators $T_\xi(k)$ are $W$-equivariant, that is

$$wT_\xi w^{-1} = T_{w\xi} \quad \text{for all } w \in W \tag{2.1}$$

where the action of $W$ on functions $f : \mathbb{R}^N \to \mathbb{C}$ is given by

$$w \cdot f(x) := f(w^{-1}x).$$

Relation (2.1) is obtained from the definition of the Dunkl operators and the $W$-invariance of $k$. Similar, one obtains the following product rule:

**2.7 Lemma.** *If $f, g \in C^1(\mathbb{R}^N)$ and at least one of them is $W$-invariant, then*

$$T_\xi(fg) \;=\; T_\xi(f)\cdot g \;+\; f\cdot T_\xi(g). \tag{2.2}$$

The most important property of the Dunkl operators, which is the basis for rich analytic structures related with them, is the following Theorem of C.F. Dunkl, [D2]:

**2.8 Theorem.** *For fixed $k$, the Dunkl operators $T_\xi = T_\xi(k), \xi \in \mathbb{R}^N$ commute.*

This result was obtained in [D2] by a clever direct argumentation. There are also alternative proofs. In [DJO], a proof relying on Koszul complex ideas is given. Another, indirect method is to deduce the commutativity of the rational Dunkl operators by a contraction limit from the corresponding result in the trigonometric case, where the Dunkl (-Cherednik) operators are simultaneously diagonalized by a certain family of trigonometric polynomials. See [O2] for the Cherednik case and [dJ4] for the contraction limit.

As a consequence of Theorem 2.8, the assignment

$$\Phi_k : x_i \to T_i(k),\ 1 \to id$$

extends to an algebra homomorphism $\Phi : \Pi \to \mathrm{End}_{\mathbb{C}}(\Pi)$. For $p \in \Pi$ we write

$$p(T(k)) := \Phi_k(p)$$

for the Dunkl operator associated with $p$. The classical case $k = 0$ will be distinguished by the notation $\Phi_0(p) =: p(\partial)$.

Let us denote by $\mathcal{P}^W$ the subalgebra of $\mathcal{P}$ consisting of those polynomials which are $W$-invariant. Suppose that $p \in \mathcal{P}^W$. Then it follows from the $W$-equivariance of the $T_\xi$ that the associated Dunkl operator $p(T) = p(T(k))$ is $W$-invariant, that is $w(p(T(k))w^{-1} = p(T(k))$. We denote by $\mathrm{Res}\, p(T(k)) : \mathcal{P}^W \to \mathcal{P}^W$ the restriction of this operator to $\mathcal{P}^W$. It has been shown by Heckman [He1] that this operator – as one expects – acts as a differential operator with coefficients from $\mathcal{P}^W$.

Of particular importance is the *Dunkl Laplacian*, which is defined by

$$\Delta_k := p(T(k)) \quad \text{with } p(x) = |x|^2.$$

As $p$ is $W$-invariant, it follows that $\Delta_k$ is $W$-invariant and

$$\Delta_k = \sum_{i=1}^N T_{\xi_i}^2$$

for any orthonormal basis $\{\xi_1, \dots, \xi_N\}$ of $\mathbb{R}^N$. The Dunkl Laplacian can be written explicitly as follows (see [D2] or [DX] for the proof):

**2.9 Proposition.**

$$\Delta_k = \Delta + 2 \sum_{\alpha \in R_+} k_\alpha \delta_\alpha \quad \textit{with} \quad \delta_\alpha f(x) = \frac{\partial_\alpha f(x)}{\langle \alpha, x \rangle} - \frac{|\alpha|^2}{2} \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2}, \tag{2.3}$$

*where $\Delta$ denotes the usual Laplacian on $\mathbb{R}^N$.*

The restriction of $\Delta_k$ is therefore given by

$$\mathrm{Res}(\Delta_k) = \Delta + 2 \sum_{\alpha \in R_+} k_\alpha \frac{\partial_\alpha}{\langle \alpha, \,.\, \rangle}.$$

Readers familiar with the theory of symmetric spaces will notice that this generalizes the radial part of the Laplace-Beltrami operator of a Riemannian symmetric space of Euclidean type, where $R$ is always crystallographic and $k$ takes only certain values (essentially the multiplicities of restricted root spaces). We shall return to this important aspect in Section 3.3.

**2.10 Examples. (1) The rank-one case.** If $N = 1$ then $R = \{\pm\alpha\}$, which is a root system of type $A_1$. The corresponding reflection group is $W = \{id, \sigma\}$ acting on $\mathbb{R}$ by $\sigma(x) = -x$. The Dunkl operator $T = T_1(k)$ with multiplicity parameter $k \in \mathbb{C}$ is given by

$$Tf(x) = f'(x) + k\, \frac{f(x) - f(-x)}{x}.$$

The restriction of $T^2$ to even functions is a singular Sturm-Liouville operator,

$$\operatorname{Res} T^2 f(x) \,=\, f''(x) + \frac{2k}{x} \cdot f'(x)\,.$$

For $k = (n-1)/2$, this is just the radial part of the usual Laplacian on $\mathbb{R}^n$ with respect to standard polar coordinates.

**(2) Dunkl operators of type $A_{N-1}$.** Consider $W = S_N$ acting on $\mathbb{R}^N$. As all transpositions are conjugate in $S_N$, the vector space of multiplicity functions is one-dimensional. The Dunkl operators associated with the multiplicity parameter $k \in \mathbb{C}$ are given by

$$T_i \,=\, \partial_i \,+\, k \cdot \sum_{j\neq i} \frac{1-\sigma_{ij}}{x_i - x_j} \quad (i = 1, \ldots, N)$$

with $\sigma_{ij}$ as in Example 2.3, and the Dunkl Laplacian is

$$\Delta_k \,=\, \Delta + 2k \sum_{1\leq i<j\leq N} \frac{1}{x_i - x_j}\Big[(\partial_i - \partial_j) - \frac{1-\sigma_{ij}}{x_i - x_j}\Big].$$

**(3) Dunkl operators of type $B_N$.** There are two conjugacy classes of reflections, leading to multiplicity functions of the form $k = (k_0, k_1)$ with $k_i \in \mathbb{C}$. The associated Dunkl operators are given by

$$T_i \,=\, \partial_i \,+\, k_1 \frac{1-\sigma_i}{x_i} \,+\, k_0 \cdot \sum_{j\neq i} \Big[\frac{1-\sigma_{ij}}{x_i - x_j} + \frac{1-\tau_{ij}}{x_i + x_j}\Big] \quad (i = 1, \ldots, N),$$

where $\tau_{ij} := \sigma_{ij}\sigma_i\sigma_j$.

In the following sections, we shall always require that the multiplicity is *non-negative*, that is $k(\alpha) \geq 0$ for all $\alpha \in R$. We write $k \geq 0$ for short.

Parts of the theory extend to a larger range of multiplicities (depending on $R$), but the condition $k \geq 0$ is essential for positivity results and probability theory.

## 2.3 A generalized Fischer pairing

In the classical theory of spherical harmonics, the following bilinear pairing on $\Pi$, sometimes called Fischer product, plays an important role:

$$[p, q]_0 := \big(p(\partial)q\big)(0), \quad p, q \in \mathcal{P}.$$

In his theory of generalized spherical harmonics, Dunkl [D4] introduced the following analogue:

**2.11 Definition.** For $p, q \in \mathcal{P}$,

$$[p, q]_k \,:=\, \big(p(T(k))q\big)(0).$$

We collect some of its basic properties:

**2.12 Lemma.** (1) *If $p \in \mathcal{P}_n$ and $q \in \mathcal{P}_m$ with $n \neq m$, then* $[p, q]_k = 0$.

(2) $[x_i\, p\,, q]_k \;=\; [p, T_i\, q]_k \quad (p, q \in \mathcal{P},\; i = 1, \ldots, N).$

(3) $[w \cdot p\,, w \cdot q]_k \;=\; [p, q]_k \quad (p, q \in \mathcal{P},\; w \in W).$

*Proof.* Parts (1) and (3) follows from the homogeneity and the $W$-equivariance of the Dunkl operators, respectively. (2) is clear from the definition. □

As before, we assume $k \geq 0$. Let $w_k$ denote the weight function on $\mathbb{R}^N$ defined by

$$w_k(x) = \prod_{\alpha \in R} |\langle \alpha, x\rangle|^{k_\alpha}.$$

It is $W$-invariant and homogeneous of degree $2\gamma$, where

$$\gamma = \gamma(k) := \sum_{\alpha \in R_+} k(\alpha). \tag{2.4}$$

(Notice that by $W$-invariance of $k$, we have $k(-\alpha) = k(\alpha)$ for all $\alpha \in R$. Hence this definition does not depend on the special choice of $R_+$). Further, we define the constant

$$c_k := \int_{\mathbb{R}^N} e^{-|x|^2/2} w_k(x) DC,$$

called a Macdonald-Mehta-Selberg integral. There exists a closed form for it which was conjectured and proved by Macdonald [M1] for the infinite series of crystallographic root systems. An extension to arbitrary crystallographic reflection groups is due to Opdam [O1], and there are computer-assisted proofs for some non-crystallographic root systems. As far as we know, a general proof for arbitrary root systems has not yet been found.
It is an important fact that the Dunkl operators are anti-symmetric with respect to the weight $w_k$:

**2.13 Proposition.** *[D5] For $f \in \mathcal{S}(\mathbb{R}^N)$ and $g \in C_b^1(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} T_\xi f(x) g(x) w_k(x) dx \;=\; -\int_{\mathbb{R}^N} f(x) T_\xi g(x) w_k(x) dx\,.$$

*Proof.* A short calculation. In order to have the appearing integrals well defined, one has to assume $k \geq 1$ first, and then extend the result to all $k$ with $\mathrm{Re} k \geq 0$ by analytic continuation. □

The paring $[\,.\,,\,.\,]_k$ is closely related to the inner product in $L^2(\mathbb{R}^N, e^{-|x|^2} w_k)$. More precisely, we have the following identity, which was first observed in the classical case $k = 0$ by Macdonald [M2] and then generalized to the Dunkl setting in [D4]:

**2.14 Proposition.** *For all $p, q \in \mathcal{P}$,*

$$[p, q]_k \;=\; c_k^{-1} \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x)\, e^{-\Delta_k/2} q(x)\, e^{-|x|^2/2}\, w_k(x) dx. \tag{2.5}$$

As the Dunkl Laplacian is homogeneous of degree $-2$, the operator $e^{-\Delta_k/2}$ is well-defined and bijective on $\mathcal{P}$, and it preserves the (total) degree. An elegant proof (on the basis of our present knowledge only) was given by M. de Jeu in his thesis [dJ2]. It is published in [dJ4].

**2.15 Corollary.** *The pairing $[\,.\,,\,.\,]_k$ on $\mathcal{P}$ is symmetric and non-degenerate.*

## 2.4 Dunkl's intertwining operator

Besides commutativity of the Dunkl operators, the second important result in rational Dunkl theory is the existence of an intertwining operator. This is an isomorphism on $\Pi$ which intertwines the commutative algebra of Dunkl operators with the algebra of partial differential operators with constant coefficients. This operator was first constructed in [D4] for non-negative multiplicities. A thorough analysis in [DJO] subsequently revealed that for general $k$, such an intertwining operator exists if and only if the common kernel of the $T_\xi(k)$, considered as linear operators on $\mathcal{P}$, contains no "singular" polynomials besides the constants. As we are only interested in non-negative multiplicities, we restrict ourselves to the results of [D4].

**2.16 Theorem.** *[D4]. Let $k \geq 0$. Then there exists a unique linear isomorphism ("intertwining operator") $V_k$ of $\mathcal{P}$ satisfying*

$$V_k(\mathcal{P}_n) = \mathcal{P}_n\,,\ \ V_k\,|_{\mathcal{P}_0} \,=\, id \quad \text{and} \quad T_\xi V_k \,=\, V_k\partial_\xi \quad \forall\, \xi \in \mathbb{R}^N. \tag{2.6}$$

The intertwining operator is defined recursively on the spaces $\mathcal{P}_n$, as follows: Let $\mathbb{K}$ be an extension field of $\mathbb{Q}$ containing the multiplicities $k_\alpha, \alpha \in R_+$. Consider the group algebra $\mathbb{K}W = \{\sum_{w\in W} c_w w : c_w \in \mathbb{K}\}$. For $w \in W$ and $0 < t \leq 1$ define the coefficient $q_w(t) = q_w(k;t) \in \mathbb{K}$ by

$$\sum_{w\in W} q_w(t) w \,=\, \exp\Big(\ln t \sum_{\alpha\in R_+} k_\alpha(1-\sigma_\alpha)\Big),$$

which is a central element in $\mathbb{K}W$. Let further

$$c_n(w) := \int_0^1 q_w(t)t^n dt \quad (n \in \mathbb{Z}_+).$$

Then $V_k$ is defined recursively on the spaces $\mathcal{P}_n$ by $V_k 1 = 1$ and

$$V_k p(x) := \sum_{w\in W} c_n(w)\Big(\sum_{i=1}^N (wx)_i V_k\big((\partial_i p)(wx)\big)\Big) \quad \text{for } p \in \mathcal{P}_n.$$

For details on this construction, we refer the interested reader to [D4] or [DX]. In contrast to the definition of $V_k$ itself, it is fairly easy to write down the inverse $U_k = V_k^{-1}$, namely

$$U_k p(x) := \big(e^{\langle x,T\rangle}p\big)(0) \,=\, \sum_{\nu\in\mathbb{Z}_+}^{\infty} \frac{x^\nu}{\nu!}(T^\nu p)(0).$$

Note that $U_k$ is well-defined, because the Dunkl operators $T_i$ are homogeneous of degree $-1$, and therefore the series is always a finite sum. Moreover, it is obvious that $U_k 1 = 1$ and that $U_k$ preserves the degree of homogeneity, that is $U_k(\mathcal{P}_n) \subseteq \mathcal{P}_n$. Finally, we have $\frac{\partial}{\partial x_i} e^{\langle x,T\rangle} \,=\, e^{\langle x,T\rangle}T_i$ on polynomials, and therefore $U_k$ satisfies the commutation relation

$$\partial_i U_k \,=\, U_k T_i \quad (i = 1, \ldots, N).$$

As a consequence of the above representation of $V_k^{-1}$ one obtains the following

**2.17 Corollary** (Taylor Formula)**.** *Let* $f \in C^m(\mathbb{R}^N)$ *for some* $m \in \mathbb{N}$. *Then*

$$f(x) = \sum_{\nu \in \mathbb{Z}_+^N, |\nu| \leq m} \frac{V_k(x^\nu)}{\nu!} (T^\nu f)(0) \,+\, o(|x|^m) \quad \text{for } x \to 0.$$

*Further, if* $f : \mathbb{R}^N \to \mathbb{C}$ *is real-analytic in a ball* $B$ *around* $0$, *then*

$$f(x) = \sum_{n=0}^\infty \sum_{|\nu|=n} \frac{V_k(x^\nu)}{\nu!} (T^\nu f)(0)$$

*where the series* $\sum_{n=0}^\infty$ *converges locally uniformly in* $B$.

*Proof.* Assume first that $f$ is a polynomial. Then $V_k^{-1} f(x) = \sum_\nu \frac{x^\nu}{\nu!} (T^\nu f)(0)$ and therefore

$$f(x) = \sum_\nu \frac{V_k(x^\nu)}{\nu!} (T^\nu f)(0).$$

The assertions now follow from the corresponding results for the classical case. □

**2.18 Lemma.** *The intertwining operator* $V_k$ *commutes with the action of* $W$:

$$w^{-1} V_k w \,=\, V_k \quad (w \in W).$$

*Proof.* The operator $\widetilde{V}_k = w^{-1} V_k w$ satisfies all the characteristic properties of $V_k$. □

Though the intertwining operator plays an important role in Dunkl's theory, an explicit representation for it is known so far only in some special cases. In the rank-one case it is given by

$$V_k(x^n) = \frac{\left(\frac{1}{2}\right)_m}{\left(k + \frac{1}{2}\right)_m} \, x^n \quad \text{with } m = \lfloor \frac{n+1}{2} \rfloor. \tag{2.7}$$

For $k > 0$, this can be written as an integral operator, namely

$$V_k \, p(x) \;=\; \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)} \int_{-1}^1 p(xt)\,(1-t)^{k-1}(1+t)^k \, dt, \tag{2.8}$$

see [D4], Theorem 5.1. Besides this case, there are explicit integral formulas for the $A_2$-case, treated in [D6], and the $B_2$-case under the additional requirement $k_1 = k_2$, treated in [D8]. They are both based on Harish-Chandra integral representations which are available for a single value of the multiplicity only.

For the further development of the theory it is crucial to extend the domain of $V_k$ away from polynomials. The basic extension is to certain normed algebras of homogeneous series. This goes back to [D4].

**2.19 Definition.** For $r > 0$, let $B_r := \{x \in \mathbb{R}^N : |x| \leq r\}$ denote the closed ball of radius $r$, and let $A_r$ be the closure of $\mathcal{P}$ with respect to the norm

$$\|p\|_{A_r} := \sum_{n=0}^\infty \|p_n\|_{\infty, B_r} \quad \text{for } p = \sum_{n=0}^\infty p_n \,,\, p_n \in \mathcal{P}_n.$$

Clearly $A_r$ is a commutative Banach-$*$-algebra under the pointwise operations and with complex conjugation as involution. Each $f \in A_r$ has a unique representation $f = \sum_{n=0}^{\infty} f_n$ with $f_n \in \mathcal{P}_n$, and is continuous on the ball $B_r$ and real-analytic in its interior. The topology of $A_r$ is stronger than the topology induced by the uniform norm on $B_r$. Notice also that $A_r \subseteq A_s$ with $\|\,.\,\|_{A_r} \geq \|\,.\,\|_{A_s}$ for $s \leq r$.

**2.20 Theorem.** *$\|V_k p\|_{\infty,B_r} \leq \|p\|_{\infty,B_r}$ for each $p \in \mathcal{P}_n$.*

The proof of this result is given in [D4] and can also be found in [DX].

**2.21 Corollary.** *$\|V_k p\|_{A_r} \leq \|p\|_{A_r}$ for every $p \in \mathcal{P}$, and $V_k$ extends uniquely to a bounded linear operator on $A_r$ by*

$$V_k f := \sum_{n=0}^{\infty} V_k f_n \quad \text{for } f = \sum_{n=0}^{\infty} f_n\,.$$

We call a linear operator $L$ *positive* on $\mathcal{P}$, if it preserves the positive cone $\mathcal{P}_+ = \{p \in \mathcal{P} : p(x) \geq 0 \ \forall\, x \in \mathbb{R}^N\}$. Formula (2.8) shows that in the rank-one case, the operator $V_k$ is positive on polynomials. This is true for general $R$ and non-negative $k$. Indeed, the following was proved in [R3]:

**2.22 Theorem.** (1) *The operator $V_k$ is positive on $\mathcal{P}$.*

(2) *For each $x \in \mathbb{R}^N$ there exists a unique probability measure $\mu_x^k$ on the Borel-$\sigma$-algebra of $\mathbb{R}^N$ such that*

$$V_k f(x) = \int_{\mathbb{R}^N} f(\xi)\, d\mu_x^k(\xi) \tag{2.9}$$

*for all $f \in A_r$ with $r \leq |x|$. The representing measures $\mu_x^k$ are compactly supported with $\operatorname{supp} \mu_x^k \subseteq \operatorname{co}(W.x)$, the convex hull of the orbit of $x$ under $W$. Moreover, they satisfy*

$$\mu_{rx}^k(B) = \mu_x^k(r^{-1}B), \quad \mu_{wx}(B) = \mu_x^k(w^{-1}(B)) \tag{2.10}$$

*for each $r > 0$, $w \in W$ and each Borel set $B$ in $R^N$.*

This result immediately allows to $V_k$ to larger function spaces such as $C(\mathbb{R}^N)$ or $L^1_{loc}(\mathbb{R}^N)$. We shall come back to extensions later.

In the following, we outline the proof of Theorem 2.22, for detail see [R3]. We begin with a lemma which is also of interest in its own.

**2.23 Lemma.** *For all $p, q \in \mathcal{P}$,*

(1) $[V_k p, q]_k = [p, q]_0$;

(2) $c_k^{-1} \int_{\mathbb{R}^N} (V_k p) q\, e^{-|x|^2} w_k(x) dx = c_0^{-1} \int_{\mathbb{R}^N} p\, \big(e^{-\Delta/2} e^{\Delta_k/2} q\big) e^{-|x|^2/2} dx.$

*Proof.* (1) Due to the orthogonality of the spaces $\mathcal{P}_n$ with respect to both pairings it suffices to consider $p, q \in \mathcal{P}_n$ for some $n$. Then

$$[V_k\, p, q]_k = [q, V_k\, p]_k = q(T)(V_k\, p) = V_k(q(\partial)p) = q(\partial)(p) = [p, q]_0\,;$$

here the characterizing properties of $V_k$ and the fact that $q(\partial)(p)$ is a constant have been used.

(2) Combining part (1) with the Macdonald-type identity (2.5), one obtains

$$\int_{\mathbb{R}^N} e^{-\Delta_k/2}(V_k p)e^{-\Delta_k/2}q\, e^{-|x|^2/2}w_k(x)dx \;=\; \frac{c_k}{c_0}\int_{\mathbb{R}^N} e^{-\Delta/2}p\, e^{-\Delta/2}q\, e^{-|x|^2/2}dx.$$

As $e^{-\Delta_k/2}(V_k\, p) \;=\; V_k\big(e^{-\Delta/2}p\big)$ , and as we may replace $p$ by $e^{\Delta/2}p$ and $q$ by $e^{\Delta_k/2}q$, this implies the claimed identity. □

*Proof of Theorem 2.22 (Sketch).* From part (2) of the above lemma it follows by standard density arguments that (1) of the Theorem is equivalent to the positivity of the operator

$$e^{-\Delta/2}e^{\Delta_k/2}$$

on $\mathcal{P}$. We consider $\Delta_k$ as a perturbation of the usual Laplacian $\Delta$,

$$\Delta_k = \Delta + L_k \quad \text{with} \quad L_k = 2\sum_{\alpha\in R_+} k_\alpha \delta_\alpha$$

as written in (2.3). An argument involving the Trotter product formula for $e^{\Delta_k/2} = e^{\Delta/2 + L_k/2}$ then shows that it suffices to verify positivity of the operators

$$e^{-\Delta}e^{tL_k}e^{\Delta} \quad (t\geq 0)$$

on $\mathcal{P}$. But

$$e^{-\Delta}e^{tL_k}e^{\Delta} \;=\; e^{tA} \quad \text{with } A = e^{-\Delta}L_k e^{\Delta}.$$

Thus, it is enough to show that the operator semigroup $(e^{tA})_{t\geq 0}$ is positive on $\mathcal{P}$. This can be achieved by verifying that its generator $A$ satisfies the positive minimum principle $(M)$ stated in Lemma 2.24 below. Indeed, it is easily checked that $A$ is degree-lowering. Further, $A$ decomposes as

$$A = 2\sum_{\alpha\in R_+} k_\alpha e^{-\partial_\alpha^2}\delta_\alpha\, e^{\partial_\alpha^2}$$

(here it is used that $\delta_\alpha$ acts in direction $\alpha$ only). Direct computation shows that the one-dimensional operators $e^{-\partial_\alpha^2}\delta_\alpha e^{\partial_\alpha^2}$ satisfy the minimum principle $(M)$. As the $k_\alpha$ are non-negative, $A$ also satisfies $(M)$. By Lemma 2.24 below, this finishes the proof.

(2) Part (1) implies that the mapping

$$\Phi_x : f \mapsto V_k f(x)$$

is a positive linear functional on the commutative Banach-*-algebra $A_{|x|}$. The Bochner representation theorem for positive functionals on commutative Banach-$*$-algebras (see for instance Theorem 21.2 of [FD]) then implies an integral representation for $\Phi_x$ with representing measures supported in the ball $B_{|x|}$. The sharper statement on the support is obtained by results of [dJ1]. The remaining statements are easy. □

**2.24 Lemma.** *Let $A$ be a degree-lowering linear operator on $\mathcal{P}$, that is $deg(Ap) < deg(p)$ for all $p\in\mathcal{P}$. Then the following statements are equivalent:*

(1) $e^{tA}$ *is positive on* $\mathcal{P}$ *for all* $t \geq 0$.

(2) $A$ *satisfies the "positive minimum principle"*

(M) *For every* $p \in \mathcal{P}_+$ *and* $x_0 \in \mathbb{R}^N$, $p(x_0) = 0$ *implies* $Ap\,(x_0) \geq 0$.

This principle is an adaption of a well-known criterion for generators of Feller semigroups, see Section 2.6.

## 2.5 The Dunkl kernel and the Dunkl transform

Again we assume that $k \geq 0$. As the Dunkl operators $T_\xi = T_\xi(k)$ commute, it is natural to consider their joint eigenvalue problem: for a fixed spectral parameter $y \in \mathbb{C}^N$, we search for a function $f$ solving

$$(E) \qquad \begin{cases} T_\xi f = \langle x, y\rangle f & \forall\, \xi \in \mathbb{R}^N \\ f(0) = 1. \end{cases}$$

Here $\langle\, .\, ,\, .\,\rangle$ denotes the *bilinear* extension of the Euclidean inner product to $\mathbb{C}^N \times \mathbb{C}^N$. If $k = 0$, then a solution to this problem is of course given by the exponential $f(x) = e^{\langle x,y\rangle}$. In the general case, we apply the intertwining operator. Notice that for fixed $y$, the function $x \mapsto e^{\langle x,y\rangle}$ belongs to each of the algebras $A_r$, $r > 0$. This justifies the following

**2.25 Definition.** [D4] For $y \in \mathbb{C}^N$, define

$$E_k(x,y) := V_k\big(e^{\langle\, .\, ,y\rangle}\big)(x), \quad x \in \mathbb{R}^N.$$

$E_k$ is called the *Dunkl-kernel* associated with $W$ and $k$.

Let us check that $f(x) = E_k(x,y)$ solves (E). We write

$$E_k(x,y) = \sum_{n=0}^{\infty} E_k^{(n)}(x,y) \quad \text{with} \quad E_k^{(n)}(x,y) = \frac{1}{n!} V_k\langle\, .\, ,y\rangle^n(x). \tag{2.11}$$

The homogeneity of $V_k$ immediately implies that $E_k(0,y) = 1.$. Further, by the intertwining property,

$$T_\xi E_k^{(n)}(\,.\,,y) \;=\; \frac{1}{n!} V_k\, \partial_\xi \langle\, .\, ,y\rangle^n \;=\; \langle \xi, y\rangle E_k^{(n-1)}(\,.\,,y).$$

This shows that (E) is satisfied.

**2.26 Remark.** From the very definition, it follows that

$$E_k(x,y) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \frac{V_k(x^\nu)y^\nu}{\nu!} \tag{2.12}$$

where the series $\sum_{n=0}^{\infty}$ converges absolutely and locally uniformly on $\mathbb{R}^N$.

**2.27 Theorem.** *Let* $y \in \mathbb{C}^N$. *Then* $f = E_k(\,.\,,y)$ *is the unique solution of the system*

$$T_\xi\, f \;=\; \langle \xi, y\rangle f \quad \textit{for all } \xi \in \mathbb{R}^N \tag{2.13}$$

*which is real-analytic on* $\mathbb{R}^N$ *and satisfies* $f(0) = 1$. *Moreover,* $E_k$ *extends to a holomorphic function on* $\mathbb{C}^N \times \mathbb{C}^N$.

This is a weakened version of a result of Opdam ([O1], Prop. 6.7) which includes complex multiplicities as well as meromorphic dependence of $E_k$ on $k$. The decisive part for the uniqueness proof is the observation that the joint kernel of the $T_\xi(k)$, considered as linear operators on $\mathcal{P}$, consists of the constants only. Details can be found in [R7], see also [O1].

**2.28 Proposition.** *For $x, y \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$ and $w \in W$,*

(1) $E_k(x,y) = E_k(y,x)$.

(2) $E_k(\lambda x, y) = E_k(x, \lambda y)$ *and* $E_k(wx, wy) = E_k(x,y)$.

(3) $\overline{E_k(x,y)} = E_k(\overline{x}, \overline{y})$.

*Proof.* Part (1) is shown in [D4]. (2) is easily obtained from the definition of $E_k$ together with the homogeneity and equivariance properties of $V_k$. For (3), notice that $f := \overline{E_k(\,.\,, y)}$, which is again real-analytic on $\mathbb{R}^N$, satisfies $T_\xi f = \langle \xi, \overline{y}\rangle f$, $f(0) = 1$. By the uniqueness part of the above Theorem, $\overline{E_k(x,y)} = E_k(x, \overline{y})$ for all real $x$. Now both $x \mapsto \overline{E_k(\overline{x}, y)}$ and $x \mapsto E_k(x, \overline{y})$ are holomorphic on $\mathbb{C}^N$ and agree on $\mathbb{R}^N$. Hence they coincide. $\square$

Just as with the intertwining operator, the kernel $E_k$ is explicitly known for some particular cases only. An important example is again the rank-one situation:

**2.29 Example.** In the *rank-one case* with $\operatorname{Re} k > 0$, the integral representation (2.8) for $V_k$ implies that for all $x, y \in \mathbb{C}$,

$$E_k(x,y) \;=\; \frac{\Gamma(k+1/2)}{\Gamma(1/2)\,\Gamma(k)} \int_{-1}^{1} e^{txy}(1-t)^{k-1}(1+t)^k\,dt \;=\; e^{xy} \cdot {}_1F_1(k, 2k+1, -2xy).$$

This can also be written as

$$E_k(x,y) \;=\; j_{k-1/2}(ixy) \;+\; \frac{xy}{2k+1}\, j_{k+1/2}(ixy) \tag{2.14}$$

where for $\alpha \geq -1/2$, $j_\alpha$ is the normalized spherical Bessel function

$$j_\alpha(z) \;=\; {}_0F_1(\alpha+1; -z^2/4) \;=\; \Gamma(\alpha+1) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n!\,\Gamma(n+\alpha+1)}\,. \tag{2.15}$$

This motivates the following

**2.30 Definition.** [O1] The (Dunkl-type) *Bessel function* associated with $R$ and $k$ is defined for $x, y \in \mathbb{C}^N$ by

$$J_k(x,y) := \frac{1}{|W|} \sum_{w \in W} E_k(wx, y). \tag{2.16}$$

Thanks to Prop. 2.28 $J_k$ is $W$-invariant in both arguments and therefore naturally considered on Weyl chambers of $W$ (or their complexifications). In the rank-one case, we have

$$J_k(x,y) = j_{k-1/2}(ixy).$$

It is a well-known fact from classical analysis that for fixed $y \in \mathbb{C}$, the function $f(x) = j_{k-1/2}(ixy)$ is the unique analytic solution of the differential equation

$$f'' + \frac{2k}{x} f' \,=\, y^2 f$$

which is even and normalized by $f(0) = 1$. This fact generalizes to the multi-variable case, as follows: Recall the algebra of $W$-invariant polynomials

$$\mathcal{P}^W = \{p \in \mathcal{P} : w \cdot p = p \quad \text{for all } \; w \in W\},$$

as well as the restriction $\operatorname{Res} p(T) : \mathcal{P}^W \to \mathcal{P}^W$ for $p \in \mathcal{P}^W$. For fixed spectral parameter $y \in \mathbb{C}^N$, $J_k(\,.\,, y)$ is a solution to the following *Bessel system:*

$$p(T)f \,=\, p(y)f \quad \text{for all } \; p \in \mathcal{P}^W, \; f(0) = 1.$$

Actually, Opdam went the converse direction in [O1]. He first proved that the Bessel system has a unique $W$-invariant analytic solution. From this solution, the generalized Bessel function, he then constructed the Dunkl kernel. The Bessel system generalizes the so-called system of invariant differential operators on a Riemannian symmetric space of Euclidean type, and the Dunkl-type Bessel functions $J_k$ generalize the associated spherical functions. We shall explain this connection in more detail in Section 3.3.

The following general Bochner-type integral representation of the Dunkl kernel is an immediate consequence of Theorem 2.22.

**2.31 Proposition.** *For each $x \in \mathbb{R}^N$, the Dunkl kernel $E_k(x, \,.\,)$ has the integral representation*

$$E_k(x,y) \,=\, \int_{\mathbb{R}^N} e^{\langle \xi,\, y\rangle} d\mu_x^k(\xi) \tag{2.17}$$

*where the $\mu_x^k$ are the representing measures from Theorem 2.22. A corresponding integral representation holds for the Bessel function $J_k$.*

**2.32 Corollary.** *The Dunkl kernel satisfies*

(1) $E_k(x,y) > 0$ *for all* $x, y \in \mathbb{R}^N$.

(2) *For all* $x \in \mathbb{R}^N, y \in \mathbb{C}^N$ *and* $\alpha \in \mathbb{Z}_+^N$,

$$|\partial_y^\alpha E_k(x,y)| \,\leq\, |x|^{|\alpha|} \max_{w \in W} e^{Re\langle wx, y\rangle}.$$

(3) $|E_k(-ix,y)| \,\leq\, 1 \quad \forall\, x, y \in \mathbb{R}^N$.

**2.33 Remarks.** (1) M. de Jeu had already an estimate on $E_k$ with slightly weaker bounds in [dJ1], differing by an additional factor $\sqrt{|W|}$.

(2) In [RV2], a completely different proof of Proposition 2.31 is given under the restriction that $R$ is crystallographic. It is based on an asymptotic relationship between the Opdam-Cherednik kernel (see [O2]) and the Dunkl kernel observed in [dJ4] (see also [BO1]), as well as on positivity results of S. Sahi for the Heckman-Opdam polynomials and their non-symmetric counterparts. In contrast to the original approach described here, in [RV2] the precise information on the support is obtained without using the exponential bounds on $E_k$ from [dJ1]. Theorem 2.22 was then, in the converse way, obtained from the integral representation for $E_k$.

We conclude this section with two important reproducing properties for the Dunkl kernel proved in [D5]. The above estimate (3) on $E_k$ assures the convergence of the involved integrals.

**2.34 Proposition.** *For all $p \in \mathcal{P}$ and $y, z \in \mathbb{C}^N$,*

$$(1) \quad \int_{\mathbb{R}^N} e^{-\Delta_k/2} p(x)\, E_k(x,y)\, e^{-|x|^2/2} w_k(x)dx \;=\; c_k\, e^{\langle y,y\rangle/2} p(y).$$

$$(2) \quad \int_{\mathbb{R}^N} E_k(x,y)\, E_k(x,z)\, e^{-|x|^2/2} w_k(x)dx \;=\; c_k\, e^{(\langle y,y\rangle + \langle z,z\rangle)/2} E_k(y,z)$$

*Proof.* We use the Macdonald-type formula (2.5). First, we show that

$$[E_k^{(n)}(x,\,.\,), p]_k \;=\; p(x) \quad \text{for all } p \in \mathcal{P}_n,\; x \in \mathbb{R}^N. \tag{2.18}$$

Indeed, if $p \in \mathcal{P}_n$ then

$$p(x) = \frac{\langle x, \partial_y\rangle^n}{n!} p(y) \quad \text{and} \quad V_k^x p(x) = E_k^{(n)}(x, \partial_y) p(y).$$

Here the uppercase index in $V_k^x$ denotes the relevant variable. Application of $V_k^y$ to both sides gives $V_k^x p(x) = E_k^{(n)}(x, T^y) V_k^y p(y)$. As $V_k$ is bijective on $\mathcal{P}_n$, this implies (2.18). For fixed $y$, let $L_n(x) := \sum_{j=0}^n E_k^{(j)}(x,y)$. If $n$ is larger than the degree of $p$, it follows from (2.18) that $[L_n, p]_k = p(y)$. Thus in view of the Macdonald formula,

$$c_k^{-1} \int_{\mathbb{R}^N} e^{-\Delta_k/2} L_n(x) e^{-\Delta_k/2} p(x) e^{-|x|^2/2} w_k(x)dx \;=\; p(y).$$

On the other hand, it is easily checked that

$$\lim_{n\to\infty} e^{-\Delta_k/2} L_n(x) \;=\; e^{-\langle y,y\rangle/2} E_k(x,y).$$

This proves (1). Identity (2) follows from (1) by homogeneous expansion of $E_k$. □

The Dunkl kernel gives rise to an integral transform, the Dunkl transform, which was introduced in [D5] for non-negative multiplicity functions and further studied in [dJ1] in the more general case $\operatorname{Re} k \geq 0$. In this article, we again restrict ourselves to $k \geq 0$.

**2.35 Definition.** The Dunkl transform associated with $R$ and $k \geq 0$ is defined on $L^1(\mathbb{R}^N, w_k)$ by

$$\widehat{f}^{\,k}(\xi) := c_k^{-1} \int_{\mathbb{R}^N} f(x) E_k(-i\xi, x)\, w_k(x)dx, \quad \xi \in \mathbb{R}^N.$$

The inverse transform is given by $f^{\vee k}(\xi) = \widehat{f}^{\,k}(-\xi)$.

Note that $\widehat{f}^{\,k}$ is continuous and bounded.

**2.36 Lemma.** *For $f, g \in L^1(\mathbb{R}^N, w_k)$,*

$$\int_{\mathbb{R}^N} \widehat{f}^{\,k}(x) g(x) w_k(x)dx \;=\; \int_{\mathbb{R}^N} f(x) \widehat{g}^{k}(x) w_k(x)dx.$$

*Proof.* This follows from Proposition 2.28 and Fubini's theorem. □

The Dunkl transform maps Dunkl operators to multiplication operators, and it therefore suggests itself to consider it on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of rapidly decreasing functions on $\mathbb{R}^N$:

**2.37 Lemma.** *Let $f \in \mathcal{S}(\mathbb{R}^N)$. Then*

(1) $\widehat{f}^k \in C^\infty(\mathbb{R}^N)$ *and* $T_j(\widehat{f}^k) = -(ix_j f)^{\wedge k}$ *for* $j = 1, \ldots, N$.

(2) $(T_j f)^{\wedge k}(\xi) = i\xi_j \widehat{f}^k(\xi)$.

(3) *The Dunkl transform leaves the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ invariant.*

*Proof.* (1) is obvious from (2.13), and (2) follows from the anti-symmetry relation (Prop. 2.13) for the Dunkl operators. For (3), it suffices to prove that $\partial_\xi^\alpha(\xi^\beta \widehat{f}^k(\xi))$ is bounded for arbitrary multi-indices $\alpha$, $\beta$. By part (2), we have $\xi^\beta \widehat{f}^k(\xi) = \widehat{g}^k(\xi)$ for some $g \in \mathcal{S}(\mathbb{R}^N)$. The assertion then follows from the definition of the Dunkl transform and the growth bounds of Corollary 2.32 on $E_k$ □

It is not hard to see that $\mathcal{S}(\mathbb{R}^N)$ is dense in $L^p(\mathbb{R}^N, w_k)$ for $1 \leq p < \infty$. Indeed, the weighted case can be reduced to the unweighted one, see [dJ1]. This immediately implies a Riemann-Lebesgue lemma for the Dunkl transform:

**2.38 Corollary.** *For $f \in L^1(\mathbb{R}^N, w_k)$, the Dunkl transform $\widehat{f}^k$ belongs to $C_0(\mathbb{R}^N)$, the space of continuous functions on $\mathbb{R}^N$ which vanish at infinity.*

Another simple consequence is the spectral resolution of the Dunkl Laplacian in $L^2(\mathbb{R}^N, w_k)$. Via the Dunkl transform, $\Delta_k$ it is unitarily equivalent with the multiplication operator $M_{-|\xi|^2}$ in $L^2(\mathbb{R}^N, w_k)$, and this gives

**2.39 Corollary.** *The Dunkl Laplacian $\Delta_k$ with domain $\mathcal{S}(\mathbb{R}^N)$ is essentially self-adjoint in $L^2(\mathbb{R}^N, w_k)$. The spectrum of its closure is $\sigma(\overline{\Delta_k}) = (-\infty, 0]$.*

The following are the main results for the Dunkl transform; they are in complete analogy to the corresponding results for the Fourier transform; for details the reader is referred to [dJ1].

**2.40 Theorem.** (1) *($L^1$-Inversion) If $f \in L^1(\mathbb{R}^N, w_k)$ with $\widehat{f}^k \in L^1(\mathbb{R}^N, w_k)$, then*

$$f = (\widehat{f}^k)^{\vee k} \quad a.e.$$

(2) *The Dunkl transform is injective on $L^1(\mathbb{R}^N, w_k)$.*

(3) *The Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^N)$ with period 4.*

(4) *(Plancherel Theorem) The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}^N, w_k)$. The extension is also denoted by $f \mapsto \widehat{f}^k$.*

*Proof.* (Sketch) The decisive part is the $L^1$-inversion. It is first proved for functions of the form $f(x) = p(x)e^{-|x|^2/2}$, $p \in \mathcal{P}$, which form a dense subalgebra of $C_0(\mathbb{R}^N)$, and then extended to arbitrary $f \in L^1(\mathbb{R}^N, w_k)$. Part (2) is immediate from (1). Together with Lemma 2.37(3), this easily implies that the Dunkl transform is a bijection of $\mathcal{S}(\mathbb{R}^N)$, and continuity in both directions follows from the closed graph theorem. Part (4) is obtained by a standard procedure (using Lemma 2.36) from the density of $\mathcal{S}(\mathbb{R}^N)$ in $L^2(\mathbb{R}^N, w_k)$. □

There have been various approaches to Paley-Wiener theorems for the Dunkl transform, see [dJ1], [dJ2], [T2] and [AdJ]. For an open subset $\Omega \subseteq \mathbb{R}^N$, we denote by $\mathcal{D}(\Omega)$ the space of compactly supported smooth functions on $\Omega$ with the usual Fréchet-space topology. The most basic variant of the Paley-Wiener theorem is as follows.

**2.41 Theorem.** *For $R > 0$ consider the ball $B_R = \{x \in \mathbb{R}^N : |x| \leq 1\}$ and let $\mathcal{H}_{B_R}$ denote the Paley-Wiener space of all entire functions $f$ on $\mathbb{C}^N$ characterized by the property that for each $M \in \mathbb{Z}_+$ there exists a constant $\gamma_M > 0$ such that*

$$|f(\lambda)| \leq \gamma_M (1 + |\lambda|)^{-M} e^{R|Im\lambda|} \quad \text{for all } \lambda \in \mathbb{C}^N.$$

*Then the Dunkl transform $f \mapsto \widehat{f}^k$ is an isomorphism from $\mathcal{D}(B_R)$ onto $\mathcal{H}_{B_R}$.*

For the original proof of this result see [dJ2] or [dJ4]. One consequence is the following important extension of the intertwining operator:

**2.42 Theorem.** *([dJ2], [T2].) The intertwining operator $V_k$ extends to a homeomorphism of $C^\infty(\mathbb{R}^N)$.*

We conclude this section with an outlook on the dual of the intertwining operator introduced in [T1]. The usual Fourier transform on $\mathbb{R}^N$ will be denoted by $f \mapsto \widehat{f}$, its inverse by $f \mapsto f^\vee$.

**2.43 Definition.** The dual of the intertwining operator $V_k$ is defined by

$${}^tV_k : \mathcal{S}(\mathbb{R}^N) \to \mathcal{S}(\mathbb{R}^N), \quad f \mapsto \big(\widehat{f}^k\big)^\vee.$$

By Theorem 2.40, ${}^tV_k$ is a homeomorphism of $\mathcal{S}(\mathbb{R}^N)$. It can be considered as an analogue of the Abel transform on Riemannian symmetric spaces, with the spherical transform being replaced by the Dunkl transform. Here are some further properties of this operator, justifying also the terminology.

**2.44 Proposition.** (1) ${}^tV_k T_\xi = \partial_\xi\, {}^tV_k$ *on* $\mathcal{S}(\mathbb{R}^N)$.

(2) *For $f \in \mathcal{S}(\mathbb{R}^N)$ and polynomials $p \in \mathcal{P}$,*

$$\int_{\mathbb{R}^N} V_k p(x) f(x) w_k(x) dx = \int_{\mathbb{R}^N} p(x)\, {}^tV_k(f)(x) dx$$

(3) *Put $\psi(x) = e^{-|x|^2/2}$. Then for all $q \in \mathcal{P}$,*

$${}^tV_k(q\psi) = \big(e^{-\Delta/2} e^{\Delta_k/2} q\big)\psi.$$

*Proof.* (1) This is immediate from Lemma 2.37.

(2) (c.f. [T1]) By definition of ${}^tV_k$,

$$\begin{aligned} \frac{1}{c_0}\int_{\mathbb{R}^N} p(x)\, {}^tV_k(f)(x) dx &= \big(p(\widehat{f}^k)^\vee\big)^\wedge(0) = \big(p(i\partial)\widehat{f}^k\big)(0) \\ &= \frac{1}{c_k} p(i\partial_\xi)\Big(\int_{\mathbb{R}^N} f(x) E_k(-ix,\xi) w_k(x) dx\Big)\Big|_{\xi=0} \\ &= \frac{1}{c_k}\int_{\mathbb{R}^N} f(x)\big(p(i\partial_\xi)E_k(-ix,\xi)\big)\big|_{\xi=0} w_k(x) dx. \end{aligned}$$

For a monomial $p(x)=x^\alpha$, we obtain from formula (2.12) the identity

$$p(i\partial_\xi)E_k(-ix,\xi)\big|_{\xi=0} \,=\, (i\partial_\xi)^\alpha \sum_\nu \frac{V(x^\nu)(-i\xi)^\nu}{\nu!}\big|_{\xi=0} \,=\, V_k(x^\alpha)=V_kp(x).$$

By linearity, this extends to all $p\in\mathcal{P}$, and the assertion follows.
(3) This results from part (2) with $f=q\psi$ and Lemma 2.23(2). □

## 2.6 Heat kernel and heat semigroup

For a fixed root system $R$ and multiplicity $k\geq 0$, the Dunkl-type heat operator is defined by

$$\Delta_k-\partial_t \quad \text{on } \mathbb{R}^n\times\mathbb{R}.$$

We consider the following initial-value problem for the generalized heat equation:

$$\text{(IVP)} \qquad \begin{cases}(\Delta_k-\partial_t)\,u \,=\, 0 & \text{on } \mathbb{R}^N\times(0,\infty)\\ u(\,.\,,0) \,=\, f\end{cases}$$

with initial data $f\in C_b(\mathbb{R}^N)$. We look for solutions $u\in C^2(\mathbb{R}^N\times(0,\infty))\cap C(\mathbb{R}^N,[0,\infty))$. We shall start with the more abstract aspect of this problem, namely the associated operator semigroup, and then determine the heat kernel and the solution of (IVP) in an explicit form. This approach will establish positivity of the heat semigroup and the heat kernel by a standard approach. As a consequence, it implies that

$$E_k(x,y)>0 \quad \forall\; x,y\in\mathbb{R}^N$$

which is already known from the positive integral representation 2.31 for $E_k$. This integral representation, however, is a much deeper result.

In the following, we consider the Dunkl Laplacian as a densely defined linear operator on the Banach space $(C_0(\mathbb{R}^n),\|\,.\,\|_\infty)$ with domain $\mathcal{S}(\mathbb{R}^N)$. In the classical case $k=0$, it is well known that $\Delta=\Delta_0$ (more precisely: its closure) generates a Feller semigroup on $C_0(\mathbb{R}^N)$, namely the heat semigroup

$$H_tf(x)=\frac{1}{(4\pi t)^{N/2}}\int_{\mathbb{R}^N} f(y)e^{-|x-y|^2/4t}dy.$$

Recall that for a locally compact Hausdorff space $\Omega$, a strongly continuous semigroup $(T_t)_{t\geq0}$ on $(C_0(\Omega),\|.\|_\infty)$ is called a *Feller semigroup*, if it is contractive and positive, that is $\|T_tf\|_\infty\leq\|f\|_\infty$ and $f\geq0$ on $\Omega$ implies that $T_tf\geq0$ on $\Omega$ for all $t\geq0$.

In order to extend the above fact to general multiplicities $k\geq0$, we employ the following useful variant of the Lumer-Phillips theorem, which characterizes Feller semigroups in terms of a "positive maximum principle", see e.g. [Kal], Thm. 17.11. In fact, this Theorem motivated the positive minimum principle 2.24 in the positivity-proof for $V_k$.

**2.45 Theorem.** *Let $A$ be a densely defined linear operator in $(C_0(\Omega),\|.\|_\infty)$ with domain $\mathcal{D}(A)$. Then $A$ is closable, and its closure $\overline{A}$ generates a Feller semigroup on $C_0(\Omega)$, if and only if the following conditions are satisfied:*

(i) *If $f\in\mathcal{D}(A)$ then also $\overline{f}\in\mathcal{D}(A)$ and $A(\overline{f})=\overline{A(f)}$.*

(ii) *The range of* $\lambda id - A$ *is dense in* $C_0(\Omega)$ *for some* $\lambda > 0$.

(iii) *If* $f \in \mathcal{D}(A)$ *is real-valued with a non-negative maximum in* $x_0 \in \Omega$, *i.e.* $0 \leq f(x_0) = \max_{x\in\Omega} f(x)$, *then* $Af(x_0) \leq 0$. *(Positive maximum principle).*

The following Lemma implies that $(\Delta_k, \mathcal{S}(\mathbb{R}^N))$ satisfies the positive maximum principle:

**2.46 Lemma.** *Let* $\Omega \subseteq \mathbb{R}^N$ *be open and* $W$ *-invariant. If a real-valued function* $f \in C^2(\Omega)$ *attains an absolute maximum at* $x_0 \in \Omega$, *i.e.* $f(x_0) = \sup_{x\in\Omega} f(x)$, *then*

$$\Delta_k f(x_0) \leq 0.$$

*Proof.* Recall the explicit expression (2.3) for $\Delta_k$. We assume first that $\langle \alpha, x_0\rangle \neq 0$ for all $\alpha \in R$. The fact that $f$ has a maximum at $x_0$ implies that $\partial_\alpha f(x_0) = 0$ for all $\alpha \in R$ and that $\Delta f(x_0) \leq 0$. Moreover, $f(x_0) \geq f(\sigma_\alpha x_0)$ for all $\alpha \in R$. Thus $\Delta_k f(x_0) \leq 0$. If $\langle \alpha, x_0\rangle = 0$ for some $\alpha \in R$, then one has to use a second order Taylor expansion of $f$. For details see [R2]. □

**2.47 Theorem.** *The operator* $(\Delta_k, \mathcal{S}(\mathbb{R}^N))$ *is closable, and its closure* $\overline{\Delta_k}$ *generates a Feller semigroup* $(H_t)_{t\geq 0}$ *on* $C_0(\mathbb{R}^N)$ *which is called the generalized heat semigroup.*

*Proof.* We have to check the conditions of Theorem 2.45. Condition (i) is obvious and (iii) is an immediate consequence of the previous lemma. (ii) is also satisfied, because for each $\lambda > 0$, the operator $\lambda id - \Delta_k$ leaves $\mathcal{S}(\mathbb{R}^N)$ invariant; this follows from the fact that the Dunkl transform is a homeomorphism of $\mathcal{S}(\mathbb{R}^N)$ and $\big((\lambda I - \Delta_k)f\big)^{\wedge k}(\xi) = (\lambda + |\xi|^2)\widehat{f}^k(\xi)$. Theorem 2.45 now implies the assertion. □

We expect that the heat semigroup $(H_t)_{t\geq 0}$ can be written explicitly in terms of a generalized heat kernel. In order to find it, we consider first a slight modification of the usual Gaussian kernel:

$$g_k(x,t) := \frac{1}{(2t)^{\gamma+N/2}c_k} e^{-|x|^2/4t} \quad (x \in \mathbb{R}^N,\ t > 0).$$

**2.48 Lemma.** (1) $g_k$ *solves the Dunkl-type heat equation* $(\Delta_k - \partial_t)u = 0$ *on* $\mathbb{R}^N \times (0,\infty)$.

(2) $\displaystyle\int_{\mathbb{R}^N} g_k(x,t)\, w_k(x)dx = 1$ *for all* $t > 0$.

(3) $\widehat{g}_k^k(\xi,t) = c_k^{-1} e^{-t|\xi|^2}$.

*Proof.* For (1), use the product rule (2.2) as well as the identity $\sum_{i=1}^N T_i(x_i) = N + 2\gamma$. Part (2) is immediate, and (3) results from the second reproducing property in Proposition 2.34. □

The Gaussian $g_k$ generalizes the fundamental solution for the classical heat equation. In the classical case $k = 0$, the heat kernel is obtained by translation from the fundamental solution. In the Dunkl setting, it is indeed also possible to define a generalized translation which matches the action of the Dunkl transform, i.e. makes it a homomorphism on suitable function spaces. For our present purposes, it will be sufficient to consider the Schwartz space.

**2.49 Definition.** On the Schwartz space $\mathcal{S}(\mathbb{R}^N)$, the Dunkl-type *generalized translation* is defined by

$$\tau_y f(x) := \frac{1}{c_k}\int_{\mathbb{R}^N} \widehat{f}^k(\xi)\, E_k(ix,\xi)E_k(iy,\xi)\, w_k(\xi)d\xi; \quad y \in \mathbb{R}^N.$$

Note that for $k=0$ our definition reduces to $\tau_y f(x) = f(x+y)$. In the rank one case, our generalized translation determines the convolution of a so-called signed hypergroup structure which was defined in [R1]; see also [Ros]. This will be discussed in more detail in Section 2.8. Similar structures are conjectured in higher-rank cases, but not established so far. Notice that $\tau_y f(x) = \tau_x f(y)$; moreover, the inversion theorem for the Dunkl transform assures that $\tau_0 f = f$ and

$$(\tau_y f)^{\wedge k}(\xi) = E_k(iy,\xi)\widehat{f}^k(\xi). \tag{2.19}$$

From this it is easy to see that $\tau_y f$ belongs to $\mathcal{S}(\mathbb{R}^N)$ again.

Let us return to the Gaussian kernel $g_k$. ¿From the definition of the generalized translation, part (3) of Lemma 2.48 and a further application of the reproducing formula (2) of Proposition 2.34, we obtain

$$\tau_{-y}F_k(x,t) \;=\; \frac{1}{(2t)^{\gamma+N/2}c_k} e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big).$$

This motivates the following

**2.50 Definition.** The generalized heat kernel $\Gamma_k$ is defined for $x, y \in \mathbb{R}^N$ and $t > 0$ by

$$\Gamma_k(t,x,y) := \frac{1}{(2t)^{\gamma+N/2}c_k}\, e^{-(|x|^2+|y|^2)/4t}\, E_k\Big(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\Big). \tag{2.20}$$

Thanks to Corollary 2.31, $\Gamma_k$ is strictly positive. Moreover, $y \mapsto \Gamma_k(t,x,y)$ belongs to $\mathcal{S}(\mathbb{R}^N)$ for fixed $x$ and $t$. We collect some further fundamental properties of this kernel which are all more or less straightforward.

**2.51 Lemma.** *The heat kernel $\Gamma_k$ has the following properties:*

(1) $\Gamma_k(t,x,y) \;=\; c_k^{-2}\displaystyle\int_{\mathbb{R}^N} e^{-t|\xi|^2}\, E_k(ix,\xi)\, E_k(-iy,\xi)\, w_k(\xi)d\xi\,.$

(2) $\displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,y)\, w_k(y)dy \;=\; 1.$

(3) $\Gamma_k(t,x,y) \;\le\; \dfrac{1}{(2t)^{\gamma+N/2}c_k}\, \max\limits_{w\in W} e^{-|wx-y|^2/4t}\,.$

(4) $\Gamma_k(t+s,x,y) \;=\; \displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,z)\, \Gamma_k(s,y,z)\, w_k(z)dz.$

(5) *For fixed $y \in \mathbb{R}^N$, the function $u(x,t) := \Gamma_k(t,x,y)$ solves the generalized heat equation $\Delta_k u = \partial_t u$ on $\mathbb{R}^N \times (0,\infty)$.*

*Proof.* (1) is clear from the definition of generalized translations. For details concerning (2) see [R2]. (3) follows from our estimates on $E_k$, (4) is obtained by inserting (1) for one of the kernels in the integral, and (5) is immediate from (1). □

**2.52 Theorem.** (1) *The heat semigroup* $(H_t)$ *on* $C_0(\mathbb{R}^N)$ *is given explicitly by*

$$H_t f(x) = \int_{\mathbb{R}^N} \Gamma_k(t,x,y) f(y)\, w_k(y) dy \quad \text{for } t > 0.$$

(2) *The heat kernel* $\Gamma_k$ *is strictly positive on* $\mathbb{R}^N \times \mathbb{R}^N \times (0,\infty)$.

(3) *For* $f \in C_0(\mathbb{R}^N)$, *the function* $u(x,t) = H_t f(x)$ *solves initial value problem (IVP).*

*Proof.* Define $u_f$ on $\mathbb{R}^N \times [0,\infty)$ by

$$u_f(x,t) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} \Gamma_k(t,x,y) f(y)\, w_k(y) dy & \text{if } t > 0, \\ f(x) & \text{if } t = 0. \end{cases}$$

¿From part (5) of Lemma 2.51 it is immediate that $u_f$ solves the generalized heat equation on $\mathbb{R}^N \times (0,\infty)$. We proceed further in several steps.

**Step 1.** Consider first initial data $f \in \mathcal{S}(\mathbb{R}^N)$. By Lemma 2.51 (1) and Fubini's theorem, we obtain that for $t > 0$,

$$u_f(x,t) \;=\; c_k^{-1} \int_{\mathbb{R}^N} e^{-t|\xi|^2} \widehat{f}^k(\xi)\, E_k(ix,\xi)\, w_k(\xi) d\xi. \tag{2.21}$$

In view of the inversion theorem for the Dunkl transform, this identity also extends to $t = 0$. This shows that $x \mapsto u_f(x,t) \in \mathcal{S}(\mathbb{R}^N)$ for all $t > 0$ and that $\|u_f(\,.\,,t) - f\|_\infty \to 0$ as $t \downarrow 0$. Thus if $f \in \mathcal{S}(\mathbb{R}^N)$, then $u_f$ solves (IVP). Next, we prove that $u_f(x,t) = H_t f(x)$ for all $x \in \mathbb{R}^N$ and $t \geq 0$. For this, recall from semigroup theory that the function $t \mapsto H_t f$ is the unique solution of the abstract Cauchy problem

$$\begin{cases} \dfrac{d}{dt} u(t) = \overline{\Delta}_k u(t) \quad \text{for } t > 0, \\ u(0) = f \end{cases}$$

within the class of all (strongly) continuously differentiable functions $u$ on $[0,\infty)$ with values in $(C_0(\mathbb{R}^N), \|.\|_\infty)$. On the other hand, if $f \in \mathcal{S}(\mathbb{R}^N)$, then also $u_f(\,.\,,t) \in \mathcal{S}(\mathbb{R}^N)$ and from formula (2.21) it is easily deduced that $t \mapsto u_f(\,.\,,t)$ also solves the abstract Cauchy problem. This proves parts (1) and (3) for initial data from $\mathcal{S}(\mathbb{R}^N)$.

**Step 2.** Part (2), which is of course also an immediate consequence of the positive integral representation for the Dunkl kernel, can be directly deduced from the positivity of the heat semigroup $(H_t)_{t\geq 0}$ on $\mathcal{S}(\mathbb{R}^N)$, which gives $\Gamma_k \geq 0$. Strict positivity then follows from formula (4) of Lemma 2.51.

**Step 3.** Consider now general initial data $f \in C_0(\mathbb{R}^N)$. By the density of $\mathcal{S}(\mathbb{R}^N)$ in $C_0(\mathbb{R}^N)$ and the positivity of $\Gamma_k$, a standard approximation arguments yield that $H_t f(x) = u_f(x,t)$ for all $f \in C_0(\mathbb{R}^N)$ (and $x \in \mathbb{R}^N, t \geq 0$.) By the strong continuity of the semigroup $(H_t)$ it further follows that $u_f$ solves (IVP). This finishes the proof of the Theorem.

□

The heat operators $(H_t)_{t\geq 0}$ naturally extend to functions $f \in C(\mathbb{R}^N)$ which satisfy the subexponential growth condition

$$\forall \epsilon > 0 \; \exists C_\epsilon > 0 : \; |f(x)| \leq \; C_\epsilon \cdot e^{\epsilon |x|^2},$$

by

$$H_t f(x) := \int_{\mathbb{R}^N} \Gamma_k(t,x,y) f(y) w_k(y) dy \quad \text{for } \; t > 0.$$

This growth condition is in particular satisfied by functions from $C_b(\mathbb{R}^N)$ and by polynomials. Based on the above results, it is checked by standard arguments that for general initial data $f \in C_b(\mathbb{R}^N)$, (IVP) is solved by $u(x,t) = H_t f(x)$. Uniqueness of the solution within classes of functions satisfying suitable exponential growth conditions is established by means of a maximum principle, just as with the classical heat equation. For details on this, the interested reader is referred to [R2].

We conclude this section with a useful observation which will be important later on:

**2.53 Proposition.** *Let $p \in \mathcal{P}$ be a polynomial. Then*

$$H_t p = e^{t\Delta_k} p.$$

*Moreover, the function $u(x,t) = e^{t\Delta_k} p(x)$ is a polynomial solution of the initial value problem (IVP) with initial data $p$.*

The polynomial $H_t p$ is of the same degree as $p$. It is called the *heat polynomial* associated with $p$.

For the proof of the proposition, the following scaling lemma is needed.

**2.54 Lemma.** *Let $p \in \mathcal{P}_n$. Then for $c \in \mathbb{C}$ and $a \in \mathbb{C} \setminus \{0\}$,*

$$\big(e^{c\Delta_k} p\big)(ax) = \; a^n \big(e^{a^{-2} c \Delta_k}\big) p(x).$$

*Proof.* This is easily checked in terms of the exponential series for $e^{c\Delta_k}$. □

*Proof of Proposition 2.53.* Part (1) of Proposition 2.34 can be written as

$$p(x) = \int_{\mathbb{R}^N} \Gamma_k\big(1/2, x, y\big) e^{-\Delta_k/2} p(y) \, w_k(y) dy \quad \text{for } \; p \in \mathcal{P}.$$

Replacing $p$ by $e^{\Delta_k/2} p$, one obtains the assertion for $t = 1/2$. The general case $t > 0$ follows by rescaling. □

## 2.7 Calogero-Moser-Sutherland models and generalized Hermite polynomials

Quantum Calogero-Moser-Sutherland (CMS) models describe quantum mechanical systems of identical particles on a circle or line which interact pairwise through potentials of inverse square type. They have gained considerable interest in mathematical physics due to their quantum-integrability. Among the broad literature in this area, we refer to [BF1]-[BF3], [He1], [HS], [vD], [K], [LV], [Pa], as well as the monograph [vDV].

The Schrödinger operator of a CMS model for $N$ particles on a line or circle is given by

$$\mathcal{H} = -\Delta + g \sum_{1\leq i<j\leq N} \frac{1}{d(x_i,x_j)^2}$$

where $g \geq -1/2$ is a coupling constant, $x_i$ is the position of particle $i$ and $d(x,y) = |x-y|$ for the linear model, while $d(x,y) = \frac{1}{\pi}\sin\bigl(\pi(x-y)\bigr)$ for a model on a circle with circumference $1$. The models on a line, initially studied by Calogero in [Ca], are closely related to rational Dunkl operators of type $A_{N-1}$, while those one a circle, going back to Sutherland [Su], are related to the trigonometric Dunkl operators of Heckman and Opdam. In order to obtain a discrete spectrum in the linear case, one has to add some external potential, the most common one being of the form $\omega^2|x|^2$ with $\omega > 0$ (harmonic confinement). Dealing with identical particles, one considers the linear CMS operator in the so-called bosonic state space

$$B = \{f \in L^2(\mathbb{R}^N) : \sigma_{ij} f = f \quad \forall\, i,j\}$$

where $\sigma_{ij}$ permutes the coordinates $x_i$ and $x_j$. After explicit spectral resolutions of CMS models had already been obtained by Calogero and Sutherland, Moser [Mo] proved complete integrability of the associated classical Hamiltonian systems. But the deeper algebraic structure of the quantum CMS models became clear only in the nineteen-nineties by independent work of [Po] and [He1]. For the free linear model, the basic idea is to consider the modification

$$\widetilde{\mathcal{H}} := -\Delta + 2k\sum_{i<j}\frac{1}{(x_i-x_j)^2}(k\cdot id - \sigma_{ij})$$

acting in $L^2(\mathbb{R}^N)$. When $g = 2k(k-1)$, then $\widetilde{\mathcal{H}}|_B = \mathcal{H}$. A short calculation, using results from [D2], gives

$$w_k^{-1/2}\widetilde{\mathcal{H}}w_k^{1/2} \;=\; -\Delta_k^S$$

where $\Delta_k^S$ denotes the Dunkl Laplacian associated with the symmetric group $S_N$, c.f. Example 2.10(2), and $w_k$ is the weight function of type $S_N$,

$$w_k(x) = \prod_{i<j}|x_i - x_j|^{2k}.$$

Now consider the algebra $\mathcal{P}^{S_N}$ of $S_N$-invariant polynomials on $\mathbb{R}^N$. It is generated by the $N$ elementary symmetric polynomials

$$s_j(x) = \sum_{1\leq i_1<\ldots<i_j\leq N} x_{i_1}\cdot\ldots\cdot x_{i_j}, \quad j = 1,\ldots,N.$$

Let $T$ stand for the Dunkl operators of type $A_{N-1}$ with multiplicity $k$. Then

$$\mathcal{A} := \{\operatorname{Res} p(T) : p \in \mathcal{P}^{S_N}\}$$

is a commutative algebra of differential operators on $\mathcal{P}^{S_N}$ containing the operator

$$\operatorname{Res}(\Delta_k^S) \;=\; -w_k^{-1/2}\mathcal{H}w_k^{1/2},$$

c.f. Section 2.2. Up to conjugation with $w_k^{1/2}$, $\mathcal{A}$ is the so-called algebra of *quantum integrals* for the CMS operator $\mathcal{H}$. It is generated by the $N$ algebraically independent elements $\mathrm{Res}(s_j(T))$, $j = 1, \ldots, N$.

There exist obvious generalizations of the classical CMS models in the context of abstract root systems: Suppose $R$ is an arbitrary (reduced) root system in $\mathbb{R}^N$ and $k$ a nonnegative multiplicity function, then the corresponding abstract Calogero Hamiltonian is given by

$$\mathcal{H} \,=\, -\Delta + 2\sum_{\alpha\in R_+} k_\alpha(k_\alpha - 1)\frac{1}{\langle\alpha, x\rangle^2}.$$

For the classical root sytems, Olshanetsky and Perelomov proved quantum integrability of this model, following the method of Moser via Lax pairs. Again, we consider a modification involving reflection terms:

$$\widetilde{\mathcal{H}} \,=\, -\Delta + 2\sum_{\alpha\in R_+} \frac{k_\alpha}{\langle\alpha, x\rangle^2}\,(k_\alpha - \sigma_\alpha)\,. \tag{2.22}$$

In this case,

$$w_k^{-1/2}\widetilde{\mathcal{H}}\,w_k^{1/2} \,=\, -\Delta_k\,,$$

see [R4]. The quantum integrals for $\mathcal{H}$ are constructed just as in the $S_N$ case. According to a classical theorem of Chevalley (see e.g. [Hu]), the algebra $\mathcal{P}^W$ of $W$-invariant polynomials is again generated by $N$ homogeneous, algebraically independent elements, providing a basis of quantum integrals in this case.

Let us now turn to the spectral analysis of abstract linear CMS operators with harmonic potential $\omega^2|x|^2$. We follow [R2], [R5] and use the normalization $\omega = 1/2$. (Other normalizations lead to results which are equivalent up to scaling). We work with the gauge-transformed version with reflection terms,

$$H_k := -\Delta_k + \frac{1}{4}|x|^2.$$

Due to the anti-symmetry of the first order Dunkl operators (Prop. 2.13), this operator is symmetric and densely defined in $L^2(\mathbb{R}^N, w_k)$ with domain $\mathcal{S}(\mathbb{R}^N)$. Note that in case $k = 0$, $H_k$ is just the Hamiltonian of the $N$-dimensional isotropic harmonic oscillator.

The next theorem contains a complete description of the spectral properties of $\mathcal{H}_k$ and generalizes well-known facts for the classical harmonic oscillator.

**2.55 Theorem.** *(Spectral Theorem for $H_k$) $L^2(\mathbb{R}^N, w_k)$ decomposes as an orthogonal Hilbert space sum according to*

$$L^2(\mathbb{R}^N, w_k) \,=\, \bigoplus_{n\in\mathbb{Z}_+} V_n$$

*where*

$$V_n := \{e^{-\Delta_k/2}p : p \in \mathcal{P}_n\} \subset \mathcal{S}(\mathbb{R}^N)$$

*is the eigenspace of $H_k$ corresponding to the eigenvalue $n+\gamma+N/2$. In particular, $H_k$ is essentially self-adjoint. The spectrum of its closure is purely discrete and given by*

$$\sigma(\overline{\mathcal{H}_k}) = \{n + \gamma + N/2, n \in \mathbb{Z}_+\}.$$

For details on the proof, the reader is referred to [R2] or [R7]. It relies on the $sl(2)$-commutation relations of the operators

$$E := \frac{1}{2}|x|^2, \;\; F := -\frac{1}{2}\Delta_k \quad \text{and } H := \sum_{i=1}^N x_i\partial_i + (\gamma + N/2)$$

observed by Heckman [He1], namely

$$\big[H, E\big] = 2E, \;\; \big[H, F\big] = -2F, \;\; \big[E, F\big] = H.$$

The first two relations are immediate from the fact that the Euler operator

$$\rho := \sum_{i=1}^N x_i\partial_i \tag{2.23}$$

satisfies $\rho(p) = np$ for each homogeneous $p \in \mathcal{P}_n$.

The eigenvalues of the CMS Hamiltonian $\mathcal{H}_k$ are highly degenerate if $N > 1$. We are now going to construct natural orthogonal bases for them. They are made up by generalizations of the classical $N$-variable Hermite polynomials and Hermite functions to the Dunkl setting. The starting point for our construction is the Macdonald-type identity: if $p, q \in \mathcal{P}$, then

$$[p, q]_k \;=\; \frac{1}{c_k}\int_{\mathbb{R}^N} e^{-\Delta_k/2}p(x)e^{-\Delta_k/2}q(x)e^{-|x|^2/2}w_k(x)dx. \tag{2.24}$$

Notice that $[\,.\,,\,.\,]_k$ is a scalar product on the $\mathbb{R}$- vector space $\mathcal{P}_\mathbb{R}$ of polynomials with real coefficients. Let $\{\varphi_\nu\,, \nu \in \mathbb{Z}_+^N\}$ be an orthonormal basis of $\mathcal{P}_\mathbb{R}$ with respect to $[.\,,.]_k$ such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$. Write $\mathcal{P}_{n,\mathbb{R}} = \mathcal{P}_n \cap \mathcal{P}_\mathbb{R}$. As homogeneous polynomials of different (total) degrees are orthogonal, the $\varphi_\nu$ with fixed $|\nu| = n$ can for example be constructed by Gram-Schmidt orthogonalization within $\mathcal{P}_{n,\mathbb{R}}$ from an arbitrary ordered real-coefficient basis. If $k = 0$, the canonical choice of the basis $\{\varphi_\nu\}$ is just $\varphi_\nu(x) := (\nu!)^{-1/2}x^\nu$.

**2.56 Definition.** The generalized Hermite polynomials $\{H_\nu\}$ and Hermite functions $\{h_\nu\}$ ($\nu \in \mathbb{Z}_+^N$) associated with the basis $\{\varphi_\nu\}$ of $\mathcal{P}_\mathbb{R}$ are defined by

$$H_\nu(x) := e^{-\Delta_k/2}\varphi_\nu(x); \quad h_\nu(x) := e^{-|x|^2/4}H_\nu(x).$$

Observe that $H_\nu$ is a polynomial of degree $|\nu|$. By the Macdonald identity (2.24), the Hermite functions $h_\nu$ form an orthogonal basis of $L^2(\mathbb{R}^N, w_k)$.

For $k = 0$ and the choice $\varphi_\nu(x) = (\nu!)^{-1/2}x^\nu$, one obtains the classical multivariable Hermite polynomials

$$H_\nu(x) \;=\; \frac{1}{\sqrt{\nu!}}\prod_{i=1}^N e^{-\partial_i^2/2}(x_i^{\nu_i}) \;=\; \frac{2^{-|\nu|/2}}{\sqrt{\nu!}}\prod_{i=1}^N \widehat{H}_{\nu_i}(x_i/\sqrt{2})$$

where the $\widehat{H}_n$ denote the classical one-variable Hermite polynomials

$$\widehat{H}_n(x) \;=\; (-1)^n\, e^{x^2}\,\frac{d^n}{dx^n}e^{-x^2}\,.$$

More interesting examples are the following:

**2.57 Examples.** (1) *The one-dimensional case.* Up to sign changes, there exists only one orthonormal basis with respect to $[\,.\,,\,.\,]_k$. The associated generalized Hermite polynomials $(H_n^k)_{n\in\mathbb{Z}_+}$ are orthogonal with respect to the weight $|x|^{2k}e^{-|x|^2}$ on $\mathbb{R}$. They can be found in [Chi] and were further studied in [Ros]. We mention that they can be written as

$$\begin{cases} H_{2n}^k(x) = (-1)^n 2^{2n} n!\, L_n^{k-1/2}(x^2), \\ H_{2n+1}^k(x) = (-1)^n 2^{2n+1} n!\, x L_n^{k+1/2}(x^2); \end{cases}$$

where the $L_n^\alpha$ are the usual Laguerre polynomials

$$L_n^\alpha(x) = \frac{1}{n!} x^{-\alpha} e^x \, \frac{d^n}{dx^n}\Big(x^{n+\alpha}e^{-x}\Big).$$

(2) *The $A_{N-1}$-case.* There exists a natural orthogonal system $\{\varphi_\nu\}$ made up by the so-called *non-symmetric Jack polynomials* $\{E_\nu = E_\nu^k,\, \nu \in \mathbb{Z}_+^N\}$. They were introduced in [O2] for arbitrary root systems (see also [KS]), and are characterized by the following conditions:

(i) $E_\nu(x) = x^\nu + \sum_{\mu <_P \nu} c_{\nu,\mu} x^\mu$ with $c_{\nu,\mu} \in \mathbb{R}$;

(ii) For all $\mu <_P \nu$, $\big(E_\nu(x), x^\mu\big)_k = 0$

Here $<_P$ is a dominance order defined within multi-indices of equal total length (see [O2]), and the inner product $(.,.)_k$ on $\mathcal{P}_\mathbb{R}$ is given by

$$(f,g)_k := \int_{\mathbb{T}^N} f(z)g(\overline{z}) \prod_{i<j} |z_i - z_j|^{2k} dz$$

with $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $dz$ being the Haar measure on $\mathbb{T}^N$. If $f, g$ have different degrees, then $(f,g)_k = 0$. The set $\{E_\nu,\, |\nu| = n\}$ forms a vector space basis of $\mathcal{P}_{n,\mathbb{R}}$. It can be shown (by use of $A_{N-1}$-type Cherednik operators) that the Jack polynomials $E_\nu$ are also orthogonal with respect to the Dunkl pairing $[\,.\,,\,.\,]_k$; for details see [R2]. The corresponding generalized Hermite polynomials and their symmetric counterparts have been studied in [La1], [La2], [vD], and in [BF1] - [BF3].

As an immediate consequence of Theorem 2.55 we obtain analogues of the classical second order differential equations for generalized Hermite polynomials and Hermite functions:

**2.58 Corollary.** (i) $\big(-\Delta_k + \sum_{i=1}^N x_i\partial_i\big)H_\nu = |\nu| H_\nu$.

(ii) $\big(-\Delta_k + \frac{1}{4}|x|^2\big)h_\nu = (|\nu| + \gamma + N/2)h_\nu$.

Various further useful properties of the classical Hermite polynomials and Hermite functions have extensions to our general setting. We conclude this section with a list of them. The proofs can be found in [R2]. For further results on generalized Hermite polynomials, one can also see for instance [vD].

**2.59 Theorem.** *Let $\{H_\nu\}$ be the Hermite polynomials and Hermite functions associated with the basis $\{\varphi_\nu\}$ on $\mathbb{R}^N$ and let $x, y \in \mathbb{R}^N$. Then*

(1) $H_\nu(x) \,=\, (-1)^{|\nu|} e^{|x|^2/2} \varphi_\nu(T) e^{-|x|^2/2}$ *(Rodrigues-Formula)*

(2) $e^{-|y|^2/2} E_k(x,y) \,=\, \sum_{\nu\in\mathbb{Z}_+^N} H_\nu(x)\varphi_\nu(y)$ *(Generating relation)*

(3) *(Mehler formula) For* $r \in \mathbb{C}$ *with* $|r| < 1$,

$$\sum_{\nu\in\mathbb{Z}_+^N} H_\nu(x)H_\nu(y) = \frac{1}{(1-r^2)^{\gamma+N/2}} \exp\left\{-\frac{r^2(|x|^2+|y|^2)}{2(1-r^2)}\right\} E_k\Big(\frac{rx}{1-r^2},\, y\Big).$$

*The sums in (2) and (3) are absolutely convergent.*

The Dunkl kernel $E_k$ in (2) and (3) replaces the usual exponential function. It comes in via the following relation with the (arbitrary!) basis $\{\varphi_\nu\}$:

$$E_k(x,y) \,=\, \sum_{\nu\in\mathbb{Z}_+^N} \varphi_\nu(x)\varphi_\nu(y).$$

**2.60 Proposition.** *The generalized Hermite functions* $\{h_\nu\,,\, \nu \in \mathbb{Z}_+^N\}$ *are an orthogonal basis of eigenfunctions for the Dunkl transform on* $L^2(\mathbb{R}^N, w_k)$ *with*

$$h_\nu^{\wedge k} \,=\, (-i)^{|\nu|} h_\nu.$$

## 2.8 Generalized translation and spherical means

We recall the definition of the generalized translation on the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ introduced by formula (2.25) of the previous section:

$$\tau_y f(x) := \frac{1}{c_k}\int_{\mathbb{R}^N} \widehat{f}^{\,k}(\xi)\, E_k(ix,\xi)E_k(iy,\xi)\, w_k(\xi)d\xi; \quad y\in\mathbb{R}^N. \tag{2.25}$$

In addition to the properties already mentioned, we state the following relation which follows from (2.19) and the Plancherel theorem for the Dunkl transform: For all $f,g\in\mathcal{S}(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} f(x *_k y)g(y)w_k(y)dy \,=\, \int_{\mathbb{R}^N} f(y)g(-x *_k y)w_k(y)dy. \tag{2.26}$$

In [T2], this translation was extended to $C^\infty(\mathbb{R}^N)$ via

$$\tau_y f(x) := V_k^x V_k^y (V_k^{-1}f)(x+y)$$

where the superscripts denote the relevant variable. Indeed, it is shown in [T2] that both definitions coincide on $\mathcal{S}(\mathbb{R}^N)$. Note that by Theorem 2.42 $\tau_y$ is continuous with respect to the usual Fréchet space topology, and that

$$\tau_0 f = f, \quad T_\xi\tau_y f = \tau_y T_\xi f \quad \text{and } \tau_y f(x) = \tau_x f(y) \quad \forall\, x,y\in\mathbb{R}^N.$$

We shall frequently use the more suggestive notion

$$f(x *_k y) = \tau_y f(x).$$

For $k = 0$, one obtains the usual group translation: $f(x *_0 y) = f(x+y)$. It is also immediate from the definition that

$$E_k(x *_k y, z) = E_k(x,z)E_k(y,z) \quad \forall z \in \mathbb{C}^N. \tag{2.27}$$

By the Plancherel theorem and the fact that $|E_k(ix,\xi)| \leq 1$ for all $x, \xi \in \mathbb{R}^N$, the translation operator $\tau_y$ extends to a continuous linear operator on $L^2(\mathbb{R}^N, w_k)$ with $\|\tau_y\| \leq 1$. It is, however, an open question in general whether $\tau_y$ is also bounded as a linear operator on the spaces $L^p(\mathbb{R}^N, w_k)$ with $1 \leq p < \infty$, $p \neq 2$. Only in rank one, this is known to be true so far. Let us briefly describe the situation in this case.

**The rank-one case.** Recall the explicit formula (2.14) for $E_k$ in terms of one-variable Bessel functions $j_\alpha$ in this case. It is well-known (see e.g. [BH], 3.5.61) that the $j_\alpha$ with $\alpha \geq -1/2$ satisfy the product formula

$$j_\alpha(xz)j_\alpha(yz) = \int_0^\infty j_\alpha(\xi z) m_\alpha(x,y,z) z^{2\alpha+1} dz \tag{2.28}$$

with the kernel

$$m_\alpha(x,y,z) = \frac{\Gamma(\alpha+1)}{\Gamma(1/2)\Gamma(\alpha+1/2)2^{2\alpha-1}} \cdot \frac{[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\alpha-1/2}}{(xyz)^{2\alpha}}.$$

We remark that formula (2.28) induces a convolution of point measures on $[0,\infty)$ according to

$$d(\delta_x *_\alpha \delta_y)(z) := m_\alpha(x,y,z)z^{2\alpha+1}dz.$$

This definition naturally extends to a weakly continuous, probability-preserving convolution on the space $M_b([0,\infty))$ of regular bounded Borel measures on $[0,\infty)$. This convolution induces the structure of a so-called commutative *hypergroup* on $[0,\infty)$ which is called the Bessel-Kingman hypergroup of index $\alpha$. There will be more on hypergroups and this important example in Section 3.

The Dunkl kernel $E_k$ itself satisfies a similar product formula which was proven in [R1], namely

$$E_k(x,z)E_k(y,z) = \int_{\mathbb{R}} E_k(\xi,z) d\mu_{x,y}^k(\xi) \quad \forall z \in \mathbb{C}$$

with the measures

$$d\mu_{x,y}^k(z) = m_{k-1/2}(|x|,|y|,|z|)|z|^{2k} \cdot \frac{1-\sigma_{x,,y,z}+\sigma_{z,x,y}+\sigma_{z,y,x}}{2}\, dz$$

where

$$\sigma_{z,x,t} = \begin{cases} \frac{z^2+x^2-t^2}{2zx} & \text{if } z,x \neq 0, \\ 0 & \text{else} \end{cases}.$$

Therefore the generalized translation on $\mathcal{S}(\mathbb{R})$ is given by

$$f(x *_k y) = \int_{\mathbb{R}} f(z) d\mu_{x,y}^k(z).$$

The measures $\mu_{x,y}^k$ are not positive in the generic case, but uniformly bounded with $\|\mu_{x,y}^k\| \leq 4$ for all $x, y \in \mathbb{R}$. The convolution of point measures defined by

$\delta_x *_k \delta_y = \mu_{x,y}^k$ extends to a weakly continuous convolution on $M_b(\mathbb{R})$ which is however not positivity-preserving. It induces the structure of a *signed* hypergroup on $\mathbb{R}$, see [R1]. Due to the uniform boundedness of the measures $\mu_{x,y}^k$, the generalized translation operators $\tau_y$ on the Schwartz space extend to bounded linear operators on the spaces $L^p(\mathbb{R}, |x|^{2k}dx)$ with $\|\tau_y f\|_p \leq 4\|f\|_p$ for all indices $p$ with $1 \leq p < \infty$. For details, the reader may see [R1] and the references cited there.

Even if no result of this kind is available in the general case as far, there is at least a useful partial result which states that the Dunkl-type generalized translation is positivity-preserving and $L^p$-bounded when restricted to radial functions. Moreover, there is a weakened form of positive product formula for $E_k$ available. The key for this is the positivity of the spherical mean operator in the Dunkl-setting. This operator was first considered in [MT]. It is defined for $f \in C^\infty(\mathbb{R}^N)$ by

$$S_f(x,t) := \frac{1}{d_k}\int_{S^{N-1}} f(x *_k ty) w_k(y) d\sigma(y) \quad (x \in \mathbb{R}^N,\, t \geq 0)$$

where $S^{N-1} = \{x \in \mathbb{R}^N : |x| = 1\}$ is the unit sphere in $\mathbb{R}^N$ with normalized Lebesgue surface measure $d\sigma$ and

$$d_k = \int_{S^{N-1}} w_k(\xi) d\sigma(\xi).$$

It is easily seen from the continuity properties of the generalized translation that $S_f \in C^\infty(\mathbb{R}^N \times [0,\infty))$. The following is the first main result of [R6].

**2.61 Theorem.** (1) *The spherical mean operator $f \mapsto S_f$ is positive on $C^\infty(\mathbb{R}^N)$, that is $f \geq 0$ on $\mathbb{R}^N$ implies that $S_f \geq 0$ on $\mathbb{R}^N \times [0,\infty)$.*

(2) *For each $x \in \mathbb{R}^N$ and $t \geq 0$, there exists a unique compactly supported probability measure $\sigma_{x,t}^k \in M^1(\mathbb{R}^N)$ such that for all $f \in C^\infty(\mathbb{R}^N)$,*

$$S_f(x,t) = \int_{\mathbb{R}^N} f(\xi) d\sigma_{x,t}^k(\xi).$$

*The measure $\sigma_{x,t}^k$ satisfies*

$$\operatorname{supp} \sigma_{x,t}^k \subseteq \bigcup_{w \in W} \{\xi \in \mathbb{R}^N : |\xi - wx| \leq t\},$$

*and the mapping $(x,t) \mapsto \sigma_{x,t}^k$ is weakly continuous. Moreover, $\sigma_{wx,t}^k(A) = \sigma_{x,t}^k(w^{-1}(A))$ and $\sigma_{rx,rt}^k(A) = \sigma_{x,t}^k(rA)$ for all $w \in W, r > 0$, and all Borel sets $A \subset \mathbb{R}^N$.*

The proof of this theorem involves, among other ingredients, the theory of $k$-spherical harmonics which was initiated in [D1]. A good introduction to this subject can be found in the monograph [DX]. The space of $k$-spherical harmonics of degree $n \geq 0$ is defined by

$$\mathcal{H}_n^k = \ker \Delta_k \cap \mathcal{P}_n.$$

As in the theory of ordinary spherical harmonics, the space $L^2(S^{N-1}, w_k d\sigma)$ decomposes as an orthogonal Hilbert space sum

$$L^2(S^{N-1}, w_k d\sigma) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n^k.$$

A decisive ingredient in the proof of Theorem 2.61 is the following special case of the Funk-Hecke formula for $k$-spherical harmonics which goes back to [X2].

**2.62 Proposition.** *Let $N \geq 2$ and put $\lambda := \gamma + N/2 - 1$. Then for all $Y \in \mathcal{H}_n^k$ and $x \in \mathbb{R}^N$,*

$$\frac{1}{d_k} \int_{S^{N-1}} E_k(ix, \xi) Y(\xi) w_k(\xi) d\sigma(\xi) = \frac{\Gamma(\lambda+1)}{2^n \Gamma(n+\lambda+1)} j_{n+\lambda}(|x|) Y(ix). \quad (2.29)$$

*In particular,*

$$\frac{1}{d_k} \int_{S^{N-1}} E_k(ix, \xi) w_k(\xi) d\sigma(\xi) = j_\lambda(|x|). \quad (2.30)$$

*Proof of Theorem 2.61 (Sketch).* The proof of part (1) is achieved by a reduction to initial data of the form $f(x) = \Gamma_k(s, x, y)$, where $\Gamma_k$ is the Dunkl-type heat kernel. Indeed, it is not hard to see by Dunkl transform methods that for each $f \in \mathcal{S}(\mathbb{R}^N)$ and $(x,t) \in \mathbb{R}^n \times [0, \infty)$,

$$S_f(x,t) = \lim_{s \downarrow 0} \int_{\mathbb{R}^N} M_{\Gamma_k(s,\,.\,,z)}(x,t) f(z) w_k(z) dz.$$

For the proof that $S_{\Gamma_k(s,\,.\,,z)}(x,t) > 0$ for all $s, t \geq 0$ and $x, z \in \mathbb{R}^N$, the Funk-Hecke formula (2.29), the positive integral representation of the intertwining operator and the positive product formula for the one-variable Bessel functions $j_\lambda$ are involved.

(2) The existence of representing measures $\sigma_{x,t}^k$ follows by standard methods from (1). For the statement on their support, one observes that $u(x,t) = S_f(x,t)$ solves the initial value problem

$$\begin{cases} (\Delta_k - A_\lambda^t) u = 0 \text{ in } \mathbb{R}^N \times (0, \infty) \\ u(x,0) = f(x), \ u_t(x,0) = 0 \end{cases}$$

with the Bessel operator

$$A_\lambda^t = \partial_t^2 + \frac{2\lambda+1}{t} \partial_t \,.$$

The operator $\Delta_k - A_\lambda^t$ is of Darboux-type and generalizes the classical wave operator. A study of the domain of dependence of its solutions leads to the claimed statement about the support of the $\sigma_{x,t}^k$. Hereby the influence of the reflection parts needs some care. □

For the special choice $f(x) = E_k(x, iz)$ with fixed $z \in \mathbb{R}^N$ we obtain from formulas (2.30) and (2.27) the simple form

$$S_f(x,t) = E_k(ix, y) j_\lambda(t|y|).$$

Therefore the Dunkl kernel satisfies the following radial product formula:

**2.63 Corollary.**

$$E_k(x, iz)\, j_\lambda(t|z|) \;=\; \int_{\mathbb{R}^N} E_k(\xi, iz) d\sigma_{x,t}^k(\xi) \quad (x, z \in \mathbb{R}^N, t \geq 0).$$

A function or measure on $\mathbb{R}^N$ is called *radial*, if it is invariant under the action of the orthogonal group $O(N, \mathbb{R})$. We shall denote subsets consisting of radial functions or measures by the subscript $rad$. As already announced, the Dunkl-type generalized translation is positive when restricted to radial functions. In fact, it can be written in a fairly simple explicit form by means of the intertwining operator. This is proven in [R6] (proof of Theorem 5.1 there) and can also be found in [DX] via a slightly different approach:

**2.64 Theorem.** *If $f \in C_{rad}^\infty(\mathbb{R}^N)$ with $f(x) = F(|x|) \geq 0 \;\forall\, x \in \mathbb{R}^N$, then also $\tau_y f(x) \geq 0$ for all $x, y \in \mathbb{R}^N$. The translate $\tau_y f$ is given explicitly by*

$$\tau_y f(x) = V_k\big(F(\sqrt{|x|^2 + |y|^2 + 2\langle x, \,.\,\rangle})\big)(y).$$

We mention that in [DX], this formula is established on the space $A_k(\mathbb{R}^N) = \{f \in L^1(\mathbb{R}^N, w_k) : \widehat{f}^{\,k} \in L^1(\mathbb{R}^N, w_k)\}$, which is a subspace of $L^2(\mathbb{R}^N, w_k)$. Let us now turn to the translation of radial measures.

**2.65 Definition.** For a subset $\mathcal{M} \subset M_b(\mathbb{R}^N)$ we call

$$\mathcal{M}_{prad} := \{\mu \in \mathcal{M} : \frac{1}{w_k}\,\mu \text{ is radial}\},$$

the space of *pseudo-radial* measures from $\mathcal{M}$.

Important examples of bounded pseudo-radial measures, i.e. measures belonging to $M_{b,prad}(\mathbb{R}^N)$, are those of the form $d\mu(x) = f(x) w_k(x) dx$ with $f \in L^1_{rad}(\mathbb{R}^N, w_k)$.

**2.66 Proposition.** *1. For $\mu \in M^1_{prad}(\mathbb{R}^N)$, the assignment*

$$\delta_x *_k \mu(\varphi) := \int_{\mathbb{R}^N} \varphi(x *_k y) d\mu(y), \quad \varphi \in \mathcal{S}(\mathbb{R}^N)$$

*defines a probability measure $\delta_x *_k \mu \in M^1(\mathbb{R}^N)$. Moreover, $*_k$ extends to a probability-preserving and bilinear convolution*

$$*_k : M_b(\mathbb{R}^N) \times M_{b,prad}(\mathbb{R}^N) \to M_b(\mathbb{R}^N),$$
$$\mu *_k \nu(\varphi) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \varphi(x *_k y) d\mu(x) d\nu(y).$$

*2. If $\mu, \nu \in M_{b,rad}(\mathbb{R}^N)$, then also $\mu *_k \nu \in M_{b,rad}(\mathbb{R}^N)$, and the convolution $*_k$ on $M_{b,rad}(\mathbb{R}^N)$ is commutative.*

For the proof of these results, see [R6] (the extension away from probability measures to bounded measures is straightforward).

In [TX], the authors extend the generalized translation $\tau_y$ to a positivity-preserving and bounded linear operator

$$\tau_y : L^p_{rad}(\mathbb{R}^N, w_k) \to L^p(\mathbb{R}^N, w_k)$$

for $1 \leq p \leq 2$. This allows further to define the convolution between functions between $L^p$-spaces such as $f *_k g$ for bounded $g \in L^1_{rad}(\mathbb{R}^N, w_k)$ and $f \in L^p(\mathbb{R}^N, w_k)$ with $1 \leq p \leq 2$. See [TX] for details and applications like the maximal operator in the Dunkl setting.

# 3 Gelfand pairs, Bessel functions, and hypergroups on matrix cones

## 3.1 Motivation

A basic motivation for the study of Dunkl operators is their close relation with the theory of Riemannian symmetric spaces and their spherical functions. Indeed, the algebra of invariant differential operators of a Riemannian symmetric space $G/K$ can be expressed in terms of the Weyl-group invariant parts of Dunkl or Cherednik operator algebras. Hereby symmetric spaces of the non-compact and compact type lead to Dunkl-Cheredenik operators, while rational Dunkl operators correspond to symmetric spaces of Euclidean type. The latter are associated with Cartan motion groups of non-compact symmetric spaces $G/K$. They are of the form $(K \ltimes V)/K$, where $V$ is a Euclidean space of finite dimension which can be identified with the tangent space of $G/K$ in the trivial coset $eK$, and $K$ is a compact Lie group acting on $V$ by orthogonal transformations. If $G/K$ is a Riemannian symmetric space of arbitrary type, then $(G,K)$ is always a *Gelfand pair*, which means that the subalgebra of $K$-biinvariant functions within the convolution algebra $L^1(G)$ is commutative.

Symmetric spaces provide many infinite discrete series of Gelfand pairs $(G,K)$ whose double coset spaces $G//K := K \setminus G/K$ can be identified with a common locally compact space $X$. For symmetric spaces of rank one (where $X$ is one-dimensional) it has been well-known since a long time that their spherical functions can be embedded into certain series of classical hypergeometric functions. These are well defined for continuous parameters, but only for certain discrete parameter values they have an interpretation as spherical functions. Nevertheless, many nice properties known for spherical functions, like integral representations and positive product formulas, extend to general parameter values, and one still obtains associated commutative convolution algebras of bounded Borel measures on $X$ which have properties similar to group convolutions and form so-called commutative hypergroups $(X,*)$. In this way, commutative hypergroups extend the theory of spherical functions in some respect, and the theory of commutative hypergroups may help to understand results for families of special functions with a continuous parameter range, which admit positive product formulas, in a systematic way.

For symmetric spaces of higher rank, the theories of Dunkl- and Cherednik operators provide a framework of hypergeometric functions in several variables which again includes the spherical functions as particular cases. In the Euclidean case, these are just Bessel functions of Dunkl type. They can be identified with spherical functions for discrete parameter values which are determined by the Cartan decomposition of the underlying Lie algebra.

One may conjecture that for arbitrary non-negative multiplicities always positive integral representations and product formulas leading to hypergroup convolutions exist. These questions are in general much harder than in rank one and remain unsolved to a major extent. There are, however, some classes of known examples which allow the extension of product formulas and commutative hypergroup algebras beyond the geometric cases. They are related with Grassmann manifolds $U(p,q;\mathbb{F})/U_p(\mathbb{F}) \times U_q(\mathbb{F})$ over one of the skew-fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and will be presented in Section 3.4. For these classes, hypergroup analysis supplements the Weyl-group invariant Dunkl theory in a satisfactory way.

The basic guiding example, which corresponds to the rank-one case of the above series of Grassmann manifolds, are the so-called Bessel-Kingman hypergroups on $X = [0,\infty)$. Their structure is precisely that of rank-one Dunkl theory in the Weyl-group invariant case. Here one takes the groups $K = O(d)$ acting on $\mathbb{R}^d$, and the spherical functions $\varphi_\lambda$ can be expressed in terms of the spherical Bessel functions $j_\alpha(x) = {}_0F_1(\alpha+1; -x^2/4)$, c.f. (2.15), with index $\alpha = d/2 - 1$, via $\varphi_\lambda(x) = j_\alpha(\lambda x)$, $\lambda \in \mathbb{C}$. The spherical product formula reads

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)} \int_{-1}^{1} \varphi_\lambda\big(\sqrt{r^2+s^2+2rst}\big)\,(1-t^2)^{\alpha-1/2}\,dt \quad (3.1)$$

for $\alpha = d/2 - 1 > -1/2$. The case $\alpha = -1/2$ is degenerate with

$$\varphi_\lambda(x)\varphi_\lambda(y) = \frac{1}{2}(\varphi_\lambda(|x-y|) + \varphi_\lambda(x+y)).$$

We shall explain this in Examples 3.9 and 3.16. In fact, this product formula extends by analytic continuation to Bessel functions of arbitrary real index $\alpha > -1/2$. It is known as Gegenbauer's product formula for Bessel functions, and (2.28) is just equivalent to (3.1). This product formula leads to a continuous family of commutative hypergroup structures on $[0,\infty)$ with parameter $\alpha \geq -1/2$, called *Bessel-Kingman hypergroups.* The associated hypergroup Fourier transforms are Hankel transforms which describe for half-integers $\alpha = d/2 - 1$ just the radial Fourier transforms on $\mathbb{R}^d$. One may thus for example investigate radial random walks on $\mathbb{R}^d$ by considering their radial parts which form random walks on Bessel-Kingman hypergroups, see [Ki].

This classical one-dimensional example was recently extended to Bessel-type hypergroups on the cones $\Pi_q = \Pi_q(\mathbb{F})$ of positive semidefinite matrices over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ in [R8] as follows: For $p \geq q$, consider the matrix space $M_{p,q} = M_{p,q}(\mathbb{F})$ with inner product $\langle x, y\rangle = \text{Re tr}(x^*y)$, where $x^* = \overline{x}^t$. The unitary group $U_p = U_p(\mathbb{F})$ acts by multiplication from the left, and these transformations are orthogonal with respect to the given inner product. The mapping $U_p x \mapsto \sqrt{x^*x}$ then establishes a homeomorphism between the space of $U_p$-orbits in $M_{p,q}$ and the cone $\Pi_q$. The spherical functions of the Gelfand pair $(U_p \ltimes M_{p,q}, U_p)$ are given by Bessel functions of matrix argument (see [FK], [Her] as standard references), and the Fourier transform of $U_p$-invariant functions on $M_{p,q}$ can be expressed in terms of a Hankel transform on the cone $\Pi_q$. These discrete series of examples can be embedded into three continuous series of matrix Bessel hypergroups on $\Pi_q$. This completes the $L^2$-theory for Hankel transforms on $\Pi_q$ as developed by [Her] and [FT] in the general framework of symmetric cones. In a second step, the connection to Dunkl theory is established, as follows: The group $U_q$ acts as a compact group of hypergroup automorphisms on $\Pi_q$ by conjugation, and the associated orbit space is canonically parametrized by

$$\Xi_q = \{\xi \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0\},$$

the set of possible spectra of positive semidefinite matrices. This is a Weyl chamber of type $B_q$. This gives three continuous series of commutative hypergroups on $\Xi_q$ whose hypergroup characters are Dunkl-type Bessel functions of type $B_q$. They interpolate the convolutions of the Gelfand pairs

$((U_p \times U_q) \ltimes M_{p,q}, U_p \times U_q)$, and one obtains explicit positive product formulas and hence commutative hypergroup structures on $\Xi_q$ for these parameters.

In order to explain these connections and their stochastic implications more precisely, we next give a short survey about Gelfand pairs of Euclidean type and commutative hypergroups.

## 3.2 Gelfand pairs, Euclidean orbit spaces, and hypergroups

In this section we give a quick introduction to certain classes of Gelfand pairs and commutative hypergroups. We do not include proofs and refer for details on Gelfand pairs to the survey of [F], and for hypergroups mainly to [J] and [BH]. We first recapitulate some notions on Gelfand pairs. Let $G$ be a locally compact group and $M_b(G)$ the Banach space of all regular bounded Borel measures on $G$. $M_b(G)$ becomes a Banach-$*$-algebra with convolution

$$\mu * \nu(f) := \int_G f(xy)\, d\mu(x)\, d\nu(y)$$

for $f \in C_b(G)$ and the involution $\mu^*(A) := \overline{\mu(A^{-1})}$ ($A \subset G$ a Borel set). The identity is $\delta_e$, the point measure in the identity $e$ of $G$. Recall also that there is a (up to normalization) unique left Haar measure $\omega_G \in M^+(G)$, that is a measure which is invariant under left translations $f \mapsto \tau_x f,\ x \in G$.

Now let $K$ be a compact subgroup of $G$. The normalized Haar measure $\omega_K \in M^1(K)$ may be regarded as measure on $G$ and satisfies $\omega_K^* = \omega_K * \omega_K = \omega_K$. It is easily checked that the space of all $K$-biinvariant bounded measures

$$\begin{aligned} M_b(G\|K) :&= \{\mu \in M_b(G) :\ \delta_x * \mu * \delta_y = \mu \text{ for all } x, y \in K\} \\ &= \{\omega_K * \mu * \omega_K :\ \mu \in M_b(G)\} \end{aligned} \tag{3.2}$$

is a Banach-$*$-subalgebra of $M_b(G)$.

**3.1 Definition.** The pair $(G, K)$ is be called a *Gelfand pair*, if $M_b(G\|K)$ is commutative.

We mention that for a Gelfand pair $(G, K)$, the group $G$ is automatically unimodular, i.e. $\omega_G$ is also a right Haar measure. Further, the space $L^1(G, \omega_G)$ is naturally identified with a subspace of $M_b(G)$ and becomes a closed subalgebra of $(M_b(G), *)$ with the convolution

$$f * g(x) := \int_G f(xy) g(y^{-1})\, d\omega_G(y)$$

and involution $f^*(x) := \overline{f(x^{-1})}$. If $(G, K)$ is a Gelfand pair, then the Banach-$*$-algebra

$$L^1(G\|K) := \{f \in L^1(G, \omega_G) :\ f(kxh) = f(x),\ x \in G,\ k, h \in K\}$$

of all biinvariant $L^1$-functions on $G$ is commutative.

There are several equivalent descriptions (e.g., using representation theory) of Gelfand pairs among which we quote the following useful criterion, see [F].

**3.2 Lemma.** *Let $K$ be a compact subgroup of a locally compact group $G$. Assume there exists a continuous involutive automorphism $\theta$ on $G$ satisfying $x^{-1} \in K\theta(x)K$ for all $x \in G$. Then $(G, K)$ is a Gelfand pair.*

We next introduce spherical functions and the spherical Fourier transform.

**3.3 Definition.** Let $(G,K)$ be a Gelfand pair. Then a function $\varphi \in C(G)$, $\varphi \not\equiv 0$, is called a *spherical function* of $(G,K)$ if $\varphi$ is $K$-biinvariant, i.e., $\varphi(kxh) = \varphi(x)$ for $k,h \in K$ and $x \in G$, and if $\varphi$ satisfies the product formula

$$\int_K \varphi(xky)\, d\omega_K(k) = \varphi(x)\varphi(y).$$

Spherical functions obviously satisfy $\varphi(e) = 1$. Denote the set of all spherical functions of $(G,K)$ by $\Sigma$.

The set $\Sigma_b := \Sigma \cap C_b(G)$ can be identified with the spectrum of the Banach-$*$-algebra $L^1(G\|K)$ via $\varphi \mapsto L_\varphi \in \Delta(L^1(G\|K))$ with $L_\varphi(f) = \int_G \varphi f\, d\omega_G$. It becomes a locally compact space with the topology of locally-uniform convergence, which corresponds to the Gelfand topology on the spectrum.

The spherical Fourier transforms of functions $f \in L^1(G\|K)$ and measures $\mu \in M_b(G\|K)$ are defined on $\Sigma$ according to

$$\widehat{f}(\varphi) = \int_G f(x)\varphi(x^{-1})\, d\omega_G(x) \quad \text{and} \quad \widehat{\mu}(\varphi) = \int_G \varphi(x^{-1})\, d\mu(x).$$

They satisfy $\widehat{f} \in C_0(\Sigma)$ ("Riemann-Lebesgue-Lemma") and $\widehat{\mu} \in C_b(\Sigma)$.

**3.4 Definition.** A function $\varphi \in C_b(G)$ is called *positive definite* if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in G$ and $c_1, \ldots, c_n \in \mathbb{C}$,

$$\sum_{i,j=1,\ldots,n} c_i \bar{c}_j \cdot \varphi(x_i x_j^{-1}) \geq 0.$$

We denote by $\Omega \subseteq \Sigma_b$ the closed subspace of all positive definite spherical functions.

We now list some major facts about the spherical Fourier transform:

**3.5 Theorem.** (1) *The spherical Fourier transform is injective.*

(2) *Bochner Theorem: $\Omega$ is the set of extreme points in the set of positive definite, biinvariant functions with $\varphi(e) = 1$, and for each biinvariant positive definite function $f$ on $G$ there is a unique measure $\mu \in M_b^+(\Omega)$ with $f(x) = \check{\mu}(x) := \int_\Omega \varphi(x)\, d\mu(\varphi)$.*

(3) *Plancherel's Theorem: There is a unique measure $\pi \in M^+(\Omega)$ such that the spherical Fourier transform of biinvariant $L^1$-functions can be extended to an isometric isomorphism from the space $L^2(G\|K)$ of all biinvariant $L^2$-functions onto $L^2(\Omega, \pi)$.*

(4) *Inversion formula: If $f \in L^1(G\|K)$ with $\widehat{f} \in L^1(\Omega, \pi)$, then $f = (\widehat{f})^\vee$.*

(5) *Inverse Riemann-Lebesgue lemma: If $f \in L^1(\Omega, \pi)$, then $\check{f} \in C_0(G)$.*

We note that $supp\, \pi \subset \Omega \subset \Sigma_b \subset \Sigma$, and that equality may fail for certain Gelfand pairs.

The theory of Gelfand pairs and the spherical Fourier transform can be considered as the origin of Fourier analysis on commutative hypergroups:

**3.6 Definition.** A *hypergroup* $(X, *)$ is a locally compact Hausdorff space $X$ with a bilinear associative convolution $*$ on $M_b(X)$ with the following properties:

(1) The map $(\mu, \nu) \mapsto \mu * \nu$ is weakly continuous.

(2) For all $x, y \in X$, the product $\delta_x * \delta_y$ of point measures is a compactly supported probability measure on $X$.

(3) The mapping $(x, y) \mapsto \text{supp}(\delta_x * \delta_y)$ from $X \times X$ into the space of nonempty compact subsets of $X$ is continuous with respect to the Michael topology, see [J].

(4) There is a neutral element $e \in X$, satisfying $\delta_e * \delta_x = \delta_x * \delta_e = \delta_x$ for all $x \in X$.

(5) There is a continuous involutive automorphism $x \mapsto \overline{x}$ on $X$ such that $\delta_{\overline{x}} * \delta_{\overline{y}} = (\delta_y * \delta_x)^-$ and $x = \overline{y} \Longleftrightarrow e \in \text{supp}(\delta_x * \delta_y)$, where for $\mu \in M_b(X)$, the measure $\mu^-$ is given by $\mu^-(A) = \mu(\overline{A})$ for Borel sets $A \subseteq X$.

A hypergroup $(X, *)$ is called commutative if $*$ is commutative. In this case, $(M_b(X), *, .^-)$ is a commutative Banach-$*$-algebra with identity $\delta_e$.

Note that by weakly continuous, bilinear extension, the convolution of a hypergroup is uniquely determined as soon as it is given for point measures.

**3.7 Examples.** (1) If $G$ is a locally compact group, then $(G, *)$ is a hypergroup with the group convolution $*$.

(2) Let $K$ be a compact subgroup of a locally compact group $G$. Consider the Banach-$*$-algebra $M_b(G\|K)$ with identity $\omega_K \in M^1(G)$, and the double coset space

$$G//K := \{KxK : \ x \in G\}$$

which is a locally compact Hausdorff space w.r.t. the quotient topology. The canonical projection $p : G \to G//K$ induces a probability preserving, isometric isomorphism $p : M_b(G\|K) \to M_b(G//K)$ of Banach spaces by taking images of measures. The transfer of the convolution on $M_b(G\|K)$ to $M_b(G//K)$ via $p$ leads to a hypergroup structure $(G//K, *)$ with identity $K \in G//K$ and involution $(KxK)^- := Kx^{-1}K$, and $p$ becomes a probability preserving, isometric isomorphism of Banach-$*$-algebras. The convolution of point measures on $G//K$ is given explicitly by

$$\delta_{KxK} * \delta_{KyK} = \int_K \delta_{KxkyK}\, d\omega_K(k) \quad (x, y \in G).$$

This *double coset hypergroup* is clearly commutative if and only if $(G, K)$ is a Gelfand pair.

**3.8 Example.** Let $(V, +)$ be a locally compact abelian group on which a compact group $K$ of automorphisms acts continuously via $(k, v) \mapsto k.v$. Consider the semidirect product $G := K \ltimes V$, and regard $K$ as a compact subgroup of $G$ in the obvious way. Then $(G, K)$ is a Gelfand pair (apply criterion 3.2 to the automorphism $\theta(k, v) := (k, -v)$). Moreover, the locally compact space

$$V^K := \{K.v : \ v \in V\}$$

of all $K$-orbits of $V$ may be identified with the double coset space $G//K$ via $K.v \in V^K \simeq K(e,v)K \in G//K$. Denoting the image measure of $\mu \in M_b(V)$ under $k \in K$ by $k(\mu)$, we see that the Banach-$*$-algebra $M_b(G\|K)$ of all biinvariant measures on $G$ may be identified with the Banach-$*$-algebra

$$M_b^K(V) := \{\mu \in M_b(V) : \ k(\mu) = \mu \text{ for all } k \in K\}$$

of $K$-invariant measures on $V$. By lifting this structure to $M_b(V^K)$, one obtains a so-called commutative *orbit hypergroup* structure on $V^K$ with the following explicit convolution of point measures:

$$\delta_{K.v} * \delta_{K.w} = \int_K \delta_{K.(v+k.w)}\, d\omega_K(k) \quad (v,w \in V).$$

The neutral element of this hypergroup is $K.0 = \{0\}$, the involution is given by $\overline{K.v} = K.(-v)$.

The most frequent examples of this type arise from Gelfand pairs of Euclidean type: Here $V$ is a finite-dimensional Euclidean vector space and $K$ is a compact subgroup of the orthogonal group $O(V)$.

**3.9 Example (Bessel-Kingman hypergroups).** Let $V = \mathbb{R}^d$ and $K = O(d)$, acting by matrix multiplication from the left. Then the orbit space consists of the spheres $\{x \in \mathbb{R}^d : |x| = r\}$ with $r \geq 0$ and can be topologically identified with $[0,\infty)$. The convolution of the orbit hypergroup $(\mathbb{R}^d)^{O(d)} \cong [0,\infty)$ becomes

$$(\delta_r * \delta_s)(f) = \int_{O(d)} f(|re_1 + kse_1|)dk \ = \int_{S^{d-1}} f(|re_1 + s\sigma|)d\sigma \qquad (3.3)$$

where $e_1 = (1,0,\ldots,0)$, $dk$ denotes the normalized Haar measure of $O(d)$, and $d\sigma$ is the normalized surface measure on the unit sphere $S^{d-1}$. For $d > 1$ we use spherical polar coordinates and the cosine theorem, thus arriving at

$$(\delta_r * \delta_s)(f) = \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)} \int_{-1}^{1} f\big(\big(\sqrt{r^2+s^2+2rst}\big)(1-t^2)^{(d-3)/2}dt.$$

For $d = 1$ the formula degenerates to

$$(\delta_r * \delta_s)(f) = \frac{1}{2}\big(f(r+s) + f(|r-s|)\big).$$

The neutral element of each of these hypergroups is $0$, and the involution is the identity.

We next collect some notations and facts on commutative hypergroups $(X,*)$.

(1) For a bounded Borel function $f: X \to \mathbb{C}$ and $x \in X$ define the translate

$$f_x(y) := f(x*y) := \int_X f\, d(\delta_x * \delta_y).$$

The spaces $C_c(X)$, $C_0(X)$ and $C_b(X)$ are preserved by such translations.

(2) By a famous result of R. Spector, there exists a (up to normalization) unique Haar measure $\omega \in M^+(X)$ which is characterized by $\omega(f) = \omega(f_x)$ for all $f \in C_c(X)$ and $x \in X$.

(3) The involution and convolution of measurable functions $f, g$ on $X$ are given by $f^*(x) := \overline{f(\bar{x})}$ and, in case of convergence,

$$f * g(x) = \int_X f(y)\, g(x * \bar{y})\, d\omega(y).$$

In particular, for $f, g \in L^1(X, \omega)$ one has $f^*, f * g \in L^1(X, \omega)$, and $L^1(X, \omega)$ becomes a commutative Banach-$*$-algebra. There are also convolutions of functions from other $L^p$-spaces, just as with groups. For example, if $f \in L^1(X, \omega)$ and $g \in L^p(X, \omega)$, then $f * g \in L^p(X, \omega)$ with $\|f * g\|_p \leq \|f\|_1 \cdot \|g\|_p$.

(4) Similar to locally compact abelian groups, one defines the spaces

(a) $\chi(X) := \{\alpha \in C(X) : \alpha \neq 0,\ \alpha(x * y) = \alpha(x)\alpha(y) \text{ for all } x, y \in X\}$;
(b) $\chi_b(X) := \chi(X) \cap C_b(B)$;
(c) $\widehat{X} := \{\alpha \in \chi_b(X) : \alpha(\bar{x}) = \overline{\alpha(x)} \text{ for all } x \in X\}$.

Elements of $\widehat{X}$ are called characters, and $\chi_b(X)$ and $\widehat{X}$ are locally compact Hausdorff spaces w.r.t. the topology of compact-uniform convergence.

(5) The Fourier(-Stieltjes) transform of a function $f \in L^1(X, \omega)$ and a measure $\mu \in M_b(X)$ are defined by

$$\widehat{f}(\alpha) := \int_X f(x)\overline{\alpha(x)}\, d\omega(x) \quad \text{and} \quad \widehat{\mu}(\alpha) := \int_X \overline{\alpha(x)} d\mu(x) \quad (\alpha \in \widehat{X}).$$

(6) The hypergroup Fourier transform satisfies $(\mu * \nu)^\wedge = \widehat{\mu} \cdot \widehat{\nu}$ and $(\mu^*)^\wedge = \overline{\widehat{\mu}}$ with analogous formulas for $L^1$-functions. Moreover, $\widehat{\mu} \in C_b(\widehat{X})$ and $\widehat{f} \in C_0(\widehat{X})$ for $f \in L^1(X, \omega)$ ("Riemann-Lebesgue-Lemma").

(7) A function $\varphi \in C_b(X)$ is called positive definite if for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and $c_1, \ldots, c_n \in \mathbb{C}$, $\sum_{i,j=1,\ldots,n} c_i \bar{c}_j \cdot \varphi(x_i * \bar{x}_j) \geq 0$.

The following results are analog to those in Theorem 3.5 for the spherical Fourier transform. For proofs we refer to [J], [BH], and references therein.

**3.10 Theorem.** (1) *The hypergroup Fourier transform is injective.*

(2) *Bochner Theorem: $\widehat{X}$ is the set of extreme points in the set of bounded positive definite functions on $X$ with normalization $\varphi(e) = 1$, and for each bounded positive definite function $f$ on $X$ there is a unique $\mu \in M_b^+(\widehat{X})$ such that $f(x) = \check{\mu}(x) := \int_{\widehat{X}} \varphi(x)\, d\mu(\varphi)$.*

(3) *Plancherel Theorem: There is a unique measure $\pi \in M^+(\widehat{X})$ such that the hypergroup Fourier transform extends to an isometric isomorphism from the space $L^2(X, \omega)$ onto $L^2(\widehat{X}, \pi)$.*

(4) *Inversion formula: If $f \in L^1(X)$ with $\widehat{f} \in L^1(\widehat{X}, \pi)$, then $f = (\widehat{f})^\vee$.*

(5) *Inverse Riemann-Lebesgue lemma: If $f \in L^1(\Omega, \pi)$, then $\check{f} \in C_0(G)$, and the space $\{\check{f} : f \in L^1(\widehat{X}, \pi)\}$ is $\|.\|_\infty$-dense in $C_0(X)$.*

Similar to Gelfand pairs we have $supp\,\pi \subset \widehat{X} \subset \chi_b(X) \subset \chi(X)$ where equality may always fail.

**3.11 Examples.** (1) Let $(G, K)$ be a Gelfand pair and $(X = G//K, *)$ the associated double coset hypergroup. Then the canonical projection of the Haar measure $\omega_G$ of $G$ onto $X$ is a Haar measure of $X$. Moreover, we may identify $\chi(X)$ and $\Sigma$ by identification of biinvariant functions on $G$ with functions on $X$. When doing so, the Plancherel measures of $(G, K)$ and $X$ are the same, and one has the inclusions

$$supp\,\pi \subset \Omega \subset \widehat{X} \subset \chi_b(X) = \Sigma_b \subset \chi(X) = \Sigma.$$

(2) For the Bessel-Kingman hypergroup $X = (\mathbb{R}^d)^{O(d)}$, the Haar measure (in standard normalization) is the image measure of the normalized Lebesgue measure $(2\pi)^{-d/2}dx$ on $\mathbb{R}^d$ under the mapping $x \mapsto |x|$ and is given by

$$d\omega_\alpha(r) = \frac{2^{-d/2}}{\Gamma(d/2)} r^{d-1} dr, \qquad \alpha = d/2 - 1.$$

In practice, for a given Gelfand pair $(G, K)$ or commutative hypergroup $(X, *)$ there usually exists a natural way of constructing examples of spherical or multiplicative functions, respectively. However some additional particular techniques then often have to be employed to prove that these examples really form ALL of them. If there exists for example a sufficiently large class of underlying invariant differential operators, then one has to show that spherical (or multiplicative) functions are eigenfunctions of these operators which then leads to a complete description of these objects. This is for instance the case for Gelfand pairs associated with Riemannian symmetric spaces (see [Hel2]) and also for so-called one-dimensional Sturm-Liouville hypergroups (see Section 3.5.23 of [BH]). In some cases, there exists a further method to determine $\Omega$ and $\widehat{X}$, which seems to be not well-known and which is as follows: Often it is possible to write down the Plancherel measure $\pi$ explicitly, i.e., $supp\pi$ is known. On the other hand there exists a growth criterion which ensures that in the Gelfand pair setting $supp\,\pi = \Omega$ and for commutative hypergroups $supp\,\pi = \widehat{X}$ holds.

**3.12 Definition.** A hypergroup $(X, *)$ with left Haar measure $\omega$ has subexponential growth, if for each compact subset $K \subset X$ and its powers $K^n$ $(n \in \mathbb{N})$, which are recursively defined by $K^1 = K$ and

$$K^{n+1} := K * K^n := \bigcup_{x \in K,\, y \in K^n} supp(\delta_x * \delta_y),$$

and for each $a > 1$, the growth rate $\omega(K^n) = o(a^n)$ holds.

**3.13 Theorem.** (1) *Let $(X, *)$ be a commutative hypergroup with subexponential growth. Then $supp\,\pi = \widehat{X} = \chi_b(X)$.*

2) *Let $(G, K)$ be a Gelfand pair where $G$ has subexponential growth. Then $supp\,\pi = \Omega = \Sigma_b$.*

*Proof.* Part (1) is due to [Vog] and [V0]; see Theorem 2.5.12 of [BH] as a standard reference. In the case of a Gelfand pair $(G, H)$, we note that $G$ has subexponential growth if and only if so has the double coset hypergroup $(G//K, *)$ such that (2) is an obvious consequence of (1). □

**3.14 Example.** Consider the Gelfand pair $(K \ltimes V, K)$ for a locally compact abelian group $V$ as above. Then $V$ is polynomially growing (see [Gr]), and thus $K \ltimes V$ as well. Theorem 3.13 therefore implies $supp\,\pi = \Omega = \Sigma_b$.

We now turn to Gelfand pairs of Euclidean type. We shall in particular determine the dual space of the Bessel Kingman orbit hypergroups.

Let $(V, \langle\,.\,,\,.\,\rangle)$ be a finite-dimensional Euclidean vector space and $K \subset O(V)$ a compact subgroup. Then by Example 3.8, $(G = K \ltimes V, K)$ is a Gelfand pair, $G/K$ may be identified with $V$, and $G//K$ with the orbit space $X := V^K$ in the obvious way. We thus have canonical projections

$$G \quad \longrightarrow \quad G/K \simeq V \quad \longrightarrow \quad G//K \simeq V^K = X.$$

A precise description of the spherical functions can be found in the recent paper of Wolf [Wo]. We introduce the complexification $V_{\mathbb{C}}$ of $V$ equipped with the extension of the scalar product as well as the complexification $K_{\mathbb{C}} := K \cdot K_{\mathbb{C}}^0 \subset O(n, \mathbb{C})$ where $K_{\mathbb{C}}^0$ is the identity component of $K_{\mathbb{C}}$ that corresponds to the complexified Lie algebra of the Lie algebra of the compact Lie group $K$.

The relevant results about the spherical functions of $(G, K)$ or, in different language, the characters of the double coset hypergroup $(X, *)$ are as follows:

**3.15 Theorem.** (1) *For $\lambda \in V_{\mathbb{C}}$, the means*

$$\varphi_\lambda(x) := \int_K e^{-i\langle k.x, \lambda\rangle}\, d\omega_K(k) = \int_K e^{-i\langle x, k.\lambda\rangle} d\omega_K(k) \quad (x \in V)$$

*are continuous, $K$-invariant functions on $V$, and thus may be regarded as biinvariant functions $\varphi_\lambda \in C(G\|K)$ on $G$ as well as functions $\varphi_\lambda \in C(X)$ on the orbit space $X$. If doing so,*

$$\{\varphi_\lambda :\ \lambda \in V_{\mathbb{C}}\} = \Sigma = \chi(X). \tag{3.4}$$

(2) *For $\lambda, \tilde\lambda \in V_{\mathbb{C}}$, the equality $\varphi_\lambda = \varphi_{\tilde\lambda}$ holds if and only if the orbit closures of $\lambda, \tilde\lambda$ under $K_{\mathbb{C}}$ satisfy $cl\, K_{\mathbb{C}}(\lambda) \cap cl\, K_{\mathbb{C}}(\tilde\lambda) \neq \emptyset$.*
*Moreover, for $\lambda, \tilde\lambda \in V$, $\varphi_\lambda = \varphi_{\tilde\lambda}$ holds if and only if $K(\lambda) = K(\tilde\lambda)$.*

(3) $\{\varphi_\lambda :\ \lambda \in V\} = supp\,\pi = \Omega = \widehat{X} = \Sigma_b = \chi_b(X)$.

*Proof.* It can be easily checked by computation that the $\varphi_\lambda$ are spherical functions; see e.g. [F] or [Wo]. For equality in (3.4) as well as for part (2) we also refer to [Wo]. For a proof of (3) we also can refer to [Wo], but we here prefer the following argument: Note first that obviously $\{\varphi_\lambda : \lambda \in V\} \subset \Omega$ holds, and that the quotient topology on $V^K$ agrees with the topology of local-uniform convergence on $\Omega$ after a suitable identification. Moreover, the $L^2$-isometry of the Fourier transform on $V$ immediately implies that the projection $\pi \in M^+(V^K)$ of the suitably normalized Lebesgue measure on $V$ is in fact the Plancherel measure on $\Omega$. Theorem 3.13 and Example 3.14 now yield the equality. □

**3.16 Example (Bessel-Kingman hypergroups).** Let $V = \mathbb{R}^d$ and $K = O(d)$ as in Example 3.9. Then we may realize the associated orbit hypergroup as $(X := [0, \infty), *)$ with the convolution there. Moreover, we have the Haar measure $\omega_\alpha \in M^b([0, \infty))$ from Example 3.11(2).

By taking spherical polar coordinates in the integrals of part (1) of the preceding theorem and a well-known integral representation of the Bessel functions $j_\alpha$ of Eq. (2.15) (see e.g. Eq. (1.71.6) of [Sz]), we see that the spherical functions are given by

$$\varphi_\lambda(x) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1)/2)\sqrt{\pi}} \int_{-1}^{1} e^{i\lambda xt}(1-t^2)^{\alpha-1/2}\, dt = j_\alpha(\lambda x) \tag{3.5}$$

for $\lambda \in \mathbb{C}$, $x \geq 0$, $\alpha = d/2-1$. Here, $\varphi_\lambda = \varphi_{\tilde{\lambda}}$ is equivalent to $\lambda^2 = \tilde{\lambda}^2$, and we conclude that $\widehat{X} = \chi_b(X) = \{\varphi_\lambda :\ \lambda \in [0,\infty)\}$ where we may identify $\widehat{X}$ with $[0,\infty)$ in this way topologically. The construction of the Plancherel measure $\pi$ in the proof above shows that $\pi \in M^+([0,\infty))$ is also the spherical projection of the Lebesgue measure up to normalization and agrees thus with the Haar measure above up to normalization. This is no accident, as it might be derived easily from the symmetry of the functions $\varphi_\lambda(x)$ in the variables $\lambda, x$ that the Bessel-Kingman hypergroups are self-dual like the underlying groups $\mathbb{R}^d$.

We next observe that the Bessel functions $j_\alpha$ and the associated function $\varphi_\lambda(x) = \varphi_\lambda^\alpha(x)$ depend analytically on $\alpha$ and that by a kind of analytical extension (see Section 3.4 where a more general case is treated), the product formula (3.3) as well as the positive integral representation (3.5) remain valid for all $\alpha \in \mathbb{R}$ with $\alpha > -1/2$ (with the degenerated case $\alpha = -1/2$). These formulas then imply that for each $\alpha \geq 1/2$, there exists a unique so-called Bessel-Kingman hypergroup on $[0,\infty)$ with the dual space $\{\varphi_\lambda^\alpha :\ \lambda \in [0,\infty)\}$, with 0 as identity, and the identity as involution. This hypergroups are also self dual, the Haar measure $\omega_\alpha$ is given as in 3.11(2), and the Plancherel measure is equal to $\omega_\alpha$ up to normalization.

We finally recapitulate that if $K$ is a compact group of automorphisms acting continuously on some locally compact abelian group $V$, then the orbit space $V^K$ becomes a commutative orbit hypergroup. This can be easily extended to the case where $K$ is a compact group of automorphisms acting continuously on some commutative hypergroup $(X,*)$. This generalization will be crucial to explain how the Bessel hypergroup structures on matrix cones lead to continuous series of commutative hypergroup structures on Weyl chambers associated with Dunkl operators of type $B_N$ in Section section-matrix-cones. We here collect some facts without proofs. For further details we refer to [J], where this concept is embedded in the more general context of orbital morphisms.

**3.17 Remark.** Let $(X,*)$ be a commutative hypergroup with dual space $\widehat{X}$.

(1) A homeomorphism $k : X \to X$ is called a hypergroup automorphism if $\delta_{k(x)} * \delta_{k(y)} = k(\delta_x * \delta_y)$ for all $x, y \in X$.

It follows readily for such an automorphism, that $k(e) = e$ and $k(\bar{x}) = \overline{k(x)}$ for $x \in K$. Moreover, $k(\mu * \nu) = k(\mu) * k(\nu)$ for all $\mu, \nu \in M_b(X)$. Furthermore, if $k_1, k_2$ are hypergroup automorphisms of $(X,*)$, then so are $k_1^{-1}$ and $k_1 k_2$.

(2) Now assume that $K$ is a compact group of hypergroup automorphisms acting continuously on $(X,*)$. Then we may form the orbit space $X^K := \{K.x :\ x \in X\}$ which is locally compact w.r.t. the quotient topology. It can be easily checked that

$$M_b(X|K) := \{\mu \in M_b(X) :\ k(\mu) = \mu \quad \text{for all} \quad k \in K\}$$

is a Banach-$*$-subalgebra of $M_b(X)$ which is isometrically isomorphic as a Banach space with the Banach space $M_b(X^K)$ by taking images of measures w.r.t. the canonical projection from $X$ onto $X^K$. A transfer of the convolution on $M_b(X|K)$ to $M_b(X^K)$ then leads to a probability preserving convolution $*$ on $M_b(X^K)$, and it can be easily checked that with this convolution, $(X^K, *)$ becomes a commutative hypergroup with identity $\{e\}$ and involution $\overline{K.x} := K.\bar{x}$ for $x \in X$. Furthermore, if we define for $\varphi \in \widehat{X}$ the $K$-invariant function $\varphi^K(x) := \int_K \varphi(k.x)\, d\omega_K(k)$ for $x \in X$, one obtains readily that $\{\varphi^K : \varphi \in \widehat{X}\} \subset (X^K)^\wedge$. Moreover, if $X$ has subexponential growth, then so also has $X^K$, and the Plancherel measure $\pi$ of $X^K$ satisfies $supp\, \pi = (X^K)^\wedge$. This implies that similar as above in the group case the equality

$$\{\varphi^K : \varphi \in \widehat{X}\} = (X^K)^\wedge$$

holds; see Section 13 of [J] or the proof of Theorem 4.1 of [R8].

## 3.3 Bessel functions associated with root systems and symmetric spaces of Euclidean type

The radial parts of invariant differential operators on a Riemannian symmetric space can be expressed in terms of commuting algebras of Dunkl or Dunkl-Cherednik operators. The rational Dunkl operators are hereby related to symmetric spaces of the Euclidean type, and their spherical functions appear as Dunkl-type Bessel functions with certain discrete multiplicity values.

To explain this connection, let us start with the underlying concepts. For more background and detail the reader may consult the monographs [Hel1], [Hel2], or also [Ko], Chapt. I, II, III.

Let $\mathfrak{g}$ be a real semisimple Lie algebra, i.e., a Lie algebra over $\mathbb{R}$ on which the Killing form $B$ is nondegenerate. An involutive automorphism $\theta$ of $\mathfrak{g}$ is called a *Cartan involution* and the corresponding eigenspace decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into $+1$ and $-1$ eigenspaces of $\theta$ is called a *Cartan decomposition* of $\mathfrak{g}$ if $B$ is negative definite on $\mathfrak{k}$ and positive definite on $\mathfrak{p}$. In particular, $(\mathfrak{p}, B)$ is a finite-dimensional Euclidean vector space. Cartan involutions exist and are all conjugate under inner automorphisms. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Such a Lie group is called semisimple. It can be shown that the Cartan involution on $\mathfrak{g}$ is the differential of a unique involutive automorphism of $G$, also denoted by $\theta$, and that the fixed point subgroup $K := \{g \in G : \theta(g) = g\}$ is just the connected Lie subgroup of $G$ with Lie algebra $\mathfrak{k}$. $K$ is compact iff $G$ has finite center; such a choice of $G$ is always possible. If $G$ has finite center and $\mathfrak{p} \neq \{0\}$ then $G/K$ is called a *Riemannian symmetric space of the non-compact type.*

**3.18 Remark.** The (irreducible) Riemannian symmetric spaces of non-compact type are completely classified, see Chapter X of [Hel1]. There are ten infinite classes of simply connected symmetric spaces, namely $SL(n,\mathbb{C})/SU(n)$, $SL(n,\mathbb{R})/SO(n)$, $SL(n,\mathbb{H})/SP(n)$, $SO(n,\mathbb{C})/SO(n)$, $Sp(n,\mathbb{C})/Sp(n)$, the Grassmann manifolds $SO_0(p,q)/SO(p)\times SO(q)$, $SU(p,q)/S(U(p)\times U(q))$ and $Sp(p,q)/Sp(p)\times Sp(q)$, as well as $Sp(n,\mathbb{R})/U(n)$ and $SO^*(2n)/U(n)$.

In the situation described above, $K$ acts on $\mathfrak{g}$ via the adjoint representation $Ad$. We denote this action with $Ad(k)X = k.X$. Recall that

$$\exp(k.X) \;=\; k\exp(X)k^{-1} \quad \text{for } X \in \mathfrak{g},\, k \in K.$$

As $[\mathfrak{k},\mathfrak{p}] \subseteq \mathfrak{p}$, $K$ leaves $\mathfrak{p}$ invariant, and acts via orthogonal transformations (with respect to the Killing form) on $\mathfrak{p}$. The semidirect product $G_0 := K \ltimes \mathfrak{p}$ is called the *Cartan motion group* associated with $G/K$, and $G_0/K \cong \mathfrak{p}$ is called a *symmetric space of Euclidean type.*

Choose now a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. All such subspaces have the same dimension which is called the rank of $G/K$. For each $\alpha$ in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ let

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

Those $\alpha$ with $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$ are called the (restricted) *roots* of $\mathfrak{g}$ w.r.t. $\mathfrak{a}$. The simultanous diagonalization of the commuting operators $ad\,H,\, H \in \mathfrak{a}$ leads to the root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha\in\Sigma} \mathfrak{g}_\alpha$$

where $\Sigma$ is the set of restricted roots. The spaces $\mathfrak{g}_\alpha$ are called *root spaces*, and $m_\alpha := \dim_{\mathbb{R}}\mathfrak{g}_\alpha$ is called the *multiplicity* of $\alpha$. We consider $\Sigma$ as a subset of $\mathfrak{a}$, identifying $\mathfrak{a}$ with its dual space via the Killing form. The set $\Sigma$ is a crystallographic, but not neccesarily reduced root system in the Euclidean space $(\mathfrak{a},B) \cong (\mathbb{R}^N,\, \langle\,.\,,\,.\,\rangle)$. Denote by $W$ the associated Weyl group. The action of $W$ on $\mathfrak{a}$ is obtained from the action of $K$ on $\mathfrak{p}$, as follows: Consider the centralizer and the normalizer of $\mathfrak{a}$ in $K$,

$$Z_K(\mathfrak{a}) := \{k \in K : k.H = H \quad \forall H \in \mathfrak{a}\}$$
$$N_K(\mathfrak{a}) := \{k \in K : k.\mathfrak{a} = \mathfrak{a}\}$$

Then $W$ is realized as the quotient $N_K(\mathfrak{a})/Z_K(\mathfrak{a})$. Each $K$-orbit in $\mathfrak{p}$ meets $\mathfrak{a}$, and the orbit space $\mathfrak{p}^K$ is homeomorphic to the closure of the Weyl chamber $\mathfrak{a}_+$ corresponding to some fixed positive subsytem of $\Sigma$,

$$\mathfrak{p}^K \cong \overline{\mathfrak{a}_+}$$

(see for instance [Hel2], Prop. I.5.18).

The *spherical functions* of the Euclidean symmetric space $G_0/K$ are defined as the spherical functions of the Gelfand pair $(G_0 = K \ltimes \mathfrak{p}, K)$. Equivalently, they can be characterized as follows: Denote by $S(\mathfrak{p})^K$ the algebra of $K$-invariant polynomials on $\mathfrak{p}$, and by $p(\partial)$ the constant coefficient differerential operator corresponding to $p \in S(\mathfrak{p})$. Then a function $\varphi : \mathfrak{p} \to \mathbb{C}$ is spherical if and only if it is smooth and satisfies $\varphi(0) = 1$ as well as

$$p(\partial)\varphi \;=\; p(\lambda)\varphi \quad \forall p \in S(\mathfrak{p})^K$$

with some $\lambda \in \mathfrak{a}_{\mathbb{C}}$. This just means that $\varphi$ is a joint eigenfunction of all $K$-invariant constant coefficient differential operators on $\mathfrak{p}$. Being $K$-invariant, $\varphi$ can also be considered a $W$-invariant function on $\mathfrak{a}$ which is a joint eigenfunction of the $K$-radial parts of the operators from the commutative algebra $\mathcal{D} = \{p(\partial),\, p \in S(\mathfrak{p})^K\}$. An important member of $\mathcal{D}$ is the Laplacian $\Delta_{\mathfrak{p}}$. Its radial part can be calculated as

$$\operatorname{Rad}(\Delta_{\mathfrak{p}}) = \Delta_{\mathfrak{a}} + \sum_{\alpha\in\Sigma_+} m_\alpha \frac{1}{\langle\alpha,\,.\,\rangle}\partial_\alpha \tag{3.6}$$

see [Hel2], Chapter II.

**3.19 Example.** For $d \geq 2$ consider the Lorentz group $G = SO_0(d,1)$ with maximal compact subgroup $K = SO(d)$. The symmetric space $G/K$ is a real hyperbolic space which can be topologically identified with the hyperboloid

$$\mathbb{H}^d = \{x \in \mathbb{R}^{d+1} : x_1^2 + \ldots + x_d^2 - x_{d+1}^2 = 1,\, x_{d+1} > 0\}.$$

In this case, $G_0 = SO(d) \ltimes \mathbb{R}^d$ is the Euclidean motion group, where $K = SO(d)$ acts in the subspace perpendicular to the $x_{d+1}$-axis. The symmetric space $G_0/K$ is naturally identified with the vector space $\mathbb{R}^d$ which is just the tangent space of $G/K$ in the base point $eK \cong (0,\ldots,0,1) \in \mathbb{H}^n$. Moreover, $\mathfrak{a} \cong \mathbb{R}$ and the rank of $G/K$ is 1. The $K$-invariant differential operators are in this case exactly those which are algebraically generated by the Laplacian $\Delta$ on $\mathbb{R}^d$. When considered as functions on $\mathbb{R}$, the spherical functions are characterized as the smooth and even eigenfunctions of the radial part of $\Delta$. These are precisely the Bessel functions $\varphi_\lambda(x) = j_{d/2-1}(\lambda x)$ with $\lambda \in [0,\infty)$, c.f. Section 2.5.

The spherical functions of a Euclidean symmetric space $G_0/K$ can be written down explicitly in terms of an integral representation of Harish-Chandra type (see [Hel2], Chap. IV). In fact, this is also an immediate consequence of Theorem 3.15:

**3.20 Theorem.** *The spherical functions of the Euclidean symmetric space $G_0/K$ are (as functions on $\mathfrak{a}$) given by*

$$\varphi_\lambda(x) = \int_K e^{iB(\lambda,k.x)}dk$$

*where $\lambda \in \mathfrak{a}_{\mathbb{C}}$ (the complexification of $\mathfrak{a}$) and $\varphi_\lambda = \varphi_{\widetilde{\lambda}}$ iff $\lambda$ and $\widetilde{\lambda}$ are in the same $W$-orbit in $\mathfrak{a}_{\mathbb{C}}$.*

By a convexity theorem of Kostant, this integral formula can be written in the form

$$\varphi_\lambda(x) = \int_{co(W.x)} e^{iB(\lambda,\xi)}d\nu_x(\xi)$$

with a probability measure $\nu_x$ on the convex hull $co(W.x)$ of the Weyl group orbit of $x$. This generalizes the Mehler integral representation (3.5) in rank one.

Let us now proceed to the interpretation of the spherical functions $\varphi_\lambda$ as Bessel functions of Dunkl type. We follow the expositions in [dJ4], [R6].

**3.21 Definition.** Let $R$ denote the reduced root system obtained from $\Sigma$ by normalizing all roots to unit length (so some roots in a component $BC_n$ of $\Sigma$ will coincide in $R$). Define the multiplicity $k$ on $R$ by $k_\alpha := \frac{1}{2}\sum_{\beta\in\mathbb{R}\alpha\cap\Sigma_+} m_\beta$. Data$(R,k)$ obtained in this way from a symmetric space are called *geometric.*

Consider the Dunkl operators associated with $R$ and $k$. Comparison of the radial part (3.6) with the $W$-invariant restriction of $\Delta_k$ shows that

$$\mathrm{Rad}(\Delta_{\mathfrak{p}}) = \mathrm{Res}(\Delta_k).$$

More general, there is a restriction isomorphism (the generalized Harish-Chandra isomorphism)

$$\Phi : S(\mathfrak{p})^K \longrightarrow S(\mathfrak{a})^W$$

where $S(\mathfrak{a})^W$ denotes the algebra of $W$-invariant polynomials on $\mathfrak{a}$ (called $\mathcal{P}^W$ in Section 2.5). It can now be shown (see [dJ4] for the details) that for each $p \in S(\mathfrak{p})^K$,

$$\mathrm{Rad}(p(\partial)) = \Phi(p)(T(k)).$$

From this and the uniqueness of the solution to both differential systems, it is not hard to see that as funtions on $\mathfrak{a} \cong \mathbb{R}^N$, the spherical functions of the flat symmetric space $G_0/K$ are given by the Dunkl-type Bessel functions associated with $R$ and $k$. That is

$$\{\varphi : \mathbb{R}^N \to \mathbb{C} : \varphi \text{ spherical for } G_0/K\} = \{J_k(\,.\,,\lambda), \lambda \in \mathbb{R}^N\}.$$

Thus, the positive integral representation of Theorem 2.31 for Bessel functions associated with arbitrary non-negative multiplicites generalizes the Harish-Chandra integral for spherical functions of Euclidean symmetric spaces.

To obtain a generalized translation with nice properties in Dunkl theory, product formulas for the Dunkl kernel and the Bessel function would be most desireable. Indeed, for the geometric cases, a positive product formula is guaranteed by the interpretation of Bessel functions as spherical functions. Let us briefly describe these matters.

Consider as above the Euclidean symmetric space $G_0/K$ associated with $G/K$ and the Bessel functions $J_k$ with corresponding geometric data $(R,k)$. As described in Section 3.2, the convolution of $K$-biinvariant functions on $G_0$ can be interpreted as the convolution of the commutative orbit hypergroup $\mathfrak{p}^K$ which is given by

$$\delta_{K.x} * \delta_{K.y} = \int_K \delta_{K.(x+K.y)} dk, \quad x, y \in \mathfrak{p}.$$

Recall that each orbit $K.x$ contains a unique element $x_+ \in \overline{\mathfrak{a}_+}$ and that the mapping $\mathfrak{p}^K \to \overline{\mathfrak{a}_+}$, $x \mapsto x_+$ is a homeomorphism. The above convolution therefore transfers to a hypergroup convolution on the closed Weyl chamber $\overline{\mathfrak{a}_+}$ which we consider as a subset of $\mathbb{R}^N$. It is given by

$$\delta_x * \delta_y = \int_K \delta_{(x+k.y)_+} dk.$$

The neutral element of the hypergroup $X = (\overline{\mathfrak{a}_+}, *)$ is 0 and the involution is given by $\overline{x_+} = (-x)_+$. Further, the space $\chi(X)$ of continuous multiplicative functions and the dual space of $X$ are

$$\chi(X) = \{J_k(\,.\,,\lambda),\, \lambda \in \mathfrak{a}_{\mathbb{C}}\}; \quad \widehat{X} = \chi_b(X) = \{J_k(\,.\,,i\lambda),\, \lambda \in \mathfrak{a}_+\},$$

c.f. Theorem 3.15. A Haar measure is given by the Dunkl-type weight function $w_k$. The restriction of the Dunkl transfrom to Weyl group invariant functions therefore coincides with the Fourier transform on the hypergroup $X$.

**3.22 Example** $(SL(N,\mathbb{C})/SU(N))$**.** Consider the semisimple connected Lie group $G = SL(N,\mathbb{C})$ with Lie algebra $\mathfrak{g} = sl(N,\mathbb{C}) = \{x \in M_N(\mathbb{C}) : trx = 0\}$. The mapping $\theta(x) = -x^*$ with $x^* = \overline{x}^t$ is a Cartan involution on $\mathfrak{g}$, and the corresponding Cartan decompositon is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ with

$$\begin{aligned} \mathfrak{k} =& \{x \in \mathfrak{g} : x^* = -x\} && \text{(skew-Hermitian matrices)} \\ \mathfrak{p} =& \{x \in \mathfrak{g} : x^* = x\} =: H_N && \text{(Hermitian matrices).} \end{aligned}$$

A possible choice for a maximal abelian subalgebra are the diagonal matrices

$$\mathfrak{a} = \{x = \operatorname{diag}(\xi_1, \ldots, \xi_N) : \xi_i \in \mathbb{R}, \sum_{i=1}^N \xi_i = 0\}$$

which we identify with a subspace of $\mathbb{R}^N$. The adjoint action of the maximal compact subgroup $K = exp\mathfrak{k} = SU(N)$ on $\mathfrak{p} = H_N$ is given by conjugation, $(u,x) \mapsto uxu^{-1}$. Each matrix $x \in H_N$ is unitarily equivalent to a unique diagonal matrix with the eigenvalues of $x$ being ordered by size. Therefore the orbit $H_N^K$ can be (actually topologically) identified with the closed set

$$\overline{C} := \{\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N : \xi_1 \geq \ldots \geq \xi_N, \sum_{i=1}^N \xi_i = 0\}.$$

Notice that $\overline{C}$ is a closed Weyl chamber of the symmetric group $S_N$, which is the Weyl group of the Riemannian symmetric space $G/K$. Indeed, the restriction of the $K$-action on $H_N$ to $\mathfrak{a}$ is realized by permutations of the diagonal entries. According to our general results, the spherical fucntions of the Euclidean symmetric space $(K \ltimes H_N)/K$ are given by the functions

$$\{\xi \mapsto J_1^A(\xi, \lambda), \ \lambda \in \mathfrak{a}_{\mathbb{C}}\}$$

where $J_1^A$ denotes the Bessel function associated with root system $A_{N-1}$ and multiplicity $k = 1$. The value of $k = 1$ results from the fact that $SL(N,\mathbb{C})$ is a complex Lie group and therefore the real dimensions of all root spaces are equal to $2$. The Bessel function $J_1^A$ is given explicitly by

$$J_1(\xi, \lambda) = c \sum_{w \in S_N} \frac{\operatorname{sign}(w)}{\pi(\xi)\pi(\lambda)} \, e^{\langle \xi, w\lambda \rangle}$$

where $c \in \mathbb{C}$ is a constant, $\langle \, . \, , \, . \, \rangle$ denotes the usual inner product in $\mathbb{R}^N$ and $\pi$ is the fundamental alternating polynomial

$$\pi(\xi) = \prod_{i<j} (\xi_i - \xi_j).$$

Indeed, this alternating sum representation for $J_1^A$ follows from the explicit formula for the spherical functions of $G/K$ in [Hel2], but it is also a special case of a general result for the Dunkl Bessel functions for arbitrary root systems with multiplicity $k = 1$. For a proof see Prop. 1.4. of [D6].
Let us finally take a look at the convolution of the orbit hypergroup $H_N^K \cong \overline{C}$. It is given by

$$\delta_\xi * \delta_\eta = \int_{SU(N)} \delta_{\sigma(\xi + u\eta u^{-1})} du \quad (\xi, \eta \in \overline{C})$$

where $\xi$ and $\eta$ are identified with the corresponding diagonal matrices, and $\sigma(x)$ denotes the eigenvalues of $x$ ordered by size. The convolution product describes the distribution of the spectra of the possible sums $x+y$ of Hermitian matrices $x, y$ with given spectra $\sigma(x) = \xi$, $\sigma(y) = \eta$. The precise description of the support of the measure $\delta_\xi * \delta_\eta$ had been a long-standing problem, formulated as Horn's conjecture, until it was completely solved only ten years ago by Klyachko, Knutson and Tao ([Kl], [KT]).

The following is a natural conjecture, which is however open so far in general:

**3.23 Remark.** Let $R$ be a reduced root system in $\mathbb{R}^N$ and $k \geq 0$ a non-negative multiplicity function. Choose a closed Weyl chamber $\overline{C}$ of $R$. Then it is conjectured that the Bessel function $J_k$ associated with $R$ and $k$ satisfies a product formula of the form

$$J_k(x,\lambda)J_k(y,\lambda) = \int_{\overline{C}} J_k(\xi,\lambda)d\mu_{x,y}^k(\xi) \quad \forall \lambda \in \mathbb{C}^N$$

with probability measures $\mu_{x,y}^k \in M^1(\overline{C})$, and that this product formula leads to a commutative hypergroup structure on $\overline{C}$ with the functions $x \mapsto J_k(x, i\lambda)$ as characters.

In Section 3.5 we shall present three continuous series of Dunkl structures where this conjecture is true, and where the convolution can be written in an explicite form. These examples are of type $B$ and are obtained by interpolation of three discrete series of orbit hypergroup convolutions related to the Cartan motion groups of Grassmann manifolds over $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$. We shall derive them from hypergroup algebras on cones of positive semidefinite matrices which have matrix Bessel fucntions as characters. These are the topic of the next following section.

## 3.4 Bessel hypergroups on matrix cones

In this section, we construct three series of Bessel hypergroups on cones of positive semidefinite matrices over one of the skew-fields $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Each of them is obtained from a discrete series of orbit hypergroup convolutions derived from radial analysis on matrix spaces with varying dimension. The main references for this section are [R8] and [FT]. For an introduction into the analysis on symmeric cones, we refer the reader to the monograph [FK]. We start with a geometric situation which generalizes the action of the orthogonal group in $\mathbb{R}^d$ to a higher rank setting:

For natural numbers $p \geq q$, consider the matrix space $M_{p,q} = M_{p,q}(\mathbb{F})$ of $p \times q$ matrices over $\mathbb{F}$. We regard $M_{p,q}$ as a real vector space of dimension $d = \dim_{\mathbb{R}}\mathbb{F}$, equipped with the inner product $\langle x, y\rangle = \mathrm{Re}\, tr(x^*y)$ (where $x^* = \overline{x}^t$ denotes the conjugate transpose) and the norm $\|x\| = \sqrt{tr(x^*x)}$. Let further $H_q = H_q(\mathbb{F}) = \{x \in M_{q,q}(\mathbb{F}) : x = x^*\}$ denote the space of Hermitian $q \times q$-matrices and

$$\Pi_q = \Pi_q(\mathbb{F}) := \{x^2 : x \in H_q\} \subset H_q$$

the closed cone of positive semidefinite matrices over $\mathbb{F}$. Its interior $\Omega_q$, consisting of the strictly positve matrices, is a symmetric cone in the sense of [FK].

The unitary group $U_p = U(p, \mathbb{F})$ over $\mathbb{F}$ acts on $M_{p,q}$ by multiplication from the left as a closed subgroup of the orthogonal group of $M_{p,q}$,

$$U_p \times M_{p,q} \to M_{p,q}, \quad (u, x) \mapsto ux.$$

It is easy to see that the orbit space $M_{p,q}^{U_p}$ for this action can be topologically identified with the cone $\Pi_q$ via $U_p.x \mapsto \sqrt{x^*x} =: |x|$. Here for $r \in \Pi_q$, $\sqrt{r}$ denotes the unique positive semidefinite square root of $r$.

The additive group structure on $M_{p,q}$ induces an orbit hypergroup convolution on $\Pi_q$. To obtain the convolution, Haar measure and dual space of this hypergroup, we apply the results of Section 3.2. The calculation of the convolution is similar as for $(\mathbb{R}^d)^{O(d)}$ in Example 3.9. For $r, s \in \Pi_q$ we obtain

$$(\delta_r * \delta_s)(f) = \int_{U_p} f\big(|\sigma_0 r + u\sigma_0 s|\big) du$$

with the block matrix $\sigma_0 := \begin{pmatrix} I_q \\ 0 \end{pmatrix} \in M_{p,q}$. The orbit of $\sigma_0$ is the Stiefel manifold

$$\Sigma_{p,q} = \{x \in M_{p,q} : x^*x = I_q\}$$

and one obtains

$$(\delta_r * \delta_s)(f) = \int_{\Sigma_{p,q}} f\big(\sqrt{r^2 + s^2 + r\widetilde{\sigma}s + (r\widetilde{\sigma}s)^*}\big) d\sigma \tag{3.7}$$

where $d\sigma$ is the normalized surface measure on $\Sigma_{p,q}$ and $\widetilde{\sigma} = \sigma_0^*\sigma$ is the $q \times q$-matrix whose rows are given by the first $q$ rows of $\sigma$. The neutral element of the orbit hypergroup $(\Pi_q, *)$ is $0$, and the involution is the identity mapping (because $x \in M_{p,q}$ and $-x$ are in the same $U_p$-orbit). A Haar measure is provided by the image measure of the Lebesgue measure on $M_{p,q}$ under the mapping $x \mapsto |x|$. Calculation in polar coordinates gives

$$\omega(f) = c \int_{\Pi_q} f(\sqrt{r}) \Delta(r)^\gamma dr$$

where $c > 0$ is a constant, $\Delta(r)$ is the determinant of $r$ (in case $\mathbb{F} = \mathbb{R}$, the Dieudonne determinant), and $\gamma = \frac{d}{2}(p - q + 1) - 1$. The dual space turns out to consist of Bessel functions associated with the symmetric cone $\Omega_q$. These are hypergeometric funtions of matrix argument defined in terms of the spherical functions of $\Omega_q$, see [FT]. The spherical polynomials of $\Omega_q$ are parameterized by partitions $\lambda = (\lambda_1 \geq \ldots \geq \lambda_q) \in \mathbb{Z}_+^q$, for which we write $\lambda \geq 0$ for short. They are given, up to normalization, by

$$Z_\lambda(x) = c_\lambda \int_{U_q} \Delta_\lambda(uxu^{-1}) du, \quad x \in H_q$$

with the power functions

$$\Delta_\lambda(x) = \Delta_1(x)^{\lambda_1 - \lambda_2} \Delta_2(x)^{\lambda_2 - \lambda_3} \cdot \ldots \cdot \Delta_q(x)^{\lambda_q}.$$

The $\Delta_i(x)$ are the principal minors of $\Delta(x)$, and the constant $c_\lambda > 0$ can be chosen such that

$$(\operatorname{tr} x)^k = \sum_{\lambda \geq 0, |\lambda| = k} Z_\lambda(x).$$

The polynomial $Z_\lambda$ is homogeneous of degree $|\lambda|$ and invariant under conjugation by $U_q$. It therefore depends only on the eigenvalues of its argument.

**3.24 Definition.** The Bessel functions $\mathcal{J}_\mu$ associated with $\Omega_q$ are defined on $H_q$ by

$$\mathcal{J}_\mu(x) = {}_0F_1(\mu; -x) := \sum_{\lambda \geq 0} \frac{(-1)^{|\lambda|}}{(\mu)_\lambda |\lambda|!} Z_\lambda(x)$$

where $\mu \in \mathbb{C}$ is an index with $\text{Re}\mu > \frac{d}{2}(q-1)$ and $(\mu)_\lambda$ denotes the generalized Pochhammer symbol

$$(\mu)_\lambda := \prod_{j=1}^{q} \big(\mu - \frac{d}{2}(j-1)\big)_{\lambda_j}.$$

If $q = 1$ then $J_\mu$ does not depend on $d$ and is given by

$$\mathcal{J}_\mu\big(\frac{x^2}{4}\big) = j_{\mu-1}(x) \quad (x \in \mathbb{R}).$$

**3.25 Lemma.** *The dual space of the orbit hypergroup $M_{p,q}^{U_p} \cong \Pi_q$ consists of the Bessel functions*

$$\varphi_s(r) = \mathcal{J}_\mu\big(\frac{1}{4} sr^2 s\big), \;\; s \in \Pi_q,$$

*with index $\mu = pd/2$.*

*Proof.* According to Theorem 3.15, the dual space is given by the functions $\varphi_s$, $s \in \Pi_q$ with

$$\varphi_s(r) = \int_{U_p} e^{-i(u\sigma_0 r|\sigma_0 s)} du \;=\; \int_{\Sigma_{p,q}} e^{-i(\sigma|\sigma_0 sr)} d\sigma. \tag{3.8}$$

By Propos. XVI.2.2 of [FK] they coincide with the stated Bessel functions. □

The hypergroup Fourier transform on $M_{p,q}^{U_p}$ is a Hankel transform involving the Bessel function $\mathcal{J}_\mu$ with $\mu = pd/2$. It coincides with the (group) Fourier transform of $U_p$-radial functions on $M_{p,q}$. An $L^2$-theory for the Hankel transform with a continuous range of real indices $\mu$ was established in [FT] in the general setting of symmetric cones. (In [Her], this had been done for $\mathbb{F} = \mathbb{R}$.) In our setting, the result is as follows:

**3.26 Theorem.** *Let $\mu > \frac{d}{2}(q-1)$ and define the measure $\omega_\mu$ on $\Pi_q$ by*

$$\omega_\mu(f) = \frac{2^{-\mu q}}{\Gamma_{\Omega_q}(\mu)} \int_{\Omega_q} f(\sqrt{r}) \Delta(r)^\gamma dr$$

*where $\gamma = \mu - \frac{d}{2}(q-1) - 1$ and $\Gamma_{\Omega_q}$ is the gamma function of the cone $\Omega_q$ (see [FK]). Put*

$$\varphi_s^\mu(r) := \mathcal{J}_\mu\big(\frac{1}{4} sr^2 s\big) = \varphi_r^\mu(s).$$

*Then the Hankel transform*

$$f \mapsto \widehat{f}^\mu, \quad \widehat{f}^\mu(s) = \int_{\Pi_q} f(r) \varphi_s^\mu(r) d\omega_\mu(r)$$

*is an isometric and self-dual isomorphism of $L^2(\Pi_q, \omega_\mu)$.*

We shall now complement this result by a hypergroup structure within a slightly smaller range of indices. The decisive obeservation is that the integrand in convolution formula (3.7) does not depend on the complete matrix $\sigma$ but depends only the truncation $\widetilde{\sigma}$. This matrix is contained in the closure of the matrix ball

$$D_q := \{v \in M_{q,q} : v^*v < I\}$$

($r < s$ means that $s - r$ is positive definite). By a corresponding splitting of coordinates on the Stiefel manifold, one obtains the convolution formula in a different form with $D_q$ as domain of integration. The dimension parameter $p$ then occurs as an exponent in the density of the integral. Let

$$\varrho := d\big(q - \frac{1}{2}\big) + 1 \quad \text{and} \quad \kappa_\mu := \int_{D_q} \Delta(I - v^*v)^{\mu-\varrho} dv$$

for $\mu \in \mathbb{R}$ with $\mu > \varrho - 1$. The decisive splitting lemma, which requires $p \geq 2q$, is as follows:

**3.27 Lemma.** *Let $p \geq 2q$ and put $\mu := pd/2$. Then for $f \in C(\Sigma_{p,q})$ of the form*

$$f(\sigma) = F(\widetilde{\sigma}), \quad \widetilde{\sigma} = \sigma_0^*\sigma$$

*one has*

$$\int_{\Sigma_{p,q}} f d\sigma = \frac{1}{\kappa_\mu} \int_{D_q} F(v)\, \Delta(I - v^*v)^{\mu-\varrho} dv.$$

*with an (explicitly known) normalization constant $\kappa_\mu > 0$.*

Rewriting the convolution formula of the orbit hypergroups on $\Pi_q$ in this way allows to extend the product formula of the Bessel functions and the associated hypergroup structure to a continuous range of indices $\mu$. The basic technique for this is analytic continuation with respect to $\mu$ from the discrete values $\mu = pd/2$ into the full half-plane $\{\mu \in \mathbb{C} : \operatorname{Re} \mu > \rho - 1\}$ by use of Carlson's Phragmen-Lindelöf-type Theorem (see [Ti], p.186). This gives three continuous series of commutative hypergroup structures on $\Pi_q$ (corresponding to $d = 1, 2, 4$) which interpolate those occuring as orbit hypergroups for the indices $\mu = pd/2$. The following theorem contains the main results of [R8]:

**3.28 Theorem.** *Let $\mu \in \mathbb{R}$ with $\mu > \varrho - 1$.*

(a) *The assignment*

$$(\delta_r *_\mu \delta_s)(f) := \frac{1}{\kappa_\mu} \int_{D_q} f\big(\sqrt{r^2 + s^2 + rvs + sv^*r}\,\big)\, \Delta(I - v^*v)^{\mu-\varrho}\, dv$$

*defines a commutative hypergroup $X_\mu = (\Pi_q\,, *_\mu)$ with neutral element $0$ and the identity mapping as involution. For $\mu = pd/2$ with $p \geq 2q$, $X_\mu$ is the orbit hypergroup $M_{p,q}^{U_p}$. The support of $\delta_r *_\mu \delta_s$ satisfies*

$$supp\,(\delta_r *_\mu \delta_s) \subseteq \{t \in \Pi_q : \|t\| \leq \|r\| + \|s\|\}.$$

(b) *A Haar measure of $X_\mu$ is given by the measure $\omega_\mu$ from Theorem 3.26.*

(c) *The dual space is given by $\widehat{X_\mu} = \{\varphi_s^\mu(r) = \mathcal{J}_\mu\big(\frac{1}{4} sr^2 s\big) :\ s \in \Pi_q\}$.*

(d) *When identifying $X_\mu$ with its dual via $s \mapsto \varphi_s^\mu$, the Plancherel measure of $X_\mu$ coincides with $\omega_\mu$.*

Part (c) is based on the analytic extension of the product formula for the Bessel functions, but one also has to make sure that there are no further characters apart from the Bessel functions $\varphi_s^\mu, s \in \Pi_q$. For this, the Plancherel Theorem 3.26 as well as subexponential growth of the hypergroup are needed. For details, see [R8].

For $\mu = \varrho - 1 = \frac{d}{2}(2q-1)$, which corresponds to the orbit hypergroup $M_{p,q}^{U_p}$ with $p = 2q-1$, the integral defining the convolution $*_\mu$ becomes singular. The degenerate version of the convolution formula can be calculated after a suitable change of coordinates; this is carried out in [R8].

## 3.5 Hypergroups for Dunkl-type Bessel functions of type $B$

In the last section we saw that the cone $\Pi_q$ carries a continuously parametrized family of commutative hypergroup structures $*_\mu$ with $\mu > \varrho - 1$, as well as additional orbit hypergroup structures for $\mu = pd/2$, $p \geq q$ an integer. Let

$$\mathcal{M}_q := \Big\{ \frac{pd}{2}, \, p = q, q+1, \dots \Big\} \cup (\rho - 1, \infty).$$

In this section we study structures depending only on the matrix spectra. Under the mapping $r \mapsto \text{spec}(r)$, the hypergroup convolutions $*_\mu$ on the matrix cone $\Pi_q$ induce a (in part) continuous series of hypergroup convolutions on a Weyl chamber of type $B_q$. The characters turn out to be Dunkl-type Bessel functions. These hypergroups extend the harmonic analysis for Cartan motion groups associated with the Grassmann manifolds $U(p,q)/U_p \times U_q$.

We start with the geometric setting. Let $G = U(p,q)$ denote the indefinite unitary group of index $(p,q)$ over $\mathbb{F}$ with $p \geq q$. Its maximal compact subgroup $K$ is naturally isomorphic with $U_p \times U_q$. We may identify $M_{p,q}$ with the tangent space of the Riemannian symmetric space $G/K$ in the coset $eK$. This identification induces an action of $U_p \times U_q$ on $M_{p,q}$ according to

$$\big((u,v), x\big) \mapsto uxv^{-1}, \quad u \in U_p, \, v \in U_q. \tag{3.9}$$

The associated orbit space is canonically parametrized by the possible singular spectra of matrices from $M_{p,q}$ and is homeomorphic to

$$\Xi_q = \{\xi \in \mathbb{R}^q : \xi_1 \geq \ldots \geq \xi_q \geq 0\}$$

which is a closed Weyl chamber of type $B_q$. The action 3.9 induces an action of $U_q$ on the cone $\Pi_q = \{x^*x : x \in M_{p,q}\}$ by conjugation $(v,r) \mapsto vrv^{-1}$, which again leads to the chamber $\Xi_q$ orbit space. From the explicit formula for the hypergroup convolution $*_\mu$ on $\Pi_q$ one readily sees that the mapping $r \mapsto vrv^{-1}$ is a hypergroup automorphism in the sense of Remark 3.17. Therefore each convolution $*_\mu$ with $\mu \in \mathcal{M}$ induces a commutative hypergroup convolution $\circ_\mu$ on $\Xi_q$ by taking image measures under the canonical mapping

$$\sigma : \Pi_q \to \Xi_q, \, r \mapsto \sigma(r),$$

where $\sigma(r)$ denotes the set of eigenvalues of $r$ ordered by size. In the next theorem, we summarize the results obtained in [R8]; for convenience of notation, $\xi \in \Xi_q$ will be identified with the associated diagonal matrix from $\Pi_q$.

**3.29 Theorem.** (1) *For fixed* $d \in \{1,2,4\}$ *and each* $\mu \in \mathcal{M}_q$ *the chamber* $\Xi_q$ *carries a commutative hypergroup structure* $Y_\mu = (\Xi_q, \circ_\mu)$ *with convolution*

$$(\delta_\xi \circ_\mu \delta_\eta)(f) = \int_{U_q} (f\circ\sigma)(\xi *_\mu v\eta v^{-1})dv.$$

*The neutral element is* $0$ *and the involution is given by the identity mapping.*

(2) *A Haar measure on* $Y_\mu$ *is given by*

$$\widetilde{\omega}_\mu = d_\mu h_\mu(\xi)d\xi \quad with \quad h_\mu(\xi) = \prod_{i=1}^q \xi_i^{2\gamma+1}\prod_{i<j}(\xi_i^2-\xi_j^2)^d$$

*It is the image measure of* $\omega_\mu$ *under the mapping* $\sigma: r\mapsto \sigma(r)$.

(3) *The dual space* $\widehat{Y_\mu}$ *is parametrized by the chamber* $\Xi_q$ *and consists of the functions*

$$\psi_\xi^\mu(\eta) = \int_{U_q}\varphi_\xi^\mu(v\eta v^{-1})dv = J_k^B(\xi, i\eta), \quad \xi\in\Xi_q$$

*where* $J_k^B$ *is the Dunkl-type Bessel function associated with* $R=B_q$ *and the multiplicity* $k$ *is given by* $k=(k_1,k_2) = \big(\mu - \frac{d}{2}(q-1)-\frac{1}{2}, \frac{d}{2}\big)$. *Here* $k_1$ *and* $k_2$ *are the parameters on the roots* $\pm e_i$ *and* $\pm e_i \pm e_j$, *respectively.*

(4) *Under the identifictation of* $Y_\mu$ *with its dual via* $\mu\mapsto\psi_\xi^\mu$, *the Plancherel measure of* $Y_\mu$ *coincides with the Haar measure* $\widetilde{\omega}_\mu$.

In particular, the Bessel function $J_k^B$ in this case satisfies the positive product formula

$$J_k^B(\xi,z)J_k^B(\eta,z) = \int_{\Xi_q} J_k^B(\zeta,z)\,d(\delta_\xi\circ_\mu\delta_\eta)(\zeta) \quad \forall\,\xi,\eta\in\Xi_q,\, z\in\mathbb{C}^q.$$

Let us briefly comment on the proof of the second identity in part (3), that is the identification of the characters of $Y_\mu$ with Dunkl-type Bessel functions of type $B_q$. Firstly, one needs the fact that the spherical polynomials $Z_\lambda$ in the series representation of $\mathcal{J}_\mu$ satisfy the product formula

$$\frac{Z_\lambda(r)Z_\lambda(s)}{Z_\lambda(I)} = \int_{U_q} Z_\lambda(\sqrt{r}usu^{-1}\sqrt{r})du \quad \forall\, r,s\in\Pi_q,$$

see [FK], Cor. XI.3.2. This leads to an expresssion of $\psi_\xi^\mu$ as a hypergometric series of tpye ${}_0F_1$ of two arguments. Secondly, the obtained series can be identified, by a result of [BF1], as a Dunkl-type Bessel function. For details see [R8].

In the geometric cases $\mu = pd/2$, the convolution of the hypergroup $Y_\mu$ coincides with the convolution of the Gelfand pair $(U_p\times U_q)\ltimes M_{p,q}/(U_p\times U_q)$, and the support of the probability measure $\delta_\xi\circ_\mu\delta_\eta$ on $\Xi_q$ describes the set of possible singular spectra of sums $x+y$ with matrices $x,y\in M_{p,q}$ having given singular spectra $\xi$ and $\eta$.

# 4 Markov processes

The main object of this chapter are Markov processes on $\mathbb{R}^N$ and closed Weyl chambers $\bar{C} \subset \mathbb{R}^N$ whose transition probabilities are related with Dunkl theory. The most prominent examples are the so-called Dunkl processes on $\mathbb{R}^N$ which are generated by the Dunkl-Laplacians $\Delta_k$ as well as their symmetrized counterparts on $\bar{C}$ which are diffusions and called Dunkl-Bessel processes.

The concept of general Markov processes on $\mathbb{R}^N$ related with Dunkl theory in [RV1] is motivated by random walks on groups which can be studied via group representations, characters and the Fourier transform, as well as random walks on hypergroups. These topics are connected in several respects.

First, under suitable symmetry assumptions and up to projections, the same processes appear as random walks on groups, as random walks on hypergroups, and as processes on Weyl chambers $\bar{C}$ on different levels after taking projections. These relations on the level are caused in fact by the algebraic relations between the underlying state spaces explained in the preceding Chapter.

Second, the very concept of Markov processes on $\mathbb{R}^N$ related with Dunkl theory as well as the concept of random walks on hypergroups may be regarded within a common frame, namely Markov processes on a state space which admits an integral transform. In Dunkl theory, this transform will be the Dunkl transform, and in case of commutative hypergroups, the hypergroup Fourier transform. This concept of an integral transform allows somehow a common diagonalization of all associated transition operators. We develop this concept in Section 4.2 and show in the further sections how this concept leads to interesting martingales and martingale characterizations. The main emphasis will be on Markov processes on $\mathbb{R}^N$ related with Dunkl theory and Dunkl processes.

## 4.1 Random walks on groups and hypergroups

In this section we briefly recall the concepts of random walks on groups and hypergroups.

**4.1 Definition.** Let $G$ be a locally compact group with identity $e$.

Let $(Y_n)_{n\geq 1}$ be a sequence of $G$-valued independent random variables with laws $\mu_n \in M^1(G)$. Then we form the right random walk $(X_n := Y_1 \dots Y_n)_{n\geq 1}$ (with the convention $X_0 = e$). This process is a Markov process on $G$ starting in $e$ with transition probabilities

$$P(X_{n+1} \in A|\, X_n = x) = (\delta_x * \mu_{n+1})(A) \quad (n \geq 0,\ x \in G,\ A \in \mathscr{B}(G)).$$

In continuous time, we proceed as follows: A $G$-valued process $(X_t)_{t\geq 0}$ is called a random walk in continuous time (or a process with right-independent increments), if for all $n \in \mathbb{N}$, $0 = t_0 < t_1 < \ldots < t_n$, the random variables $X_{t_0}, X_{t_0}^{-1}X_{t_1}, \ldots, X_{t_{n-1}}^{-1}X_{t_n}$ are independent. Denoting the laws of $X_s^{-1}X_t$ by $\mu_{s,t}$ for $0 \leq s \leq t$, we obtain that $(X_t)_{t\geq 0}$ is a Markov process on $G$ with initial distribution $P_{X_0}$ and transition probabilities

$$P(X_t \in A|\, X_s = x) = (\delta_x * \mu_{s,t})(A) \quad (0 \leq s \leq t,\ x \in G,\ A \in \mathscr{B}(G)).$$

This characterization of random walks on groups in discrete or continuous time as Markov process with translation invariant transition kernels motivates:

**4.2 Definition.** Let $(X, *)$ be a (always second countable) hypergroup, and let $I = \mathbb{Z}_+$ or $I = [0, \infty)$. A $X$-valued Markov process $(X_t)_{t \in I}$ is called a random walk on $X$, if for all $s \leq t \in I, x \in X$, and $A \in \mathscr{B}(X)$,

$$P(X_t \in A|\, X_s = x) = (\delta_x * \mu_{s,t})(A)$$

for a suitable family $(\mu_{s,t})_{s \leq t \in I}$ of probability measures on $X$.

It can be easily checked that the $\mu_{s,t}$ form a so-called hemigroup, i.e., for $s \leq t \leq u \in I$, we have $\mu_{s,t} * \mu_{t,u} = \mu_{s,u}$. Moreover, for $t \in I$, the law of $X_t$ is given by $P_{X_0} * \mu_{0,t}$ as in the group case. Moreover, standard arguments on the construction of Markov processes ensure that for a given initial law and a given hemigroup $(\mu_{s,t})_{s \leq t \in I} \subset M^1(X)$ there always exists an associated random walk on a suitable probability space.

We next turn to relations between random walks on different groups or hypergroups.

**4.3 Example.** Let $(G, K)$ be a Gelfand pair and $(X_t)_{t \in I}$ a random walk on $G$ starting in $e$ such that all transition probabilities $\mu_{s,t}$ are $K$-biinvariant, i.e., contained in $M^1(G||K)$. We then say that $(X_t)_{t \in I}$ is a $K$-biinvariant random walk on $G$. It is well-known that $K$-biinvariant random walks on $G$ may be analyzed via spherical Fourier transform. In particular, central limit theorems are often derived in this way similar to the classical approach on the group $\mathbb{R}^N$.

We may regard this approach also as follows: Consider the double coset hypergroup $(X = G//K, *)$ and the canonical projection $p : G \to G//K$. It can be easily checked (see below) that the process $(p(X_t))_{t \in I}$ then is a random walk on the commutative hypergroup $(X = G//K, *)$. In this way, an investigation of this random walk on $(X = G//K, *)$ via the hypergroup Fourier transform corresponds precisely to the study of $K$-biinvariant random walks on $G$ via the spherical Fourier transform.

**4.4 Lemma.** *In the preceding setting, $(p(X_t))_{t \in I}$ is a random walk on the commutative hypergroup $(X = G//K, *)$ associated with the hemigroup $(p(\mu_{s,t}))_{s \leq t \in I}$.*

*Proof.* Let $(\mathcal{F})_{t \in I}$ be the canonical filtration of the Markov process $(X_t)_{t \in I}$ on $G$ defined on some probability space $(\Omega, \mathcal{A}, P)$, and let $(\tilde{\mathcal{F}})_{t \in I}$ be the canonical filtration of $(p(X_t))_{t \in I}$. Fix $s \leq t \in I$ and $A \in \mathscr{B}(X)$. We first note that the function $x \longmapsto (\delta_x * \mu_{s,t})(p^{-1}(A))$ on $G$ is $K$-biinvariant. This implies

$$(\delta_x * \mu_{s,t})(p^{-1}(A)) = (\delta_{p(x)} * p(\mu_{s,t}))(A) \quad \text{for all} \quad x \in G.$$

Therefore, by the Markov property of $(X_t)_{t \in I}$,

$$\begin{aligned} P(p(X_t) \in A|\, \mathcal{F}_s) &= P(X_t \in p^{-1}(A)|\, X_s) = (\delta_{X_s} * \mu_{s,t})(p^{-1}(A)) \\ &= (\delta_{p(X_s)} * p(\mu_{s,t}))(A) \end{aligned}$$

a.s.. This implies that

$$P(p(X_t) \in A|\, \tilde{\mathcal{F}}_s) = P(p(X_t) \in A|\, p(X_s)) = (\delta_{p(X_s)} * p(\mu_{s,t}))(A)$$

a.s. as claimed. □

There exists the following variant of the preceding example:

**4.5 Example.** Let $V$ be a locally compact abelian group on which a compact group $K$ of automorphisms acts continuously. Let $(X_t)_{t\in I}$ be a random walk on $V$ starting in the identity $e$ such that its transition probabilities are $K$-invariant, i.e., for all $0 \leq s \leq t$ and $k \in K$, $k(\mu_{s,t}) = \mu_{s,t}$. We then say that $(X_t)_{t\in I}$ is $K$-invariant.

Consider the commutative orbit hypergroup $(V^K, *)$ and the associated canonical projection $p : V \to V^K$. In the same way as in the preceding lemma it may be checked that then $(p(X_t))_{t\in I}$ is a random walk on $(V^K, *)$ starting in the identity with transition probabilities $(p(\mu_{s,t}))_{s\leq t\in I} \subset M^1(V^K)$.

**4.6 Examples.** (1) Let $V = \mathbb{R}^N$ and $K = O(N)$. Then $V^k \simeq [0, \infty)$, and the radial parts of $O(N)$-invariant random walks on $\mathbb{R}^N$ are random walks on the Bessel-Kingman hypergroup of index $\alpha = N/2 - 1$. In particular, this holds for Bessel processes as radial parts of Brownian motions.

(2) Let $V = M_{p,q}(\mathbb{F})$ with $p, q \in \mathbb{N}$ and $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Let $K := U_p(\mathbb{F})$ act on $V$ from the left. Then, by Section 3.4 ), $V^K \simeq \Pi_q$, where the canonical projection $p : V \to V^K$ is given by $p(x) = (x^*x)^{1/2}$. Therefore, if $(X_t)_{t\in I}$ is an $K$-invariant random walk on $M_{p,q}(\mathbb{F})$, then $((X_t^* X_t)^{1/2})_{t\in I}$ is a random walk on the matrix Bessel hypergroup $(M_{p,q}(\mathbb{F}), *)$ with index $\mu = pd/2$. In particular, Wishart processes with these shape parameters may be regarded as random walks on these hypergroups. The reader should be careful with a different norming of these processes in the literature; see [B] and [V2].

(3) Let $V = H_n(\mathbb{F})$ the space of all Hermitian matrices on which $K = U_n(\mathbb{F})$ acts by conjugation. Then $V^K$ may be identified with the Weyl chamber $\bar{C}_n$ of type $A_{n-1}$ by taking the ordered spectrum. If $(X_t)_{t\in I}$ is an $U_n(\mathbb{F})$-conjugation invariant random walk on $H_n(\mathbb{F})$, then its spectral part forms a random walk on the associated orbit hypergroup $(\bar{C}_n, *)$ which is associated with Dunkl theory of type $A_{n-1}$ with multiplicity $k = 1$.

Similar interpretations exist for the root systems $B_n, C_n, D_n$.

The concept of Example 4.5 may be easily transfered to the case where $V$ is a commutative hypergroup: Let $(X, *)$ be a commutative hypergroup on which a compact group of hypergroup automorphisms acts continuously. Let $(X_t)_{t\in I}$ be a random walk on $(X, *)$ starting in the identity $e$ such that its transition probabilities are $K$-invariant, i.e., for all $0 \leq s \leq t$ and $k \in K$, $k(\mu_{s,t}) = \mu_{s,t}$. We then say that $(X_t)_{t\in I}$ is $K$-invariant. Consider the commutative orbit hypergroup $(X^K, *)$ and the associated canonical projection $p : X \to X^K$. As above, $(p(X_t))_{t\in I}$ then is a random walk on $(V^K, *)$ starting in the identity with transition probabilities $(p(\mu_{s,t}))_{s\leq t\in I} \subset M^1(V^K)$.

**4.7 Example.** Consider the matrix Bessel hypergroup $(\Pi_q(\mathbb{F}), *)$ of some admissable index $\mu$ as in Section 3.4. As explained there, the group $U_q(\mathbb{F})$ acts on $\Pi_q(\mathbb{F})$ by conjugation as a group of hypergroup automorphisms, and the associated orbit space can be identified with the Weyl chamber $\bar{C}_q$ of type $B_q$. Therefore, the spectrum of $U_q(\mathbb{F})$-conjugation invariant random walks on the Bessel hypergroup $(\Pi_q(\mathbb{F}), *)$ are random walks on the associated orbit hypergroup structures on $\bar{C}_q$ which belong, by Section 3.4, to Dunkl theory with multiplicities $k_1 = \mu - d(q-1)/2 - 1/2$ and $k_2 = d/2$ with $d = dim_{\mathbb{R}}\mathbb{F}$.

In summary, projections of sufficiently symmetric classical random walks on the Euclidean space $M_{p,q}(\mathbb{F})$ appear as random walks on different levels: As random walks on the matrix Bessel hypergroup $(\Pi_q(\mathbb{F}),*)$ of index $\mu = pd/2$, as random walks on the commutative hypergroup on the Weyl chamber $\bar{C}_q$ of type $B_q$ which is associated with Dunkl theory with the multiplicities above, and finally as random walks on the one-dimensional Bessel-Kingman hypergroup $[0,\infty)$ of index $\alpha = dpq/2 - 1$.

## 4.2 Markov processes related with integral transforms

In this section we introduce a concept which allows to study many features of random walks on commutative hypergroups as well as of Markov processes on $\mathbb{R}^N$, which are related with the Dunkl transform, within of a common frame. The concept is that of processes related with an abstract integral transform on the underlying state space. This transform will be either the Dunkl transform or the Fourier transform on a commutative hypergroup. Further integral transforms related with suitable families of special functions are also possible.

The following definition is suitable for our purposes:

**4.8 Definition.** Let $X$ be a second countable, locally compact space, and $(\varphi_\lambda)_{\lambda\in\widehat{X}}$ a family of functions in $C_b(X)$ labeled by some further locally compact space $\widehat{X}$ with $|\varphi_\lambda(x)| \le 1$ for $x \in X, \lambda \in \widehat{X}$. The triple $T = (X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$ will be called an abstract integral transform, if the following holds:

(1) The mapping $\lambda \mapsto \varphi_\lambda$ is continuous w.r.t. compact-uniform topology on $C_b(X)$.

(2) There exists $e \in X$ with $\varphi_\lambda(e) = 1$ for all $\lambda \in \widehat{X}$, and $\widehat{e} \in \widehat{X}$ with $\varphi_{\widehat{e}}(x) = 1$ for all $x \in X$.

(3) Riemann-Lebesgue Lemma: There exists $\pi \in M^+(\widehat{X})$ such that for all $f \in L^1(\widehat{X},\pi)$, the function $\check{f}(x) := \int_{\widehat{X}} f(\lambda)\varphi_\lambda(x)\, d\pi(\lambda)$ satisfies $\check{f} \in C_0(X)$, and $\{\check{f} : f \in L^1(\widehat{X},\pi)\}$ is $\|.\|_\infty$-dense in $C_0(X)$.

**4.9 Examples.** (1) Let $X = \widehat{X} = \mathbb{R}^N$, $e = \widehat{e} = 0$, $d\pi(x) = w_k(x)dx$, and $\varphi_\lambda(x) := E_k(-ix,\lambda)$ the Dunkl kernel for some multiplicity $k \ge 0$. Then all axioms above are satisfied by Section 2.5.

(2) Let $X$ be a (from now on always second countable) commutative hypergroup with neutral element $e$, dual space $\widehat{X}$, and Plancherel measure $\pi$. Taking $\varphi_\lambda(x) := \overline{\lambda(x)}$ for $\lambda \in \widehat{X}$, $x \in X$ and $\widehat{e} \equiv 1$, we obtain all axioms above from Section 3.2.

(3) By interchanging the roles of the commutative hypergroup $X$ and its dual $\widehat{X}$ and taking the Haar measure instead of $\pi$, one also obtains also all axioms above from Section 3.2.

We collect a few obvious properties of abstract integral transforms:

**4.10 Proposition.** (1) *The integral transform* $M_b(X) \to C_b(\widehat{X})$, $\mu \mapsto \widehat{\mu}$ *with* $\widehat{\mu}(\lambda) = \int_X \varphi_\lambda(x)\, d\mu(x)$ *is injective.*

(2) *For* $\mu \in M_b(X)$ *and* $f \in L^1(\widehat{X},\pi)$, $\int_X \check{f}\, d\mu = \int_{\widehat{X}} f\widehat{\mu}\, d\pi$.

(3) *The $\varphi_\lambda$ separate points of $X$, and $e$ is unique.*

*Proof.* (2) is clear and leads together with 4.8(1) and 4.8(3) to (1). Part (3) is then clear. □

Up to a further technical restriction, the axioms are also strong enough in order to yield Lévy's continuity theorem in a strong version:

**4.11 Theorem.** *Let $(\mu_n)_{n\in\mathbb{N}} \subset M_b^+(X)$.*

(1) *If $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to $\mu \in M_b^+(X)$, then $(\widehat{\mu}_n)_{n\in\mathbb{N}}$ converges to $\widehat{\mu}$ pointwise.*

(2) *If $(\widehat{\mu}_n)_{n\in\mathbb{N}}$ tends pointwise to a $\mathbb{C}$-valued function $\varphi$ on $\widehat{X}$ continuous at $\widehat{e}$ and if $\widehat{e} \in supp\,\pi$ holds, then there is a unique $\mu \in M_b^+(X)$ with $\widehat{\mu} = \varphi$, and $(\mu_n)_{n\in\mathbb{N}}$ tends weakly to $\mu$.*

Notice that the additional condition in (2) holds in the Dunkl setting and precisely for the commutative hypergroups where the identity character is contained in $supp\,\pi$. This is the case for all commutative hypergroups with subexponential growth and in particular for all double coset hypergroups of Euclidean type in Chapter 3.2 as well as for all Bessel hypergroups on matrix cones.

*Proof.* Part (1) is obvious. For the proof of (2) we use the well-known approach of Siebert which works for commutative groups and hypergroups; see e.g. Section 4.2 of [BH]: By the assumptions, $\lim_n \|\mu_n\| = \lim_n \widehat{\mu}_n(\widehat{e}) = \varphi(\widehat{e})$. Therefore, by a compactness argument, there is a $\sigma(M_b(X), C_0(X))$-convergent subsequence $(\mu_{n_k})_k$ with some limit $\mu \in M_b^+(X)$. Therefore, by Proposition 4.10(2), we obtain for all $g \in L^1(\widehat{X}, \pi)$ that $\lim_k \int_{\widehat{X}} g\widehat{\mu}_{n_k}\, d\pi = \int_{\widehat{X}} g\widehat{\mu}\, d\pi$. Moreover, as $\|\widehat{\mu}_n\|_\infty$ remains bounded, this limit is also equal to $\int_{\widehat{X}} g\varphi\, d\pi$. As $\varphi$ and $\widehat{\mu}$ are continuous at $\widehat{e} \in supp\,\pi$, we conclude that $\varphi(\widehat{e}) = \widehat{\mu}(\widehat{e})$. This shows that $\lim_k \|\mu_{n_k}\| = \|\mu\|$, i.e., $(\mu_{n_k})_k$ tends weakly to $\mu$. Therefore, $\widehat{\mu} = \varphi$, i.e., $\mu$ is determined uniquely and independent of the subsequence. As the closed unit ball in $M_b^+(X)$ is compact and metrizable w.r.t. $\sigma(M_b^+(X), C_0(X))$, we readily obtain that $(\mu_n)_n$ converges to $\mu$. □

We notice that for commutative hypergroups $X$ the following weaker continuity result holds (see Section 4.2 of [BH]): If for measures $\mu_n, \mu \in M_b^+(X)$, $\widehat{\mu}_n$ tends pointwise to $\widehat{\mu}$ on $\widehat{X}$, then $\mu_n$ tends weakly to $\mu$.

In any case, all versions of Lévy's continuity theorem are strong enough for application in probability, e.g., in order to derive CLTs. We do not go into details and refer to [BH] for applications to random walks on hypergroups.

We next turn to Markov kernels associated with abstract integral transforms. For this we fix an abstract integral transform $T = (X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$.

**4.12 Definition.** A Markov kernel $P : X \times \mathcal{B}(X) \to [0,1]$ is called related with $T$ if

$$P(x,.)^\wedge(\lambda) = P(e,.)^\wedge(\lambda) \cdot \varphi_\lambda(x) \qquad \text{for all } x \in X, \lambda \in \widehat{X}. \tag{4.1}$$

**4.13 Examples.** (1) Let $X$ be a commutative hypergroup with dual $\widehat{X}$. Let $\mu \in M^1(X)$. It can be easily checked that $P_\mu(x, A) := \mu * \delta_x(A)$ establishes a "translationinvariant" Markov kernel on $X$. This kernel is obviously related with the hypergroup Fourier transform.

(2) For $X = \widehat{X} = \mathbb{R}^N$ and the Dunkl transform, condition (4.1) can be translated easily. We then obtain the notion of a kernel $P$ related with the Dunkl transform. For multiplicity $k = 0$, i.e., the group case, this notion agrees by the injectivity of the classical Fourier transform with the usual notion of translation invariant kernels on $\mathbb{R}^N$. We notice that it is an open problem, for which probability measures $\mu \in M^1(\mathbb{R}^N)$ precisely there exists a Markov kernel $P$ on $\mathbb{R}^N$ with $\mu = P(0,.)$. On the other hand there exist many examples like all pseudo-radial probability measures on $\mathbb{R}^N$ in the sense of Section 2.8, which in particular includes the Dunkl Gaussians.

We next collect basic properties of such Markov kernels.

**4.14 Lemma.** *Let $P$ and $Q$ be Markov kernels related with the abstract integral transform as above.*

(1) $Pf(x) := \int_X f(y)\, P(x,dy)$ *defines a bounded linear operator on* $C_0(X)$.

(2) *The composition* $P \circ Q$ *with* $P \circ Q(x,A) = \int_X Q(z,A) P(x,dz)$ *is a Markov kernel on $X$ related with $T$, and*

$$((P\circ Q)(x,.))^\wedge(\lambda) = Q(e,.)^\wedge(\lambda)\cdot P(e,.)^\wedge(\lambda)\cdot \varphi_\lambda(x) \;\; for\;\; x\in X, \lambda\in\widehat{X}. \tag{4.2}$$

*In particular,* $P\circ Q = Q\circ P$.

*Proof.* For (1), it suffices to check $Pf \in C_0(X)$ for $f \in C_0(X)$. As $\{\check{g} : g \in L^1(\widehat{X},\pi)\}$ is $\|.\|_\infty$-dense in $C_0(X)$, it suffices by an $\epsilon$-estimate, to do this for $f = \check{g}$, $g \in L^1(\widehat{X},\pi)$. In this case we obtain from Proposition 4.10(2)

$$\begin{aligned} Pf(x) &= \int_X \check{g}(y)\, P(x,dy) = \int_{\widehat{X}} g(\lambda)\cdot P(x,.)^\wedge(\lambda)\, d\pi(\lambda) \\ &= \int_{\widehat{X}} g(y)\cdot \varphi_\lambda(x) P(e,.)^\wedge(\lambda)\, d\pi(\lambda) = (g\cdot P(e,.)^\wedge)^\vee(x) \end{aligned}$$

which is a function in $C_0(X)$ by our axioms. This implies (1). (2) is easy to check; compare e.g. with Lemma 4.2 of [RV1]. □

We now turn to families of Markov kernels on $X$ and associated Markov processes related to an abstract integral transform $T$. For this fix a time area $I = [0,\infty)$ or $I = \mathbb{Z}_+$, and put $S := \{(s,t) : s,t \in I,\ s \le t\}$.

**4.15 Definition.** A family $(P_{s,t})_{s\le t\in I}$ of Markov kernels on $X$ related to $T$ is called a (continuous) hemigroup of Markov kernels related with $T$, if

(1) for all $s \le t \le u \in I$, $P_{s,t} \circ P_{t,u} = P_{s,u}$,

(2) $P_{t,t}$ is the trivial kernel for $t \in I$, and

(3) the mapping $S \to M^1(X)$, $(s,t) \mapsto P_{s,t}(0,.)$, is weakly continuous.

In particular, if in addition $P_{s,t} = P_{s+u,t+u}$ for all $(s,t) \in S$ and $u \in I$, then we may put $P_t := P_{0,t}$ for $t \in I$ and obtain a continuous semigroup $(P_t)_{t\in I}$ of Markov kernels related with $T$.

Assume that a hemigroup $(P_{s,t})_{s\leq t\in I}$ of Markov kernels on $X$ related to $T$ as well as a starting probability $\mu\in M^1(X)$ are given. Then we can construct in a canonical way an associated Markov process on $X$. Such processes will be called Markov processes on $X$ related with the integral transform $T$. In the case of a semigroup we obtain time-homgoeneous Markov processes on $X$ related with $T$; they are analogs of Lévy processes.

**4.16 Examples.** (1) Let $(\mu_t)_{t\geq 0}\subset M^1(X)$ be a convolution semigroup on a commutative hypergroup $X$, i.e., $\mu_s*\mu_t=\mu_{s+t}$ for $s,t\geq 0$ with $\mu_0=\delta_e$, and $[0,\infty)\to M^1(X)$, $t\mapsto\mu_t$ is weakly continuous. Then the kernels $P_t(x,A):=\mu_t*\delta_x(A)$ $(x\in X,\ A\in\mathcal{B}(X))$ form a semigroup of Markov kernels related to the hypergroup Fourier transform.

(2) For any root system and any multiplicity $k\geq 0$, the Dunkl heat kernels $P_t(x,dy)=\Gamma_k(t,x,y)w_k(y)dy$ on $\mathbb{R}^N$ of Chapter 2.6 with

$$\Gamma_k(t,x,y):=\frac{1}{(2t)^{\gamma+N/2}c_k}\,e^{-(|x|^2+|y|^2)/4t}\,E_k\Big(\frac{x}{\sqrt{2t}},\frac{y}{\sqrt{2t}}\Big),\quad x,y\in\mathbb{R}^N,$$

and $t>0$ form (together with the trivial kernel $P_0$) a semigroup of Markov kernels related to the Dunkl transform.

**4.17 Remark.** If $\widehat{X}$ is connected (which is for instance the case in Dunkl theory), then condition (2) of Definition 4.15 holds automatically. In fact, $P_{t,t}=P_{t,t}\circ P_{t,t}$ and Lemma 4.14(2) ensure that $P_{t,t}(e,.)^\wedge$ is a $\{0,1\}$-valued function with $P_{t,t}(e,.)^\wedge(\widehat{e})=1$, and hence $P_{t,t}(e,.)^\wedge\equiv 1$ which implies together with the injectivity of the integral transform the assertion.

**4.18 Lemma.** *Let $(P_{s,t})_{s\leq t\in[0,\infty)}$ be a hemigroup of Markov kernels on $X$ related with $T$. Then for all $(s,t)\in S$ and $\lambda\in\widehat{X}$, $P_{s,t}(e,.)^\wedge(\lambda)\neq 0$.*

*Proof.* For $\lambda\in\widehat{X}$, the mapping $(s,t)\mapsto P_{s,t}(e,.)^\wedge(\lambda)\neq 0$ is continuous with $P_{t,t}(e,.)^\wedge(\lambda)=1$ and $P_{s,t}(e,.)^\wedge(\lambda)\cdot P_{t,u}(e,.)^\wedge(\lambda)=P_{s,u}(e,.)^\wedge(\lambda)$. Therefore, $P_{s,t}(e,.)^\wedge(\lambda)=0$ immediately would lead to a contradiction. □

We now restrict our attention to the time-homogeneous case.

**4.19 Remark.** The arguments of the proof of 4.18 imply for a semigroup $(P_t)_{t\geq 0}$ of Markov kernels on $X$ that there exists a function $\psi:\widehat{X}\to\mathbb{C}$ with $P_t(e,.)^\wedge(\lambda)=e^{-t\psi(\lambda)}$ for $t\geq 0$, $\lambda\in\widehat{X}$. The function $\psi$ satisfies $\mathrm{Re}\,\psi\geq 0$ and

$$\psi(\lambda)=\lim_{t\downarrow 0}\frac{1}{t}(1-P_t(0,.)^\wedge(\lambda))\qquad(\lambda\in\widehat{X}).\tag{4.3}$$

Moreover, $\psi$ is continuous because of

$$\Big(\int_0^\infty e^{-t}\,P_t(e,.)\,dt\Big)^\wedge(\lambda)=\int_0^\infty e^{-t}\,e^{-t\psi(\lambda)}\,dt=(1+\psi(\lambda))^{-1}.$$

$\psi$ is called the negative definite function associated with $(P_t)_{t\geq 0}$.

**4.20 Proposition.** *Each semigroup $(P_t)_{t\geq 0}$ of kernels on $X$ related to $T$ is a Feller semigroup, i.e., for $f\in C_0(X)$ and $t\geq 0$, $P_tf\in C_0(X)$, and*

$$\lim_{t\to 0}\|P_tf-f\|_\infty=0\quad\textit{for}\quad f\in C_0(X).\tag{4.4}$$

*Proof.* It suffices to check (4.4). Taking $f = \check{g}$, $g \in L^1(\widehat{X}, \pi)$, we have

$$|P_t f(x) - f(x)| = \Big| \int_{\widehat{X}} g(\lambda) \Big( P_t(e,0)^\wedge(\lambda) - 1 \Big) \psi_\lambda(x) \, d\pi(\lambda) \Big|$$
$$\leq \int_{\widehat{X}} |g(\lambda)| \, |P_t(e,0)^\wedge(\lambda) - 1| \, d\pi(\lambda)$$

which tends for $t \to 0$ to 0 independent of $x$. In other words, (4.4) holds for $f = \check{g}$, $g \in L^1(\widehat{X}, \pi)$. As the space of these functions is $\|.\|_\infty$-dense in $C_0(X)$, and the operators $P_t$ on $C_0(X)$ satisfy $\|P_t\| \leq 1$, (4.4) follows by an $\epsilon$-argument for $f \in C_0(X)$. □

Proposition 4.20 together with the theorem of Dynkin-Kinney-Blumenthal (see, e.g., [Dy]) imply:

**4.21 Corollary.** *Each time-homogeneous Markov process related with an abstract integral transform admits a càdlàg modification, i.e. a modification with right continuous paths and left limits.*

Let $(P_t)_{t \geq 0}$ be a semigroup of Markov kernels on $X$ related to some abstract integral transform $T$. As this semigroup forms a positive contraction semigroup on $C_0(X)$, we introduce the generator

$$Lf := \lim_{t \to 0} \frac{1}{t}(P_t f - f)$$

which is a closed operator with a $\|.\|_\infty$-dense domain in $C_0(X)$ by the Hille-Yoshida theory. We also define the following extended domains:

$$D(L) := \{ f \in C(X) : \ \frac{1}{t}(P_t f - f) \ \text{converges uniformly on } X \text{ for } t \to 0 \},$$

and

$$D_b(L) := D(L) \cap C_b(X), \quad D_0(L) := D(L) \cap C_0(X).$$

$D_0(L)$ is the domain of $L$ on $C_0(X)$, and $Lf \in C_0(X)$ for $f \in D_0(L)$.

Moreover, for $t > 0$, $x \in X$ and $\lambda \in \widehat{X}$,

$$P_t \varphi_\lambda(x) = P_t(x,.)^\wedge(\lambda) = P_t(e,.)^\wedge(\lambda) \varphi_\lambda(x) = P_t \varphi_\lambda(e) \cdot \varphi_\lambda(x)$$

and hence

$$\lim_{t \to 0} \frac{1}{t}(P_t \varphi_\lambda(x) - \varphi_\lambda(x)) = -\psi(\lambda) \varphi_\lambda(x)$$

uniformly with the negative definite function $\psi$ of Remark 4.19. In particular, we have $\{\varphi_\lambda : \lambda \in \widehat{X}\} \subset D_b(L)$ with $L\varphi_\lambda = -\psi \cdot \varphi_\lambda$.

We next introduce a notion of Gaussian semigroups analog to locally compact groups, which also works for commutative hypergroups.

**4.22 Definition.** Let $(P_t)_{t \geq 0}$ be a semigroup of Markov kernels on $X$ related with some abstract integral transform with the generator $L$ on $C_0(X)$.

(1) $(P_t)_{t \geq 0}$ is called Gaussian if

$$\lim_{t \to 0} \frac{1}{t} P_t(e, X \setminus U) = 0 \quad \text{for all open subsets } U \subset X \text{ with } e \in U.$$

(2) $L$ is called of local type if $supp(Lf) \subset supp\, f$ for all $f \in D_c(L)$.

It is well-known from the theory of Feller processes (see e.g. [Dy]) that its generator $L$ is of local type if and only if each associated Feller process admits an a.s. continuous modification. Moreover, the following characterization of Gaussian processes on commutative hypergroups, which is analog to the group case, can be found in [ReV].

**4.23 Theorem.** *Let $(\mu_t)_{t\geq 0} \subset M^1(X)$ be a convolution semigroup on a commutative hypergroup $X$. Then the following are equivalent:*

*(1) $(\mu_t)_{t\geq 0}$ is Gaussian, i.e., the associated semigroup of kernels is Gaussian.*

*(2) $L$ is of local type.*

*(3) Each Lévy process on $X$ associated with $(\mu_t)_{t\geq 0}$ admits an a.s. continuous modification.*

This equivalence is not longer valid for time-homogeneous Markov processes related with abstract integral transforms. In fact, for multiplicities $k \geq 0$, the heat kernels of Dunkl type satisfy

$$P_t(0, dy) = \frac{1}{(2t)^{\gamma+N/2} c_k} e^{-|y|^2/4t} w_k(y)\, dy,$$

i.e., like classical Gaussian kernels they are Gaussian in the sense of 4.22(1). On the other hand, the generator of this heat semigroup is the Dunkl Laplacian, which is obviously of local type if and only if $k \equiv 0$ holds. Therefore, taking Theorem 4.23 into account, we conclude:

**4.24 Corollary.** *For any root system and any multiplicity $k \geq 0$ with $k \not\equiv 0$, there exists no commutative hypergroup structure on $\mathbb{R}^N$ with the Dunkl transform as hypergroup Fourier transform.*

We notice that for $N = 1$, explicit formulas for the Dunkl convolution are known ([R1] and [Ros]; see Chapter 2.8), which immediately show that this convolution is not positivity preserving.

## 4.3 Martingales associated with integral transforms

Let $T = (X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$ be an abstract integral transform as introduced in the preceding section. We here construct martingales related to Markov processes on $X$ related with $T$ in a canonical way by using the functions $\varphi_\lambda$ which play the role of characters.

For the classical Brownian motion on the group $\mathbb{R}^N$, the following martingale characterization can be found in many textbooks on stochastic analysis as a preparation to the Lévy characterization of Brownian motion. Extensions of this characterization to Lévy processes on locally compact groups or commutative hypergroups can be found e.g. in [ReV], [V1], [HP]. Moreover, a version for processes on $\mathbb{R}^N$ related with the Dunkl transform, can be found in [RV1].

**4.25 Theorem.** *Let $I = [0, \infty)$ or $I = \mathbb{Z}_+$. Let $(P_{s,t})_{s \le t \in I}$ a hemigroup of Markov kernels on $X$ related to $T$. Then an arbitrary stochastic process $(X_t)_{t \in I}$ on $X$ is a Markov process related with the hemigroup $(P_{s,t})_{s \le t \in I}$ if and only if*

$$\Big( \frac{1}{P_{0,t}(e,.)^\wedge(\lambda)} \cdot \varphi_\lambda(X_t) \Big)_{t \ge 0} \tag{4.5}$$

*is a martingale for each $\lambda \in \widehat{X}$ w.r.t. the canonical filtration $(\mathcal{F}_t)_{t \in I}$ of $(X_t)_{t \in I}$.*

*Proof.* Notice first that $P_{s,t}(x,.)^\wedge(\lambda) \neq 0$ for all $t \ge s \ge 0$, $x \in X$, $\lambda \in \widehat{X}$ by Lemma 4.18. This ensures that the processes above are well-defined. Let $(\Omega, \mathcal{A}, P)$ be the probability space on which the process $(X_t)_{t \in I}$ is defined.

To check the only-if-part, take $s, t \in I$ and $\lambda \in \widehat{X}$. Then for a.e. $\omega \in \Omega$,

$$\begin{aligned} E(\varphi_\lambda(X_{s+t})|\mathcal{F}_s)(\omega) = E(\varphi_\lambda(X_{s+t})|X_s)(\omega) &= \int_X \varphi_\lambda(x)\, P_{s,s+t}(X_s(\omega), dx) \\ &= \varphi_\lambda(X_s(\omega)) \cdot P_{s,s+t}(e,.)^\wedge(\lambda). \end{aligned}$$

Hence, as $P_{0,s+t}(e,.)^\wedge = P_{0,s}(e,.)^\wedge \cdot P_{s,s+t}(e,.)^\wedge$, the process (4.5) is a martingale.

To check the if-part, take again $s, t \in I$ and $\lambda \in \widehat{X}$. Then, by our assumption and the preceding equation,

$$E(\varphi_\lambda(X_{s+t})|\mathcal{F}_s) = \varphi_\lambda(X_s) \cdot P_{s,s+t}(e,.)^\wedge(\lambda). \quad \text{a.s..}$$

Now take $F \in \mathcal{F}_s$ with $P(F) > 0$. Define the probability measure $P_F$ on $(\Omega, \mathcal{A})$ by $P_F(A) := \frac{P(A \cap F)}{P(F)}$. The distributions $\mu_s^F, \mu_{s+t}^F \in M^1(X)$ of $X_s$ and $X_{s+t}$ w.r.t. $P_F$ satisfy

$$\begin{aligned} (\mu_{s+t}^F)^\wedge(\lambda) &= \int_X \varphi_\lambda(y)\, d\mu_{s+t}^F(y) = \frac{1}{P(F)} \int_F \varphi_\lambda(X_{s+t})\, dP \\ &= \frac{1}{P(F)} \int_F E(\varphi_\lambda(X_{s+t})|\mathcal{F}_s)\, dP \\ &= \frac{1}{P(F)} \int_F P_{s,s+t}(e,.)^\wedge(\lambda) \cdot \varphi_\lambda(X_s)\, dP \\ &= P_{s,s+t}(e,.)^\wedge(\lambda) \cdot (\mu_s^F)^\wedge(\lambda) = (P_{s,s+t} \circ \mu_s^F)^\wedge(\lambda). \end{aligned}$$

As this holds for all $\lambda \in \widehat{X}$, the injectivity of our integral transform yields that $\mu_{s+t}^F = P_{s,s+t} \circ \mu_s^F$. Hence, for each Borel set $B \subset X$ and each $F \in \mathcal{F}_s$,

$$\begin{aligned} \int_F 1_{\{X_{s+t} \in B\}}\, dP &= P(\{X_{s+t} \in B\} \cap F) = P(F) \cdot \mu_{s+t}^F(B) \\ &= P(F) \cdot (P_{s,s+t} \circ \mu_s^F)(B) = \int_F P_{s,s+t}(X_s(\omega), B)\, dP(\omega). \end{aligned}$$

As $\omega \mapsto P_t(X_s(\omega), B)$ is $\sigma(X_s)$-measurable and $\mathcal{F}_s \supset \sigma(X_s)$, we obtain

$$P(X_{s+t} \in B|\mathcal{F}_s) = P(X_{s+t} \in B|X_s) = P_{s,s+t}(X_s, B) \quad \text{a.s.}$$

for Borel sets $B \subset \mathbb{R}^N$. Hence, $(X_t)_{t \in I}$ is a Markov process associated with the hemigroup $(P_{s,t})$ as claimed. □

We next rewrite Theorem 4.25 in the time-homogeneous case, where we shall employ the negative definite function $\psi \in C(\widehat{X})$ of a semigroup $(P_t)_{t\geq 0}$ of kernels related to the integral transform $T$.

**4.26 Lemma.** *Let $(P_t)_{t\geq 0}$ be a semigroup of Markov kernels on $X$ related with the integral transform $(X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$ and with negative definite function $\psi \in C(\widehat{X})$. Let $(X_t)_{t\geq 0}$ be a càdlàg process on $X$. Then, for each $\lambda \in \widehat{X}$, $\left(\frac{1}{P_t(e,.)^\wedge(\lambda)} \cdot \varphi_\lambda(X_t)\right)_{t\geq 0}$ is a martingale if and only if so is*

$$\Big(X_t^\lambda := \varphi_\lambda(X_t) + \psi(\lambda) \cdot \int_0^t \varphi_\lambda(X_s)\, ds\Big)_{t\geq 0}.$$

*Proof.* By a boundedness argument, both processes are martingales if and only if the are local $L^2$–martingale.

Assume now that $\left((P_t(e,.)^\wedge(-\lambda))^{-1} \cdot \varphi_\lambda(X_t)\right)_{t\geq 0}$ is a local $L^2$-martingale. Then $(\varphi_\lambda(X_t))_{t\geq 0}$ is a semimartingale, and Ito integration yields

$$\begin{aligned} d(\varphi_\lambda(X_t)e^{t\psi(\lambda)}) &= e^{t\psi(\lambda)} d\varphi_\lambda(X_t) + \varphi_\lambda(X_{t-}) de^{t\psi(\lambda)} \\ &= e^{t\psi(\lambda)} \cdot (d\varphi_\lambda(X_t) + \psi(\lambda)\varphi_\lambda(X_t)dt). \end{aligned} \tag{4.6}$$

Therefore, $d\varphi_\lambda(X_t)+\psi(\lambda)\varphi_\lambda(X_t)dt = e^{-t\psi(\lambda)} \cdot d(\varphi_\lambda(X_t)e^{t\psi(\lambda)})$ is the differential of a local $L^2$-martingale as claimed.

The converse direction is similar. □

We next give a martingale characterization of time-homogeneous Markov processes $(X_t)_{t\geq 0}$ on $X$ associated with a specific semigroup $(P_t)_{t\geq 0}$ with generator $L$ in the spirit of the martingale problem of Stroock and Varadhan [SV]. For this, we define for any càdlàg process $(X_t)_{t\geq 0}$ on $X$ and $f \in D(L)$ the $\mathbb{C}$–valued process

$$\Pi_X^{L,f} = \Big(f(X_t) - f(X_0) - \int_0^t L(f)(X_s) ds\Big)_{t\geq 0}. \tag{4.7}$$

**4.27 Theorem.** *Let $(P_t)_{t\geq 0}$ be a semigroup of Markov kernels on $X$ related with $(X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$ and with negative definite function $\psi$ and generator $L$. Then the following are equivalent for any càdlàg process $(X_t)_{t\geq 0}$ on $X$.*

(1) *$(X_t)_{t\geq 0}$ is Markov process associated with the semigroup $(P_t)_{t\geq 0}$.*

(2) *For each $\lambda \in \widehat{X}$, the process $(\frac{1}{P_t(e,.)^\wedge(-\lambda)} \cdot \varphi_\lambda(X_t))_{t\geq 0}$ is a martingale.*

(3) *$\Big(\varphi_\lambda(X_t) + \psi(\lambda) \cdot \int_0^t \varphi_\lambda(X_s)\, ds\Big)_{t\geq 0}$ is a martingale for each $\lambda \in \widehat{X}$ .*

(4) *$\Pi_X^{L,\varphi_\lambda}$ is a martingale for each $\lambda \in \widehat{X}$.*

(5) *$\Pi_X^{L,f}$ is a martingale for each $f \in D_b(L)$.*

*Proof.* The equivalence of (1), (2) and (3) follows from 4.25 and 4.26. Moreover, (3) $\Leftrightarrow$ (4) $\Leftarrow$ (5) is obvious, and (1) $\Rightarrow$ (5) is just the well-known Dynkin formula, see e.g. Prop. 4.1.7 of [EK]. □

Now consider a arbitrary root system on $\mathbb{R}^N$ with multiplicity $k \geq 0$, and the generator $\Delta_k$ of the heat semigroup $(P_t)_{t\geq 0}$ of Dunkl type on $\mathbb{R}^N$. As $\Delta_k$ is a second-order differential-difference operator, it is more convenient here to restrict our attention to the subspaces $C_0^2(\mathbb{R}^N)$, $C_b^2(\mathbb{R}^N)$, and $C^2(\mathbb{R}^N)$ instead of the domains $D_0(\Delta_k)$, $D_b(\Delta_k)$ and $D(\Delta_k)$. Moreover, it is possible to take test functions $f$ above which depend on the time. If we define

$$\Pi_X^{\Delta_k,f} = \Big(f(X_t,t) - f(X_0,0) - \int_0^t \big(\frac{\partial}{\partial s} + \Delta_k\big) f(X_s,s)\, ds\Big)_{t\geq 0}$$

for $f \in C^{2,1}(\mathbb{R}^N \times [0,\infty))$, the characterization above may be rewritten as follows by using standard techniques (see Theorem 6.4 of [RV1]):

**4.28 Theorem.** *Let $(P_t)_{t\geq 0}$ be the heat semigroup of Dunkl type on $\mathbb{R}^N$ with generator $\Delta_k$. Then the following statements are equivalent for any càdlàg process $(X_t)_{t\geq 0}$ on $\mathbb{R}^N$ whose radial part $(|X_t|)_{t\geq 0}$ is continuous.*

(1) *$X$ is a Markov process associated with $(P_t)_{t\geq 0}$.*

(5) *$\Pi_X^{\Delta_k,f}$ is a martingale for each $f \in C_c^2(\mathbb{R}^N)$.*

(5′) *$\Pi_X^{\Delta_k,f}$ is a martingale for each $f \in C_c^{2,1}(\mathbb{R}^N \times [0,\infty))$.*

(6) *$\Pi_X^{\Delta_k,f}$ is a local martingale for each $f \in C^2(\mathbb{R}^N)$.*

(6′) *$\Pi_X^{\Delta_k,f}$ is a local martingale for each $f \in C^{2,1}(\mathbb{R}^N \times [0,\infty))$.*

For a detailed discussion of Dunkl processes, i.e., Markov processes on $\mathbb{R}^N$ associated with the Dunkl heat semigroup $(P_t)_{t\geq 0}$, we refer to [Chy], [GY1], [GY2], [GY3], and to the surveys [CGY],[De] in this volume.

We finally note that a similar result is available for Gaussian processes on Sturm-Liouville hypergroups, see [ReV].

## 4.4 Moment functions

In this section we consider a further approach to functions $f \in C(X)$ which lead to martingales for Markov processes on $X$ related to abstract integral transforms. Remember, that in Theorem 4.25 we used the functions $\varphi_\lambda$ which have multiplicative properties and which replace exponentials on the group $\mathbb{R}^N$. We now introduce analogs of polynomials, i.e. functions with additive properties. We begin with an example:

**4.29 Example.** Consider the usual group $X = \mathbb{R}^N$ with characters $\varphi_\lambda(x) = e^{-i<\lambda,x>}$, $\lambda \in \mathbb{R}^n = \widehat{X}$. Then for $\nu \in \mathbb{Z}_+$, the monomials $m_\nu(x) := x^\nu$ satisfy

$$m_\nu(x) = i^{|\nu|} \cdot \partial_\lambda^\nu \varphi_\lambda(x)\big|_{\lambda=0} \quad \text{and} \quad \varphi_\lambda(x) = \sum_{\nu\in\mathbb{Z}_+} (-i)^{|\nu|} \frac{m_\nu(x)}{\nu!}\lambda^\nu$$

as well as the Leibniz rule

$$m_\nu(x+y) = \sum_{\rho\in\mathbb{Z}_+,\rho\leq\nu} \binom{\nu}{\rho} m_\rho(x)\cdot m_{\nu-\rho}(y).$$

The Leibniz rule can be used to introduce so-called moment functions on commutative hypergroups; see, e.g., [Z1], [Z2], [ReV], and Ch. 7.2 of [BH]:

**4.30 Definition.** Let $X$ be a commutative hypergroup. Define $m_0 :\equiv 1$.

(1) A finite sequence $(m_i)_{i=1,\ldots,n} \subset C(X)$ is called a sequence of moment functions of length $n \in \mathbb{N}$ if

$$(\delta_x * \delta_y)(m_i) = \sum_{j=0}^{i} \binom{i}{j} m_j(x) m_{i-j}(y) \qquad (i = 1, \ldots, n;\ x, y \in X). \quad (4.8)$$

(2) For a fixed sequence $(m_i)_{i=1,\ldots,n}$ of moment functions, define the space

$$M_n^1(X) := \{\mu \in M^1(X) :\ m_i \in L^1(X, \mu) \quad \text{for all } 0 \le i \le n\}$$

of probability measures for which all moments up to the $n$-th exist.

We collect a few basic results for moment functions on commutative hypergroups $X$ from [ReV]:

**4.31 Remark.** (1) If $(m_i)_{i=1,\ldots,n}$ is a sequence of measurable, locally bounded functions on $X$ satisfying (4.8), then all $m_i$ are continuous.

(2) By induction, moment functions satisfy $m_i(e) = 0$ for $i = 1, \ldots, n$.

(3) If $\mu, \nu \in M_n^1(X)$, then $\mu * \nu \in M_n^1(X)$ and

$$\mu * \nu(m_i) = \sum_{j=0}^{i} \binom{i}{j} \mu(m_j) \cdot \nu(m_{i-j}) \quad \text{for } 0 \le i \le n. \qquad (4.9)$$

(4) If $(\mu_t)_{t \ge 0} \subset M_n^1(X)$ is a convolution semigroup on $X$, then the functions $f_i : [0, \infty) \to \mathbb{R}$, $t \mapsto \int_X m_i \, d\mu_t$ satisfy

$$f_i(s + t) = \sum_{j=0}^{i} \binom{i}{j} f_j(s) \cdot f_{i-j}(t) \qquad (0 \le i \le n,\ \ s, t \ge 0). \qquad (4.10)$$

Moreover, if the $f_i$ are continuous at $t = 0$, induction yields that the $f_i$ are polynomials with

$$f_i(t) = \sum_{j=1}^{i} c_{i,j} t^{i+1-j} \quad (1 \le i \le n,\ t \ge 0)$$

for unique $c_{i,j} \in \mathbb{R}$. Moreover, the generator $L$ of the reflected semigroup $(\mu_t^-)_{t \ge 0}$ satisfies

$$L m_i(x) = \sum_{j=0}^{i-1} m_j(x) \binom{i}{j} c_{i-j, i-j} \quad \text{for } 1 \le i \le n,\ x \in X.$$

In practice, first and second moments play the most prominent role. Under the conditions of Remark 4.31(4) we have for $m_1, m_2$ and a convolution semigroup $(\mu_t^-)_{t\geq 0}$ with $\lim_{t\to 0}\int_X m_i\, d\mu_t = 0$ for $i = 1, 2$ that

$$\int_X m_1\, d\mu_t = c_1 t \quad \int_X m_2\, d\mu_t = c_1^2 t^2 + c_2 t \quad (t \geq 0), \quad \text{and} \tag{4.11}$$

$$Lm_1(x) = c_1, \quad Lm_2(x) = 2c_1 m_1(x) + c_2 \quad \text{for } x \in X. \tag{4.12}$$

Moreover, we can construct martingales from moment functions. Here is a result for the first and second moments due to [Z2] which can be easily checked:

**4.32 Lemma.** *Let $(X_t)_{t\geq 0}$ be a Lévy process on $X$ associated with $(\mu_t)_{t\geq 0} \subset M_2^1(X)$ in the setting above. Then $(m_1(X_t) - E(m_1(X_t))_{t\geq 0}$ and*

$$\big(m_2(X_t) - 2m_1(X_t) \cdot E(m_1(X_t)) - E(m_2(X_t)) + 2E(m_1(X_t))^2\big)_{t\geq 0}$$

*are martingales.*

Clearly, these result may be extended to higher moments. We mention that Lemma 4.32 together with martingale convergence theorems can be used to derive strong limit theorems for the processes $(X_t)_{t\geq 0}$ for $t \to \infty$. For details see the monograph [BH].

We next turn to the question, how we can construct examples moment functions on given hypergroups, and how the concept of moment functions can be lifted to abstract integral transforms. Motivated by the observations for the group $\mathbb{R}^N$ in 4.29, we introduce the following notion:

**4.33 Definition.** An abstract integral transform $T = (X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$ is called differentiable, if the following holds:

(1) $\widehat{X}$ is a closed subset on $\mathbb{R}^N$ with $\widehat{e} = 0 \in \widehat{X}$.

(2) There exist $\zeta_1, \ldots, \zeta_N \in \mathbb{R}^N$ and $\epsilon > 0$ with $t_1\zeta_1 + \ldots + t_N\zeta_N \in \widehat{X}$ for $t_1, \ldots, t_N \in [0, \epsilon]$.

(3) For $\nu \in \mathbb{Z}_+^n$ and $x \in X$ the directional derivatives w.r.t. $\lambda$

$$m_\nu(x) := i^{|\nu|} \cdot \partial_\zeta^\nu \varphi_\lambda(x)\big|_{\lambda=0} := i^{|\nu|} \cdot \partial_{\zeta_1}^{\nu_1} \ldots \partial_{\zeta_N}^{\nu_N} \varphi_\lambda(x)\big|_{\lambda=0}$$

exist, and $m_\nu$ is continuous on $X$.

For an abstract integral transform, the functions $(m_\nu)_{\nu\in\mathbb{Z}_+^n}$ are called moment functions.

**4.34 Examples.** (1) For the group $\mathbb{R}^N$, we have $\varphi_\lambda(x) = e^{-i<\lambda,x>}$, $\lambda \in \mathbb{R}^n = \widehat{X}$. Taking the $\zeta_k$ as unit vectors $e_k$, we obtain $m_\nu(x) = x^\nu$.

(2) Consider the Bessel-Kingman hypergroup on $X = [0, \infty)$ of index $\alpha \geq -1/2$ with $\widehat{X} = [0, \infty)$ and the characters $\varphi_\lambda(x) := j_\alpha(\lambda x)$ with the normalized Bessel function

$$j_\alpha(z) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\alpha+1)}{2^{2k} k!\, \Gamma(\alpha+k+1)}\, z^{2k} \quad (z \in \mathbb{C}). \tag{4.13}$$

Then we obtain the moment functions $m_{2n+1} \equiv 0$ and

$$m_{2n}(x) := \frac{\Gamma(\alpha+1)(2n)!}{2^{2n} n!\, \Gamma(\alpha+n+1)} x^{2n} \tag{4.14}$$

for $n \in \mathbb{Z}_+$ in the sense of the preceding definition.

(3) Consider the Bessel hypergroups on the matrix cones $\Pi_q$ introduced in Chapter 3.4. A complete system of moment functions is given here in [V2]. Moreover, applications to strong laws of large numbers for random walks in this setting are also given there.

Before turning to moment functions associated with Dunkl kernels, we notice that the Leibniz rule for partial derivatives leads to:

**4.35 Lemma.** *Let $X$ be a commutative hypergroup with dual $\widehat{X}$ such that the hypergroup Fourier transform is differentiable in the sense of 4.33. Then, for any nonnegative linear combination $\zeta$ of the vectors $\zeta_1, \ldots, \zeta_N \in \mathbb{R}^N$ in 4.33(2), the functions $m_n(x) := \partial_\zeta^n \varphi_\lambda|_{\lambda=0}$ ($n \in \mathbb{Z}_+$) with partial derivatives w.r.t. $\lambda$ form moment functions in the sense of Definition 4.30(1).*

For Markov kernels associated with a differentiable integral transform, this additivity reads as follows:

**4.36 Lemma.** *Let $P, Q$ be Markov kernels on $X$ related with the differentiable integral transform $(X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$. Assume that for some $n \in \mathbb{N}$ and all $x \in X$, the measures $P(x,.), Q(x,.), Q \circ P(x,.)$ are contained in*

$$M_n^1(X) := \{\mu \in M^1(X) :\ {}_\nu \in L^1(X,\mu) \text{ for } |\nu| \le n\}. \tag{4.15}$$

*Then, for all $x \in X$ and $\nu \in \mathbb{Z}_+^N$ with $|\nu| \le n$ and with the notion $m_\nu(\mu) := \int_X m_\nu \, d\mu$,*

(1) $m_\nu(P(x,.)) = \sum_{\rho\le\nu} \binom{\nu}{\rho} m_\rho(P(e,.)) \cdot m_{\nu-\rho}(x)$,

(2) $m_\nu(Q \circ P(x,.)) = \sum_{\rho\le\nu} \binom{\nu}{\rho} m_\rho(P(x,.)) \cdot m_{\nu-\rho}(Q(e,.))$.

*Proof.* (1) follows from the Leibniz rule; in fact, as we may interchange differentiation and integration, we observe by differentiation w.r.t. $\lambda$ that

$$\begin{aligned} m_\nu(P(x,.)) &= i^{|\nu|} \cdot \partial_\zeta^\nu \Big( P(e,.)^\wedge(\lambda) \cdot \varphi_\lambda(x) \Big) \Big|_{\lambda=0} \\ &= \sum_{\rho\le\nu} \binom{\nu}{\rho} i^{|\rho|} \cdot \partial_\zeta^\rho (P(e,.)^\wedge(\lambda))|_{\lambda=0} \cdot i^{|\nu|-|\rho|} \cdot \partial_\zeta^{\nu-\rho} \varphi_\lambda(x)|_{\lambda=0} \\ &= \sum_{\rho\le\nu} \binom{\nu}{\rho} m_\rho(P(e,.)) \cdot m_{\nu-\rho}(x). \end{aligned}$$

Part (2) can be checked in the same way by using Lemma 4.14(2). □

As in Lemma 4.32, we can construct martingales:

**4.37 Proposition.** *Let $(P_t)_{t\geq 0}$ be a semigroup of Markov kernels related with the differentiable integral transform $(X, \widehat{X}, (\varphi_\lambda)_{\lambda\in\widehat{X}})$ with negative definite function $\psi$, and let $(X_t)_{t\geq 0}$ be an associated Markov process on $X$ with canonical filtration on the underlying probability space $(\Omega, \mathcal{A}, P)$. Then the moment functions of Section 4.33 satisfy:*

(1) *If the measures $P_{X_0}$ and $P_t(x,.)$ are contained in $M_1^1(X)$ for $t > 0$ and $x \in X$ in the sense of (4.15), then for $l = 1, \ldots, N$, the process $(m_{e_l}(X_t) - E(m_{e_l}(X_t)))_{t\geq 0}$ is a martingale with*

$$E(m_{e_l}(X_t)) = E(m_{e_l}(X_0)) - it \cdot \partial_{\zeta_l}\psi(0) \qquad \textit{for} \quad t \geq 0.$$

(2) *If the measures $P_{X_0}$ and $P_t(x,.)$ are contained in $M_2^1(X)$ for $t > 0$ and $x \in X$, then for $l, j = 1, \ldots, N$,*

$$\Big(m_{e_l+e_j}(X_t) - m_{e_l}(X_t)\, E(m_{e_j}(X_t)) - m_{e_j}(X_t)\, E(m_{e_l}(X_t)) \\ + E(m_{e_l}(X_t))\, E(m_{e_j}(X_t)) - E(m_{e_l+e_j}(X_t))\Big)_{t\geq 0}$$

*is a martingale. In particular, the "modified variances"*

$$V^l(X_t) := E(m_{2e_l}(X_t)) - E(m_{e_l}(X_t))^2$$

*satisfy $V_k^l(X_t) = V_k^l(X_0) + t \cdot \partial_{\zeta_l}^2 \psi(0)$ for $t \geq 0$.*

*Proof.* (1) We have $\psi(0) = 0$ and $P_t(e,.)^\wedge = e^{-t\psi}$ for $t \geq 0$. Therefore, by the dominated convergence theorem,

$$m_{e_l}(P_t(e,.)) = i \cdot \partial_{\zeta_l}(P_t(e,.)^\wedge)(0) = -it\, \partial_{\zeta_l}\psi(0).$$

Now take $s, t \geq 0$. The preceding lemma ensures that for a.a. $\omega \in \Omega$,

$$\begin{aligned} E(m_{e_l}(X_{s+t})|\, \mathcal{F}_t)(\omega) &= \int_X m_{e_l}\, dP_t(X_s(\omega), .) \\ &= m_{e_l}(P_t(e,.)) + m_{e_l}(X_s(\omega)) \\ &= -it \cdot \partial_l\psi(0) + m_{e_l}(X_s(\omega)). \end{aligned} \qquad (4.16)$$

If we take the usual expectation of both sides with $s = 0$, we obtain the claimed formula for $E(m_{e_l}(X_t))$, and that $(m_{e_l}(X_t) - E(m_{e_l}(X_t)))_{t\geq 0}$ is a martingale.

(2) can be shown in a similar way; c.f. Proposition 7.5 of [RV1].

□

We now turn to moment functions in the sense of 4.33 for the Dunkl transform. Recall that the Dunkl kernel $E_k$ is analytic on $\mathbb{C}^{N\times N}$, i.e., there there exist unique analytic functions $m_\nu$ ($\nu \in \mathbb{Z}_+^N$) with

$$E_k(x,y) = \sum_{\nu\in\mathbb{Z}_+^N} \frac{m_\nu(x)}{\nu!} y^\nu \qquad (x, y \in \mathbb{C}^N) \qquad (4.17)$$

with

$$m_\nu(x) = (\partial_y^\nu E_k(x,y))|_{y=0} = i^{|\nu|}\, (\partial_y^\nu E_k(x,-iy))|_{y=0}. \qquad (4.18)$$

Therefore, the $m_\nu$ are moment functions in the sense of 4.33.

We denote the $j$-th unit vector by $e_j \in \mathbb{Z}_+^N$. and the the moment functions of order 1 and 2 by $m_{e_j}$ and $m_{e_j+e_k}$ $(j,k = 1,\ldots,N)$.

¿From the description of the Dunkl kernel $E_k$ via the intertwiner $V_k$ (see the definition of the Dunkl kernel!) we immediately obtain

$$m_\nu(x) = V_k(x^\nu) \qquad \text{for} \quad \nu \in \mathbb{Z}_+^N. \tag{4.19}$$

In particular, for each $n \in \mathbb{Z}_+$ the moment functions $m_\nu$ with $|\nu| = n$ form a basis of the space $\mathcal{P}_n$ of all homogeneous polynomials of degree $n$. This in particular implies that a measure $\mu \in M^1(\mathbb{R}^N)$ is contained in $M_n^1(\mathbb{R}^N)$ (i.e., it has moments up to order $n$ in the meaning above) if and only if it has usual moments up to order $n$.

Moreover, via the recurrence relation for $V_k$ in the beginning of Chapter 2.4 (see [D4],[DX]) it is possible to compute the moment functions $m_\nu$.

**4.38 Examples.** (1) If $k = 0$, then $E_k(x,y) = e^{<x,y>}$ and $m_\nu(x) = x^\nu$.

(2) If $N = 1$, $W = \mathbb{Z}_2$ and $k \geq 0$, then the explicit form of $E_k$ in terms of Bessel functions (see Example 2.29 and compare with (2.7)) implies

$$m_{2n}(x) = \frac{\Gamma(k+1/2)\,(2n)!}{\Gamma(n+k+1/2)\,2^{2n}\,n!}x^{2n}$$
$$m_{2n+1}(x) = \frac{\Gamma(k+1/2)\,(2n+1)!}{\Gamma(n+k+3/2)\,2^{2n+1}\,n!}x^{2n+1}.$$

(3) **The $A_{N-1}$-case:** For the symmetric group $W = S_N$ and multiplicity $k \in [0,\infty)$, some computation with the intertwiner yields

$$m_{e_l}(x) = V_k x_l = \frac{1}{1+kN}\Big(x_l + k\sum_{i=1}^N x_i\Big). \tag{4.20}$$

for the moment functions of first order and similar formulas for that of second order; see Section 7.1 of [RV1] for details.

(4) **The $B_N$-case:** Here the multiplicity consists of two parameters $k_0, k_1 \geq 0$, and it follows (see [D7] and [RV1]) that for $l, j \in \{1,\ldots,N\}$,

$$m_{e_l}(x) = V_k x_l = \frac{x_l}{1+2k_1+2k_0(N-1)},$$
$$m_{e_l+e_j} = V_k(x_l x_j) = \frac{x_l x_j}{1+2k_1+2k_0(N-1)}, \qquad \text{for} \quad l \neq j,$$
$$m_{2e_l}(x) = V_k x_l^2 = \frac{x_l^2 + k_0\sum_{i=1}^N x_i^2}{(1+Nk_0)(1+2(N-1)k_0+2k_1)}.$$

We next collect some properties of moment functions. We mention that similar results are also available for Sturm-Liouville hypergroups on $[0,\infty)$; see [BH], [ReV], [Z1], [Z2].

**4.39 Proposition.** *For all $x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}_+^N$, and $l \in \{1,\ldots,N\}$:*

(1) $T_l m_{\nu+e_l} = (\nu_l + 1)\cdot m_\nu$.

(2) $|m_\nu(x)| \le |x|^{|\nu|}$ *and* $0 \le m_\nu(x)^2 \le m_{2\nu}(x)$.

(3) *Taylor formula: If* $f \in C^{(n)}(\mathbb{R}^N)$ *for* $n \in \mathbb{N}$, *then*

$$f(y) = \sum_{\nu \in \mathbb{Z}_+^N,\, |\nu| \le n} \frac{m_\nu(y)}{\nu!}\, T^\nu f(0) + o(|y|^n) \qquad \text{for} \quad y \to 0.$$

*Moreover, if* $f : \mathbb{C}^N \to \mathbb{C}$ *is analytic in a neighborhood of 0, then*

$$f(y) = \sum_{n=0}^{\infty} \sum_{|\nu|=n} \frac{m_\nu(y)}{\nu!}\, T^\nu f(0)$$

*where the series converges absolutely and locally uniformly.*

*Proof.* (1) The properties of $V_k$ and (4.19) yield

$$T_l m_{\nu+e_l} = T_l V_k x^{\nu+e_l} = V_k \partial_l x^{\nu+e_l} = (\nu_l + 1) \cdot V_k x^l = (\nu_l + 1) \cdot m_\nu.$$

(2) The positive integral representation of $E_k$ in Theorem 2.22 together with (4.18) imply that for $x \in \mathbb{R}^N$ there exists $\mu_x \in M^1(\mathbb{R}^N)$ with *supp* $\mu_x \subset \{z \in \mathbb{R}^N : |z| \le |x|\}$ and

$$m_\nu(x) = \int_{\mathbb{R}^N} y^\nu \, d\mu_x(y) \qquad \text{for all} \quad \nu \in \mathbb{Z}_+^N,\ x \in \mathbb{R}^N.$$

The first inequality is now clear from the support condition on $\mu_x$ while the second one follows from Jensen's inequality.

(3) See Corollary 2.17.

□

**4.40 Example.** Let $(X_t)_{t\ge 0}$ be a Dunkl process on $\mathbb{R}^N$ with $X_0 = 0$ associated with the Dunkl heat semigroup $(P_t^\Gamma)_{t\ge 0}$. In this case, all moments exists, and

$$P_t^\Gamma(0,.)^\wedge(y) = e^{-t|y|^2} = \sum_{\nu \in \mathbb{Z}_+^N} \frac{(-t)^{|\nu|}}{\nu!}\, y^{2\nu} \qquad \text{for} \quad t \ge 0,\ y \in \mathbb{R}^N.$$

This yields that

$$E(m_{2\nu}(X_t)) = m_{2\nu}(P_t^\Gamma(0,.)) = \frac{(2\nu)!}{\nu!}\, t^{|\nu|} \qquad (\nu \in \mathbb{Z}_+^N,\ t \ge 0) \tag{4.21}$$

and $E(m_\nu(X_t)) = 0$ whenever at least one component of $\nu$ is odd.

Proposition 4.37(1) now implies that the processes $(m_{e_l}(X_t))_{t\ge 0}$ are martingales for $l \in \{1, \dots, N\}$. Moreover, as the $m_{e_l}$ form a basis of the space $\mathcal{P}_1$ of all homogeneous polynomials of degree 1, $(X_t)_{t\ge 0}$ itself is an $N$-dimensional martingale.

Moreover, Proposition 4.37(2) and $E(m_{e_l}(X_t)) = 0$ show that

$$(m_{e_l+e_j}(X_t) - E(m_{e_l+e_j}(X_t)))_{t\ge 0}$$

is a martingale for $l, j \in \{1, \dots, N\}$. As the moment functions $m_{e_j+e_l}$ form a basis of $\mathcal{P}_2$, it follows that for all $l, j \in \{1, \dots, N\}$, the processes

$$(X_t^l \cdot X_t^j - E(X_t^l \cdot X_t^j))_{t\ge 0}$$

are martingales. For higher moments, results of this type are more complicated and will be considered in the next section.

## 4.5 General Appell characters

Let $(X_t)_{t\geq 0}$ be time-homogeneous Markov process on $X$ associated with some abstract integral transform as above. Based on concept of moment functions and certain generating functions, we construct a system $(R_\nu)_{\nu\in\mathbb{Z}_+^N}$ of functions on $\mathbb{R}\times X$ associated with $(X_t)_{t\geq 0}$ such that the processes $(R_\nu(t,X_t))_{t\geq 0}$ become martingales. These systems, called Appell characters, generalize the well-known heat polynomials, which are connected with Brownian motion and given in terms of classical Hermite polynomials.

The concept of Appell characters is quite old and has its origin in umbral calculus; see for instance [FS], [Rom].

We begin with a general definition. Later we shall restrict our attention mainly to Dunkl processes.

**4.41 Definition.** Let $(P_t)_{t\geq 0}$ be a semigroup of Markov kernels related with the differentiable integral transform $(X,\widehat{X},(\varphi_\lambda)_{\lambda\in\widehat{X}})$ with negative definite function $\psi\in C(\widehat{X})$: we use the notions of 4.33 and assume that $P_t(x,.)\in M_n^1(X)$ for $t\geq 0$ and $x\in X$, i.e., that all moments up to order $n\geq 1$ exist. We know from the preceding section that $P_t(e,.)^\wedge = e^{-t\psi}\in C^n(\widehat{X})$ for $t\geq 0$. Therefore,

$$\lambda\longmapsto \frac{\varphi_\lambda(x)}{P_t(e,.)^\wedge(\lambda)} = \varphi_\lambda(x)\cdot e^{t\psi(\lambda)}$$

is $n$-times continuously differentiable for $t\geq 0$, $x\in X$. By Taylor's formula,

$$\varphi_\lambda(x)\cdot e^{t\psi(\lambda)} = \sum_{\nu\in\mathbb{Z}_+^N,\,|\nu|\leq n}\frac{(-i\lambda)^\nu}{\nu!}R_\nu(t,x)+o(|\lambda|^n)\quad\text{for } y\to 0; \tag{4.22}$$

with unique functions

$$\begin{aligned}R_\nu(t,x) &= i^{|\nu|}\partial_\zeta^\nu\Big(\varphi_\lambda(x)\cdot e^{t\psi(\lambda)}\Big)\Big|_{\zeta=0}\\ &= i^{|\nu|}\sum_{\rho\in\mathbb{Z}_+^N,\,\rho\leq\nu}\binom{\nu}{\rho}\partial_\zeta^\rho(\varphi_\lambda(x))\Big|_{\lambda=0}\cdot\partial_\zeta^{\nu-\rho}(e^{t\psi(\lambda)})\Big|_{y=0}\\ &= \sum_{\rho\in\mathbb{Z}_+^N,\,\rho\leq\nu}\binom{\nu}{\rho}m_\rho(x)\cdot a_{\nu-\rho}^\psi(t)\end{aligned}\tag{4.23}$$

by Definition 4.33 with the polynomials

$$a_\rho^\psi(t) := i^{|\rho|}\cdot\partial_\zeta^\rho(e^{t\psi(y)})\big|_{\lambda=0}\qquad(\rho\in\mathbb{Z}_+^N,\ |\rho|\leq n)\tag{4.24}$$

in $t$ of degree at most $|\rho|$. Note that $a_\nu^\psi(-t) = m_\nu(P_t(e,.))\in\mathbb{R}$ holds for $t\geq 0$. The functions $R_\nu$ will be called *Appell characters* associated with the semigroup $(P_t)_{t\geq 0}$.

We next collect some basic properties of Appell characters:

**4.42 Lemma.** *In the setting above, the following holds for $\nu\in\mathbb{Z}_+^N$ with $|\nu|\leq n$:*

(1) *Inversion formula: For all $x\in X$ and $t\in\mathbb{R}$,*

$$m_\nu(x) = \sum_{\rho\in\mathbb{Z}_+^N,\,\rho\leq\nu}\binom{\nu}{\rho}R_\rho(t,x)\cdot a_{\nu-\rho}^\psi(-t).$$

(2) *For* $x \in X$ *and* $t \geq 0$, $\displaystyle\int_X R_\nu(t,y)\, dP_t(x,dy) = m_\nu(x).$

*Proof.* (1) Write down the Taylor expansion of order $n$ of

$$\varphi_\lambda(x) = e^{-t\psi(\lambda)} \cdot (\varphi_\lambda(x) \cdot e^{t\psi(\lambda)})$$

as above in two ways and compare the coefficients; see Lemma 8.1 of [RV1].

(2) Recall that $a_\nu^\psi(-t) = m_\nu(P_t(e,.))$. Eq. (4.23) and Lemma 4.36(1) yield

$$\begin{aligned}
\int_X R_\nu(t,y)\, dP_t(x,dy) &= \sum_{\rho\leq\nu} \binom{\nu}{\rho} a^\psi_{\nu-\rho}(t) \cdot \int_X m_\rho(y)\, dP_t(dy) \\
&= \sum_{\rho\leq\nu} \binom{\nu}{\rho} a^\psi_{\nu-\rho}(t) \cdot \Big(\sum_{\varphi\leq\rho} \binom{\rho}{\varphi} m_\varphi(P_t(e,.)) \cdot m_{\rho-\varphi}(x)\Big) \\
&= \sum_{\rho\leq\nu} \binom{\nu}{\rho} a^\psi_{\nu-\rho}(t) \cdot \Big(\sum_{\varphi\leq\rho} \binom{\rho}{\varphi} a^\varphi_\psi(-t) \cdot m_{\rho-\varphi}(x)\Big).
\end{aligned}$$

The assertion now follows from Part (1).

□

**4.43 Remark.** For Dunkl theory on $X = \mathbb{R}^N$, part (1) of the preceding result implies that for $t \in \mathbb{R}$ and suitable $l \in \mathbb{N}$, the $(R_\nu(t,.))_{\nu\in\mathbb{Z}_+^N,\, |\nu|\leq l}$ form a basis of the space $\bigoplus_{j=0}^l \mathcal{P}_j$ of all polynomials of degree at most $l$.

Moreover, for all $x \in \mathbb{R}^N$, $t \in \mathbb{R}$, and $j \in \{1,\ldots,N\}$,

$$T_j R_{\nu+e_j}(t,x) = (\nu_j + 1) \cdot R_\nu(t,x) \tag{4.25}$$

where $T_j$ acts with respect to the variable $x$.

In fact, the expansion of $R_\nu$ and Proposition 4.39(1) yield

$$\begin{aligned}
T_j R_{\nu+e_j} &= \sum_{\rho\leq\nu+e_j} \binom{\nu+e_j}{\rho} T_j m_\rho \cdot a^\psi_{\nu+e_j-\rho} = \sum_{\rho\leq\nu} \binom{\nu+e_j}{\rho+e_j} (\rho_j+1) m_\rho \cdot a^\psi_{\nu-\rho} \\
&= (\nu_j+1) \cdot \sum_{\rho\leq\nu} \binom{\nu}{\rho} m_\rho \cdot a^\psi_{\nu-\rho} = (\nu_j+1) \cdot R_\nu\,.
\end{aligned}$$

We return to the general setting:

**4.44 Theorem.** *Let* $n \geq 1$ *and* $(P_t)_{t\geq 0}$ *a semigroup of Markov kernels on* $X$ *as in Definition 4.41 above. Let* $(X_t)_{t\geq 0}$ *be an associated Markov process with the following property:*

(*) *There exists* $\epsilon > 0$ *with* $\zeta = \sum_{j=1}^n t_j \zeta_j \in \widehat{X}$ *for all* $0 \leq t_j \leq \epsilon$ *(*$j = 1,\ldots,N$*) and the directions* $\zeta_1,\ldots,\zeta_N \in \mathbb{R}^n$ *in 4.33, and for all* $\nu \in \mathbb{Z}_+^N$ *with* $|\nu| \leq n$*, the functions*

$$x \mapsto \sup_{\lambda=\sum_{j=1}^n t_j\zeta_j,\, 0\leq t_j\leq\epsilon} |\partial_\zeta^\nu \varphi_\lambda(x)|$$

*are integrable w.r.t. the distributions* $P_{X_t}$ *of all random variables* $X_t$.

*Then for each* $\nu \in \mathbb{Z}_+^N$ *with* $|\nu| \le n$, $(R_\nu(t, X_t))_{t\ge 0}$ *is a martingale.*

*Proof.* We prove more generally by induction on $|\nu|$ that for each $\lambda \in \widehat{X}$ and $\nu \in \mathbb{Z}_+^N$ with $|\nu| \le n$, the process

$$\Big(W_t^{\nu,\lambda} := \partial_\zeta^\nu(\varphi_\lambda(X_t) \cdot e^{t\varphi(\lambda)})\Big)_{t\ge 0} \tag{4.26}$$

is a martingale for $\lambda = \sum_{j=1}^n t_i \zeta i$ with $0 \le t_j < \epsilon$. The theorem then follows for $\lambda = 0$.

In fact, the case $\nu = 0$ follows from Proposition 4.25. For the induction step, consider some direction $\zeta_j$ as in Definition and assume that $W_t^{\nu,\lambda}$ is a martingale for all $\lambda \in \mathbb{R}^N$ as above and some $\nu \in \mathbb{Z}_+^N$ with $|\nu| \le n$. To prove that $(W_t^{\nu+e_j,\lambda})_{t\ge 0}$ is a martingale for these $\lambda$, we observe that for $t \ge 0$,

$$\lim_{h\to 0} \frac{1}{h}\Big(W_t^{\nu,\lambda} - W_t^{\nu,\lambda+h\cdot\zeta_j}\Big) = W_t^{\nu+e_j,y} \qquad \text{pointwise.}$$

Moreover, by the mean value theorem, we find $r \in [0, h]$ with

$$\Big|\frac{1}{h}\Big(W_t^{\nu,\lambda} - W_t^{\nu,\lambda+h\cdot\zeta_j}\Big)\Big| = \big|W_t^{\nu+e_j,\lambda+r\cdot\zeta_j}\big|. \tag{4.27}$$

Condition (*) ensures that the dominated convergence theorem may be applied to the limit above, and hence

$$\lim_{h\to 0}\Big\|\frac{1}{h}\Big(W_t^{\nu,\lambda} - W_t^{\nu,\lambda+h\cdot\zeta_j}\Big) - W_t^{\nu+e_j,\lambda}\Big\|_1 = 0 \qquad \text{for all } t \ge 0.$$

It follows for the canonical filtration $(\mathcal{F}_t)_{t\ge 0}$ of $(X_t)_{t\ge 0}$ that for $s, t \ge 0$,

$$E\Big(\frac{1}{h}\Big(W_{s+t}^{\nu,\lambda} - W_{s+t}^{\nu,\lambda+h\cdot\zeta_j}\Big)\Big|\mathcal{F}_t\Big) \longrightarrow E(W_{s+t}^{\nu+e_j,\lambda}|\mathcal{F}_t) \qquad \text{a.s}$$

Hence, $(W_t^{\nu+e_j,\lambda})_{t\ge 0}$ is a martingale. □

**4.45 Remarks.** (1) For concrete abstract integral transforms, condition (*) above can be simplified considerably.

For instance, using bounds for derivatives of Dunkl kernels (see Corollary 2.32), it can be easily checked that in the Dunkl setting condition (*) holds if and only if all distributions $P_{X_t}$ of the process admit moments up to order $n$; c.f. also the proof of Theorem 8.2 of [RV1].

Moreover, for Sturm-Liouville hypergroups on $[0, \infty)$, an analog result is available; we refer to [Z1], [Z2], [ReV], and Section 7.2 of [BH]. In particular, for the Bessel hypergroups on $[0, \infty)$, the same result is available as in the Dunkl setting; see also below.

(2) For $|\nu| = 1, 2$, the martingales $R_\nu(t, X_t)$ of the theorem above agree with the martingales of Proposition 4.37.

(3) Consider the Dunkl setting. Then for each polynomial $f \in \mathcal{P}$, the polynomial function $u(x,t) := e^{t\Delta_k} f(x)$ satisfies $u_t = \Delta_k u$ on $\mathbb{R}^N \times \mathbb{R}$ (see e.g. Theorem 3.1(2) of [RV1]). This fact together with Proposition 4.55

below thus imply that the Appell characters $R_\nu^\Gamma$ for the Dunkl heat semigroup satisfy

$$(\partial_t + \Delta_k)R_\nu^\Gamma = 0,$$

i.e., they form so-called heat polynomials for the Dunkl heat semigroup. This again reflects the close connection between Theorems 4.44 and 4.27.

(4) In the Dunkl setting there is a close connection between general Appell characters $R_\nu$ and the intertwiner $V_k$: Let $n \geq 0$ and $(P_t^k)_{t\geq 0}$ a semigroup of Dunkl-Markov kernels on $\mathbb{R}^N$ such that $P_t^k(x,.) \in M_n^1(\mathbb{R}^N)$ for $t \geq 0$ and $x \in \mathbb{R}^N$. Let $R_\nu^k$ be the associated Appell characters for $|\nu| \leq n$. By Theorem 2.22(2), there exist probability measures $\mu_x \in M^1(\mathbb{R}^N)$ such that the negative definite function $\psi$ associated with $(P_t^k)_{t\geq 0}$ satisfies

$$e^{-t\psi(\lambda)} = P_t^k(0,.)^\wedge(\lambda) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{\langle z,-i\lambda\rangle}\, d\mu_x(z)\, dP_t^k(0,.)(x)$$

for $t \geq 0$, $\lambda \in \mathbb{R}^N$. Hence, the $e^{-t\psi}$ are positive definite in the classical sense, and by Bochner's theorem, there is a semigroup $(P_t^0)_{t\geq 0}$ of group-translation invariant Markov kernels. If the associated Appell characters are denoted by $R_\nu^0$, we obtain

$$R_\nu^k(t,x) = \sum_{\rho\leq\nu}\binom{\nu}{\rho} m_\rho(x)\, a_{\nu-\rho}^\psi(t) = \sum_{\rho\leq\nu}\binom{\nu}{\rho}(V_k x^\rho)\, a_{\nu-\rho}^\psi(t) = V_k R_\nu^0(t,x).$$

We next discuss Theorem 4.44 for Bessel processes on $[0,\infty)$:

**4.46 Example.** For $\alpha \geq -1/2$, consider the Bessel hypergroup on $[0,\infty)$. On this hypergroup there exists up to time normalization a unique Gaussian convolution semigroup in the sense of Theorem 4.23, namely

$$d\rho_t^\alpha(x) = \frac{1}{\Gamma(\alpha+1)}\,\frac{2^\alpha}{t^{\alpha+1}}\; x^{2\alpha+1}\, e^{-x^2/(2t)}\, dx \quad \text{on } [0,\infty) \quad \text{for } t>0 \qquad (4.28)$$

with generator $Lf := \frac{1}{2}\big(f'' + \frac{2\alpha+1}{x}f'\big)$. The associated Markov processes are Bessel processes $(X_t^\alpha)_{t\geq 0}$ of order $\alpha$. In particular, for the $N$-dimensional Brownian motion $(B_t)_{t\geq 0}$, $(|B_t|)_{t\geq 0}$ is a Bessel process of index $\alpha = N/2-1$. Using the moment functions of Example 4.34(2) and Theorem 4.44, we obtain the following well-known martingale connection between Bessel processes and the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{1}{n!}x^{-\alpha}e^x \cdot \frac{d^n}{dx^n}\big(x^{n+\alpha}e^{-x}\big) = \sum_{j=0}^n \binom{n+\alpha}{n-j}\frac{(-x)^j}{j!} \quad (n\geq 0) \qquad (4.29)$$

of index $\alpha \geq 0$ (see [Sz]).

**4.47 Lemma.** *If $R_n^{(\alpha)}(t,x) := t^n L_n^{(\alpha-1/2)}(x^2/(2t))$ $(t,x\in\mathbb{R},\ n\geq 0)$ is the "$n$-th heat polynomial of Laguerre-type", then for $n \geq 0$, $(R_n^{(\alpha)}(t,X_t))_{t\geq 0}$ is a martingale.*

This observation was the motivation in Section 7.2 of [ReV] for the following Lévy-type characterizations of Bessel processes:

**4.48 Theorem.** *Let $\alpha \geq -1/2$. Then an a.s. continuous process $X$ on $[0,\infty)$ is a Bessel process of order $\alpha$ if and only if $(X_t^2 - 2(\alpha+1)t)_{t\geq 0}$ and $(X_t^4 - 4(\alpha+2)tX_t^2 + 4(\alpha+1)(\alpha+2)t^2)_{t\geq 0}$ are martingales (or local martingales).*

*Proof.* By the (local) martingale conditions it is possible to compute the quadratic variation $[X_t^2 - 2(\alpha+1)t]_t$ which allows to verify one of the conditions of the martingale characterization 4.27. □

**4.49 Theorem.** *For $\alpha \geq -1/2$, let $(X_t)_{t\geq 0}$ be an arbitrary process on $[0,\infty)$ such that $(R_n^{(\alpha)}(t, X_t))_{t\geq 0}$ is a martingale for $n = 1,2,3,4$. Then $(X_t)_{t\geq 0}$ is a Bessel process of index $\alpha$.*

*Proof.* A straightforward calculation yields

$$E((X_t^2 - X_s^2)^4) = 16(\alpha+1)(\alpha+2)(t-s)^2 \cdot g_\alpha(s,t) \tag{4.30}$$

for some concrete polynomial $g_\alpha$. Kolmogorov's criterion now ensures that $(X_t^2)_{t\geq 0}$, and hence $(X_t)_{t\geq 0}$, admits an a.s. continuous modification. Theorem 4.48 completes the proof. □

A similar result for Brownian motion on $\mathbb{R}$ and Hermite polynomials can be found in Wesolowski [We], and for Brownian motions on compact Lie groups in [V1]. For further polynomial martingale relations of concrete processes we also refer to [Scho]. An extension to one-dimensional Dunkl processes is given below.

We finally notice that the characterizations 4.48 and 4.48 can be also extended to the (a.s. continuous) Dunkl-Bessel processes on Weyl chambers as well as to Wishart processes on the matrix cones $\Pi_q$, which are the Gaussian processes on the Bessel-type hypergroups on $\Pi_q$; see [V2].

## 4.6 Appell characters associated with Dunkl processes

In this section we restrict our attention to the Dunkl Gaussian semigroup $(P_t^\Gamma)_{t\geq 0}$ for some multiplicity $k \geq 0$. Here, all moments exist, and the Taylor expansion (4.22) actually becomes a power series. The coefficients $a_\nu^\Gamma(t)$ of the associated Appell characters $R_\nu^\Gamma$ satisfy $a_\nu^\Gamma(-t) = m_\nu(P_t^\Gamma(0,.))$ for $t \geq 0$. It follows from 4.40 that for all $t \in \mathbb{R}$,

$$a_{2\nu}^\Gamma(t) = \frac{(2\nu)!}{\nu!} \cdot (-t)^{|\nu|} \qquad \text{and} \qquad a_\lambda^\Gamma(t) = 0 \qquad \text{otherwise,}$$

i.e., if at least one component of $\lambda \in \mathbb{Z}_+^N$ is odd. Therefore,

$$R_\nu^\Gamma(t,x) = \sum_{\rho \in \mathbb{Z}_+^N,\ 2\rho \leq \nu} \frac{\nu!}{(\nu - 2\rho)!\, \rho!} (-t)^{|\rho|}\, m_{\nu-2\rho}(x) \qquad \text{for } \nu \in \mathbb{R}_+^N. \tag{4.31}$$

In particular the homogeneity of the $m_\nu$ yields that

$$R_\nu^\Gamma(t,x) = \sqrt{t}^{|\nu|} \cdot R_\nu^\Gamma(1, x/\sqrt{t}) \qquad (x \in \mathbb{R}^N,\ t > 0). \tag{4.32}$$

**4.50 Examples.** (1) In the group case $k = 0$ with $m_\nu(x) := x^\nu$, Eq. (4.31) implies $R_\nu^\Gamma(t,x) = \sqrt{t}^{|\nu|} \cdot \widetilde{H}_\nu\Big(\frac{x}{2\sqrt{t}}\Big)$ for $x \in \mathbb{R}^N$, $\nu \in \mathbb{Z}_+^N$, $t \in \mathbb{R}$ with the classical $N$-dimensional Hermite polynomials

$$\widetilde{H}_\nu(x) = \prod_{i=1}^N H_{\nu_i}(x_i) \qquad \text{with} \qquad H_n(y) = \sum_{j=0}^{\lfloor n/2 \rfloor} \frac{(-1)^j\, n!}{j!\,(n-2j)!}\,(2y)^{n-2j};$$

cf. Section 5.5 of [Sz] for $N = 1$.

(2) For $N = 1$, $W = \mathbb{Z}_2$ and $k \geq 0$, Example 4.38(2) shows

$$\begin{aligned} R_{2n}^{\Gamma,k}(t,x) &= (-1)^n 2^{2n} n!\, t^n\, L_n^{(k-1/2)}(x^2/4t), \\ R_{2n+1}^{\Gamma,k}(t,x) &= (-1)^n 2^{2n+1} n!\, t^n\, x\, L_n^{(k+1/2)}(x^2/4t) \qquad (n \in \mathbb{Z}_+), \end{aligned} \tag{4.33}$$

with the Laguerre polynomials in (4.29). The $(R_n^{\Gamma,k})_{n\geq 0}$ are called often generalized Hermite or heat polynomials (see e.g. [Ros]). For each $t > 0$ the $(R_n^{\Gamma,k}(t,.))_{n\geq 0}$ are orthogonal w.r.t.

$$dP_t^\Gamma(0,.)(x) = \frac{\Gamma(k+1/2)}{(4t)^{k+1/2}}\, |x|^{2k}\, e^{-x^2/4t}\, dx.$$

Theorem 4.44 implies that for an one-dimensional Dunkl process $(X_t)_{t\geq 0}$ associated with the Dunkl heat semigroup of index $k \geq 0$ the $(R_n^{\Gamma,k}(t,X_t))_{t\geq 0}$ are martingales for $n \in \mathbb{N}$. We can generalize the Lévy-type characterization 4.49 of Bessel processes as follows:

**4.51 Theorem.** *For $k \geq 0$, let $(X_t)_{t\geq 0}$ be an arbitrary process on $\mathbb{R}$ such that $(R_n^{\Gamma,k}(t,X_t))_{t\geq 0}$ is a martingale for $n = 1, 2, 4, 6, 8$. Then $(X_t)_{t\geq 0}$ is a Dunkl process of index $k$.*

*Proof.* The process $(X_t^2)_{t\geq 0}$ satisfies the conditions of Theorem 4.49, i.e., it is a Bessel process in distribution and a submartingale. On the other hand, as $(X_t)_{t\geq 0}$ is a martingale, Theorem 3 of [CGY] yields the claim. □

One might suggest from the examples in 4.50 that the $R_\nu^\Gamma(t,.)$ are always orthogonal w.r.t. $dP_t^\Gamma(0,.)$ for $t > 0$. We shall see below that this is not correct in many cases. For this we introduce so-called Appell cocharacters, which turn out to form a biorthogonal system for the Appell characters.

**4.52 Definition.** Consider the quotient

$$\theta_t(x,y) := \frac{\Gamma_k(t,x,y)}{\Gamma_k(t,0,y)} = e^{-|x|^2/4t}\, E_k(x, y/2t) = \sum_{\nu \in \mathbb{Z}_+^n} \frac{m_\nu(x)}{\nu!}\, S_\nu^\Gamma(t,y) \tag{4.34}$$

where, by Proposition 4.39(3), the coefficients satisfy

$$S_\nu^\Gamma(t,y) = T_x^\nu \big(e^{-|x|^2/4t}\, E_k(x, y/2t)\big)\Big|_{x=0}.$$

The $S_\nu^\Gamma(t,.)$ are also polynomials of degree $|\nu|$ and will be called the *Appell cocharacters* of the Dunkl heat semigroup.

Using the homogeneity of $m_\nu$, we obtain

$$S_\nu^\Gamma(t,y) = \Big(\frac{1}{\sqrt{t}}\Big)^{|\nu|} \cdot S_\nu^\Gamma(1, y/\sqrt{t}\,) \qquad (y \in \mathbb{R}^N,\ t>0). \tag{4.35}$$

A comparison of the homogeneous parts of degree $n$ in the expansions (4.34) and (4.22) shows that the linear spaces generated by $(S_\nu^\Gamma(t,.))_{|\nu|=n}$ and $(R_\nu^\Gamma(t,.))_{|\nu|=n}$ are equal for $t>0$. Hence, $(S_\nu^\Gamma(t,.))_{|\nu|\le n}$ is also a basis of $\bigoplus_{j=0}^n \mathcal{P}_j$.

The $S_\nu^\Gamma(t,.)$ and $R_\nu^\Gamma(t,.)$ are related by the following biorthogonality:

**4.53 Theorem.** *Let $t>0$, $\nu,\rho \in \mathbb{Z}_+^N$, and $p\in\mathcal{P}$ with $deg\, p < |\nu|$. Then:*

(1) $\displaystyle\int_{\mathbb{R}^N} R_\nu^\Gamma(t,y)\cdot S_\rho^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y) = \nu!\,\delta_{\nu,\rho};$

(2) $\displaystyle\int_{\mathbb{R}^N} p(y)\cdot S_\nu^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y) = \int_{\mathbb{R}^N} p(y)\cdot R_\nu^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y) = 0.$

*Proof.* The definition of $\theta_t$ and Lemma 4.42(2) yield

$$\begin{aligned} m_\nu(x) &= \int_{\mathbb{R}^N} R_\nu^\Gamma(t,y)\theta_t(x,y)\ dP_t^\Gamma(0,.)(y) \qquad (4.36)\\ &= \int_{\mathbb{R}^N} \sum_{n=0}^\infty \sum_{|\rho|=n} R_\nu^\Gamma(t,y) S_\rho^\Gamma(t,y)\,\frac{m_\rho(x)}{\rho!}\ dP_t^\Gamma(0,.)(y)\\ &= \sum_{n=0}^\infty \sum_{|\rho|=n} \frac{m_\rho(x)}{\rho!} \int_{\mathbb{R}^N} R_\nu^\Gamma(t,y)\, S_\rho^\Gamma(t,y)\, dP_t^\Gamma(0,.)(y) \end{aligned}$$

where we must justify that summation and integration commute. For this, we may restrict our attention to the case $t=1/4$ by normalization and decompose $\theta_{1/4}(x,y)$ into its $x$-homogeneous parts

$$\theta_{1/4}(x,y) = \sum_{n=0}^\infty L_n(y,x) \qquad \text{with} \qquad L_n(y,x) = \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!} S_\nu^\Gamma(1/4,y).$$

The estimations of Corollary 2.32 imply

$$|L_{2n}(y,x)| \le \frac{|x|^{2n}}{n!}\cdot(1+2|y|^2)^n \quad \text{for } n\in\mathbb{Z}_+,$$

and a similar estimation for odd indices (see the proof of 3.8 in [R2]). Therefore,

$$\sum_{n=0}^\infty \int_{\mathbb{R}^N} |L_n(y,x)|\, R_\nu^\Gamma(1/4,y)\, dP_{1/4}^\Gamma(0,.)(y) < \infty.$$

The dominated convergence theorem now justifies the last step in the equation above for $t=1/4$, which yields Part (1). Part (2) follows from Part (1). □

**4.54 Remark.** For $t=1/2$, the preceding result shows that $(R_\nu^\Gamma(1/2,.))_{\nu\in\mathbb{Z}_+^N}$ is orthogonal w.r.t $dP_{1/2}^\Gamma(0,.)$ if and only if $R_\nu^\Gamma(1/2,x) = c_\nu S_\nu^\Gamma(1/2,x)$ with suitable constants $c_\nu$. A comparison of (4.34) and (4.22) shows that this is equivalent to $m_\nu(x) = c_\nu x^\nu$ which holds for the examples in 4.50. On the other hand, this is not correct for the $A_{N-1}$- and the $B_N$-cases for $N\ge 3$.

The following result reflects the dual nature of Appell characters and cocharacters.

**4.55 Proposition.** *Let $t \in \mathbb{R}$, $x \in \mathbb{R}^N$, and $\nu \in \mathbb{Z}_+^N$. Then*

$$R_\nu^\Gamma(t,x) = e^{-t\Delta_k} m_\nu(x) \qquad \textit{and} \qquad S_\nu^\Gamma(t,x) = \big(\frac{1}{2t}\big)^{|\nu|} \cdot e^{-t\Delta_k} x^\nu.$$

*Proof.* By Proposition 2.53, $\Delta_k$ is also the generator of the heat semigroup acting on $\mathcal{P}$ (instead of $C_0(\mathbb{R}^N)$). Therefore, by Lemma 4.42(2), $e^{t\Delta_k} R_\nu^\Gamma(t,x) = m_\nu(x)$ for $t \geq 0$. This yields the first statement for $t \geq 0$. As both sides are polynomials in $t$, this holds in general.

Let $\Delta_k^y$ be the Dunkl Laplacian acting on the variable $y$, and $V_x$ be the intertwiner w.r.t. $x$. Then

$$\begin{aligned} e^{t\Delta_k^y}\Big(e^{-|x|^2/4t}\, E_k(x, y/2t)\Big) &= e^{-|x|^2/4t} \cdot e^{|x|^2/4t}\, E_k(x, y/2t) \\ &= E_k(x, y/2t) \;=\; V_x(e^{\langle x,\, y/2t\rangle}). \end{aligned}$$

Consider on both sides the homogeneous part $W_n$ of degree $n$ in the variable $x$. Using the left hand side, we obtain from (4.34) that

$$W_n = e^{t\Delta_k^y}\Big(\sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!}\, S_\nu^\Gamma(t,y)\Big) = \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!}\, e^{t\Delta_k^y}\, S_\nu^\Gamma(t,y).$$

Moreover, using the right hand side, we conclude from Section 2.2 and $V_x(x^\nu) = m_\nu(x)$ that

$$W_n = V_x\Big(\sum_{|\nu|=n} \frac{x^\nu}{\nu!}\, (y/2t)^\nu\Big) = \sum_{|\nu|=n} \frac{m_\nu(x)}{\nu!}\, (y/2t)^\nu.$$

A comparison of the coefficients leads to the second statement. □

We give a further application of Theorem 4.53 for $t = 1/2$. For this, we employ the adjoint operator $T_j^*$ of the Dunkl operator $T_j$ ($j = 1, \ldots, N$) in $L^2(\mathbb{R}^N, dP_{1/2}^\Gamma(0,.))$ which is given by

$$T_j^* f(x) = x_j f(x) - T_j f(x) = -e^{|x|^2/2} \cdot T_j\Big(e^{-|x|^2/2}\, f(x)\Big) \qquad (f \in \mathcal{P}); \quad (4.37)$$

see Lemma 3.7 of [D4] and use for the second equation the product rule 2.7.)

**4.56 Corollary.** *For $\nu \in \mathbb{Z}_+^N$, $j = 1, \ldots, N$, $x \in \mathbb{R}^N$, and $t > 0$,*

(1) $S_{\nu+e_j}^\Gamma(1/2, x) = T_j^* S_\nu^\Gamma(1/2, x)$;

(2) *Rodriguez formula:* $S_\nu^\Gamma(t,x) = (-1)^{|\nu|}\, e^{|x|^2/4t}\, T^\nu\big(e^{-|x|^2/4t}\big)$.

*Proof.* For simplicity, we suppress the time parameter $t = 1/2$ in (1). Theorem 4.53(1) and Remark 4.43 yield that for all $\rho \in \mathbb{Z}_+^N$,

$$\begin{aligned} \int_{\mathbb{R}^N} R_{\rho+e_j}^\Gamma \cdot T_j^* S_\nu^\Gamma \; dP^\Gamma &= \int_{\mathbb{R}^N} T_j R_{\rho+e_j}^\Gamma \cdot S_\nu^\Gamma \; dP^\Gamma \\ &= (\rho_j + 1) \int_{\mathbb{R}^N} R_\rho^\Gamma \cdot S_\nu^\Gamma \; dP^\Gamma = \delta_{\rho,\nu} \cdot (\rho + e_j)! \\ &= \int_{\mathbb{R}^N} R_{\rho+e_j}^\Gamma \cdot S_{\nu+e_j}^\Gamma \; dP^\Gamma. \end{aligned}$$

As $\mathcal{P}$ is dense in $L^2(\mathbb{R}^N, dP^\Gamma_{1/2}(0,.))$, Part (1) is clear. Part (2) for $t = 1/2$ follows now from (4.37), and the general case is a consequence of the homogeneity of the $S^\Gamma_\nu$. □

Theorem 4.53 and orthogonalization within the spaces

$$V_n := e^{-\Delta_k/2}\mathcal{P}_n \subset \mathcal{P}$$

leads to systems of orthogonal polynomials on $\mathbb{R}^N$ w.r.t $P^\Gamma_{1/2}(0,.)$, namely the generalized Hermite polynomials $(H_\nu)_{\nu\in\mathbb{Z}^N_+}$ from Definition 2.56. In this way results for generalized Hermite polynomials can be transfered to Appell characters and cocharacters and conversely. Here is a list of a few facts in this direction:

**4.57 Proposition.** *For all $t \in \mathbb{R}$, $x, y \in \mathbb{R}^N$, $n \in \mathbb{N}$, and $\nu \in \mathbb{Z}^N_+$:*

(1) $R^\Gamma_\nu(t,x) = e^{-t\Delta_k} m_\nu(x)$ *and* $S^\Gamma_\nu(t,x) = \big(\frac{1}{2t}\big)^{|\nu|} e^{-t\Delta_k} x^\nu$;

(2) *Rodriguez formula for $R^\Gamma_\nu$: Let $m_\nu(T)$ denote the operators which is obtained from $m_\nu(x)$, $\varphi_\nu(x)$ by replacing the $x_j$ by the Dunkl operators $T_j$. Then*

$$R^\Gamma_\nu(t,x) = (-2t)^{|\nu|} e^{|x|^2/4t}\, m_\nu(T) e^{-|x|^2/4t}.$$

(3) *Eigenfunctions of a CMS-type Schrödinger operator: $R^\Gamma_\nu(t,.)$ and $S^\Gamma_\nu(t,.)$ satisfy*

$$\big(2t\Delta_k - \sum_{l=1}^N x_l\partial_l\big) f = -|\nu| \cdot f.$$

(4) *The functions $e^{-|x|^2/8t}\, R^\Gamma_\nu(t,x)$ and $e^{-|x|^2/8t}\, S_\nu(t,x)$ satisfy*

$$(4t\,\Delta_k - |x|^2) f = -(2|\nu| + 2\gamma + N) f.$$

(5) *Mehler formula: For $r \in \mathbb{C}$ with $|r| < 1$,*

$$\sum_{\nu\in\mathbb{Z}^N_+} \frac{R^\Gamma_\nu(t,x)\, S^\Gamma_\nu(t,y)}{\nu!}\, r^{|\nu|} = \frac{1}{(1-r^2)^{\gamma+N/2}} \exp\Big\{-\frac{tr^2(|x|^2+|y|^2)}{1-r^2}\Big\} E_k\Big(\frac{2trx}{1-r^2}, y\Big).$$

*Proof.* (1) is Proposition 4.55 and (2) follows from Corollary 4.56. Moreover, the generalized Hermite polynomials satisfy the equations in (3) and (4) for $t = 1/2$ (see Corollary 2.58) where this equation only depends on the degree $|\nu|$. Therefore, (3) and (4) hold for $R^\Gamma_\nu(t,.)$ and $S^\Gamma_\nu(t,.)$ for $t = 1/2$. Renormalization by Eq. (4.32) and (4.35) then leads to the case $t > 0$ and analytic continuation to the general case. For (5) we may again assume $t = 1/2$ by (4.32) and (4.35). In this case, we first note that for any orthonormal basis $\{\varphi_\nu, \nu \in \mathbb{Z}^N_+\}$ of $\mathcal{P}_\mathbb{R}$ with respect to $[.,.]_k$ with $\varphi_\nu \in \mathcal{P}_{|\nu|}$, we have

$$\sum_\nu \varphi_\nu(x)\varphi_\nu(y) = E_k(x,y) = \sum_\nu \frac{m_\nu(x) y^\nu}{\nu!}$$

where the second equation follows from the definition of moment functions. Therefore, by Proposition 4.55 and the definition of the $H_\nu$,

$$\sum_\nu H_\nu(x)H_\nu(y) = \sum_\nu \frac{R^\Gamma_\nu(1/2,x)\, S^\Gamma_\nu(1/2,y)}{\nu!},$$

which leads together with Theorem 2.59 to (6) for $t = 1/2$. □

## 5 Notation

We denote by $\mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ the sets of integer, real and complex numbers respectively. Further, $\mathbb{Z}_+ = \{n \in \mathbb{Z} : n \geq 0\}$. For a locally compact Hausdorff space $X$, we denote by $C(X), C_b(X), C_c(X), C_0(X)$ the spaces of continuous complex-valued functions on $X$, those which are bounded, those with compact support, and those which vanish at infinity, respectively. Further, $M_b(X)$, $M_b^+(X)$, $M^1(X)$ are the spaces of regular bounded Borel measures on $X$, those which are positive, and those which are probability-measures, respectively. Finally, $\mathscr{B}(X)$ stands for the $\sigma$-algebra of Borel sets on $X$. Moreover $\mathcal{P}$ stands for the vector space of all poynomials in $N$ variables, and $\mathcal{P}_n$ is the subspace of those, which are homogeneous of degree $n$. Finally, $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$.

## References


[AS] Abramowitz, M., Stegun, I.A., *Pocketbook of Mathematical Functions.* Verlag Harri Deutsch, Frankfurt/Main, 1984.

[AdJ] Andersen, N.B., de Jeu, M., Elementary proofs of Paley-Wiener theorems for the Dunkl transform on the real line. *Int. Math. Res. Not.* 30 (2005), 1817-1831.

[BF1] Baker, T.H., Forrester, P.J., The Calogero-Sutherland model and generalized classical polynomials. *Comm. Math. Phys.* 188 (1997), 175–216.

[BF2] Baker, T.H., Forrester, P.J., The Calogero-Sutherland model and polynomials with prescribed symmetry. *Nucl. Phys. B* 492 (1997), 682–716.

[BF3] Baker, T.H., Forrester, P.J., Non-symmetric Jack polynomials and integral kernels. *Duke Math. J.* 95 (1998), 1–50.

[BO1] Ben Said, S., Ørsted, B.: Bessel functions for root systems via the trigonometric setting. *Int. Math. Res. Not.* 9 (2005), 551-585.

[BO2] Ben Said, S., Ørsted, B.: Segal-Bargmann transforms associated with finite Coxeter groups. *Math. Ann.* 334 (2006), 281-323.

[BBC1] Biane, P, Bougerol, P., O'Connell, N.: Littlemann paths and Brownian paths. *Duke Math. J.* 130 (2005), 127–167.

[BBC2] Biane, P, Bougerol, P., O'Connell, N.: Continuous crystal and Duistermaat-Heckman measure for Coxeter groups. Preprint 2008.

[BH] Bloom, W.R., Heyer, H., *Harmonic analysis of probability measures on hypergroups.* de Gruyter, 1995.

[B] Bru, M., Wishart processes, *J. Theoret. Probab.* 4 (1991), 725–751.

[Ca] Calogero, F., Solution of the one-dimensional N-body problems with quadratic and/or inversely quadratic pair potentials. *J. Math. Phys.* 12 (1971), 419–436.

[Chi] Chihara, T.S., *An Introduction to Orthogonal Polynomials.* Gordon and Breach, 1978.

[Chy] Chybiryakov, O., Processus de Dunkl et relation de Lamperti, PhD Thesis, Universite de Paris VI, 2006.

[CGY] Chybiryakov, O., Gallardo, L., Yor, M., Dunkl processes and their radial parts relative to a root system. This volume.

[Da] Davies, E.B.: *Spectral Theory and Differential Operators.* Cambridge University Press, 1995.

[De] Demni, N., A guided tour in the world of radial Dunkl processes. This volume.

[Di] Dib, H., Fonctions de Bessel sur une algébre de Jordan. *J. math. pures et appl.* 69 (1990), 403–448.

[vD] van Diejen, J.F., Confluent hypergeometric orthogonal polynomials related to the rational quantum Calogero system with harmonic confinement. *Comm. Math. Phys.* 188 (1997), 467–497.

[vDV] van Diejen, J.F., Vinet, L. *Calogero-Sutherland-Moser Models.* CRM Series in Mathematical Physics, Springer-Verlag, 2000.

[DDMY] Donati-Martin, C., Doumerc, Y., Matsumoto, H., Yor, M., Some properties of the Wishart processes and a matrix extension of the Hartman-Watson laws. *Publ Math. RIMS Kyoto* 40 (2004), 1385–1412.

[D1] Dunkl, C.F., Reflection groups and orthogonal polynomials on the sphere. *Math. Z.* 197 (1988), 33–60.

[D2] Dunkl, C.F., Differential-difference operators associated to reflection groups. *Trans. Amer. Math. Soc.* 311 (1989), 167–183.

[D3] Dunkl, C.F., Operators commuting with Coxeter group actions on polynomials. In: Stanton, D. (ed.), *Invariant Theory and Tableaux*, Springer, 1990, pp. 107–117.

[D4] Dunkl, C.F., Integral kernels with reflection group invariance. *Canad. J. Math.* 43 (1991), 1213–1227.

[D5] Dunkl, C.F., Hankel transforms associated to finite reflection groups. In: *Proc. of the special session on hypergeometric functions on domains of positivity, Jack polynomials and applications.* Proceedings, Tampa 1991, Contemp. Math. 138 (1992), pp. 123–138.

[D6] Dunkl, C.F., Intertwining operators associated to the group $S_3$. *Trans. Amer. Math. Soc.* 347 (1995), 3347–3374.

[D7] Dunkl, C.F., Symmetric Functions and $B_N$-invariant spherical harmonics. J. Phys. A, Math. Gen. 35, No. 48 (2002), 10391-10408.

[D8] Dunkl, C.F., An intertwining operator for the group $B_2$. *Glasgow Math. J.* 49 (2007), 291–319.

[DJO] Dunkl, C.F., de Jeu M.F.E., Opdam, E.M., Singular polynomials for finite reflection groups. *Trans. Amer. Math. Soc.* 346 (1994), 237–256.

[DX] Dunkl, C.F., Xu, Y., *Orthogonal Polynomials of Several Variables.* Cambridge Univ. Press, 2001.

[Dy] Dynkin, E.B., *Markov Processes I.* Springer, 1965.

[E] Eastham, M.S.P., *The Asymptotic Solution of Linear Differential Systems: Applications of the Levinson Theorem.* Clarendon Press, Oxford 1989.

[EK] Ethier, S.N., Kurtz, T.G., *Markov Processes. Characterizations and convergence.* Wiley 1986.

[F] Faraut, J., Analyse harmonique sur les pairs de Gelfand et les espaces hyperboliques. In: *J.-L. Clerc et al.. Analyse Harmonique*, C.I.M.P.A., Nice 1982, Ch. IV.

[FT] Faraut, J., Travaglini, G., Bessel functions associated with representations of formally real Jordan algebras, *J. Funct. Anal.* 71 (1987), 123–141.

[FD] Fell, J.M.G., Doran, R.S., *Representations of ∗-Algebras, Locally Compact Groups , and Banach-∗-Algebraic Bundles*, Vol. 1. Academic Press, 1988.

[FK] Faraut, J., Korányi, A., *Analysis on symmetric cones.* Oxford Science Publications, 1994.

[FS] Feinsilver, Ph., Schott, R., *Algebraic Structures and Operator Calculus. Vol. III. Representations of Lie groups.* Kluwer: Dordrecht 1996.

[GY1] Gallardo, L., Yor, M., Some new examples of Markov processes which enjoy the time inversion property. *Probab. Theor. Rel. Fields* 132 (2005), 150-162.

[GY2] Gallardo, L., Yor, M., Some remarkable properties of the Dunkl martingales. In: Seminaire de Probabilites XXXIX, pp. 337–356, Springer LNM 1874 (2006).

[GY3] Gallardo, L., Yor, M., A chaotic representation property of the Dunkl martingales. *Ann. Probab.* 34 (2006), 1530–1549.

[Go] Godefroy, L., Frontière de Martin sur les hypergroupes et principe d'invariance relative au processus de Dunkl. Thèse, Universite de Tours, 2003.

[Gr] Greenleaf, F., *Invaraint means on topological groups and their applications,* Van Nostrand Reinhold Co., 1969.

[GB] Grove, L.C., Benson, C.T., *Finite Reflection Groups;* Second edition. Springer, 1985. *Correlation effects in low-dimensional electron systems.* Springer, 1995, pp. 3–20.

[He1] Heckman, G.J., A remark on the Dunkl differential-difference operators. In: Barker, W., Sally, P. (eds.) *Harmonic analysis on reductive groups.* Progress in Math. **101**, Birkhäuser, 1991. pp. 181 – 191.

[He2] Heckman, G.J., Dunkl operators. Séminaire Bourbaki 828, 1996–97; *Astérisque* 245 (1997), 223–246.

[HS] Heckman, G.J., Schlichtkrull, H., *Harmonic Analysis and Special Functions on Symmetric Spaces.* Perspectives in Mathematics **16**, Academic Press, California, 1994.

[Hel1] Helgason, S., *Differential Geometry, Lie Groups, and Symmetric Spaces.* Academic Press, New York, 1978.

[Hel2] Helgason, S., *Groups and Geometric Analysis.* American Mathematical Society, 1984.

[Her] Herz, C.S., Bessel functions of matrix argument. *Ann. Math.* 61 (1955), 474–523.

[HP] Heyer, H., Pap, G., Martingale characterizations of increment processes in a locally compact group. *Inf. Dim. Analysis, Quantum Probab. Related Topics* 6 (2003), 43–87.

[Hu] Humphreys, J.E., *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1990.

[dJ1] de Jeu, M.F.E., The Dunkl transform. *Invent. Math.* 113 (1993), 147–162.

[dJ2] de Jeu, M.F.E., Dunkl operators. Thesis, University of Leiden, 1994.

[dJ4] de Jeu, M.F.E., Paley-Wiener theorems for the Dunkl transform. *Trans. Amer. Math. Soc.* 358 (2006), no. 10, 4225–4250.

[J] Jewett, R.I., Spaces with an abstract convolution of measures, *Adv. Math.* 18 (1975), 1–101.

[K] Kakei, S., Common algebraic structure for the Calogero-Sutherland models. *J. Phys. A* 29 (1996), L619–L624.

[Kal] Kallenberg, O., *Foundations of Modern Probability.* Springer-Verlag, 1997.

[Ki] Kingman, J.F.C., Random walks with spherical symmetry. *Acta Math.* 109 (1963), 11-53.

[Kl] Klyachko, A., Stable bundles, representation theory, and Hermitian operators, *Selecta Math.* (new Series) 4 (1998), 419-445.

[Kn] Knapp, A.W., *Lie Groups Beyond an Introduction.* 2nd ed., Birkhäuser Verlag, 2005.

[KT] Knutson, K., Tao, T., The honeycomb model of $GL_n(\mathbb{C})$ tensor product I. Proof of the saturation conjecture, *J. Amer. Math. Soc.* 12 (1999), 1055-1090.

[Ko] Koornwinder, T.H. (ed.), *The Structure of Real Semisimple Lie Groups.* MC Syllabus 49, Math. Centrum, Amsterdam 1982.

[KS] Knop, F., Sahi, S., A recursion and combinatorial formula for Jack polynomials. *Invent. Math.* 128 (1997), 9–22.

[LV] Lapointe L., Vinet, L., Exact operator solution of the Calogero-Sutherland model. *Comm. Math. Phys.* 178 (1996), 425–452.

[La1] Lassalle, M., Polynômes de Laguerre généralisés. *C.R. Acad. Sci. Paris* t. 312 Série I (1991), 725–728

[La2] Lassalle, M., Polynômes de Hermite généralisés. *C.R. Acad. Sci. Paris* t. 313 Série I (1991), 579–582.

[M1] Macdonald, I.G., Some conjectures for root systems. *SIAM J. Math. Anal.* 13 (1982), 988–1007.

[M2] Macdonald, I.G., The Volume of a Compact Lie Group. *Invent. Math.* 56 (1980), 93–95.

[M3] Macdonald, I.G., Commuting differential operators and zonal spherical functions, in: Algebraic groups (Utrecht 1986), eds. M. Cohen et al, *Lecture Notes in Mathematics* 1271, Springer-Verlag, Berlin, 1987.

[MT] Mejjaoli, H., Trimèche ,K., On a mean value property associated with the Dunkl Laplacian operator and applications. *Integral Transform. Spec. Funct.* 12 (2001), 279–302.

[Mo] Moser, J., Three integrable Hamiltonian systems connected with isospectral deformations, *Adv. in Math.* 16 (1975), 197–220.

[Mu] Muirhead, R.J., *Aspects of multivariate statistical theory.* Wiley, 1982.

[OP] Olshanetsky, M.A., Perelomov, A.M., Completely integrable Hamiltonian systems connected with semisimple Lie algebras. *Invent. Math.* 37 (1976), 93–108.

[O1] Opdam, E.M., Dunkl operators, Bessel functions and the discriminant of a finite Coxeter group. *Compositio Math.* 85 (1993), 333–373.

[O2] Opdam, E.M., Harmonic analysis for certain representations of graded Hecke algebras. *Acta Math.* 175 (1995), 75 – 121

[O3] Opdam, E.M., *Lecture Notes on Dunkl Operators for Real and Complex Reflection Groups.* MSJ Memoirs, vol. 8, Math. Soc. of Japan, Tokyo, 2000.

[Pa] Pasquier, V.: A lecture on the Calogero-Sutherland models. In: Integrable models and strings (Espoo, 1993), *Lecture Notes in Phys.* 436, Springer, 1994, pp. 36–48.

[Po] Polychronakos, A.P., Exchange operator formalism for integrable systems of particles. *Phys. Rev. Lett.* 69 (1992), 703–705.

[Re1] Rentzsch, C., *Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmaßen auf einer kommutativen Hypergruppe und ihre kanonische Darstellung.* Dissertation, Universität Tübingen, 1997.

[Re2] Rentzsch, C., A Lèvy Khintchine representation of convolution semigroups on commutative hypergroups. *Probab. Math. Stat.* **18** (1998), 185 – 198.

[ReV] Rentzsch, C., Voit, M., Lévy processes on commutative hypergroups. *Contemp. Math.* 261 (2000), 83–105.

[RY] Revuz, D., Yor, M., *Continuous Martingales and Brownian Motion.* Springer, 3rd ed., 1999.

[Rom] Roman, S.: *The Umbral Calculus.* Academic Press 1984.

[R1] Rösler, M., Bessel-type signed hypergroups on $\mathbb{R}$. In: Heyer, H., Mukherjea, A. (eds.) Probability measures on groups and related structures XI. Proceedings, Oberwolfach 1994. World Scientific 1995, 292–304.

[R2] Rösler, M., Generalized Hermite polynomials and the heat equation for Dunkl operators. *Comm. Math. Phys.* 192 (1998), 519–542.

[R3] Rösler, M., Positivity of Dunkl's intertwining operator. *Duke Math. J.* 98 (1999), 445–463.

[R4] Rösler, M., Short-time estimates for heat kernels associated with root systems, in *Special Functions,* Conf. Proc. Hong Kong June 1999, eds.C. Dunkl et al., World Scientific, Singapore, 2000, 309–323.

[R5] Rösler, M., One-parameter semigroups related to abstract quantum models of Calogero type. In: Infinite Dimensional Harmonic Analysis (Kyoto, Sept. 1999, eds. H. Heyer et al.) Gräbner-Verlag 2000, 290–305.

[R6] Rösler, M., A positive radial product formula for the Dunkl kernel. *Trans. Amer. Math. Soc.* 355 (2003), 2413–2438.

[R7] Rösler, M., Dunkl operators: Theory and applications. In: "Orthogonal polynomials and special functions", Leuven 2002, *Springer Lecture Notes in Math.* 1817 (2003), 93–135.

[R8] Rösler, M., Bessel convolutions on matrix cones, *Compos. Math.* 143 (2007), 749–779.

[RV1] Rösler, M., Voit, M., Markov Processes related with Dunkl operators. *Adv. Appl. Math.* 21 (1998), 575–643.

[RV2] Rösler, M., Voit, M., Positivity of Dunkl's intertwining operator via the trigonometric setting. *Internat. Math. Res. Notices* 2004, no 63, 3379-3389.

[RV3] Rösler, M., Voit, M., Limit theorems for radial random walks on $p \times q$ matrices as $p$ tends to infinity. ArXiv math.CA/0703520 (2007), to appear in Math. Nachr..

[Ros] Rosenblum, M., Generalized Hermite polynomials and the Bose-like oscillator calculus. In: Operator Theory: Advances and Applications, Vol. 73, Basel, Birkhäuser Verlag 1994, 369–396.

[Sh] Schapira, B., The Heckman-Opdam Markov-processes. Probab. Theor. Rel. Fields, to appear.

[Scho] Schoutens, W., *Stochastic processes and orthogonal polynomials.* Lecture Notes in Statistics 146. Springer, 2000.

[SV] Stroock, D., Varadhan, S.R.S., *Multidimensional Diffusion Processes.* Springer, 1979.

[Su] Sutherland, B., Exact results for a quantum many-body problem in one dimension. *Phys. Rep.* A5 (1972), 1372–1376.

[Sz] Szegö, G.: *Orthogonal Polynomials.* Amer. Math. Soc., 1959.

[TX] Thangavelu, S, Xu, Y., Convolution operator and maximal function for the Dunkl transform. *J. Anal. Math.* 97 (2005), 25–55.

[Ti] Titchmarsh, E.C., *The Theory of Functions.* 2nd ed., Oxford University Press, 1950.

[T1] Trimèche, K., The Dunkl intertwining operator on spaces of functions and distributions and integral representations of its dual. *Integral Transform. Spec. Funct.* 12 (2001), 349–374.

[T2] Trimèche, K., Paley-Wiener Theorems for the Dunkl transform and Dunkl translation operators. *Integral Transform. Spec. Funct.* 13 (2002), 17–38.

[Vog] Vogel, M., Spectral synthesis on algebras of orthogonal polynomial series. *Math. Z.* 194 (1987), 99–116.

[V0] Voit, M., Positive characters on commutative hypergroups and some applications. *Math. Z.* 198 (1988), 405–421.

[V1] Voit, M., Martingale characterizations of stochastic processes on compact groups. *Probab. Math. Stat.* 19 (1999), 389–405.

[V2] Voit ,M., Bessel convolutions on matrix cones: Algebraic properties and random walks. arXiv: math.CA/0603017, to appear in J. Theor. Prob..

[V3] Voit ,M., Central limit theorems for radial random walks on $p{\times}q$ matrices for $p \to \infty$. Preprint, 2008.

[We] Wesolowski, J., A martingale characterization of the Poisson process. *Bull. Polish Acad. Sci. Math.* 38 (1990), 49–53.

[Wo] Wolf, J. A., Spherical functions on Euclidean space. *J. Funct. Anal.* 239 (2006), 127–136.

[X1] Xu, Y., Integration of the intertwining operator for $h$-harmonic polynomials associated to reflection groups. *Proc. Amer. Math. Soc.* 125 (1997), 2963–2973.

[X2] Xu, Y., Funk-Hecke formula for orthogonal polynomials on spheres and on balls. *Bull. London Math. Soc.* 32 (2000), 447–457.

[Z1] Zeuner, Hm., Laws of large numbers for hypergroups on $\mathbb{R}_+$. *Math. Ann.* 283 (1989), 657–678.

[Z2] Zeuner, Hm., Moment functions and laws of large numbers on hypergroups. *Math. Z.* 211 (1992), 369–407.