

THE DUNKL-LAPLACE TRANSFORM AND MACDONALD'S HYPERGEOMETRIC SERIES

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ABSTRACT. We continue a program generalizing classical results from the analysis on symmetric cones to the Dunkl setting for root systems of type A . In particular, we prove a Dunkl-Laplace transform identity for Heckman-Opdam hypergeometric functions of type A and more generally, for the associated Opdam-Cherednik kernel. This is achieved by analytic continuation from a Laplace transform identity for non-symmetric Jack polynomials which was stated, for the symmetric case, as a key conjecture by Macdonald in [M13]. Our proof for the Jack polynomials is based on Dunkl operator techniques and the raising operator of Knop and Sahi. Moreover, we use these results to establish Laplace transform identities between hypergeometric series in terms of Jack polynomials. Finally, we conclude with a Post-Widder inversion formula for the Dunkl-Laplace transform.

1. INTRODUCTION

The Laplace transform is an important tool in various areas of harmonic analysis and forms a cornerstone in the analysis on symmetric cones, see [FK94]. In particular, there are important Laplace transform identities between ${}_pF_q$ -hypergeometric functions on a symmetric cone, which are given as expansions with respect to the associated spherical polynomials, c.f. [FK94, Chap.XV]. For cones of positive definite matrices, such hypergeometric series trace back to ideas of Bochner and were studied in detail by Herz [He55], where they were actually defined recursively by means of the Laplace transform. For important further developments see for instance [C63, GR89, Kan93]. Multivariable hypergeometric series have found many applications in multivariate statistics [Mu82], but also in number theory and mathematical physics.

In his unpublished manuscript [M13] from the 1980ies, Macdonald introduced hypergeometric series in terms of Jack polynomials, which include the hypergeometric functions on symmetric cones as special cases. In this context, he also introduced a generalization of the Laplace transform for radial functions on symmetric cones, but many statements in [M13] remained at a formal level.

Radial analysis on symmetric cones is closely related to Dunkl theory for root systems of type A , and also Macdonald's concepts have a natural interpretation within Dunkl theory, because the ${}_0F_0$ -hypergeometric function, which replaces the exponential kernel in the Macdonald's Laplace transform, is just a Dunkl-Bessel

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function of type A . The connection of the concepts in [M13] to Dunkl theory was already observed by Baker and Forrester in their seminal papers [BF97, BF98] related to the study of Calogero-Moser-Sutherland models.

To become more precise on these connections, consider a symmetric cone $\Omega = G/K$ inside a simple Euclidean Jordan algebra V of rank n and with Peirce constant d , which takes only specific integer values. Let $F \in L^1_{loc}(\Omega)$ be K -invariant, that is of the form $F(x) = f(\text{spec}(x))$, where $\text{spec}(x) \in \mathbb{R}_+^n$ with $\mathbb{R}_+ =]0, \infty[$ denotes the set of eigenvalues of x ordered by size. Then for $y \in \Omega$, the Laplace transform

$$LF(y) = \int_{\Omega} e^{-(x|y)} F(x) dx$$

depends only on $\eta = \text{spec}(y) \in \mathbb{R}_+^n$ and can be written as

$$LF(y) = \text{const} \cdot \int_{\mathbb{R}_+^n} J_{d/2}(-\xi, \eta) f(\xi) \omega_{d/2}(\xi) d\xi \quad (1.1)$$

where for $k \geq 0$,

$$\omega_k(\xi) = \prod_{i < j} |\xi_i - \xi_j|^{2k}$$

and

$$J_k(z, w) = \sum_{\lambda \in \Lambda_n^+} \frac{1}{|\lambda|!} \frac{C_\lambda(z; \alpha) C_\lambda(w; \alpha)}{C_\lambda(\mathbf{1}; \alpha)} = {}_0F_0^\alpha(z, w), \quad \alpha = \frac{1}{k} \in]0, \infty].$$

Here Λ_n^+ denotes the set of partitions with at most n parts and the $C_\lambda(\cdot; \alpha)$ are the (symmetric) Jack polynomials of index α in C -normalization as in Lemma 6.1 below. See e.g. [R20] for some details. On the other hand, it is well-known (c.f. [BF98] and Remark 6.6) that for arbitrary $k \geq 0$, the function J_k coincides with a Dunkl-Bessel function of type A_{n-1} with multiplicity k .

Macdonald [M13] considered the Laplace transform (1.1) with the Bessel function $e(z, w) = {}_0F_0^\alpha(z, w)$ for arbitrary indices $\alpha > 0$. Many of his calculations were of a formal nature and rested on the following "Conjecture (C)" about the Laplace transform of Jack polynomials: For $k \geq 0$, let

$$\mu_0 := k(n-1) \text{ and } \Delta(z) := \prod_{j=1}^n z_j \text{ for } z \in \mathbb{C}^n$$

and write $C_\lambda(z) := C_\lambda(z; \frac{1}{k})$ for abbreviation. Then for all $\lambda \in \Lambda_n^+$, $y \in \mathbb{R}_+^n$ and all $\mu \in \mathbb{C}$ with $\text{Re } \mu > \mu_0$,

$$\int_{\mathbb{R}_+^n} J_k(-y, x) C_\lambda(x) \Delta(x)^{\mu - \mu_0 - 1} \omega_k(x) dx = \Gamma_n(\lambda + \underline{\mu}) C_\lambda(\frac{1}{y}) \Delta(y)^{-\mu}. \quad (1.2)$$

Here $\Gamma_n(\lambda) = \Gamma_n(\lambda; k)$ is Macdonald's multivariate gamma function defined in formula (3.7) below. In [R20], we gave a rigorous treatment of the Dunkl-Laplace transform

$$\mathcal{L}f(z) = \int_{\mathbb{R}_+^n} E_k(-z, x) f(x) \omega_k(x) dx$$

where compared to Macdonald's version, the Bessel function is replaced by the Dunkl kernel E_k of type A_{n-1} and multiplicity k . This transform was already considered by Baker and Forrester [BF98] and later used in [SZ07], but convergence issues had remained open for a long time. Formula (1.2) generalizes a Laplace

transform identity for spherical polynomials on a symmetric cone, which is in turn a consequence of the following important Laplace transform identity for the generalized power functions Δ_s , $s = (s_1, \dots, s_n) \in \mathbb{C}^n$ with $\operatorname{Re} s_j > \frac{d}{2}(j-1)$ (see [FK94, Chapter VII]):

$$\int_{\Omega} e^{-(y|x)} \Delta_s(x) \Delta(x)^{-m/n} dx = \Gamma_{\Omega}(s) \Delta_s(y^{-1}) \quad (y \in \Omega), \quad (1.3)$$

where m is the dimension of the Jordan algebra, Δ denotes the Jordan determinant and Γ_{Ω} is the Gindikin gamma function associated with Ω . Taking K -means in (1.3), one gets the same Laplace transform identity for the spherical functions of Ω , which may be parametrized as

$$\varphi_{\lambda}(x) = \int_K \Delta_{\lambda}(kx) dk, \quad \lambda \in \mathbb{C}^n$$

and depend only on the eigenvalues of their argument. More precisely, for $y \in \Omega$ and $s \in \mathbb{C}^n$ as above,

$$\int_{\Omega} e^{-(y|x)} \varphi_s(x) \Delta(x)^{-m/n} dx = \Gamma_{\Omega}(s) \varphi_s(y^{-1}). \quad (1.4)$$

For parameters $\lambda \in \Lambda_n^+$, the φ_{λ} are just the spherical polynomials of Ω given by

$$\varphi_{\lambda}(x) = \frac{C_{\lambda}(\operatorname{spec}(x), \frac{2}{d})}{C_{\lambda}(\underline{1}, \frac{2}{d})}.$$

Rewriting (1.4) by means of identity (1.1), one gets formula (1.2) for the particular multiplicities $k = d/2$. It is well-known that the spherical functions of Ω can be expressed in terms of Heckman-Opdam hypergeometric functions of type A_{n-1} and with multiplicity $k = d/2$, see e.g. [RKV13] for cones of positive definite matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

In the present paper, we shall establish a generalization of formula (1.4) to the Dunkl setting of type A_{n-1} with arbitrary multiplicity $k \geq 0$. Namely, we obtain in Corollary 5.4 the following Laplace transform identity for Heckman-Opdam hypergeometric functions of type A_{n-1} and, more generally, for the associated Opdam-Cherednik kernel \mathcal{G}_k : For $\lambda \in \mathbb{C}^n$ with $\operatorname{Re} \lambda \geq \mu_0 \cdot \underline{1}$ and $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$,

$$\int_{\mathbb{R}_+^n} E_k(-z, x) \mathcal{G}_k(\lambda, x) \Delta(x)^{-\mu_0-1} \omega_k(x) dx = \Gamma_n(\lambda + \rho) \mathcal{G}_k(\lambda, \frac{1}{z}). \quad (1.5)$$

The first step towards the proof of (1.5) will be a rigorous proof of Macdonald's Conjecture (C). More generally, we shall prove Dunkl-Laplace transform identities for the non-symmetric Jack polynomials in the sense of [O95, KS97], from which (1.2) then follows by symmetrization. These non-symmetric identities were already stated in [BF98], but the proof given there in terms of Laguerre expansions is involved and not fully carried out. The proof we are presenting here is completely different and very natural; it is based on a reformulation via Dunkl operators and is carried out by induction, using the raising operator of Knop and Sahi [KS97] for the non-symmetric Jack polynomials. The statement for the Cherednik kernel is then obtained via analytic continuation with respect to the spectral variable, and for the hypergeometric function it follows by symmetrization.

Based on the Laplace transform identities for Jack polynomials, we then study hypergeometric series in terms of Jack polynomials of the form

$${}_pF_q(\mu; \nu; z, w) := \sum_{\lambda \in \Lambda_n^+} \frac{[\mu_1]_\lambda \cdots [\mu_p]_\lambda}{[\nu_1]_\lambda \cdots [\nu_q]_\lambda} \frac{C_\lambda(z)C_\lambda(w)}{|\lambda|! C_\lambda(\mathbf{1})} \quad (\mu \in \mathbb{C}^p, \nu \in \mathbb{C}^q)$$

as well as their non-symmetric analogues, and we establish Laplace transform identities between them. This generalizes known results on symmetric cones and settles several conjectural Laplace transform formulas in [M13]. As a further application, we finally prove a Post-Widder inversion theorem for the Dunkl-Laplace transform, which complements a result by Faraut and Gindikin in [FG90] for the Laplace transform on symmetric cones.

The organization of this paper is as follows: Section 2 provides the necessary background on the type A Dunkl setting, both in the rational and trigonometric case. In Section 3, we collect results on the symmetric and non-symmetric Jack polynomials which will be relevant in the sequel, and we prove the Dunkl-Laplace transform identities for Jack polynomials. Section 4 contains a digression on some useful properties of the Opdam-Cherednik kernel for arbitrary root systems. These will be employed (for type A_{n-1}) in Section 5, where the Laplace transform identities for Jack polynomials are extended to the Opdam-Cherednik kernel and to the hypergeometric function. Section 6 is devoted to the study of Jack-hypergeometric series, and Section 7 contains the Post-Widder inversion formula in the Dunkl setting.

To avoid notational overload, we shall always suppress in our notations the dependence on the fixed multiplicity parameter $k \geq 0$ on the root system A_{n-1} .

2. THE TYPE A DUNKL SETTING

For a general background, the reader is referred to [D89, dJ93, R03, DX14, O95, R20, HO21].

We consider the root system $A_{n-1} = \{\pm(e_i - e_j) : 1 \leq i < j \leq n\}$ in the Euclidean space \mathbb{R}^n with inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i$ and norm $|x| = \sqrt{\langle x, x \rangle}$, where the e_i denote the standard basis vectors. The inner product $\langle \cdot, \cdot \rangle$ is extended to \mathbb{C}^n in a bilinear way. The reflection group generated by A_{n-1} is the symmetric group \mathcal{S}_n on n elements. It acts on functions $f : \mathbb{R}^n \rightarrow \mathbb{C}$ by $\sigma f := f(\sigma^{-1} \cdot)$. The (rational) Dunkl operators associated with A_{n-1} and a multiplicity parameter $k \in \mathbb{C}$ are given by

$$T_\xi = T_\xi(k) = \partial_\xi + \frac{k}{2} \cdot \sum_{\alpha \in A_{n-1}} \langle \alpha, \xi \rangle \frac{1 - s_\alpha}{\langle \alpha, x \rangle} \quad (\xi \in \mathbb{R}^n)$$

where s_α denotes the orthogonal reflection in the root α . For fixed k , the operators $T_\xi(k)$, $\xi \in \mathbb{R}^n$ commute. Let $\mathcal{P} := \mathbb{C}[\mathbb{R}^n]$ denote the space of polynomial functions on \mathbb{R}^n . The assignment $\langle \cdot, \xi \rangle \mapsto T_\xi$ has a unique extension to a unital morphism of algebras $\phi : \mathcal{P} \rightarrow \text{End}(\mathcal{P})$, and we shall write $\phi(p) =: p(T)$.

The Dunkl operators satisfy a product rule: if $f, g \in C^1(\mathbb{R}^n)$ and one of them is symmetric, then $T_\xi(fg) = (T_\xi f)g + fT_\xi g$.

In this paper, we shall always assume that $k \geq 0$. Let $E(\lambda, \cdot) := E_k(\lambda, \cdot) : \mathbb{C}^n \rightarrow \mathbb{C}$, $\lambda \in \mathbb{C}^n$ be the associated Dunkl kernel of type A_{n-1} , i.e. the unique holomorphic

solution of the system

$$\begin{cases} T_\xi E(\lambda, \cdot) = \langle \lambda, \xi \rangle E(\lambda, \cdot), & \text{for all } \xi \in \mathbb{R}^n \\ E(\lambda, 0) = 1 \end{cases}.$$

The associated Bessel function is defined by

$$J(\lambda, z) := J_k(\lambda, z) := \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} E(\lambda, \sigma z).$$

The Dunkl kernel is positive on $\mathbb{R}^n \times \mathbb{R}^n$ and for $\lambda, z \in \mathbb{C}^n, s \in \mathbb{C}, \sigma \in \mathcal{S}_n$ one has

$$E(\lambda, z) = E(z, \lambda), \quad E(s\lambda, z) = E(\lambda, sz), \quad E(\sigma\lambda, \sigma z) = E(\lambda, z).$$

Moreover, it satisfies

$$E(\lambda, z + \underline{s}) = e^{\langle \lambda, \underline{s} \rangle} \cdot E(\lambda, z)$$

for $s \in \mathbb{C}$, where

$$\underline{s} := s \cdot (1, \dots, 1) \in \mathbb{C}^n.$$

Let

$$\omega(x) = \prod_{1 \leq i < j \leq n} |x_i - x_j|^{2k}.$$

Dunkl analysis associated with the root system A_{n-1} generalizes the radial analysis on symmetric cones, which just corresponds to the multiplicity values $k = d/2$, where d is the Peirce constant of the cone. There is a well-behaved Laplace transform of functions $f \in L^1_{loc}(\mathbb{R}^n_+)$, for arbitrary $k \geq 0$, which is given by

$$\mathcal{L}f(z) = \int_{\mathbb{R}^n_+} f(x) E(-z, x) \omega(x) dx,$$

and was first considered in [BF98]. For $x \in \mathbb{R}^n_+$ and $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq a$ for some $a \in \mathbb{R}^n$ (which is understood componentwise), the type A Dunkl kernel satisfies the exponential bound

$$|E(-z, x)| \leq \exp(-\|x\|_1 \cdot \min_{1 \leq i \leq n} a_i), \quad (2.1)$$

see [R20]. Here $\|x\|_1 = \sum_{i=1}^n |x_i|$. This estimate, which seems to be exclusive in type A , guarantees good convergence properties of the Laplace integral. In particular, we recall the following

Lemma 2.1 ([R20]). *Suppose that $f : \mathbb{R}^n_+ \rightarrow \mathbb{C}$ is measurable and exponentially bounded according to $|f(x)| \leq C e^{s\|x\|_1}$ with some constants $C > 0$ and $s \in \mathbb{R}$. Then $\mathcal{L}f(z)$ exists and is holomorphic on $\{z \in \mathbb{C}^n : \operatorname{Re} z > \underline{s}\}$. Moreover, for each polynomial $p \in \mathcal{P}$, $p(-T)(\mathcal{L}f) = \mathcal{L}(fp)$ on $\{\operatorname{Re} z > \underline{s}\}$.*

Let us turn to the trigonometric setting. Here root systems are required to be spanning, and therefore we consider A_{n-1} as a subset of

$$\mathbb{R}_0^n := \{x \in \mathbb{R}^n : x_1 + \dots + x_n = 0\}.$$

We fix the positive subsystem $A_{n-1}^+ = \{e_j - e_i : i < j\}$. The (trigonometric) Cherednik operators on \mathbb{R}_0^n associated with A_{n-1}^+ and a multiplicity parameter $k \geq 0$ are given by

$$D_\xi = D_\xi(A_{n-1}^+, k) := \partial_\xi - \langle \rho(R_+), \xi \rangle + k \sum_{\alpha \in A_{n-1}^+} \langle \alpha, \xi \rangle \frac{1 - s_\alpha}{1 - e^{-\langle \cdot, \alpha \rangle}}, \quad \xi \in \mathbb{R}^n,$$

with the Weyl vector

$$\rho(R_+) = \rho(R_+, k) := \frac{k}{2} \sum_{\alpha \in A_{n-1}^+} \alpha = -\frac{k}{2}(n-1, n-3, \dots, -n+1).$$

The operators D_ξ , $\xi \in \mathbb{R}_0^n$ commute. Let $\mathbb{C}_0^n = \mathbb{R}_0^n \oplus i\mathbb{R}_0^n$. Due to [O95], there exist an \mathcal{S}_n -invariant tubular open neighborhood U of \mathbb{R}_0^n in \mathbb{C}_0^n and a unique holomorphic function $G = G_k$ on $\mathbb{C}_0^n \times (\mathbb{R}_0^n + iU)$ which satisfies the joint eigenvalue problem

$$\begin{cases} D_\xi G(\lambda, \cdot) = \langle \lambda, \xi \rangle G(\lambda, \cdot), & \text{for all } \xi \in \mathbb{R}_0^n \\ G(\lambda, 0) = 1. \end{cases}$$

The function G is called the Opdam-Cherednik kernel associated with A_{n-1} and multiplicity k . By symmetrization, one obtains the associated Heckman-Opdam hypergeometric function

$$F(\lambda, z) = F_k(\lambda, z) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} G_k(\lambda, \sigma z).$$

It is actually \mathcal{S}_n -invariant in both λ and z . According to [KO08, Theorem 13.15] (see also [HO21, Cor. 8.6.2]), F extends to a holomorphic function on $\mathbb{C}_0^n \times (\mathbb{R}_0^n + i\Omega)$ with

$$\Omega = \{x \in \mathbb{R}_0^n : |x_i - x_j| < \pi \text{ for all } 1 \leq i < j \leq n\},$$

and the proof of [O95, Theorem 3.15] shows that G extends holomorphically to the same domain. We mention that For $k = d/2$ with $d = 1, 2, 4$, the functions $t \mapsto F(\lambda, 2t)$ on \mathbb{R}_0^n are naturally identified with the spherical functions of $SL_n(\mathbb{F})/SU_n(\mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$; cf. [O95, RKV13].

In this paper, we shall work with natural extensions of G and F . In order to define them, we consider the Cherednik operators D_ξ as operators on \mathbb{R}^n , for arbitrary $\xi \in \mathbb{R}^n$.

Lemma 2.2. *Consider the orthogonal projection $\pi : \mathbb{R}^n \rightarrow \mathbb{R}_0^n$, $x \mapsto x - \frac{1}{n}\langle x, \mathbf{1} \rangle \cdot \mathbf{1}$. Then for all $\xi \in \mathbb{R}^n$ and $f, g \in C^1(\mathbb{R}^n)$,*

- (1) $D_\xi(f \circ \pi) = D_{\pi(\xi)}f \circ \pi$.
- (2) *If f or g is \mathcal{S}_n -invariant, then $D_\xi(fg) = f(D_\xi g) + (D_\xi f)g + \langle \rho(R_+), \xi \rangle fg$.*

Proof. Part (1) is obtained by a short calculation, using that $\partial_\xi(f \circ \pi) = \partial_{\pi(\xi)}f \circ \pi$ and that ρ and all roots are contained in \mathbb{R}_0^n . Part (2) is straightforward. \square

In order to extend the Opdam-Cherednik kernel, put $V := \Omega + \mathbb{R}\mathbf{1} \subset \mathbb{R}^n$, which is \mathcal{S}_n -invariant with $\pi(V) = \Omega$. Keeping the notation, we define

$$G : \mathbb{C}^n \times (\mathbb{R}^n + iV) \rightarrow \mathbb{C}, \quad G(\lambda, z) := e^{\frac{1}{n}\langle z, \mathbf{1} \rangle \langle \lambda, \mathbf{1} \rangle} G(\pi(\lambda), \pi(z)), \quad (2.2)$$

where π is canonically extended to a linear mapping $\pi : \mathbb{C}^n \rightarrow \mathbb{C}_0^n$. The extended hypergeometric function F is obtained in the same way.

Proposition 2.3. *The extended Opdam-Cherednik kernel $G(\lambda, \cdot)$, $\lambda \in \mathbb{C}^n$ is the unique holomorphic solution on $\mathbb{R}^n + iV$ to the system*

$$\begin{cases} D_\xi G(\lambda, \cdot) = \langle \lambda, \xi \rangle G(\lambda, \cdot) & \text{for all } \xi \in \mathbb{R}^n \\ G(\lambda, 0) = 1. \end{cases} \quad (2.3)$$

Proof. Since $z \mapsto e^{\langle z, \underline{1} \rangle \langle \lambda, \underline{1} \rangle / n}$ is \mathcal{S}_n -invariant, it is immediate from Lemma 2.2(2) that $G(\lambda, \cdot)$ solves (2.3). Assume that f is a further solution in some neighborhood U' of $0 \in \mathbb{C}^n$. Then for $\xi \in \mathbb{R}_0^n$, Lemma 2.2 (1) gives

$$D_\xi(f \circ \pi) = D_\xi f \circ \pi = \langle \lambda, \xi \rangle (f \circ \pi) = \langle \pi(\lambda), \xi \rangle (f \circ \pi).$$

The uniqueness of the Opdam-Cherednik kernel on \mathbb{R}_0^n thus implies that f coincides with $G(\pi(\lambda), \cdot)$ on $\pi(U')$. Finally, for $\xi = \underline{1}$ we have $D_{\underline{1}} = \partial_{\underline{1}}$, and the eigenvalue equation $D_{\underline{1}}f = \langle \lambda, \underline{1} \rangle f$ leads to

$$\partial_{\underline{1}}(e^{-\langle \cdot, \underline{1} \rangle \langle \lambda, \underline{1} \rangle / n} \cdot f) \equiv 0 \quad \text{on } \mathbb{R}^n.$$

Therefore $x \mapsto e^{-\frac{\langle x, \underline{1} \rangle \langle \lambda, \underline{1} \rangle}{n}} f(x)$ is in each point constant in direction $\underline{1}$, i.e.

$$e^{-\frac{\langle x, \underline{1} \rangle \langle \lambda, \underline{1} \rangle}{n}} f(x) = e^{-\frac{\langle \pi(x), \underline{1} \rangle \langle \lambda, \underline{1} \rangle}{n}} f(\pi(x)) = f(\pi(x)) = G(\pi(\lambda), \pi(x)).$$

The result follows by analytic extension. \square

3. JACK POLYNOMIALS AND MACDONALD'S CONJECTURE

We first recall some well-known facts about Jack polynomials from [KS97, 10, St89]. Let $\Lambda_n^+ = \{\lambda \in \mathbb{N}_0^n : \lambda_1 \geq \dots \geq \lambda_n\}$ denote the set of partitions of length at most n . The dominance order on Λ_n^+ is given by

$$\mu \leq_D \lambda \quad \text{iff} \quad |\lambda| = |\mu| \quad \text{and} \quad \sum_{j=1}^r \mu_j \leq \sum_{j=1}^r \lambda_j \quad \text{for all } r = 1, \dots, n,$$

where $|\lambda| = \lambda_1 + \dots + \lambda_n$. The dominance order is extended from Λ_n^+ to \mathbb{N}_0^n as follows: For each composition $\eta \in \mathbb{N}_0^n$ denote by $\eta_+ \in \Lambda_n^+$ the unique element in the \mathcal{S}_n -orbit of η . Then the dominance order on \mathbb{N}_0^n is defined by

$$\kappa \preceq \eta \quad \text{iff} \quad \begin{cases} \kappa_+ \leq_D \eta_+, & \kappa_+ \neq \eta_+ \\ w_\eta \leq w_\kappa, & \kappa_+ = \eta_+ \end{cases},$$

where $w_\eta \in \mathcal{S}_n$ is the shortest element with $w_\eta \eta_+ = \eta$, and \leq refers to the Bruhat order on \mathcal{S}_n . Consider the rational Cherednik operators

$$\mathcal{D}_j = \mathcal{D}_j(k) := x_j T_j + k(1 - n) + k \sum_{i>j} s_{ij}, \quad 1 = 1, \dots, n$$

where the $T_j := T_{e_j}(k)$ are the type A Dunkl operators with multiplicity k and s_{ij} denotes the reflection in the root $e_i - e_j$, which acts by interchanging x_i and x_j . We remark that our notion differs by a factor k from that in [10]. This facilitates the handling of the case $k = 0$. The operators \mathcal{D}_j are closely related to the usual Cherednik operators $D_j := D_{e_j}(k)$. Indeed, consider $f \in C^1(U)$ for some open $U \subseteq \mathbb{R}^n$ and define $g : \exp^{-1}(U) \subseteq \mathbb{R}^n \rightarrow \mathbb{C}$ by $g(x) := f(e^x)$, where e^x is understood componentwise. Then a short calculation gives

$$\left(D_j - \frac{k}{2}(n-1)\right)g(x) = (D_j f)(e^x). \quad (3.1)$$

The operators \mathcal{D}_j are upper triangular with respect to \preceq on $\mathcal{P} = \mathbb{C}[\mathbb{R}^n]$. More precisely,

$$\mathcal{D}_j x^\eta = \bar{\eta}_j x^\eta + \sum_{\kappa \prec \eta} d_{\kappa\eta} x^\kappa$$

with some $d_{\kappa\eta} \in \mathbb{R}$ and

$$\bar{\eta}_j = \eta_j - k\#\{i < j \mid \eta_i \geq \eta_j\} - k\#\{i > j \mid \eta_i > \eta_j\}.$$

The non-symmetric Jack polynomials of index $\alpha = 1/k$ with $k \in [0, \infty)$ can be characterized as the unique basis $(E_\eta = E_\eta(\cdot; \alpha))_{\eta \in \mathbb{N}_0^n}$ of \mathcal{P} satisfying

- (1) $E_\eta(x) = x^\eta + \sum_{\kappa \prec \eta} c_{\eta\kappa} x^\kappa$ with $c_{\eta\kappa} \in \mathbb{C}$,
- (2) $\mathcal{D}_j E_\eta = \bar{\eta}_j E_\eta$ for all $j = 1, \dots, n$.

By definition, E_η is homogeneous of degree $|\eta| = \eta_1 + \dots + \eta_n$, and for $k = 0$ we have $E_\eta(x; \infty) = x^\eta$. Property (2) together with Proposition 2.3 and identity (3.1) show that the polynomials E_η are related to the (extended) Opdam-Cherednik kernel via

$$\frac{E_\eta(e^x)}{E_\eta(\mathbf{1})} = G(\bar{\eta} + \frac{k}{2}(n-1)\mathbf{1}, x), \quad \bar{\eta} = (\bar{\eta}_1, \dots, \bar{\eta}_n). \quad (3.2)$$

Following [10], we denote by $P_\lambda(x) = P_\lambda(x; \alpha)$, $\lambda \in \Lambda_n^+$ the symmetric Jack polynomials in n variables of index $\alpha = \frac{1}{k}$ in monomial normalization. In the limiting case $k = 0$, they coincide with the monomial symmetric functions

$$m_\lambda(x) = \sum_{\eta \in \mathcal{S}_n \lambda} x^\eta.$$

The non-symmetric and symmetric Jack polynomials of the same index are related via symmetrization: for $\lambda \in \Lambda_n^+$ and $\eta \in \mathbb{N}_0^n$ with $\eta_+ = \lambda$,

$$\frac{P_\lambda(x)}{P_\lambda(\mathbf{1})} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \frac{E_\eta(\sigma x)}{E_\eta(\mathbf{1})}. \quad (3.3)$$

The Jack polynomials P_λ satisfy a binomial formula:

$$\frac{P_\lambda(\mathbf{1} + x)}{P_\lambda(\mathbf{1})} = \sum_{\mu \subseteq \lambda} \binom{\lambda}{\mu} \frac{P_\mu(x)}{P_\mu(\mathbf{1})}, \quad (3.4)$$

where $\mu \subseteq \lambda$ for $\lambda, \mu \in \Lambda_n^+$ means $\mu_i \leq \eta_i$ for all i , and $\binom{\lambda}{\mu} = \binom{\lambda}{\mu}_k \geq 0$ is a generalized binomial coefficient. Symmetrization in (3.2) yields a relation between the (extended) hypergeometric function $F = F_k$ and the symmetric Jack polynomials: If $\lambda \in \Lambda_n^+$, then

$$\bar{\lambda} + \frac{k}{2}(n-1) \cdot \mathbf{1} = \lambda - \rho \quad (3.5)$$

and therefore

$$\frac{P_\lambda(e^x)}{P_\lambda(\mathbf{1})} = F(\lambda - \rho, x). \quad (3.6)$$

In the following lemma, we collect some further useful properties of the non-symmetric Jack polynomials $E_\eta = E_\eta(\cdot; \frac{1}{k})$, which can be found in [10, KS97, 31] for $k > 0$ and are obvious for $k = 0$. Here we consider the Jack polynomials as functions on \mathbb{C}^n .

Lemma 3.1. (1) For all $p \in \mathbb{N}_0$,

$$\Delta(z)^p E_\eta(z) = E_{\eta+p}(z).$$

By this property, the non-symmetric Jack polynomials uniquely extend to indices $\eta \in \mathbb{Z}^n$.

(2) Let $z \in \mathbb{C}^n$ with $z_i \neq 0$ for all $i = 1, \dots, n$. Then

$$E_\eta\left(\frac{1}{z}\right) = E_{-\eta^R}(z^R),$$

where $\eta^R = (\eta_n, \dots, \eta_1)$, $z^R = (z_n, \dots, z_1)$.

(3) Let Φ be the so-called raising operator, which acts on functions $f : \mathbb{C}^n \rightarrow \mathbb{C}$ by

$$\Phi f(z) = z_n f(z_n, z_1, \dots, z_{n-1})$$

and on \mathbb{N}_0^n by

$$\Phi \eta = (\eta_2, \dots, \eta_m, \eta_1 + 1).$$

Then the non-symmetric Jack polynomials satisfy

$$\Phi E_\eta = E_{\Phi \eta} \quad (\eta \in \mathbb{N}_0^n).$$

According to part (1) this identity extends to all $\eta \in \mathbb{Z}^n$, because $\Phi(\Delta E_\eta) = \Delta \cdot \Phi E_\eta$ and $\Phi(\eta + \underline{p}) = \Phi(\eta) + \underline{p}$ for all $\underline{p} \in \mathbb{N}$.

(4) The coefficients $c_{\eta\kappa}$ in the monomial expansion of E_η , $\eta \in \mathbb{N}_0^n$, are non-negative.

Lemma 3.2. *There exists a polynomial $Q \in \mathcal{P}$ such that for alle $\eta \in \mathbb{N}_0^n$ and $\lambda \in \Lambda_n^+$,*

$$0 \leq E_\eta(\underline{1}) \leq Q(\eta), \quad 0 \leq P_\lambda(\underline{1}) \leq Q(\lambda).$$

Proof. By [10, Prop. 12.3.2],

$$E_\eta(\underline{1}) = \prod_{(i,j) \in \eta} \frac{j + kn - k\ell'(\eta, i, j)}{\eta_i - j + 1 + k\ell(\eta, i, j) + k}$$

with the leg length and coleg length $\ell(\eta, i, j), \ell'(\eta, i, j) \in \{0, \dots, n\}$. Therefore

$$E_\eta(\underline{1}) \leq \prod_{i=1}^n \prod_{j=1}^{\eta_i} \frac{j + kn}{\eta_i - j + 1} = \prod_{i=1}^n \frac{(\eta_i + kn)!}{(kn)! \eta_i!},$$

which is polynomially bounded in η by Stirling's formula. Similarly (c.f. [10, Prop. 12.6.2]),

$$P_\lambda(\underline{1}) = \prod_{(i,j) \in \lambda} \frac{j - 1 + kn - k\ell'(\lambda, i, j)}{\lambda_i - j + k\ell(\eta, i, j) + k} \leq \prod_{i=1}^n \prod_{j=1}^{\lambda_i} \frac{j - 1 + kn}{\lambda_i - j + k},$$

which is also polynomially bounded in λ . □

To formulate the main results of this section, we introduce the gamma function

$$\Gamma_n(\lambda) = \Gamma_n(k; \lambda) := d_n(k) \cdot \prod_{j=1}^n \Gamma(\lambda_j - k(j-1)) \quad (\lambda \in \mathbb{C}^n), \quad (3.7)$$

with

$$d_n(k) := \prod_{j=1}^n \frac{\Gamma(1 + jk)}{\Gamma(1 + k)},$$

as well as the generalized Pochhammer symbol

$$[\mu]_\eta := \prod_{j=1}^n (\mu - k(j-1))_{\eta_j} = \frac{\Gamma_n(\underline{\mu} + \eta)}{\Gamma_n(\underline{\mu})} \quad (\mu \in \mathbb{C}, \eta \in \mathbb{N}_0^n).$$

For abbreviation, we also write

$$\Gamma_n(\mu) := \Gamma_n(\underline{\mu}) \quad \text{for } \mu \in \mathbb{C}.$$

Note that Γ_n differs by the factor $d_n(k)$ from the notion in [R20, M13], but is in accordance with the notion for the gamma function on symmetric cones. We shall obtain the master theorem as a consequence of the following result, which involves the type A Dunkl operators $T = T(k)$ with multiplicity k .

Theorem 3.3. *Fix $k \geq 0$, and consider the non-symmetric Jack polynomials $(E_\eta)_{\eta \in \mathbb{N}_0^n}$ and the symmetric Jack polynomials $(P_\lambda)_{\lambda \in \Lambda_n^+}$ of index $1/k$. Then for all $\mu \in \mathbb{C}$ and all $x \in \mathbb{R}^n$ with $x_i \neq 0$ for all $i = 1, \dots, n$,*

- (1) $E_\eta(T)\Delta^{-\mu}(x) = (-1)^{|\eta|} [\mu]_{\eta_+} E_\eta\left(\frac{1}{x}\right) \Delta(x)^{-\mu};$
- (2) $P_\lambda(T)\Delta^{-\mu}(x) = (-1)^{|\lambda|} [\mu]_\lambda P_\lambda\left(\frac{1}{x}\right) \Delta(x)^{-\mu}.$

For the proof, we first note

Lemma 3.4. *The set \mathbb{N}_0^n can be recursively constructed from $0 \in \mathbb{N}_0^n$ by a chain of the following operations:*

- (i) *apply the raising operator Φ to $\eta \in \mathbb{N}_0^n$,*
- (ii) *apply a simple permutation $s_i = (i, i+1)$ to $\eta \in \mathbb{N}_0^n$ with $\eta_i < \eta_{i+1}$.*

Proof. This is easily verified by induction on the weight $|\eta|$. Indeed, assume that all elements of weight at most r are already constructed and take $\eta \in \mathbb{N}_0^n$ with $|\eta| = r+1$. Consider the maximal index $j = 1, \dots, n$ with $\eta_j \neq 0$ and $\eta_k = 0$ for $j < k \leq n$. Then

$$\eta = (\eta_1, \dots, \eta_j, 0, \dots, 0) = s_j \cdots s_{n-1}(\eta_1, \dots, \eta_{j-1}, 0, \dots, 0, \eta_j) = s_j \cdots s_{n-1} \Phi \hat{\eta}$$

with $\hat{\eta} = (\eta_j - 1, \eta_1, \dots, \eta_{j-1}, 0, \dots, 0)$, which is already constructed by induction hypothesis. \square

Proof of Theorem 3.3. Part (2) is obtained from (1) by symmetrization. Part (1) is clear for $\eta = 0$, since $E_0 = 1$. In view of the above observation, it therefore suffices to consider the following two cases:

Case 1. Assume formula (1) is correct for some $\eta \in \mathbb{N}_0^n$ with $\eta_i < \eta_{i+1}$, and consider $E_{s_i \eta}$. According to [10, Proposition 12.2.1] there exists a constant $d_i^\eta \in \mathbb{R}$ such that

$$E_{s_i \eta} = d_i^\eta E_\eta + s_i E_\eta.$$

The Dunkl operators are \mathcal{S}_n -equivariant, i.e. $\sigma T_\xi \sigma^{-1} = T_{\sigma \xi}$, $\sigma \in \mathcal{S}_n$. Hence the symmetry of $\Delta(x)$ leads to

$$\begin{aligned} (s_i E_\eta)(T)\Delta(x)^{-\mu} &= (s_i E_\eta(T)(s_i \Delta)^{-\mu})(x) = E_\eta(T)\Delta^{-\mu}(s_i x) \\ &= (-1)^{|\eta|} [\mu]_{\eta_+} E_\eta\left(\frac{1}{s_i x}\right) \Delta(s_i x)^{-\mu} \\ &= (-1)^{|\eta|} [\mu]_{\eta_+} (s_i E_\eta)\left(\frac{1}{x}\right) \Delta(x)^{-\mu} \end{aligned}$$

As $|s_i \eta| = |\eta|$ and $(s_i \eta)_+ = \eta_+$, the formula follows for $s_i \eta$ by linear combination.

Case 2. Assume that formula (1) is correct for some $\eta \in \mathbb{N}_0^n$, and consider $\Phi \eta$. Using the identity $\Phi E_\eta = E_{\Phi \eta}$ from Lemma 3.1 and the product rule for the Dunkl

operators, we calculate

$$\begin{aligned}
E_{\Phi\eta}(T)\Delta(x)^{-\mu} &= T_n E_\eta(T_n, T_1, \dots, T_{n-1})\Delta(x)^{-\mu} \\
&= T_n \left((-1)^{|\eta|} [\mu]_{\eta_+} E_\eta\left(\frac{1}{x_n}, \frac{1}{x_1}, \dots, \frac{1}{x_{n-1}}\right) \Delta(x)^{-\mu} \right) \\
&= (-1)^{|\eta|} [\mu]_{\eta_+} \left((T_n \Delta(x)^{-\mu}) E_\eta\left(\frac{1}{x_n}, \frac{1}{x_1}, \dots, \frac{1}{x_{n-1}}\right) \right. \\
&\quad \left. + \Delta(x)^{-\mu} (T_n E_\eta\left(\frac{1}{x_n}, \frac{1}{x_1}, \dots, \frac{1}{x_{n-1}}\right)) \right). \tag{3.8}
\end{aligned}$$

As T_n acts on symmetric functions as the partial derivative $\frac{\partial}{\partial x_n}$, we have

$$T_n \Delta(x)^{-\mu} = -\mu x_n^{-1} \Delta(x)^{-\mu}.$$

Parts (1) and (2) of Proposition 3.1 show that

$$E_\eta\left(\frac{1}{x_n}, \frac{1}{x_1}, \dots, \frac{1}{x_{n-1}}\right) = \Delta^{-p}(x) E_{\eta^*}(x_{n-1}, \dots, x_1, x_n)$$

with $\eta^* = -\eta^R + p$, where $p \in \mathbb{N}$ is so large that $-\eta^R + p \in \mathbb{N}_0^n$. Note further that $\frac{1}{x_n} E_\eta\left(\frac{1}{x_n}, \frac{1}{x_1}, \dots, \frac{1}{x_{n-1}}\right) = E_{\Phi\eta}\left(\frac{1}{x}\right)$. Thus formula (3.8) reduces to

$$\begin{aligned}
&E_{\Phi\eta}(T)\Delta(x)^{-\mu} \\
&= (-1)^{|\eta|} [\mu]_{\eta_+} \Delta(x)^{-\mu} \left(-\mu E_{\Phi\eta}\left(\frac{1}{x}\right) + T_n(\Delta^{-p}(x) E_{\eta^*}(x_{n-1}, \dots, x_1, x_n)) \right). \tag{3.9}
\end{aligned}$$

Again by the product rule for T_n and that fact that T_n commutes with s_1, \dots, s_{n-2} , we further obtain

$$\begin{aligned}
&T_n(\Delta^{-p}(x) E_{\eta^*}(x_{n-1}, \dots, x_1, x_n)) \\
&= -p \Delta(x)^{-p} E_{\eta^*}(x_{n-1}, \dots, x_1, x_n) + \Delta(x)^{-p} (T_n E_{\eta^*}(x_{n-1}, \dots, x_1, x_n)) \\
&= -p x_n^{-1} \Delta(x)^{-p} E_{\eta^*}(x_{n-1}, \dots, x_1, x_n) + \Delta(x)^{-p} (T_n E_{\eta^*})(x_{n-1}, \dots, x_1, x_n) \\
&= x_n^{-1} \Delta(x)^{-p} \left(-p E_{\eta^*}(x_{n-1}, \dots, x_1, x_n) + (x_n T_n E_{\eta^*})(x_{n-1}, \dots, x_1, x_n) \right).
\end{aligned}$$

As $x_n T_n = \mathcal{D}_n + k(n-1)$, we have

$$x_n T_n E_{\eta^*}(x_{n-1}, \dots, x_1, x_n) = (\overline{\eta}_n^* + n - 1) E_{\eta^*}(x_{n-1}, \dots, x_1, x_n)$$

with $\overline{\eta}_n^* = \eta_n^* - k \# \{ \ell < n \mid \eta_\ell^* \geq \eta_n^* \}$, so that

$$\begin{aligned}
&T_n(\Delta^{-p}(x) E_{\eta^*}(x_{n-1}, \dots, x_1, x_n)) \\
&= (-p + (\overline{\eta}_n^* + k(n-1))) \Delta(x)^{-p} \frac{1}{x_n} E_{\eta^*}(x_{n-1}, \dots, x_1, x_n) \\
&= (-p + (\overline{\eta}_n^* + k(n-1))) \frac{1}{x_n} E_\eta\left(\frac{1}{x_n}, \frac{1}{x_1}, \dots, \frac{1}{x_{n-1}}\right) \\
&= (-p + (\overline{\eta}_n^* + k(n-1))) E_{\Phi\eta}\left(\frac{1}{x}\right).
\end{aligned}$$

Thus equation (3.9) reduces to

$$E_{\Phi\eta}(T)\Delta(x)^{-\mu} = (-1)^{|\eta|} [\mu]_{\eta_+} (-\mu - p + \overline{\eta}_n^* + k(n-1)) E_{\Phi\eta}\left(\frac{1}{x}\right) \Delta(x)^{-\mu}. \tag{3.10}$$

Let $1 \leq j \leq n$ be minimal such that the entry j in η_+ is equal to η_1 , i.e.

$$j - 1 = \# \{ \ell > 1 \mid \eta_1 < \eta_\ell \}. \tag{3.11}$$

Now at position j in $(\Phi\eta)_+$ there is $\eta_1 + 1$. Thus, by definition of j we have

$$\begin{aligned}\overline{\eta}_n^* &= (\eta^R + p)_n - k\#\{\ell < n \mid -\eta_\ell^R + p \geq -\eta_n^R + p\} \\ &= p + \eta_1 - k\#\{\ell < n \mid \eta_{n-\ell+1} \leq \eta_1\} \\ &= p + \eta_1 - k\#\{\ell > 1 \mid \eta_\ell \leq \eta_1\} \\ &= p + \eta_1 - k(n - j).\end{aligned}$$

So finally, since j is the position of $\eta_1 + 1 = (\Phi\eta)_n$ in $(\Phi\eta)_+$ we have that $(\Phi\eta)_+$ is exactly η_+ plus an 1 at position j . Therefore

$$\begin{aligned}(-1)^{|\eta|}[\mu]_{\eta_+}(-\mu - p + \overline{\eta}_n^* + k(n - 1)) \\ &= (-1)^{|\eta|}[\mu]_{\eta_+}(-\mu + \eta_1 - k(n - j) + k(n - 1)) \\ &= (-1)^{|\eta|+1}[\mu]_{\eta_+}(\mu - k(j - 1) + (\eta_1 + 1) - 1) \\ &= (-1)^{|\Phi\eta|}[\mu]_{(\Phi\eta)_+}.\end{aligned}$$

Plugging this into (3.10) we obtain the assertion. \square

Theorem 3.5 (Laplace transform identities for Jack polynomials). *Let $(E_\eta)_{\eta \in \mathbb{N}_0^n}$ and $(P_\lambda)_{\lambda \in \Lambda_n^+}$ be the non-symmetric and symmetric Jack polynomials of index $1/k$, $k \geq 0$. Then for all $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > \mu_0$ and $z \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$,*

$$\begin{aligned}(1) \int_{\mathbb{R}_+^n} E(-z, x) E_\eta(x) \Delta(x)^{\mu - \mu_0 - 1} \omega(x) dx &= \Gamma_n(\eta_+ + \underline{\mu}) E_\eta\left(\frac{1}{z}\right) \Delta(z)^{-\mu}; \\ (2) \int_{\mathbb{R}_+^n} J(-z, x) P_\lambda(x) \Delta(x)^{\mu - \mu_0 - 1} \omega(x) dx &= \Gamma_n(\lambda + \underline{\mu}) P_\lambda\left(\frac{1}{z}\right) \Delta(z)^{-\mu}.\end{aligned}$$

Part (2) is just Macdonald's [M13] Conjecture (C), and part (1) corresponds to formula (4.38) in [BF98] (there is a misprint: the Laguerre polynomial $E_\eta^{(L)}$ has to be replaced by E_η).

Proof. The integrals converge by Lemma 2.1. According to [R20],

$$\Delta(z)^{-\mu} = \frac{1}{\Gamma_n(\mu)} \mathcal{L}(\Delta^{\mu - \mu_0 - 1})(z),$$

and for each polynomial $p \in \mathcal{P}$,

$$p(-T)\Delta^{-\mu}(z) = \frac{1}{\Gamma_n(\mu)} \mathcal{L}(p \Delta^{\mu - \mu_0 - 1})(z).$$

Now part (1) is immediate from Theorem 3.3(1) and part (2) follows by symmetrization. \square

4. SOME GENERAL PROPERTIES OF THE OPDAM- CHEREDNIK KERNEL

We want to extend the Laplace transform identities for Jack polynomials to the Opdam-Cherednik kernel of type A . For this, we shall need some relations for this kernel which are of a general nature and seem not to be stated in the literature. In this section, we therefore consider an arbitrary crystallographic root system R with Weyl group W in a Euclidean space \mathfrak{a} . In particular, it is required that R spans \mathfrak{a} . The inner product in \mathfrak{a} is extended to $\mathfrak{a}_\mathbb{C}$ in a bilinear way. A W -invariant function $k : R \rightarrow \mathbb{C}$, $\alpha \mapsto k_\alpha$ is called a multiplicity function. The Cherednik operators

associated with multiplicity function k and some positive subsystem R_+ of R are defined by

$$D_\xi(R_+) = D_\xi(R_+, k) = \partial_\xi - \langle \rho(R_+), \xi \rangle + \sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{1 - s_\alpha}{1 - e^{-\alpha}}.$$

where $\rho(R_+) = \rho(R_+, k) = \frac{1}{2} \sum_{\alpha \in R_+} k_\alpha \alpha$ is the generalized Weyl vector. The $D_\xi(R_+)$, $\xi \in \mathfrak{a}$ commute. According to [O95], there exist a W -invariant tubular neighborhood U of \mathfrak{a} in $\mathfrak{a}_\mathbb{C}$ such that for fixed multiplicity k with $\operatorname{Re} k \geq 0$ and each $\lambda \in \mathfrak{a}_\mathbb{C}$ there exists a unique holomorphic function $f = G(\lambda, \cdot) = G_k(R_+, \lambda, \cdot)$, called the Opdam-Cherednik kernel associated with R_+ and k , which satisfies the eigenvalue system

$$\begin{cases} D_\xi(R_+)f = \langle \lambda, \xi \rangle f & \text{for all } x \in \mathfrak{a}; \\ f(0) = 1. \end{cases}$$

Proposition 4.1. *The Cherednik operators and the Opdam-Cherednik kernel associated with the positive system R_+ of R have the following properties.*

- (1) $wD_\xi(R_+)w^{-1} = D_{w\xi}(wR_+)$ for all $w \in W$.
- (2) $D_\xi(R_+)f^- = -(D_\xi(R_-)f)^-$, where $f^-(x) = f(-x)$ and $R_- := -R_+$.
- (3) $G(R_+, \lambda, z) = G(wR_+, w\lambda, wz)$. In particular, the hypergeometric function

$$F(\lambda, z) = F_k(\lambda, z) := \frac{1}{|W|} \sum_{w \in W} G(R_+, \lambda, wz)$$

does not depend on the choice of R_+ .

- (4) $G(R_+, \lambda, -z) = G(R_+, -w_0\lambda, w_0z)$, where w_0 is the longest element of W .
- (5) $F(\lambda, -z) = F(-\lambda, z)$.

Proof. (1) This follows from the identities

$$w(\partial_\xi - \langle \rho(R_+), \xi \rangle)w^{-1} = \partial_{w\xi} - \langle \rho(wR_+), w\xi \rangle$$

and

$$\sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle w \frac{1 - s_\alpha}{1 - e^{-\alpha}} w^{-1} = \sum_{\beta \in wR_+} k_\beta \langle \beta, \xi \rangle \frac{1 - s_\beta}{1 - e^{-\beta}}.$$

(2) Note that $(\partial_\xi - \langle \rho(R_+), \xi \rangle)f^- = -(\partial_\xi - \langle \rho(-R_+), \xi \rangle f)^-$. As $k_\alpha = k_{-\alpha}$, we also have

$$\sum_{\alpha \in R_+} k_\alpha \langle \alpha, \xi \rangle \frac{f^- - s_\alpha f^-}{1 - e^{-\alpha}} = - \left(\sum_{\beta \in -R_+} k_\beta \langle \beta, \xi \rangle \frac{1 - s_\beta}{1 - e^{-\beta}} f \right)^-.$$

Thus $D_\xi(R_+)f^- = -(D_\xi(-R_+)f)^-$.

(3) In view of part (1), the defining eigenvalue equation $D_\xi(R_+)f = \langle \lambda, \xi \rangle f$ is equivalent to

$$D_{w\xi}(wR_+)(wf) = wD_\xi(R_+)f = \langle \lambda, \xi \rangle (wf) = \langle w\lambda, w\xi \rangle (wf).$$

Hence $G(R_+, \lambda, w^{-1}z) = G(wR_+, w\lambda, z)$.

(4) From part (2) we observe that

$$D_\xi(R_-)G(\lambda, -\cdot) = -(D_\xi(R_+)G_k(\lambda, \cdot))^- = -\langle \lambda, \xi \rangle G(\lambda, -\cdot).$$

Therefore $G(R_+, \lambda, -z) = G(R_-, -\lambda, z)$. The longest element $w_0 \in W$ satisfies $w_0 R_- = R_+$. Hence by part (3),

$$G(R_+, \lambda, -z) = G(w_0 R_-, -w_0 \lambda, w_0 z) = G(R_+, -w_0 \lambda, w_0 z).$$

(5) This is clear from part (4). \square

If $k \geq 0$, then according to [Sch08] and [O95], $G(\lambda, x) > 0$ for all $\lambda, x \in \mathfrak{a}$ and

$$|G(\lambda, x)| \leq G(\operatorname{Re} \lambda, x) \leq \sqrt{|W|} e^{\max_{w \in W} \langle \operatorname{Re} \lambda, wx \rangle} \quad \text{for all } \lambda \in \mathfrak{a}_{\mathbb{C}}, x \in \mathfrak{a}. \quad (4.1)$$

The following result generalizes the estimate of the Opdam-Cherednik kernel stated in [RKV13, Theorem 3.3]. Notice that the eigenvalue characterization of G implies that $G(-\rho, \cdot) \equiv 1$ for $\rho = \rho(R_+)$.

Proposition 4.2. *Suppose that $k \geq 0$. Then the Cherednik kernel $G = G_k(R_+, \cdot, \cdot)$ satisfies the following estimate for $x \in \mathfrak{a}$ and all $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}$:*

$$|G(\lambda + \mu, x)| \leq G(\operatorname{Re} \mu, x) \cdot e^{\max_{w \in W} \langle \operatorname{Re} \lambda, wx \rangle}.$$

Moreover, since $G(-\rho, \cdot) \equiv 1$, we in particular have

$$|G(\lambda - \rho, x)| \leq e^{\max_{w \in W} \langle \operatorname{Re} \lambda, wx \rangle} \quad \text{for all } \lambda \in \mathfrak{a}_{\mathbb{C}}, x \in \mathfrak{a}.$$

Proof. In [RKV13, Theorem 3.3] it was proven by a Phragmen-Lindelöf argument that for all $\lambda \in \mathfrak{a}$, $\mu \in \overline{\mathfrak{a}_+}$ and $x \in \mathfrak{a}$,

$$G(\lambda + \mu, x) \leq G(\mu, x) \cdot e^{\max_{w \in W} \langle \lambda, wx \rangle}. \quad (4.2)$$

An inspection of the proof in loc.cit. shows that it can be carried out in exactly the same way when $\overline{\mathfrak{a}_+}$ is replaced by an arbitrary closed Weyl chamber $C \subset \mathfrak{a}$. Therefore estimate (4.2) extends to all $\mu \in \mathfrak{a}$, and the claim follows from (4.1). \square

Remark 4.3. For root system $R = A_{n-1}$, items (3)–(5) of Proposition 4.1 as well as Proposition 4.2 are easily checked to remain valid for the extensions of G and F as defined in (2.2).

5. LAPLACE TRANSFORM OF THE OPDAM-CHEREDNIK KERNEL

In this section, we return to A_{n-1} and resume the notations from Sections 1–3. We shall extend the statements of Theorem 3.5 to the Opdam-Cherednik kernel and the hypergeometric function of type A_{n-1} . Formulas (3.2) and (3.6) suggest that it will be convenient to work with suitable modifications of the (extended) kernels G and F . According to our construction in Section 2, the extended Opdam-Cherednik kernel G is in particular holomorphic on the set

$$\mathbb{C}^n \times (\mathbb{R}^n + i\Omega') \quad \text{with } \Omega' = \{x \in \mathbb{R}^n : |x_i| < \frac{\pi}{2} \text{ for all } i = 1, \dots, n\}.$$

The exponential mapping $\exp : z \mapsto e^z$ (understood componentwise) maps $\mathbb{R}^n + i\Omega'$ biholomorphically onto $H := \{z \in \mathbb{C}^n : \operatorname{Re} z > 0\}$ with inverse \log . The modified kernels

$$\mathcal{G}(\lambda, z) := G(\lambda, \log(z)),$$

$$\mathcal{F}(\lambda, z) := F(\lambda, \log(z))$$

are therefore holomorphic on $\mathbb{C}^n \times H$. We call them the rational version of the Opdam-Cherednik kernel and the hypergeometric function, respectively. Obviously \mathcal{F} is \mathcal{S}_n -invariant in each argument. We collect some properties of \mathcal{G} and \mathcal{F} which shall be needed lateron.

Lemma 5.1. (1) For all $z \in H, \lambda \in \mathbb{C}^n$ and $\mu \in \mathbb{C}$,

$$\Delta(z)^\mu \mathcal{G}(\lambda, z) = \mathcal{G}(\lambda + \underline{\mu}, z), \quad \Delta(z)^\mu \mathcal{F}(\lambda, z) = \mathcal{F}(\lambda + \underline{\mu}, z).$$

(2) For all $z \in H$ and $\lambda \in \mathbb{C}^n$,

$$\mathcal{G}(\lambda, \frac{1}{z}) = \mathcal{G}(-\lambda^R, z^R), \quad \mathcal{F}(\lambda, \frac{1}{z}) = \mathcal{F}(-\lambda, z),$$

(3) For all $x \in \mathbb{R}_+^n$ and all $\lambda, \mu \in \mathbb{C}^n$,

$$|\mathcal{G}(\lambda + \mu, x)| \leq \mathcal{G}(\operatorname{Re} \lambda, x) \cdot \max_{\sigma \in \mathcal{S}_n} x^{\sigma(\operatorname{Re} \mu)}.$$

In particular, for all $x \in \mathbb{R}_+^n$ and $\lambda \in \mathbb{C}^n$,

$$|\mathcal{G}(\lambda - \rho, x)| \leq \max_{\sigma \in \mathcal{S}_n} x^{\sigma(\operatorname{Re} \lambda)}$$

and

$$|\mathcal{G}(\lambda, x)| \leq \sqrt{n!} \max_{\sigma \in \mathcal{S}_n} x^{\sigma(\operatorname{Re} \lambda)}.$$

The same estimates hold for \mathcal{F} .

(4) For partitions $\lambda \in \Lambda_n^+$,

$$\mathcal{G}(\lambda - \rho, z) = \frac{E_\lambda(z)}{E_\lambda(\mathbf{1})}.$$

Proof. Part (1) is clear from the definitions. Part (2) follows from Proposition 4.1, since the longest element $w_0 \in \mathcal{S}_n$ acts by $w_0 \lambda = \lambda^R = (\lambda_n, \dots, \lambda_1)$. Part (3) is immediate from Proposition 4.2 (and the subsequent remark). Finally, part (4) follows from identities (3.2) and (3.5). \square

The extension of Theorem 3.5 will be carried out by analytic extension with respect to the spectral parameter, which is based on the following generalization of the classical Carlson theorem [T76, p.186].

Lemma 5.2. Let $U \subseteq \mathbb{C}^n$ be an open neighborhood of $\{\operatorname{Re} z \geq 0\} \subseteq \mathbb{C}^n$ and let $f : U \rightarrow \mathbb{C}$ be holomorphic. Put $\|z\|_1 := \sum_{i=1}^n |z_i|$. If f satisfies

$$(*) \quad f(z) = \mathcal{O}(e^{c\|z\|_1}) \text{ for some } c < \pi \text{ and } f|_{\Lambda_n^+} \equiv 0,$$

then $f \equiv 0$.

Proof. We proceed by induction on n . The case $n = 1$ is Carlson's classical theorem. To achieve step $n - 1 \rightarrow n$, consider for fixed $\lambda \in \Lambda_n^+$ the holomorphic function

$$f_\lambda : U' \rightarrow \mathbb{C}, \xi \mapsto f(\xi + \lambda_1, \lambda_2, \dots, \lambda_n)$$

where $U' \subseteq \mathbb{C}$ is a suitable neighborhood of $\{\operatorname{Re} \xi + \lambda_1 \geq 0\} \subseteq \mathbb{C}$. Then $f_\lambda|_{\mathbb{N}_0} \equiv 0$ and

$$f_\lambda(\xi) = \mathcal{O}(e^{c|\xi|})$$

with c as in (*). Therefore f_λ vanishes identically. From this we conclude that for fixed $\xi \in \mathbb{C}$ with $\operatorname{Re} \xi \geq 0$, the function

$$g_\xi : \tilde{U} \rightarrow \mathbb{C}, \quad w \mapsto f(\xi, w)$$

vanishes on Λ_{n-1}^+ for some suitable neighborhood $\tilde{U} \subseteq \mathbb{C}^{n-1}$ of $\{\operatorname{Re} w \geq 0\}$. Moreover

$$g_\xi(w) = \mathcal{O}(e^{c\|(\xi, w)\|_1}) = \mathcal{O}(e^{c\|w\|_1}),$$

and by the induction hypothesis we obtain that g_ξ vanishes identically. As ξ was arbitrary, we obtain $f \equiv 0$. \square

We are now in the position to prove the following generalization of Theorem 3.5.

Theorem 5.3. *Let $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > \mu_0 = k(n-1)$, $\lambda \in \mathbb{C}^n$ with $\operatorname{Re} \lambda \geq 0$ and $z \in H$. Then*

$$(1) \int_{\mathbb{R}_+^n} E(-z, x) \mathcal{G}(\lambda, x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \Gamma_n(\lambda + \rho + \underline{\mu}) \mathcal{G}(\lambda, \frac{1}{z}) \Delta(z)^{-\mu}.$$

$$(2) \int_{\mathbb{R}_+^n} J(-z, x) \mathcal{F}(\lambda, x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \Gamma_n(\lambda + \rho + \underline{\mu}) \mathcal{F}(\lambda, \frac{1}{z}) \Delta(z)^{-\mu}.$$

In view of Lemma 5.1, the above Theorem can be equivalently reformulated as follows:

Corollary 5.4. *Suppose that $\operatorname{Re} \lambda \geq \underline{\mu}_0$. Then for all $z \in H$,*

$$(1) \int_{\mathbb{R}_+^n} E(-z, x) \mathcal{G}(\lambda, x) \Delta(x)^{-\mu_0-1} \omega(x) dx = \Gamma_n(\lambda + \rho) \mathcal{G}(\lambda, \frac{1}{z});$$

$$(2) \int_{\mathbb{R}_+^n} J(-z, x) \mathcal{F}(\lambda, x) \Delta(x)^{-\mu_0-1} \omega(x) dx = \Gamma_n(\lambda + \rho) \mathcal{F}(\lambda, \frac{1}{z}).$$

The second formula generalizes the Laplace transform identity (1.4) for spherical functions on a symmetric cone.

Proof of Theorem 5.3. It suffices to check part (1). By Carlson's theorem, we shall prove that

$$\frac{1}{\Gamma_n(\lambda + \rho + \underline{\mu})} \int_{\mathbb{R}_+^n} E(-z, x) \mathcal{G}(\lambda, x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \mathcal{G}(\lambda, \frac{1}{z}) \Delta(z)^{-\mu}. \quad (5.1)$$

Note first that (5.1) holds for all $\lambda \in \Lambda_n^+ - \rho$, by Theorem 3.5 and Lemma 5.1(4). The right hand side of (5.1) is holomorphic in (λ, z, μ) on $\mathbb{C}^n \times H \times \mathbb{C}$. Moreover, the left hand side exists and is continuous on $\overline{H} \times H \times \{\operatorname{Re} \mu > \mu_0\}$ and holomorphic on $H \times H \times \{\operatorname{Re} \mu > \mu_0\}$. Indeed, suppose that $\operatorname{Re} z \geq \underline{s}$ for some $s > 0$. Then by [R20],

$$|E(-z, x)| \leq E(-\operatorname{Re} z, x) \leq e^{-\langle \underline{s}, x \rangle}.$$

Together with Lemma 5.1 we obtain for $x \in \mathbb{R}_+^n$

$$|E(-z, x) \mathcal{G}(\lambda, x) \Delta(x)^{\mu-\mu_0-1}| \leq \sqrt{n!} e^{-\langle \underline{s}, x \rangle} \Delta(x)^{\operatorname{Re} \mu - \mu_0 - 1} \max_{\sigma \in \mathcal{S}_n} x^{\sigma(\operatorname{Re} \lambda)}.$$

Hence the integral on the left hand side of formula (5.1) exists and is (by standard arguments) continuous respectively holomorphic as stated. It therefore suffices to check (5.1) for $z \in \mathbb{R}^n$ with $z > \underline{1}$ and $\mu \in \mathbb{R}$ with $\mu > \mu_0$. We want to apply Carlson's Theorem 5.2 with respect to λ . As $z > \underline{1}$, the right hand side of (5.1) is bounded in λ according to Lemma 5.1, and it remains to control the growth of the left hand side. For $\lambda \in H$, define $\eta(\lambda) := ([\operatorname{Re} \lambda_1], \dots, [\operatorname{Re} \lambda_n])_+ \in \Lambda_n^+$. Then for arbitrary $x \in \mathbb{R}_+^n$,

$$\max_{\sigma \in \mathcal{S}_n} x^{\sigma(\operatorname{Re} \lambda)} \leq \max_{\sigma \in \mathcal{S}_n} (\underline{1} + x)^{\sigma(\operatorname{Re} \lambda)} \leq P_{\eta(\lambda)}(\underline{1} + x),$$

because the coefficients of $P_{\eta(\lambda)}$ in its monomial expansion are nonnegative. Now recall the binomial formula (3.4) for the Jack polynomials as well as the identity

$$\int_{\mathbb{R}_+^n} e^{-\langle \underline{1}, x \rangle} P_{\kappa}(x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx = \Gamma_n(\kappa + \underline{\mu}) P_{\kappa}(\underline{1})$$

from [M13, (6.18)] (c.f. also [R20, Lemma 5.1]). We may therefore estimate

$$\begin{aligned}
& \int_{\mathbb{R}_+^n} |E(-z, x) \mathcal{G}(\lambda, x) \Delta(x)^{\mu-\mu_0-1}| \omega(x) dx \\
& \leq \int_{\mathbb{R}_+^n} e^{-\langle \underline{1}, x \rangle} P_{\eta(\lambda)}(\underline{1} + x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx \\
& = \sum_{\kappa \subseteq \eta(\lambda)} \binom{\eta(\lambda)}{\kappa} \int_{\mathbb{R}_+^n} e^{-\langle \underline{1}, x \rangle} P_\kappa(x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx \\
& = \sum_{\kappa \subseteq \eta(\lambda)} \binom{\eta(\lambda)}{\kappa} P_\kappa(\underline{1}) \Gamma_n(\kappa + \underline{\mu}).
\end{aligned}$$

By monotonicity of the classical gamma function,

$$\Gamma_n(\kappa + \underline{\mu}) \leq \Gamma_n(\eta(\lambda) + \underline{\mu}) \leq \Gamma_n((\operatorname{Re} \lambda)_+ + \underline{1} + \underline{\mu}).$$

Moreover, by Remark 3.2,

$$\sum_{\kappa \subseteq \eta(\lambda)} \binom{\eta(\lambda)}{\kappa} P_\kappa(\underline{1}) = P_{\eta(\lambda)}(\underline{2}) = 2^{|\eta(\lambda)|} P_{\eta(\lambda)}(\underline{1}) \leq 2^{\|\lambda\|_1} \cdot Q(\lambda)$$

with some polynomial $Q \in \mathcal{P}$. Therefore

$$\begin{aligned}
I_{z, \mu}(\lambda) & := \left| \frac{1}{\Gamma_n(\lambda + \rho + \underline{\mu})} \int_{\mathbb{R}_+^n} E(-z, x) \mathcal{G}(\lambda, x) \Delta(x)^{\mu-\mu_0-1} \omega(x) dx \right| \\
& \leq Q(\lambda) \cdot \frac{\Gamma_n((\operatorname{Re} \lambda)_+ + \underline{1} + \underline{\mu})}{|\Gamma_n(\lambda + \rho + \underline{\mu})|} \cdot 2^{\|\lambda\|_1}.
\end{aligned}$$

Choose $\sigma \in \mathcal{S}_n$ with $\sigma(\operatorname{Re} \lambda) = (\operatorname{Re} \lambda)_+$. Then

$$\frac{\Gamma_n((\operatorname{Re} \lambda)_+ + \underline{1} + \underline{\mu})}{|\Gamma_n(\lambda + \rho + \underline{\mu})|} = \prod_{j=1}^n \frac{\Gamma(((\operatorname{Re} \lambda)_+)_j + \mu + 1 - k(j-1))}{|\Gamma(\lambda_j + \mu + \rho(k)_j - k(j-1))|} = F_1(\lambda) \cdot F_2(\lambda)$$

with

$$\begin{aligned}
F_1(\lambda) & = \prod_{j=1}^n \frac{\Gamma(\operatorname{Re} \sigma(\lambda)_j + \mu + 1 - k(j-1))}{\Gamma(\operatorname{Re} \sigma(\lambda)_j + \mu + 1 - \frac{k}{2}(n-1))}, \\
F_2(\lambda) & = \prod_{j=1}^n \frac{|\lambda_j + \mu - \frac{k}{2}(n-1)| \cdot \Gamma(\operatorname{Re} \lambda_j + \mu + 1 - \frac{k}{2}(n-1))}{|\Gamma(\lambda_j + \mu + 1 - \frac{k}{2}(n-1))|}.
\end{aligned}$$

By Stirling's formula, $F_1(\lambda)$ is polynomially bounded, i.e. $F_1(\lambda) = \mathcal{O}(e^{\epsilon \|\lambda\|_1})$ for arbitrary $\epsilon > 0$. For F_2 , we employ the estimate ([N10, Formula 5.6.7])

$$\frac{\Gamma(x)}{|\Gamma(x + iy)|} \leq \sqrt{\cosh(\pi y)} = \mathcal{O}(e^{\frac{\pi}{2}|y|}), \quad x > \frac{1}{2}, y \in \mathbb{R},$$

which leads to

$$F_2(\lambda) = \mathcal{O}(e^{(\epsilon + \frac{\pi}{2}) \|\lambda\|_1})$$

with arbitrary $\epsilon > 0$. Putting things together, we obtain that $I_{z, \mu}(\lambda)$ satisfies the growth condition of Carlson's Theorem 5.2, which finishes the proof. \square

6. MACDONALD'S HYPERGEOMETRIC SERIES AND THEIR LAPLACE TRANSFORM

In the setting of symmetric cones, the Laplace transform establishes important identities between hypergeometric series. Analogous formulas were formally stated by Macdonald [M13] for general Jack-hypergeometric series, as consequences of his conjecture (C). With Theorem 3.5 at hand, we shall make these identities precise, and extend them to hypergeometric expansions in terms of non-symmetric Jack polynomials. We start with the appropriate normalization of the symmetric and non-symmetric Jack polynomials.

Lemma 6.1. (i) *There are numbers $c_\eta > 0$ for $\eta \in \mathbb{N}_0^n$ such that the renormalized Jack polynomials $C_\lambda := c_\lambda P_\lambda$ and $L_\eta := c_\eta E_\eta$ satisfy*

$$\sum_{\lambda \in \Lambda_n^+ : |\lambda| = m} C_\lambda(z) = \sum_{\eta \in \mathbb{N}_0^n : |\eta| = m} L_\eta(z) = (z_1 + \dots + z_n)^m \quad \text{for all } m \in \mathbb{N}_0;$$

$$(ii) \quad C_\lambda = \sum_{\eta \in \mathcal{S}_n \lambda} L_\eta \quad \text{for all } \lambda \in \Lambda_n^+.$$

$$(iii) \quad c_\lambda \leq \frac{|\lambda|!}{\lambda!} \quad \text{for all } \lambda \in \Lambda_n^+.$$

Proof. We may assume that $k > 0$. Part (i) for the symmetric Jack polynomials is well-known (see e.g. [10, (12.135)]), with

$$c_\lambda = \frac{|\lambda|!}{k^{|\lambda|} d'_\lambda}.$$

Here the constants d'_η for $\eta \in \mathbb{N}_0^n$ are given by

$$d'_\eta = \prod_{(i,j) \in \eta} \left(\frac{1}{k} (\eta_i - j + 1) + \ell(\eta, i, j) \right) > 0,$$

with the leg length $\ell(\eta, i, j) = \#\{\ell > i \mid j \leq \eta_\ell \leq \eta_i\} + \#\{\ell < i \mid j \leq \eta_\ell + 1 \leq \eta_i\}$. In particular, for each partition $\lambda \in \Lambda_n^+$ we have

$$c_\lambda = \frac{|\lambda|!}{\prod_{(i,j) \in \lambda} ((\lambda_i - j + 1) + k\ell(\lambda, i, j))} \leq \frac{|\lambda|!}{\prod_{1 \leq j \leq \lambda_i} (\lambda_i - j + 1)} = \frac{|\lambda|!}{\lambda!},$$

which is part (iii). From [10, Proposition 12.6.1] it is further known that

$$P_\lambda = d'_\lambda \sum_{\eta \in \mathcal{S}_n \lambda} \frac{1}{d'_\eta} E_\eta. \quad (6.1)$$

Hence, we put

$$c_\eta := c_{\eta^+} \frac{d'_{\eta^+}}{d'_\eta} = \frac{|\eta|!}{k^{|\eta|} d'_\eta}$$

for $\eta \in \mathbb{N}_0^n$, and part (i) for the non-symmetric Jack polynomials follows. Finally, part (ii) is immediate from the definition of c_η and relation (6.1). \square

On the space $\mathcal{P}_\mathbb{R} = \mathbb{R}[\mathbb{R}^n]$ of real polynomials on \mathbb{R}^n there exists an \mathcal{S}_n -invariant inner product $[\cdot, \cdot] = [\cdot, \cdot]_k$ called the Dunkl pairing (cf. [D91]), which is defined by

$$[p, q] := (p(T)q)(0).$$

Here the Dunkl operators are again those of type A_{n-1} with multiplicity k . Polynomials with different homogeneous degree are orthogonal with respect to this pairing,

and $[T_\xi p, q] = [p, \langle \cdot, \xi \rangle q]$. This property and the invariance under the action of \mathcal{S}_n show that the Cherednik operators \mathcal{D}_j are symmetric with respect to the Dunkl pairing. In particular, the non-symmetric Jack polynomials $(E_\eta)_{\eta \in \mathbb{N}_0^n}$ form an orthogonal basis of $\mathcal{P}_{\mathbb{R}}$ with respect to $[\cdot, \cdot]$. More precisely, their renormalizations $L_\eta = c_\eta E_\eta$ satisfy

$$[L_\eta, L_\kappa] = |\eta|! L_\eta(\underline{1}) \cdot \delta_{\eta, \kappa} \quad (6.2)$$

which is obtained by combining Proposition 3.18 and formula (2.4) of [BF98].

Lemma 6.2. *The Dunkl kernel of type A_{n-1} with multiplicity $k \geq 0$ satisfies*

$$E(z, w) = \sum_{\eta \in \mathbb{N}_0^n} \frac{L_\eta(z) L_\eta(w)}{|\eta|! L_\eta(\underline{1})}.$$

The series converges locally uniformly on $\mathbb{C}^n \times \mathbb{C}^n$.

Proof. This is immediate from [R98, Lemma 3.1] together with identity (6.2). Alternatively, the stated expansion follows from [10, Propos. 13.3.4]. \square

Definition 6.3. Consider the Jack polynomials $(L_\eta)_{\eta \in \mathbb{N}_0^n}$ and $(C_\lambda)_{\lambda \in \Lambda_n^+}$ of index $\alpha = \frac{1}{k}$, respectively (normalized as above). Following [M13], [Kan93] and [BF98], we define for indices $\mu \in \mathbb{C}^p$ and $\nu \in \mathbb{C}^q$ with $p, q \in \mathbb{N}_0$ the non-symmetric hypergeometric series

$${}_p K_q(\mu; \nu; z, w) := \sum_{\eta \in \mathbb{N}_0^n} \frac{[\mu_1]_{\eta_+} \cdots [\mu_p]_{\eta_+}}{[\nu_1]_{\eta_+} \cdots [\nu_q]_{\eta_+}} \frac{L_\eta(z) L_\eta(w)}{|\eta|! L_\eta(\underline{1})}$$

as well as the symmetric hypergeometric series

$${}_p F_q(\mu; \nu; z, w) := \sum_{\lambda \in \Lambda_n^+} \frac{[\mu_1]_\lambda \cdots [\mu_p]_\lambda}{[\nu_1]_\lambda \cdots [\nu_q]_\lambda} \frac{C_\lambda(z) C_\lambda(w)}{|\lambda|! C_\lambda(\underline{1})}.$$

More common in the literature are hypergeometric series in one variable, which are obtained as functions in z by setting $w = \underline{1}$. For abbreviation, we write for $\lambda \in \Lambda_n^+$

$$[\mu]_\lambda := [\mu_1]_\lambda \cdots [\mu_p]_\lambda; \quad [\nu]_\lambda := [\nu_1]_\lambda \cdots [\nu_q]_\lambda.$$

Note that for $p = 0$ or $q = 0$, an empty product occurs. For those values of k for which the $C_\lambda = C_\lambda(\cdot; \frac{1}{k})$ are the spherical polynomials of a symmetric cone, the convergence properties of ${}_p F_q$ -hypergeometric series in one variable are well-known, see [FK94, GR89]. For general $k > 0$, partial results on the domain of convergence of ${}_p F_q$ were obtained in [Kan93]. For some values of p and q , the nonsymmetric series ${}_p K_q$ were considered in [BF98]. But to our knowledge, their convergence properties have not been studied so far.

Lemma 6.4. *The non-symmetric and symmetric hypergeometric functions are related by*

$$\frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} {}_p K_q(\mu; \nu; \sigma z, w) = {}_p F_q(\mu; \nu; z, w).$$

Proof. By identity (3.3) and Lemma 6.1 we have

$$\begin{aligned} \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} {}_p K_q(\mu; \nu; \sigma z, w) &= \sum_{\eta \in \mathbb{N}_0^n} \frac{[\mu]_{\eta_+}}{[\nu]_{\eta_+}} \frac{1}{|\eta|!} L_\eta(w) \frac{C_{\eta_+}(z)}{C_{\eta_+}(\underline{1})} \\ &= \sum_{\lambda \in \Lambda_n^+} \frac{[\mu]_\lambda}{[\nu]_\lambda} \frac{1}{|\lambda|!} \left(\sum_{\eta \in \mathcal{S}_{n,\lambda}} L_\eta(w) \right) \frac{C_\lambda(z)}{C_\lambda(\underline{1})} \\ &= {}_p F_q(\mu; \nu; w, w). \end{aligned}$$

□

Theorem 6.5. *Let $\mu \in \mathbb{C}^p$ and $\nu \in \mathbb{C}^q$ with $\nu_i \notin \{0, k, \dots, k(n-1)\} - \mathbb{N}_0$ for all $i = 1, \dots, n$ (i.e. $[\nu]_\lambda \neq 0$ for all $\lambda \in \Lambda_n^+$).*

- (1) *If $p \leq q$, the series ${}_p K_q(\mu; \nu; \cdot, \cdot)$ and ${}_p F_q(\mu; \nu; \cdot, \cdot)$ are entire functions.*
- (2) *If $p = q + 1$, the series ${}_p K_q(\mu; \nu; \cdot, \cdot)$ and ${}_p F_q(\mu; \nu; \cdot, \cdot)$ are holomorphic on the domain $\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^n : \|z\|_\infty \|w\|_\infty < 1\}$.*

Moreover, the hypergeometric series are holomorphic in the parameters (μ, ν) on the domain

$$\{(\mu, \nu) \in \mathbb{C}^p \times \mathbb{C}^q \mid \nu_i \notin \{0, k, \dots, k(n-1)\} - \mathbb{N}_0 \text{ for all } i = 1, \dots, n\}.$$

Proof. It suffices to verify the statements for ${}_p K_q$. From Lemma 3.1(4) we have $|L_\eta(z)| \leq L_\eta(|z|) \leq L_\eta(\underline{1}) \|z\|_\infty^{|\eta|}$ and therefore

$$\begin{aligned} S(\mu, \nu; z, w) &:= \sum_{\eta \in \mathbb{N}_0^n} \left| \frac{[\mu]_{\eta_+}}{[\nu]_{\eta_+}} \right| \cdot \left| \frac{L_\eta(z) L_\eta(w)}{|\eta|! L_\eta(\underline{1})} \right| \leq \sum_{\eta \in \mathbb{N}_0^n} \left| \frac{[\mu]_{\eta_+}}{[\nu]_{\eta_+}} \right| \cdot \frac{\|z\|_\infty^{|\eta|} \|w\|_\infty^{|\eta|}}{|\eta|!} L_\eta(\underline{1}) \\ &= \sum_{\lambda \in \Lambda_n^+} \left| \frac{[\mu]_\lambda}{[\nu]_\lambda} \right| \cdot \frac{\|z\|_\infty^{|\lambda|} \|w\|_\infty^{|\lambda|}}{|\lambda|!} C_\lambda(\underline{1}), \end{aligned}$$

where for the last identity, Lemma 6.1(ii) was used. From Lemmata 3.2 and 6.1 we know that

$$C_\lambda(\underline{1}) = c_\lambda P_\lambda(\underline{1}) \leq \frac{|\lambda|!}{\lambda!} Q(\lambda) \quad (6.3)$$

with some polynomial $Q \in \mathcal{P}$. Therefore, we can find to each $\epsilon > 1$ a constant $C_\epsilon > 0$ such that $Q(\lambda) \leq C_\epsilon e^{|\lambda|}$. This gives

$$S(\mu, \nu; z, w) \leq C_\epsilon \sum_{\lambda \in \Lambda_n^+} \left| \frac{[\mu]_\lambda}{[\nu]_\lambda} \right| \cdot \frac{(\epsilon \|z\|_\infty \|w\|_\infty)^{|\lambda|}}{\lambda!}. \quad (6.4)$$

To prove part (1), consider the case $p \leq q$. In this case, the quotient

$$\frac{[\mu]_\lambda}{[\nu]_\lambda} = \frac{\prod_{i=1}^p [\mu_i]_\lambda}{\prod_{i=1}^q [\nu_i]_\lambda}$$

is of polynomial growth in λ . To see this, write

$$\frac{[\mu]_\lambda}{[\nu]_\lambda} = \prod_{i=1}^p \prod_{j=1}^n \frac{\Gamma(\nu_i - k(j-1)) \Gamma(\mu_i + \lambda_j - k(j-1))}{\Gamma(\mu_i - k(j-1)) \Gamma(\nu_i + \lambda_j - k(j-1))}.$$

By Stirling's formula we have, locally uniformly in μ and ν ,

$$\frac{\Gamma(\mu_i + \lambda_j - k(j-1))}{\Gamma(\nu_i + \lambda_j - k(j-1))} \sim (\nu_i + \lambda_j - k(j-1))^{\nu_i - \mu_i} \text{ for } \lambda_j \rightarrow \infty.$$

Moreover, $|\nu_i]_\lambda| \geq 1$ for large λ . Thus, for each $\epsilon > 1$ there are constant $D > 0$ and a compact neighborhood $K \subseteq \mathbb{C}^p \times \mathbb{C}^q$ of (μ, ν) , such that

$$\left| \frac{[\mu]_\lambda}{[\nu]_\lambda} \right| \leq D\epsilon^{|\lambda|} \text{ for all } (\mu, \nu) \in K.$$

Hence, for each $\epsilon > 1$, we find a constant $C_\epsilon > 0$ such that

$$\begin{aligned} S(\mu, \nu; z, w) &\leq C_\epsilon \sum_{\lambda \in \Lambda_n^+} \frac{(\epsilon \|z\|_\infty \|w\|_\infty)^{|\lambda|}}{\lambda!} \leq C_\epsilon \sum_{\lambda \in \mathbb{N}_0^g} \frac{(\epsilon \|z\|_\infty \|w\|_\infty)^{|\lambda|}}{\lambda!} \\ &\leq C_\epsilon e^{n\epsilon \|z\|_\infty \|w\|_\infty}. \end{aligned}$$

Therefore the ${}_pK_q$ -series is converges locally uniformly on $\mathbb{C}^n \times \mathbb{C}^n$ and also locally uniformly on the stated domain of parameters μ and ν , which proves part (1).

For part (2), observe that for $p = q + 1$, we have

$$\frac{[\mu]_\lambda}{[\nu]_\lambda} = \frac{\prod_{i=1}^q [\mu_i]_\lambda}{\prod_{i=1}^q [\nu_i]_\lambda} \cdot [\mu_p]_\lambda.$$

As in part (1), the first factor is of polynomial growth. Moreover,

$$\frac{[\mu_p]_\lambda}{\lambda!} = \prod_{j=1}^n \frac{\Gamma(\mu_p - k(j-1) + \lambda_j)}{\Gamma(\lambda_j + 1) \Gamma(\mu_p - k(j-1))},$$

which is of polynomial growth as well. Starting from estimate (6.4), we therefore obtain that for each $\epsilon > 1$, there is a constant $C_\epsilon > 0$ with

$$S(\mu, \nu; z, w) \leq C_\epsilon \sum_{\lambda \in \Lambda_n^+} (\epsilon \|z\|_\infty \|w\|_\infty)^{|\lambda|} \leq C_\epsilon \frac{1}{(1 - \epsilon \|z\|_\infty \|w\|_\infty)^n}.$$

This yields the claim. \square

Note that part (2) of this theorem improves, in the case $w = 1$, the results of [Kan93].

Remark 6.6. For $p = q = 0$, one gets the Dunkl kernel and Bessel function of type A_{n-1} , respectively. Indeed, Lemma 6.2 just says that

$$E(z, w) = {}_0K_0(z, w),$$

and symmetrization yields

$$J(z, w) = {}_0F_0(z, w),$$

which was already noted in [BF98].

Remark 6.7. The proof of Theorem 6.5 shows that for $p \leq q$ and arbitrary $\epsilon > 1$ there is a constant $C_\epsilon > 0$ such that

$$|{}_pK_q(\mu; \nu; z, w)| \leq S(\mu, \nu; z, w) \leq C_\epsilon e^{n\epsilon \|z\|_\infty \|w\|_\infty}. \quad (6.5)$$

Taking a closer look at the above proof, we see that for $p < q$ this estimate can be improved. Indeed, consider the quotient

$$\frac{[\mu]_\lambda}{[\nu]_\lambda} = \prod_{j=1}^p \frac{[\mu_j]_\lambda}{[\nu_j]_\lambda} \cdot \prod_{j=p+1}^q \frac{1}{[\nu_j]_\lambda}.$$

By Stirling's formula, the first factor is of polynomial growth, and thus of order $\mathcal{O}(\epsilon_1^{|\lambda|})$ for arbitrary $\epsilon_1 > 1$, while the second factor is of order $\mathcal{O}(\epsilon_2^{-|\lambda|})$ for arbitrary $\epsilon_2 > 1$. Under the assumption $p < q$ we therefore obtain the estimate

$$|{}_pK_q(\mu; \nu; z, w)| \leq S(\mu, \nu; z, w) \leq C_\epsilon e^{\epsilon \|z\|_\infty \|w\|_\infty}. \quad (6.6)$$

for arbitrary $\epsilon > 0$, with some constant $C_\epsilon > 0$.

The domain of convergence of the hypergeometric series ${}_pK_q$ and ${}_pF_q$ and their growth estimates (6.5), (6.6) are important to obtain from the Laplace transform identities for Jack polynomials in Theorem 3.5 similar Laplace transform identities for the hypergeometric series.

Theorem 6.8. *Let $\mu \in \mathbb{C}^p$, $\nu \in \mathbb{C}^q$ with $\nu_i \notin \{0, k, \dots, k(n-1)\} - \mathbb{N}_0$ for all $i = 1, \dots, n$ and let $\mu' \in \mathbb{C}$ with $\operatorname{Re} \mu' > \mu_0$.*

(1) *If $p < q$, then for all $z, w \in \mathbb{C}^n$ with $\operatorname{Re} z > 0$,*

$$\begin{aligned} \int_{\mathbb{R}_+^n} E(-z, x) {}_pK_q(\mu; \nu; w, x) \Delta(x)^{\mu' - \mu_0 - 1} \omega(x) dx \\ = \Gamma_n(\mu') \Delta(z)^{-\mu'} {}_{p+1}K_q((\mu', \mu); \nu; w, \frac{1}{z}). \end{aligned}$$

(2) *If $p = q$, then part (1) is valid under the condition $\|w\|_\infty \cdot \left\| \frac{1}{\operatorname{Re} z} \right\|_\infty < \frac{1}{n}$.*

Moreover, both parts remain valid if ${}_pK_q$ is replaced by ${}_pF_q$.

Proof. (1) By expanding ${}_pK_q$ into its defining series, this is immediate from the Laplace transform identity of Theorem 3.5 by interchanging the order of summation and integration. We have to justify this interchange. Choose $\epsilon > 0$ such that $\|w\|_\infty \cdot \|1/\operatorname{Re} z\|_\infty < \frac{1}{\epsilon}$. Under these conditions the estimates (2.1), (6.5) and (6.6) show that

$$|E(-x, z)| S(\mu, \nu; w, x) \leq C_\epsilon e^{-d_\epsilon \|x\|_\infty}$$

with $d_\epsilon = \min_{i=1 \dots n} \operatorname{Re} z_i - \epsilon \|w\|_\infty > 0$. Hence, we can apply the dominated convergence theorem to justify the interchange of summation and integration, so that part (1) is proven since $\epsilon > 0$ was chosen arbitrarily. Part (2) is obtained in the same way, by choosing $\epsilon > 1$ such that $\|w\|_\infty \cdot \|1/\operatorname{Re} z\|_\infty < \frac{1}{n\epsilon}$. By symmetrization we get the same identities for ${}_pF_q$. \square

We continue with an integral representation which was already observed in [M13, p.39] for the symmetric case, i.e. for ${}_1F_0$, but only at a formal level and without any statement on convergence.

Corollary 6.9. *Let $\mu \in \mathbb{C}$ with $\operatorname{Re} \mu > \mu_0$. Then the hypergeometric series ${}_1K_0(\mu; -z, w)$ has an analytic continuation to $D := \{\operatorname{Re} z > 0\} \times \{\operatorname{Re} w > 0\}$ which is given by*

$${}_1K_0(\mu; -z, w) = \frac{\Delta(z)^{-\mu}}{\Gamma_n(\mu)} \int_{\mathbb{R}_+^n} E(-\frac{1}{z}, x) E(-w, x) \Delta(x)^{\mu - \mu_0 - 1} \omega(x) dx.$$

By symmetrization, the same formula is valid if one replaces ${}_1K_0$ by ${}_1F_0$ and the Dunkl kernel by the Bessel function.

Proof. Recall that $E(z, w) = {}_0K_0(z, w)$. Then, by Theorem 6.8, the stated integral formula holds on a suitable open subset of D . Moreover, estimate (2.1) for the Dunkl kernel shows that the integral exists and defines a holomorphic function

on D by standard theorems on holomorphic parameter integrals. Hence, analytic continuation finishes the proof. \square

The following proposition is a generalization of the Euler integral for hypergeometric functions on symmetric cones ([FK94, Proposition XV.1.4]) and can be found as a formal statement in [M13, formula (6.21)]. It will be obtained from Kadell's [Kad97] generalization of the Selberg integral,

$$\int_{[0,1]^n} \frac{C_\lambda(x)}{C_\lambda(\underline{1})} \Delta(x)^{\mu-\mu_0-1} \Delta(\underline{1}-x)^{\nu-\mu_0-1} \omega(x) dx = \frac{\Gamma_n(\mu) \Gamma_n(\nu)}{\Gamma_n(\mu+\nu)} \frac{[\mu]_\lambda}{[\mu+\nu]_\lambda} \quad (6.7)$$

for all $\lambda \in \Lambda_+^n$ and $\mu, \nu \in \mathbb{C}$ with $\operatorname{Re} \mu, \operatorname{Re} \nu > \mu_0$.

Proposition 6.10. *Consider $p \leq q+1$ and $\mu', \nu' \in \mathbb{C}$ with $\operatorname{Re} \mu', \operatorname{Re}(\nu' - \mu') > \mu_0$. Moreover, let $\mu \in \mathbb{C}^p$ and $\nu \in \mathbb{C}^q$ with $\nu_i \notin \{0, k, \dots, k(n-1)\} - \mathbb{N}_0$ for all $i = 1, \dots, n$. Then for arbitrary $w \in \mathbb{C}^n$ with the additional condition $\|w\|_\infty < 1$ in the case $p = q+1$, one has*

$$\begin{aligned} \int_{[0,1]^n} {}_pF_q(\mu; \nu; w, x) \Delta(x)^{\mu' - \mu_0 - 1} \Delta(\underline{1} - x)^{\nu' - \mu' - \mu_0 - 1} \omega(x) dx \\ = \frac{\Gamma_n(\mu) \Gamma_n(\nu - \mu)}{\Gamma_n(\nu)} {}_{p+1}F_{q+1}((\mu', \mu); (\nu', \nu); w, \underline{1}). \end{aligned}$$

Proof. This is immediate from Kadell's integral (6.7) after expanding ${}_pF_q$ into its defining series and changing the order of integration and summation. The latter is justified since the series ${}_pF_q(\mu; \nu; w; \cdot)$ is absolutely bounded on $[0, 1]^n$ by the estimates in the proof of Theorem 6.5 for $w \in \mathbb{C}^n$ and $\|w\|_\infty < 1$ if $p = q+1$. \square

The following theorem generalizes Proposition XV.1.2. of [FK94] for hypergeometric series on symmetric cones.

Theorem 6.11. *The Jack polynomials and the hypergeometric series have the following properties under the action of the Dunkl operator $\Delta(T)$ associated to the polynomial Δ .*

- (1) $\Delta(T)E_\eta = c_\eta E_{\eta - \underline{1}}$ with some constant $c_\eta \in \mathbb{R}$. Moreover, $c_\eta = 0$ if $\eta_i = 0$ for some $i \in \{1, \dots, n\}$.
- (2) $\eta \mapsto c_\eta$ is \mathcal{S}_n -invariant.
- (3) $\Delta(T)L_\eta = d_\eta L_{\eta - \underline{1}}$ and $\Delta(T)C_\lambda = d_\lambda C_{\lambda - \underline{1}}$, where

$$d_\eta = \begin{cases} \frac{|\eta|!}{|\eta - \underline{1}|!} & \text{if } \eta_i \neq 0 \text{ for all } i = 1, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

- (4) If $p \leq q+1$, then

$$\Delta(T) {}_pK_q(\mu; \nu; w, \cdot) = \frac{[\mu]_{\underline{1}}}{[\nu]_{\underline{1}}} \Delta(w) {}_pK_q(\mu + \underline{1}, \nu + \underline{1}; w, \cdot)$$

for all $w \in \mathbb{C}^n$, with the understanding that $[\mu]_{\underline{1}} = 1$ if $p = 0$ and $[\nu]_{\underline{1}} = 1$ if $q = 0$. The same is valid for ${}_pK_q$ instead of ${}_pF_q$.

Proof. (1) From the properties of the Dunkl pairing together with Lemma 3.1 (1) we can conclude that for compositions $\eta, \kappa \in \mathbb{N}_0^n$,

$$[\Delta(T)E_\eta, E_\kappa] = [E_\eta, \Delta E_\kappa] = [E_\eta, E_{\kappa + \underline{1}}] = \begin{cases} 0 & \text{if } \eta \neq \kappa + \underline{1}; \\ [E_\eta, E_\eta] > 0 & \text{if } \eta = \kappa + \underline{1}. \end{cases}$$

Hence, $\Delta(T)E_\eta$ must be a scalar multiple of $E_{\eta-\underline{1}}$ if $\eta_i \geq 1$ for all $i = 1, \dots, n$ and vanishes otherwise.

(2) Denote again by $\bar{\eta}_i$ the eigenvalue of E_η under the Cherednik operator \mathcal{D}_i . It suffices to show that $c_\eta = c_{s_i\eta}$ if $\eta_i < \eta_{i+1}$. Due to [10, Proposition 12.2.1], we then have

$$E_{s_i\eta} = d_i^\eta E_\eta + s_i E_\eta \quad (6.8)$$

with the constant

$$d_i^\eta = \frac{k}{\bar{\eta}_{i+1} - \bar{\eta}_i}.$$

It is immediate that $\overline{\eta + \underline{1}} = \bar{\eta} + \underline{1}$ and therefore $d_i^{\eta+\underline{1}} = d_i^\eta$. Applying $\Delta(T)$ to equation (6.8), using $d_i^{\eta+\underline{1}} = d_i^\eta$ and the \mathcal{S}_n -equivariance of the Dunkl operators, we obtain from part (1) that $c_{s_i\eta} = c_\eta$.

(3) As L_η is a renormalization of E_η , there is a constant d_η such that $\Delta(T)L_\eta = d_\eta L_{\eta-\underline{1}}$, and $d_\eta = 0$ if $\eta_i = 0$ for some i . Since $\eta \mapsto \frac{|\eta|!}{|\eta-\underline{1}|!}$ is \mathcal{S}_n -invariant and $C_\lambda = \sum_{\eta \in \mathcal{S}_n\lambda} L_\eta$, it suffices to verify the stated value of d_η with $|\eta| \geq n$. Recall that T_i acts as ∂_i on symmetric polynomials. We therefore conclude that for $m \in \mathbb{N}_0$ with $m \geq n$,

$$\begin{aligned} m \cdots (m-n+1) \sum_{\substack{\eta \in \mathbb{N}_0^n: \\ |\eta|=m-n}} L_\eta(x) &= m \cdots (m-n+1)(x_1 + \dots + x_n)^{m-n} \\ &= \Delta(T)(x_1 + \dots + x_n)^m = \sum_{\substack{\eta \in \mathbb{N}_0^n: \\ |\eta|=m}} \Delta(T)L_\eta = \sum_{\substack{\eta \in \mathbb{N}_0^n: \\ |\eta|=m}} d_\eta L_{\eta-\underline{1}}. \end{aligned}$$

Equating the coefficients proves the stated formula for d_η .

(4) This is an immediate consequence of part (3) by expanding the hypergeometric series. One has to perform an index shift $\eta \mapsto \eta + \underline{1}$ after applying $\Delta(T)$ and the identity of part (3) together with

$$\frac{L_\eta(w)}{L_\eta(\underline{1})} = \Delta(w) \frac{L_{\eta-\underline{1}}(w)}{L_{\eta-\underline{1}}(w)}$$

and $[\theta]_{\eta_+} = [\theta + 1]_{\eta_+ - \underline{1}} [\theta]_{\underline{1}}$. □

7. A POST-WIDDER INVERSION FORMULA FOR THE DUNKL-LAPLACE TRANSFORM

We consider again the Dunkl setting of type A_{n-1} with multiplicity $k \geq 0$ and keep our previous notations. As shown in [R20], the Dunkl-Laplace transform

$$\mathcal{L}f(z) = \int_{\mathbb{R}_+^n} f(x) E(-z, x) \omega(x) dx$$

satisfies the following Cauchy inversion theorem: Let $f \in L_{loc}^1(\mathbb{R}_+^n)$ such that $\mathcal{L}f(\underline{s})$ exists for some $s \in \mathbb{R}$ (then $\mathcal{L}f(z)$ also exists for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z = \underline{s}$). Assume further that $y \mapsto \mathcal{L}f(\underline{s} + iy) \in L^1(\mathbb{R}^n, \omega)$. Then f has a continuous representative f_0 , and

$$\frac{(-i)^n}{c_k^2} \int_{\operatorname{Re} z = \underline{s}} \mathcal{L}f(z) E(x, z) \omega(z) dz = \begin{cases} f_0(x) & \text{for } x \in \mathbb{R}_+^n; \\ 0 & \text{otherwise,} \end{cases}$$

with the constant $c_k = \int_{\mathbb{R}^n} e^{-|x|^2/2} \omega(x) dx$. For the classical Laplace transform

$$Lf(z) = \int_0^\infty f(x) e^{-zx} dx, \quad f \in L^1_{loc}(\mathbb{R}_+),$$

a further well-known inversion theorem is the Post-Widder inversion formula (see e.g. [ABHN01]): Assume that $f \in L^1_{loc}(\mathbb{R}_+)$ has a finite abscissa of convergence and is continuous in $\xi \in \mathbb{R}_+$. Then

$$f(\xi) = \lim_{\nu \rightarrow \infty} \frac{(-1)^\nu}{\nu!} \left(\frac{\nu}{\xi}\right)^{\nu+1} (Lf)^\nu \left(\frac{\nu}{\xi}\right).$$

In this section, we prove a Post-Widder inversion formula for the Dunkl-Laplace transform, which is the counterpart to a result of Faraut and Gindikin [FG90] in the setting of symmetric cones.

Theorem 7.1 (Post-Widder inversion formula for \mathcal{L}). *Let $f : \mathbb{R}_+^n \rightarrow \mathbb{C}$ be measurable and bounded, and suppose that f is continuous at $\xi \in \mathbb{R}_+^n$. Then*

$$f(\xi) = \lim_{\nu \rightarrow \infty} \frac{(-1)^{n\nu}}{\Gamma_n(\nu + \mu_0 + 1)} \Delta\left(\frac{\nu}{\xi}\right)^{\nu + \mu_0 + 1} (\Delta(T)^\nu(\mathcal{L}f))\left(\frac{\nu}{\xi}\right).$$

The idea of proof for this theorem is similar to [FG90]. It was elaborated to some extent by Frederik Hoppe in his master thesis [Ho20], which was supervised by the second author of this paper. A fundamental ingredient is Levy's continuity theorem for the Dunkl transform. Let us recall this for the reader's convenience. Denote by $M_b^+(\mathbb{R}^n)$ the space of positive bounded Borel measures on \mathbb{R}^n . The Dunkl transform of $\mu \in M_b^+(\mathbb{R}^n)$ (associated with A_{n-1} and multiplicity k) is given by

$$\widehat{\mu}(\xi) = \widehat{\mu}^k(\xi) = \int_{\mathbb{R}^n} E(-i\xi, x) d\mu(x), \quad \xi \in \mathbb{R}^n.$$

Note that $\widehat{\mu} \in C_b(\mathbb{R}^n)$, since $|E(-i\xi, x)| \leq 1$ for all $\xi, x \in \mathbb{R}^n$. The Dunkl transform is injective on $M_b^+(\mathbb{R}^n)$, see [RV98]. The following is the essential part of Levy's continuity theorem for the Dunkl transform.

Lemma 7.2 ([RV98]). *Let $(\mu_\nu)_{\nu \in \mathbb{N}} \subseteq M_b^+(\mathbb{R}^n)$ such that the sequence $(\widehat{\mu}_\nu)_{\nu \in \mathbb{N}}$ converges pointwise to a function $\varphi : \mathbb{R}^n \rightarrow \mathbb{C}$ which is continuous at 0. Then there exists a unique $\mu \in M_b^+(\mathbb{R}^n)$ with $\widehat{\mu} = \varphi$, and $(\mu_\nu)_{\nu \in \mathbb{N}}$ converges to μ weakly.*

Proof of Theorem 7.1. We consider on \mathbb{R}_+^n the functions

$$h_\nu(x) := E\left(-\frac{\nu}{\xi}, x\right) \Delta(x)^\nu, \quad \nu \in \mathbb{N}.$$

By estimate (2.1), the Laplace transform

$$\mathcal{L}h_\nu(z) = \int_{\mathbb{R}_+^n} E(-z, x) E\left(-\frac{\nu}{\xi}, x\right) \Delta(x)^\nu \omega(x) dx$$

exists for all $z \in \mathbb{C}^n$ with $\operatorname{Re} z \geq 0$. For such z , put $\nu(z) := \max_i [\|z\|_\infty \xi_i] \in \mathbb{N}$. Then for $\nu > \nu(z)$, we calculate

$$\begin{aligned} \mathcal{L}h_\nu(z) &= \int_{\mathbb{R}_+^n} \left(\sum_{\eta \in \mathbb{N}_0^n} \frac{L_\eta(-z)L_\eta(x)}{|\eta|! L_\eta(\underline{1})} \right) E(-\frac{\nu}{\xi}, x) \Delta(x)^\nu \omega(x) dx \\ &= \sum_{\eta \in \mathbb{N}_0^n} \frac{L_\eta(-z)}{|\eta|! L_\eta(\underline{1})} \int_{\mathbb{R}_+^n} L_\eta(x) E(-\frac{\nu}{\xi}, x) \Delta(x)^\nu \omega(x) dx \\ &= \sum_{\eta \in \mathbb{N}_0^n} \frac{L_\eta(-z)}{|\eta|! L_\eta(\underline{1})} \Gamma_n(\eta_+ + \nu + \mu_0 + 1) L_\eta(\frac{\xi}{\nu}) \Delta(\frac{\nu}{\xi})^{-\nu - \mu_0 - 1}. \end{aligned}$$

Here the interchange of the sum and the integral is justified by the dominated convergence theorem, because $|L_\eta(-z)| \leq L_\eta(\|z\|_\infty \cdot \underline{1})$ and therefore

$$\begin{aligned} E(-\frac{\nu}{\xi}, x) \sum_{\eta \in \mathbb{N}_0^n} \frac{|L_\eta(-z)L_\eta(x)|}{|\eta|! L_\eta(\underline{1})} &\leq E(-\frac{\nu}{\xi}, x) E(\|z\|_\infty \cdot \underline{1}, x) \\ &= E(-\frac{\nu}{\xi} + \|z\|_\infty \cdot \underline{1}, x). \end{aligned}$$

This decays exponentially on \mathbb{R}_+^n , since $-\nu/\xi + \|z\|_\infty \cdot \underline{1} < 0$ by our assumption on ν . Thus for $\nu \geq \nu(z)$,

$$f_\nu(z) := \frac{\Delta(\frac{\nu}{\xi})^{\nu + \mu_0 + 1}}{\Gamma_n(\nu + \mu_0 + 1)} \mathcal{L}h_\nu(z) = \sum_{\eta \in \mathbb{N}_0^n} c_\nu(\eta) \cdot \frac{L_\eta(-z)L_\eta(\xi)}{|\eta|! L_\eta(\underline{1})} \quad (7.1)$$

with the coefficients

$$c_\nu(\eta) = \frac{[\nu + \mu_0 + 1]_{\eta_+}}{\nu^{|\eta|}} = \prod_{j=1}^n \left(1 + \frac{1 + k(n-j)}{\nu} \right)_{\lambda_j}, \quad \lambda = \eta_+.$$

They satisfy

$$\lim_{\nu \rightarrow \infty} c_\nu(\eta) = 1 \quad \text{for fixed } \eta,$$

and it follows that

$$\lim_{\nu \rightarrow \infty} f_\nu(z) = \sum_{\eta \in \mathbb{N}_0^n} \frac{L_\eta(-z)L_\eta(\xi)}{|\eta|! L_\eta(\underline{1})} = E(-z, \xi). \quad (7.2)$$

We still have to justify that the limit $\nu \rightarrow \infty$ may be taken inside the sum in (7.1). For this, note that $\nu \mapsto c_\nu(\eta)$ is monotonically decreasing. Hence for $\nu \geq \nu(z)$, the series on the right-hand side of (7.1) is dominated by the convergent series

$$\sum_{\eta \in \mathbb{N}_0^n} c_{\nu(z)}(\eta) \frac{L_\eta(\|z\|_\infty \cdot \underline{1}) L_\eta(\xi)}{|\eta|! L_\eta(\underline{1})} = f_{\nu(z)}(-\|z\|_\infty \cdot \underline{1}) < \infty,$$

which justifies the above limit. We now consider the measures

$$dm_\nu(x) := \frac{\Delta(\frac{\nu}{\xi})^{\nu + \mu_0 + 1}}{\Gamma_n(\nu + \mu_0 + 1)} \cdot \mathbf{1}_{\mathbb{R}_+^n}(x) E(-\frac{\nu}{\xi}, x) \Delta(x)^\nu \omega(x) dx \in M_b^+(\mathbb{R}^n).$$

Due to Theorem 3.5, m_ν is actually a probability measure on \mathbb{R}^n . Formula (7.2), considered for arguments $z \in i\mathbb{R}^n$, shows that the Dunkl transforms satisfy

$$\widehat{m}_\nu \rightarrow \widehat{\delta}_\xi \quad \text{pointwise on } \mathbb{R}^n,$$

where δ_ξ denotes the point measure in ξ . Levy's continuity theorem (Lemma 7.2) now implies that $m_\nu \rightarrow \delta_\xi$ weakly. Thanks to the Portemanteau theorem ([Kle14]) we even get

$$\lim_{\nu \rightarrow \infty} \int_{\mathbb{R}^n} g dm_\nu = \int_{\mathbb{R}^n} g d\delta_\xi = g(\xi)$$

for all measurable bounded functions $g : \mathbb{R}^n \rightarrow \mathbb{C}$ which are continuous at ξ . Now suppose $f : \mathbb{R}_+^n \rightarrow \mathbb{C}$ is measurable, bounded and continuous at ξ . Extend f by zero to \mathbb{R}^n . Then

$$\frac{\Delta\left(\frac{\nu}{\xi}\right)^{\nu+\mu_0+1}}{\Gamma_n(\nu+\mu_0+1)} \int_{\mathbb{R}_+^n} f(x) E\left(-\frac{\nu}{\xi}, x\right) \Delta(x)^\nu \omega(x) dx = \int_{\mathbb{R}^n} f dm_\nu \rightarrow f(\xi).$$

But in view of to Lemma 2.1 the integral on the left-hand side can be written as

$$\mathcal{L}(\Delta^\nu f)\left(\frac{\nu}{\xi}\right) = (\Delta(-T))^\nu (\mathcal{L}f)\left(\frac{\nu}{\xi}\right),$$

which finishes the proof. \square

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