The computation of an unstable invariant set inside a cylinder containing a knotted flow

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Abstract

We perform a numerical study of a knotted flow through a cylinder. In this example – which is due to Conley [1] – based on certain assumptions on the flow Wazewski's Theorem guarantees the existence of an unstable invariant set inside the cylinder. We explicitly construct a flow with the desired properties and approximate the corresponding invariant set using set oriented multilevel subdivision techniques.

1 Introduction

It is now well known that index theory yields a powerful methodology for proving the existence of invariant sets of dynamical systems inside certain regions of state space. See e.g. [6, 7] and references therein. However, the underlying techniques are based on results from algebraic topology, and these methods are typically not constructive in the sense that they do not give insight into the precise structure and location of the invariant sets. Thus, one has to rely on numerical methods if one is interested in more detailed information on the geometric structure of these objects.

In this article we review a prominent example in this area and describe a numerical approach which allows to reliably approximate the corresponding unstable invariant set. Concretely we consider the following scenario and conclusion – the latter one is an application of the Wazewski Theorem (see Section 2) – which goes back to Conley [1]:

> Let φ^t denote a flow of an ordinary differential equation on $\mathbb{R}^3$ with the following properties: there is a cylinder of finite length such that outside the cylinder trajectories run vertically downward with respect to the cylinder. Assume further that there is some solution running through the cylinder which makes a knot as it goes from top to bottom. Then there must be a nontrivial invariant set inside the cylinder.

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A more detailed outline of the paper is as follows. In Section 2 we describe the theoretical background taken from [1]. In Section 3 we explicitly construct a flow with the properties described above. The underlying numerical methods for the approximation of this object are developed in Section 4. Finally we present the specific numerical example which has been used to produce the cover image of these proceedings (Section 5).

2 Theoretical Background

We now describe the theoretical background showing the existence of an invariant set inside the cylinder for the flow described in the introduction. In what follows we essentially follow the exposition in [1].

We begin with the definition of a so-called *Wazewski set* for a flow $\phi^t : \mathbb{R}^n \to \mathbb{R}^n$.

DEFINITION 2.1 Let $W \subset \mathbb{R}^n$ be a subset of phase space and let W° be the set of points $x \in W$ such that $\phi^t(x) \notin W$ for some positive t. Moreover, let W^- be the set of points $x \in W$ such that $\phi^t(x) \notin W$ for all $t > 0$. Then $W^- \subset W^\circ$ is called the *exit set* of W.

Moreover, W is called a *Wazewski set* if the following conditions are satisfied:

(a) If $x \in W$ and $\phi^s(x) \in \mathrm{cl}(W)$ for all $s \in [0, t]$ with $t \geq 0$, then $\phi^s(x) \subset W$ for all $s \in [0, t]$;

(b) W^- is closed relative to W°.

With this definition we have the following result. The proof of this theorem can be found in e.g. [1].

THEOREM 2.2 (WAZEWSKI) *If $W \subset \mathbb{R}^n$ is a Wazewski set then the exit set W^- is a strong deformation retract of W° and W° is open relative to W.*

REMARK 2.3 As an immediate consequence of this theorem we have:

> *If W is a Wazewski set and the exit set W^- is not a strong deformation retract of W then $W \setminus W^\circ \neq \emptyset$, that is, there exist solutions which stay inside W for all positive time.*

Using this observation we obtain the following corollary.

COROLLARY 2.4 *Consider the flow φ^t through the cylinder as described in the introduction. Then there is a nonempty invariant set inside the cylinder.*

Proof: Let W be the cylinder minus the knotted trajectory. Then the exit set W^- of W is the bottom of the cylinder minus a point. Obviously W is a Wazewski set. Since the fundamental group of W is not the same as that of the punctured disk W^- it follows that W^- cannot be a strong deformation retract of W. Thus, $W \setminus W^\circ \neq \emptyset$, and there exist solutions which stay inside W for all positive time. ∎

3 Creation of the Vector Field

In this section we explicitly construct a vector field v such that the corresponding flow φ^t has the desired properties.

We consider a cylinder $C = C(r,h) = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2 \le r^2, |z| \le h\}$ and a path (the knot) $\gamma\colon [0,T] \to C$, with

$$\gamma(0) \in C \cap \{z = h\} \quad \text{and} \quad \gamma(T) \in C \cap \{z = -h\}.$$

The aim is to define γ and a continuous vector field $v\colon C \to S^2$ such that the following conditions are satisfied:

(1) the knot γ describes a trajectory of v,

(2) $v(c) = (0,0,-1)$ for all $c \in M = \{(x,y,z) \in C : x^2+y^2 = r^2\}$.

Recall that the stereographic projection $\sigma\colon S^2 \to \overline{\mathbb{R}}^2$ ($\overline{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$) is given by

$$\sigma(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

The inverse of σ is given by

$$\sigma^{-1}(\alpha,\beta) = \left(\frac{2\alpha}{1+\alpha^2+\beta^2}, \frac{2\beta}{1+\alpha^2+\beta^2}, 1 - \frac{2}{1+\alpha^2+\beta^2}\right).$$

The idea is to construct the vector field v by defining a suitable map $w\colon C \to \mathbb{R}^2$ and by setting $v = \sigma^{-1} \circ w$.

We define

$$w(c) = \frac{\displaystyle\int_0^T \|\gamma'(t)\|\, d(c,\gamma(t))\, \sigma\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right) dt}{\displaystyle\int_0^T \|\gamma'(t)\|\, d(c,\gamma(t))\, dt + d(c,M)} \tag{3.1}$$

with a continuous weight function $d\colon C \times C \to \overline{\mathbb{R}}$ where $d^{-1}(\infty) = \{(c,c) \mid c \in C\}$. The first term in the denominator simply normalizes the weight function, whereas the second leads to the desired values on the mantle M of the cylinder. Observe that the use of the stereographic projection in the definition of w requires that $\gamma'(t)/\|\gamma'(t)\| \ne (0,0,1)$ for all t.

We now verify the fact that the knot γ describes a trajectory (i.e. condition (1)) and that v is indeed continuous. Away from $\operatorname{im}(\gamma)$ and M the function w is obviously continuous and so is v. If we consider a sequence $c_i \to \gamma(t_0)$, $c_i \notin \operatorname{im}(\gamma)$, $i = 0,1,\ldots,$ for some fixed $t_0 \in [0,T]$, then

$$\begin{aligned} w(c_i) &= \int_0^T \frac{\|\gamma'(t)\|\, d(c_i,\gamma(t))}{\int_0^T \|\gamma'(s)\|\, d(c_i,\gamma(s))\, ds + d(c_i,M)}\, \sigma\left(\frac{\gamma'(t)}{\|\gamma'(t)\|}\right) dt \\ &\to \delta\left(\sigma\left(\frac{\gamma'(\cdot + t_0)}{\|\gamma'(\cdot + t_0)\|}\right)\right) = \sigma\left(\frac{\gamma'(t_0)}{\|\gamma'(t_0)\|}\right) \quad \text{as } i \to \infty, \end{aligned}$$

where δ is the Dirac distribution. Hence we obtain

$$v(c_i) \quad \to \quad \frac{\gamma'(t_0)}{\|\gamma'(t_0)\|} \quad \text{as } i \to \infty$$

as desired.

Condition (2) is satisfied because $c_i \to M$ implies

$$d(c_i, M) \to \infty \quad \Rightarrow \quad w(c_i) \quad \to \quad 0 \quad \Rightarrow \quad v(c_i) \to (0,0,-1) \quad \text{as } i \to \infty.$$

Hence we have satisfied the conditions (1) and (2).

4 Computation of the Invariant Set

The vector field v as defined in the previous section does not possess an equilibrium inside C. However topological considerations (cf. Section 2) show that there must be an unstable invariant set contained in C. In this section we describe how to compute approximations to such a set. In fact we show how to compute approximations to the chain recurrent set of some time-τ-map $f = \varphi^\tau$ in C.

DEFINITION 4.1 A point $c \in C$ belongs to the *chain recurrent set* of f in C if for every $\epsilon > 0$ there is an ϵ-pseudoperiodic orbit containing c, that is, there exists $\{c = c_0, c_1, \dots, c_{\ell-1}\} \subset C$ such that

$$\|f(c_i) - c_{i+1 \bmod \ell}\| \le \epsilon \text{ for } i = 0, \dots, \ell - 1.$$

The chain recurrent set is closed and invariant.

In [2] multilevel subdivision techniques have been developed for the approximation of global attractors relative to some compact subset Q of state space. Roughly speaking this set should be viewed as the union of all the invariant sets within Q together with their unstable manifolds. We now present a modification of this algorithm which allows to approximate chain recurrent sets.

Let Q be a compact subset of the underlying state space, say, Q is part of the cylinder C. Then we construct a sequence $\mathcal{B}_0, \mathcal{B}_1, \dots$ of collections of compact subsets of Q as follows. First set $\mathcal{B}_0 = \{Q\}$. Then for $k \ge 1$ the collection $\mathcal{B}_k$ is obtained from $\mathcal{B}_{k-1}$ in two steps:

(i) **Subdivision:** construct a collection $\hat{\mathcal{B}}_k$, such that

$$\bigcup_{B \in \hat{\mathcal{B}}_k} B = \bigcup_{B \in \mathcal{B}_{k-1}} B \quad \text{and} \quad \operatorname{diam} \hat{\mathcal{B}}_k \le \theta \operatorname{diam} \mathcal{B}_{k-1}$$

for some $0 < \theta < 1$.

(ii) **Selection:** construct a directed graph $G = (V, E)$, where the vertices V and the edges E are given by

$$V = \hat{\mathcal{B}}_k \text{ and } E = \{(B, B') \in \hat{\mathcal{B}}_k \times \hat{\mathcal{B}}_k : f(B) \cap B' \neq \emptyset\}.$$

Compute the strongly connected components $S_1, \dots, S_r$ of G and set

$$\mathcal{B}_k = \{B \in \hat{\mathcal{B}}_k : B \in S_i \text{ for some } i \in \{1 \dots, r\}\}.$$

REMARK 4.2 Recall that a subset $W \subset V$ of the nodes of a directed graph $G = (V, E)$ is called a *strongly connected component* of G, if for all $w, \tilde{w} \in W$ there is a path from w to $\tilde{w}$ (i.e. if there is a sequence $(w_i, w_{i+1}) \in E$, $i = 0, \dots, m-1$, such that $w = w_0$ and $\tilde{w} = w_m$). The set of all strongly connected components of a given directed graph can be computed in linear time [5].

Intuitively it is plausible that the sequences of box coverings $\mathcal{B}_k$ converge to the chain recurrent set of f. Indeed, under mild assumptions on the box coverings one can prove convergence, see [4, 8].

5 Numerical Example

In this section we apply the algorithm described above to a specific knotted flow. We consider a cylinder $C(r, h)$ with radius $r = 2$ and height $2h = 4$. Then one way to obtain parameterized knots is to take an epicycloid in the (x, y)-plane

$$\begin{aligned} x(t) &= \cos(t) - \frac{LA}{A+B} \cdot \cos\left(\frac{A+B}{A} \cdot t\right) \\ y(t) &= \sin(t) - \frac{LA}{A+B} \cdot \sin\left(\frac{A+B}{A} \cdot t\right), \end{aligned}$$

and to modulate the third coordinate $z(t)$ appropriately

$$z(t) = -\sin\left(\frac{B}{A} \cdot t\right) + h\left(1 - \frac{2t}{T}\right)^p.$$

In the computations we have set

$$L = 0.9, \quad h = 2 \quad \text{and} \quad p = 3.$$

For $0 \le t \le T = 2\pi A$ we obtain various knots by varying the integers A and B. We have chosen

$$A = 2 \quad \text{and} \quad B = 1.$$

Finally the function d in (3.1) is defined by

$$d(x, y) = \left(\sum_{i=1}^{3} |x_i - y_i|^q\right)^{-1},$$

where we have chosen $q = 4$.

The numerical integration for the construction of w is done by an adaptive Gaussian quadrature. It turned out to be computationally too demanding to compute these integrals for every evaluation of the vector field. We thus decided to precompute the vector field on an equidistant mesh, to store the resulting data and to use a linear interpolation between mesh points for the evaluation of v. Certainly the original knot is not a trajectory of this discretized vector field any more, but the actual trajectories are very similar and still knotted in the same way.

In the computation of the time-τ-map $f = \varphi^\tau$ we have chosen $\tau = 1.5$, and for the numerical integration of the ordinary differential equation we have used a fourth order Runge-Kutta scheme with constant step size $h = 0.1$. For the construction of the directed graph we use a grid P of test points in each box $B \in \hat{\mathcal{B}}_k$ with 10 points per coordinate direction (i.e. 1000 points per box) and compute

$$E = \{(B, B') \in \hat{\mathcal{B}}_k \times \hat{\mathcal{B}}_k : f(P) \cap B' \neq \emptyset\}.$$

Now we apply 30 steps of the algorithm described in Section 4 beginning with $\mathcal{B}_0 = \{Q\} = \{C\}$ and ending up with a collection of 175476 boxes covering the chain recurrent set. On the front cover of this volume we show a visualization of the resulting box collection (blue) together with the knot (red). Moreover, in Figure 1 we present a couple of alternative illustrations of the object.

The computation has been done using the software package GAIO[1] and for the visualization on the cover image we have used the software platform GRAPE [9, 3].

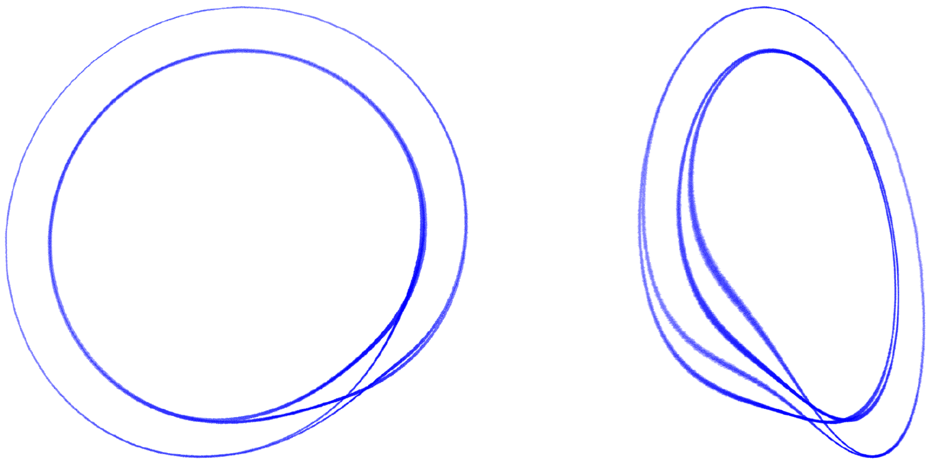

Figure 1: Chain recurrent set for the knotted flow.

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[1]See `http://math-www.uni-paderborn.de/~agdellnitz/gaio/`

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