A critical exponent in a degenerate parabolic equation

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Abstract

We consider positive solutions of the Cauchy problem in $\mathbb{R}^n$ for the equation

$$u_t = u^p \Delta u + u^q, \qquad p \geq 1, q \geq 1,$$

and show that concerning global solvability, the number $q = p+1$ appears as a critical growth exponent which is, in contrast to the case $0 \leq p < 1$, independent of the space dimension.

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Introduction

When investigating the Cauchy problem for the semilinear heat equation $u_t = \Delta u + u^q$, $q > 1$, the authors in [Fu] and [We] discovered that there is a critical exponent $q_{c,heat} = 1 + \frac{2}{n}$ having the property that

(C1)

- for $1 \leq q \leq q_{c,heat}$, there is no positive global (in time) solution and
- for $q > q_{c,heat}$, there are both global (small data) and non-global (large data) positive solutions, where the latter ones become unbounded in finite time.

Among the numerous answers to challenging questions on critical exponents in different situations studied since these pioneering works (see [Le] and [DL] for a survey) there are also some concerning *degenerate* parabolic equations such as the forced porous medium equation

$$v_t = \Delta v^{\sigma+1} + v^\beta, \quad \sigma > 0, \beta > 1. \tag{0.1}$$

After the transformation $u(x,t) := av^{\sigma+1}(bx,t)$, $a := (\sigma+1)^{\frac{\sigma+1}{\beta-1}}$, $b := (\sigma+1)^{\frac{\beta-\sigma-1}{2(\beta-1)}}$, this equation translates to

$$u_t = u^p \Delta u + u^q \tag{0.2}$$

with $p = \frac{\sigma}{\sigma+1} \in (0,1)$ and $q = \frac{\sigma+\beta}{\sigma+1} \in (1,\infty)$, and one of the results derived in [GKMS] reads as follows:

(C2)

- For $1 \leq q < q_{c,pme} := p + 1 + \frac{2}{n}(1-p)$ there are no global (positive) solutions.
- for $q > q_{c,pme}$, there are both global solutions and solutions blowing up in finite time.

The aim of the present work is to see what happens if we drop the restriction $p \in (0,1)$ in (0.2) by considering general positive p and thereby allowing the diffusion coefficient u^p in (0.2) to decrease more rapidly as $u \searrow 0$. More precisely, we shall examine the Cauchy problem

$$\begin{aligned} u_t &= u^p \Delta u + u^q \quad \text{in } \mathbb{R}^n \times (0,T), \\ u|_{t=0} &= u_0 \end{aligned} \tag{0.3}$$

with $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ positive and $p \geq 1$ as well as $q \geq 1$. That the exponent $p = 1$ indeed appears as some kind of turning point for the diffusion coefficient in degenerate parabolic equations not in divegence form is already indicated in [LDalP] and [Win1] where it is e.g. proved that the spatial support of (weak) solutions to (0.3) does not increase with time if $p \geq 1$. This behavior, drastically contrary to the case $p < 1$, may be interpreted as a consequence of the fact that near points where u is small, diffusion is weakened more effectively when $p \geq 1$. We will therefore not be too much surprised if the corresponding assertion on the interaction between the source term u^q and global solvability of (0.3) essentially differs from (C2) in that it shows for small q a significant tendency towards global existence. Roughly speaking, our main results can be formulated as follows:

(C3)

- For $1 \leq q < p+1$ (resp. $1 \leq q < \frac{3}{2}$ if $p = 1$), all positive solutions of (0.3) are global but unbounded, provided that u_0 decreases sufficiently fast in space (cf. Lemma 2.1 and Theorem 2.7).
- For $q = p+1$, all positive solutions blow up in finite time (Theorem 3.1).
- For $q > p+1$, there are both global and non-global positive solutions, depending on the size of u_0 (see Theorem 4.1).

It follows from (C3) that as in the previous cases there is a critical growth exponent $q_c = p+1$ for (0.3) which now, however, has a slightly different meaning and is independent of the space dimension n. Moreover, unlike the forced heat and porous medium equations, (0.2) for $p \geq 1$ has the property that this critical exponent would be the same if we replaced $\mathbb{R}^n$ with any smooth bounded domain $\Omega \subset \mathbb{R}^n$; namely, in this case the results in [Wi2] and in [Win1] imply global existence for $1 \leq q < p+1$ and the proof of Theorem 4.1 will show that both global existence and finite time blow-up may occur in Ω if $q > p+1$. The critical case $q = p+1$ in bounded domains is more subtle (cf. [FMcL], [Wi1], [Wi2], or [Win1]).
Unfortunately, returning to the Cauchy problem, we are not able to close the gap appearing for $p = 1$ between $q = \frac{3}{2}$ and $q = p+1$; we believe, however, that this is mainly due to technical difficulties, and that for $p = 1$ the behavior is actually the same as for larger p.

1 Existence and approximation of solutions

In this section we briefly collect some results on existence and approximation of a local-in-time solution to (0.3) under the assumption that

$(H0)$ $\quad u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive.

In order to approximate a solution, we write $B_R := B_R(0)$ for $R > 0$ and let, for $k \in \mathbb{N}$, $u_{0,k} \in C^1(\bar{B}_k)$ be such that $0 < u_{0,k} < u_{0,k+1}$ in B_k, $u_{0,k}|_{\partial B_k} = 0$ and $u_{0,k} \nearrow u_0$ in $\mathbb{R}^n$ as $k \nearrow \infty$. Then we have

Lemma 1.1 *There is $T \in (0, \infty]$ such that the problem*

$$\begin{aligned} \partial_t u_k &= u_k^p \Delta u_k + u_k^q \quad \text{in } B_k \times (0,T), \\ u_k|_{\partial B_k} &= 0, \\ u_k|_{t=0} &= u_{0,k} \end{aligned} \tag{1.1}$$

is uniquely solvable in $C^0(\bar{B}_k \times [0,T)) \cap C^{2,1}(B_k \times (0,T))$. The solution can be obtained as the $C^0_{loc}(\bar{B}_k \times [0,T)) \cap C^{2,1}_{loc}(B_k \times (0,T))$–limit of a decreasing sequence of solutions $u_{k,\varepsilon}$ of (1.1) with $u_{k,\varepsilon}|_{\partial B_k} = \varepsilon$ and $u_{k,\varepsilon}|_{t=0} = u_{0,k} + \varepsilon$, $\varepsilon \searrow 0$.
If $q < p+1$, we can choose $T = \infty$.

For $q < p+1$, the assertion is proved in [Wi2], Thm. 3.2, while for $q \geq p+1$ the proof is nearly identical to that of Lemma 1.1 in [Win2].
Throughout, we shall assume the u_k to be extended by zero to all of $\mathbb{R}^n \times [0,T)$. Taking $k \to \infty$ now yields a solution to (0.3), according to

Lemma 1.2 *There is $T_{max} \in (0, \infty]$ such that (0.3) admits a positive classical solution $u \in C^0(\mathbb{R}^n \times [0,T_{max})) \cap C^{2,1}(\mathbb{R}^n \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); L^\infty(\mathbb{R}^n))$. If u_k denotes the solution of (1.1), we have $u_k \to u$ in $C^0_{loc}(\mathbb{R}^n \times [0,T_{max})) \cap C^{2,1}_{loc}(\mathbb{R}^n \times (0,T_{max}))$.*

The proof can be carried out in almost exactly the same way as that of Lemma 1.2 in [Win2].

Concerning the question of uniqueness, we do not know a satisfactory answer covering all the cases we wish to consider below. However, let us at least remark that using the same ideas as in Lemma 1.4 in [Win2], one can achieve uniqueness of u (within suitable function classes), provided that

- $n \leq 2$, or
- $\sup_{t \in (0,T)} \int_{\mathbb{R}^n} u^\alpha(t) < \infty$ for all $T < T_{max}$ and some $\alpha = \alpha(T) > 0$, or
- $\lim_{R \to \infty} \|u(t)\|_{L^\infty(\partial B_R)} = 0$ for all $t \in [0, T_{max})$.

In particular, all the solutions to be discussed in Section 2 as well as the global solutions in Section 4 are unique. Unless otherwise stated we mean by 'the' solution u the limit $u = \lim_{k\to\infty} u_k$ from Lemma 1.2 which clearly is actually a *minimal solution* in the sense that $u \leq v$ for any positive classical solution v of (0.3).

2 The subcritical case $q < p + 1$

In a smooth bounded domain Ω, every positive solution of the initial-boundary value problem corresponding to (0.3) with zero Dirichlet data on $\partial\Omega$ exists globally and, as $t \to \infty$, approaches a u_0-independent steady state W which solves $\Delta W + W^{q-p} = 0$ in Ω, $W|_{\partial\Omega} = 0$ (cf. [Wi2] and [Win1]). In $\mathbb{R}^n$, however, (0.3) has no nontrivial equilibria, so that it seems to be a plausible guess that global positive solutions, if existing at all, must be unbounded. Indeed, we have

Lemma 2.1 *Every global positive solution u to (0.3) is unbounded in the sense that as $t \to \infty$, $u(t) \to \infty$ uniformly on compact subsets of $\mathbb{R}^n$.*

PROOF. For any $R > 0$, let Θ be the principal Dirichlet eigenfunction of $-\Delta$ in B_R corresponding to the first eigenvalue $\lambda_1 = \lambda_1(B_R) > 0$, with $\max\Theta = \frac{1}{2}$. Letting $v(x,t) := c_0\Theta(x)$ with $c_0 > 0$ small such that $\lambda_1 c_0^{p+1-q} \le 1$ and $c_0 < u_0$ in B_R, we have $v < u$ on ∂B_R and at $t = 0$, while

$$\begin{aligned} v_t - v^p\Delta v - v^q &= \lambda_1 c_0^{p+1}\Theta^{p+1} - c_0^q\Theta^q \\ &\le c_0^q\Theta^q(\lambda_1 c_0^{p+1-q} - 1) \\ &\le 0 \quad \text{in } B_R \times (0,\infty), \end{aligned}$$

hence $v \le u$ in $B_R \times (0,\infty)$ by comparison. It follows that for all $R > 0$ there is $c_0(R) > 0$ such that $u \ge c_0(R)$ in $B_R \times (0,\infty)$.

Next, defining $y_\infty(R) := \left(\frac{1}{2\lambda_1(B_R)}\right)^{\frac{1}{p+1-q}}$, we let $\delta \in (0, \frac{1}{2})$ be such that $\delta \le \frac{c_0(R)}{y_\infty(R)}$, and $y(t)$ be in $C^1([0,\infty))$ and fulfil $y(0) \le u_0$ in B_R, $0 \le y' \le \frac{\delta^{q-1}}{2}y^q$ in $(0,\infty)$ as well as $y(t) \to y_\infty(R)$ as $t \to \infty$. Then the function $w(x,t) := y(t)(\Theta + \delta)$ lies below u at $t = 0$ and, as $y\delta \le c_0(R)$, also on ∂B_R, while $y \le y_\infty(R)$ implies $\lambda_1 y^{p+1} \le \frac{1}{2}y^q$, so that

$$\begin{aligned} w_t - w^p\Delta w - w^q &= \Big(y' + \lambda_1 y^{p+1}(\Theta+\delta)^{p-1}\Theta - y^q(\Theta+\delta)^{q-1}\Big)(\Theta+\delta) \\ &\le \Big(y' + (\lambda_1 y^{p+1} - y^q)(\Theta+\delta)^{q-1}\Big)(\Theta+\delta) \\ &\le (y' - \frac{\delta^{q-1}}{2}y^q)(\Theta+\delta) \\ &\le 0 \quad \text{in } B_R \times (0,\infty). \end{aligned}$$

Thus, $w \le u$ in $B_R \times (0,\infty)$ by comparison, which shows $u(t) \ge \frac{1}{4}y_\infty(R)$ in the set $\{\Theta \ge \frac{1}{4}\}$ for t large enough. But as $\Theta(x) = f(|x|)$ with $f''(r) = -\frac{n-1}{r}f'(r) - \lambda_1 f(r) \ge -\lambda_1 f(r)$, it is easy to see that $\Theta(x) \ge \frac{1}{4}$ if $r \le \frac{1}{\sqrt{\lambda_1(B_R)}} =: r_0(R)$. Consequently, for any $K \subset\subset \mathbb{R}^n$ and $M > 0$ we can find $R > 0$ large such that $K \subset B_{r_0(R)}$ and $\frac{1}{4}y_\infty(R) \ge M$ and conclude by the above arguments that $u(t) \ge M$ in K for large t. ////

Accordingly, although each of the u_k exists for all times and converges to some W, it does not seem to be too promising to look for global bounds on u_k (or u); hence, the best we can hope for is that some quantity involving $u_k(t)$ does not increase too rapidly with t, uniformly in k. In spite of the absence of divergence structure in (0.3) (resp. (1.1), a testing procedure will turn out to be the key to success and show that an adequate quantity for our purpose is $\|u_k(t)\|_{L^\alpha(\mathbb{R}^n)}$ for *small* $\alpha > 0$, where we have set $\|v\|_{L^\alpha(\Omega)}^\alpha := \int_\Omega |v|^\alpha$ for measurable v. We shall therefore require small summability powers in the Gagliardo-Nirenberg interpolation inequality which for our purpose reads as follows.

Lemma 2.2 *Suppose $r_0 \in (0,2]$. Then there is a constant $c_0 = c_0(n, r_0) > 0$ such that for all $r \in [r_0, 2]$, any $s \in (0, \min\{1, r\})$ and all $\varphi \in L^s(\mathbb{R}^n)$ with $\nabla\varphi \in L^2(\mathbb{R}^n)$, the estimate*

$$\|\varphi\|_{L^r(\mathbb{R}^n)} \le c_0 \|\nabla\varphi\|^a_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-a}_{L^s(\mathbb{R}^n)} \tag{2.1}$$

holds, where the number $a \in (0,1)$ is defined by

$$-\frac{n}{r} = (1 - \frac{n}{2})a - \frac{n}{s}(1-a). \tag{2.2}$$

Proof. As $r < s \le 2$ and $s < 1 < 2$, Hölder's inequality gives

$$\|\varphi\|_{L^r(\mathbb{R}^n)} \le \|\varphi\|^b_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-b}_{L^s(\mathbb{R}^n)} \qquad \text{with} \qquad b = \frac{2(r-s)}{r(2-s)}$$

and

$$\|\varphi\|_{L^1(\mathbb{R}^n)} \le \|\varphi\|^c_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-c}_{L^s(\mathbb{R}^n)} \qquad \text{with} \qquad c = \frac{2(1-s)}{2-s}.$$

Using the standard Gagliardo-Nirenberg inequality (cf. [Ta], Ch. 3.4.), we infer that

$$\|\varphi\|_{L^2(\mathbb{R}^n)} \le c_1 \|\nabla\varphi\|^d_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-d}_{L^1(\mathbb{R}^n)}, \qquad \text{where} \quad d = \frac{n}{n+2}.$$

Combining these relations, we obtain

$$\|\varphi\|_{L^r(\mathbb{R}^n)} \le c_1^{\frac{b}{1-(1-d)c}} \|\nabla\varphi\|^{\frac{bd}{1-(1-d)c}}_{L^2(\mathbb{R}^n)} \|\varphi\|^{1-b+\frac{b(1-c)(1-d)}{1-(1-d)c}}_{L^s(\mathbb{R}^n)}. \tag{2.3}$$

As $\frac{1-s}{2-s} \le \frac{1}{2}$, we estimate

$$\begin{aligned}\frac{b}{1-(1-d)c} &= \frac{2}{s}\frac{r-s}{2-s}\Big(1 - \frac{4}{n+2}\frac{1-s}{2-s}\Big)^{-1} \\ &\le \frac{2}{s_0}\frac{n+2}{n},\end{aligned}$$

hence the constant in (2.3) is independent of $r \in [r_0, 2]$ and $s \in (0, \min\{1, r\})$. Now an elementary calculation shows that $\frac{bc}{1-(1-d)c}$ coincides with a and thus (2.1) follows. ////

The next auxiliary assertion on an integral inequality is elementary.

Lemma 2.3 *Let $T > 0$ and suppose $y \in C^0([0,T])$ satisfies*

$$y(t) \le y_0 + c_0 \int_0^t y^{1+\lambda}(s)ds \quad \forall\, t \in [0,T] \tag{2.4}$$

with positive numbers y_0, c_0 and λ. Then

$$y(t) \le y_0 \cdot (1 - \lambda y_0^\lambda c_0 t)^{-\frac{1}{\lambda}} \quad \forall\, t \in [0,T]. \tag{2.5}$$

Proof. The assertion will follow as soon as we have shown that for all $\varepsilon > 0$ and all $t \in [0,T]$,

$$y(t) < (y_0 + \varepsilon) \cdot (1 - \lambda(y_0+\varepsilon)^\lambda c_0 t)^{-\frac{1}{\lambda}} =: y_\varepsilon(t). \tag{2.6}$$

Indeed, for $t = 0$ this is obvious, hence if (2.6) were false there were $t_0 \in (0, T]$ such that $y(t) < y_\varepsilon(t)$ for all $t < t_0$ and $y(t_0) = y_\varepsilon(t_0)$. Noting that $y'_\varepsilon = c_0 y_\varepsilon^{1+\lambda}$, we thus obtain

$$\begin{aligned} y_\varepsilon(t_0) &= y_0 + \varepsilon + c_0 \int_0^{t_0} y_\varepsilon(s)^{1+\lambda} ds \\ &> y_0 + \varepsilon + c_0 \int_0^{t_0} y(s)^{1+\lambda} ds \\ &> y(t_0), \end{aligned}$$

a contradiction. ////

Basing upon the last two lemmas, the following one will be the main ingredient in Theorem 2.7. Before formulating it, we now state the announced decay condition on u_0 (see (C3)) which will finally imply global existence.

(H1) There is a radially symmetric $\varphi \in C^\infty(\mathbb{R}^n)$ with $R \mapsto \|\varphi\|_{L^\infty(\partial B_R)}$ nonincreasing such that $u_0 \le \varphi$ and

$$\int_{\mathbb{R}^n} \varphi^\alpha \le c\alpha^{-\Lambda_0 + \nu}$$

for some $\nu > 0$ and all sufficiently small $\alpha > 0$, where

$$\Lambda_0 := \begin{cases} \frac{n}{2} \cdot \frac{p+1-q}{q-1} & \text{if } p > 1, \\ \frac{n}{2} \cdot \frac{3-2q}{q-1} & \text{if } p = 1. \end{cases}$$

Remark. Hypothesis (H1) is fulfilled if $\Lambda_0 > 0$ (that is, $p > 1$ or $p = 1$ and $q < \frac{3}{2}$) and e.g.

$$u_0(x) \le c_1 e^{-c_2|x|^l} \quad \text{in } \mathbb{R}^n$$

for some $l > \frac{n}{\Lambda_0}$ and positive numbers c_1 and c_2.

Lemma 2.4 *Suppose $q > 1$ and (H1) holds. Then for all $T_0 > 0$ there exists $\alpha > 0$ and $C_0 > 0$ such that*

$$\sup_{t \in (0, T_0)} \|u_k(t)\|_{L^\alpha(\mathbb{R}^n)} \le C_0 \quad \forall k \in \mathbb{N}. \tag{2.7}$$

Proof. We multiply the equation defining $u_{k,\varepsilon}$ by $u_{k,\varepsilon}^{\alpha-1}$, $0 < \alpha < 1$ to be chosen later, and integrate over $B_k \times (\tau, t)$, $0 < \tau < t \le T_0$, to obtain

$$\begin{aligned} \frac{1}{\alpha} \int_{B_k} u_{k,\varepsilon}^\alpha(t) &+ (p + \alpha - 1) \int_\tau^t \int_{B_k} u_{k,\varepsilon}^{p+\alpha-2} |\nabla u_{k,\varepsilon}|^2 - \int_\tau^t \int_{\partial B_k} u_{k,\varepsilon}^{p+\alpha-1} \partial_N u_{k,\varepsilon} \\ &= \frac{1}{\alpha} \int_{B_k} u_{k,\varepsilon}^\alpha(\tau) + \int_\tau^t \int_{B_k} u_{k,\varepsilon}^{q+\alpha-1}. \end{aligned} \tag{2.8}$$

As $u_{k,\varepsilon} \geq \varepsilon$ in $B_k \times (0,\infty)$ by comparison and $u_{k,\varepsilon}|_{\partial B_k} = \varepsilon$, the third term on the left is nonnegative, while the second equals $\frac{4(p+\alpha-1)}{(p+\alpha)^2} \int_\tau^t \int_{B_k} |\nabla u_{k,\varepsilon}^{\frac{p+\alpha}{2}}|^2$. We now let τ and then ε tend to zero; taking into account that $u_{k,\varepsilon} \to u_k$ uniformly in $\bar{B}_k \times [0,t]$ and $\nabla u_{k,\varepsilon}^{\frac{p+\alpha}{2}} \to \nabla u_k^{\frac{p+\alpha}{2}}$ a.e. in $B_k \times (0,t)$, we gain from Fatou's Lemma that

$$\frac{1}{\alpha}\int_{B_k} u_k^\alpha(t) \quad + \quad (p+\alpha-1)\int_0^t \int_{B_k} u_k^{p+\alpha-2}|\nabla u_k|^2 \leq \frac{1}{\alpha}\int_{B_k} u_{0,k}^\alpha + \int_0^t \int_{B_k} u_k^{q+\alpha-1}. \tag{2.9}$$

If we define $v := u_k^{\frac{p+\alpha}{2}}$ then $u_k^\alpha = v^\gamma$ with $\gamma = \frac{2\alpha}{p+\alpha}$ and $u_k^{q+\alpha-1} = v^\delta$ with $\delta = \frac{2(q+\alpha-1)}{p+\alpha}$, and (2.9) reads

$$\frac{1}{\alpha}\|v(t)\|_{L^\gamma(B_k)}^\gamma + \frac{4(p+\alpha-1)}{(p+\alpha)^2}\int_0^t \|\nabla v(s)\|_{L^2(B_k)}^2 ds \leq \frac{1}{\alpha}\|v(0)\|_{L^\gamma(B_k)}^\gamma + \int_0^t \|v(s)\|_{L^\delta(B_k)}^\delta ds. \tag{2.10}$$

In order to take advantage from the gradient term on the left, we estimate by the Gagliardo-Nirenberg inequality, Lemma 2.2,

$$\|v(s)\|_{L^\delta(B_k)}^\delta \leq c_0^\delta \|\nabla v(s)\|_{L^2(B_k)}^{a\delta} \|v(s)\|_{L^\gamma(B_k)}^{(1-a)\delta}, \tag{2.11}$$

where

$$a = \frac{\frac{1}{\gamma} - \frac{1}{\delta}}{\frac{1}{n} - \frac{1}{2} + \frac{1}{\gamma}}.$$

Let us continue with the case $p > 1$ first. Then the coefficient of the gradient term in (2.10) is bounded below by $c_p := \frac{4(p-1)}{(p+1)^2} > 0$. To the right hand side of (2.11) we apply Young's inequality in the form

$$AB \leq \eta A^r + c(r,\eta) B^{\frac{1}{1-\frac{1}{r}}}, \quad \forall\, A, B > 0, \quad \text{where } c(r,\eta) := \frac{r-1}{r}(r\eta)^{-\frac{1}{r-1}}, \tag{2.12}$$

with $\eta := \frac{c_p}{c_0^\delta}$, $r := \frac{2}{a\delta}$. If $\alpha \to 0$ then also $\gamma \to 0$ and thus $a\delta \to \frac{2(q-1)}{p}$ and $r \to \frac{p}{q-1} > 1$, so that we may assume α to be small enough such that $c(r,\eta) \leq c_1 < \infty$, whence we have altogether

$$\|v(s)\|_{L^\delta(B_k)}^\delta \leq c_p \|\nabla v(s)\|_{L^2(B_k)}^2 + c_1 \Big(\|v(s)\|_{L^\gamma(B_k)}\Big)^{\frac{(1-a)\delta}{1-\frac{a\delta}{2}}}. \tag{2.13}$$

Inserting this into (2.10) and writing $y(t) := \|v(t)\|_{L^\gamma(B_k)}^\gamma \equiv \|u_k(t)\|_{L^\alpha(B_k)}^\alpha$, $t \in [0, T_0]$, we obtain

$$y(t) \leq y(0) + c_1\alpha \int_0^t y^{1+\lambda(\alpha)}(s) ds \tag{2.14}$$

with $\lambda(\alpha) := \frac{(1-a)\delta}{1-\frac{a\delta}{2}}\frac{1}{\gamma} - 1$. An elementary calculation reveals that

$$\lambda(\alpha) = \frac{q-1}{\frac{n}{2}(p+1-q)+\alpha} > 0,$$

and Lemma 2.3 guarantees

$$y(t) \leq y(0)\Big(1 - \lambda(\alpha) y^{\lambda(\alpha)}(0) \cdot c_1 \alpha t\Big)^{-\frac{1}{\lambda(\alpha)}}. \tag{2.15}$$

Since $\lambda(\alpha) \nearrow \lambda_0 := \frac{q-1}{\frac{n}{2}(p+1-q)}$ as $\alpha \searrow 0$, we can now choose $\alpha > 0$ fulfilling

$$T := \frac{1}{c^{\lambda(\alpha)} c_1 \lambda(\alpha) \alpha^{1-\frac{\lambda(\alpha)}{\lambda_0}+\nu\lambda(\alpha)}} \geq 2T_0, \tag{2.16}$$

where ν and c have been taken from hypothesis (H1). With this value of α fixed henceforth, (2.15) yields for $t \in [0, T_0]$ and all k

$$\begin{aligned}\int_{B_k} u_k^\alpha(t) &\leq \int_{B_k} u_0^\alpha \cdot \Big[1 - \frac{1}{2}\lambda(\alpha)\Big(\int_{B_k} u_0^\alpha\Big)^{\lambda(\alpha)} c_1 \alpha T\Big]^{-\frac{1}{\lambda(\alpha)}} \\ &\leq \int_{B_k} u_0^\alpha \cdot \Big[1 - \frac{1}{2}\lambda(\alpha)\Big(c\alpha^{-\frac{1}{\lambda_0}+\nu}\Big)^{\lambda(\alpha)} c_1 \alpha T\Big]^{-\frac{1}{\lambda(\alpha)}} \\ &\leq \int_{B_k} u_0^\alpha \cdot 2^{\frac{1}{\lambda(\alpha)}},\end{aligned}$$

which proves (2.7).

If $p = 1$, however, letting $\alpha \searrow 0$ means also taking the second coefficient $\frac{4\alpha}{(1+\alpha)^2}$ on the left of (2.10) to zero. We have to respect this in the choice of η in (2.12), so that we use $\eta := \frac{4\alpha}{(1+\alpha)^2 c_0^\delta}$. But then $c(r,\eta) \leq c_1 \alpha^{-\frac{1}{r-1}}$ needs no longer be bounded as $\alpha \searrow 0$. Fortunately, $r - 1 \to \frac{2-q}{q-1}$ so that at least for any $\xi > 0$ and all sufficiently small $\alpha < \alpha_0(\xi)$, we have $c(r,\eta) \leq c_1 \alpha^{-\frac{q-1}{2-q}-\xi}$. Let us set $\kappa := \frac{3-2q}{2-q}$ and fix $\xi \in (0, \frac{\nu\lambda_0}{2})$. Then with an obvious change in (2.13), we obtain (2.14) in the modified form

$$y(t) \leq y(0) + c_1 \alpha^{\kappa-\xi} \int_0^t y^{1+\lambda(\alpha)}(s) ds.$$

Accordingly, the final choice of α will be such that

$$\frac{1}{c^{\lambda(\alpha)} \cdot c_1 \lambda(\alpha) \alpha^{-\lambda(\alpha)\Lambda_0 + \nu\lambda(\alpha) + \kappa - \xi}} \geq 2T_0,$$

which is possible since $-\lambda(\alpha)\Lambda_0 + \nu\lambda(\alpha) + \kappa - \xi \to -\lambda_0\Lambda_0 + \nu\lambda_0 + \kappa - \xi > \frac{\nu\lambda_0}{2} > 0$ as $\alpha \to 0$. Now the remaining part of the proof is as before. ////

Unfortunately, using Lemma 2.4 alone we cannot exclude the case that e.g. u blows up at single points in finite time. However, if u_0 is radially symmetric and decreasing in $|x|$, Lemma 2.6 will show that this is impossible. Its proof relies on

Lemma 2.5 *For all $k \in \mathbb{N}$, we have*

$$\frac{\partial_t u_k}{u_k} \geq -\frac{1}{pt} \quad \textit{in } \mathbb{R}^n \times (0,\infty). \tag{2.17}$$

Proof. For fixed $\tau > 0$, classical regulartiy theory tells us that the approximate solutions $u_{k,\varepsilon}$ are in $C^{2,1}(\bar{B}_k \times [\tau,\infty))$, hence the function $z_{k,\varepsilon} := \frac{\partial_t u_{k,\varepsilon}}{u_{k,\varepsilon}} = u_{k,\varepsilon}^{p-1}\Delta u_{k,\varepsilon} + u_{k,\varepsilon}^{q-1}$ is in $C^0(\bar{B}_k \times [\tau,\infty))$ and fulfils

$$\begin{aligned}\partial_t z_{k,\varepsilon} &= (p-1)u_{k,\varepsilon}^{p-1}(\Delta u_{k,\varepsilon} + u_{k,\varepsilon}^{q-p}) z_{k,\varepsilon} + u_{k,\varepsilon}^{p-1}\Big(\Delta(u_{k,\varepsilon} z_{k,\varepsilon}) + (q-p)u_{k,\varepsilon}^{q-p} z_{k,\varepsilon}\Big) \\ &= p z_{k,\varepsilon}^2 + (q-p-1)u_{k,\varepsilon}^{q-1} z_{k,\varepsilon} + u_{k,\varepsilon}^{p-1}(2\nabla u_{k,\varepsilon} \cdot \nabla z_{k,\varepsilon} + u_{k,\varepsilon}\Delta z_{k,\varepsilon}).\end{aligned}$$

$z_{k,\varepsilon}$ vanishes at $\partial B_k \times [\tau, \infty)$, while at $t = \tau$, $z_{k,\varepsilon} \geq -M$ for all $M \geq M_\varepsilon$ and some sufficiently large $M_\varepsilon > 0$. Hence, by comparison, $z_{k,\varepsilon} \geq f_M$ on $B_k \times (\tau, \infty)$ for all $M \geq M_\varepsilon$, where $f_M(t)$ is the solution of $f_M' = pf_M^2$ on (τ, ∞), $f_M(\tau) = -M$, i.e. $f_M(t) = -\frac{1}{p(t-\tau)+M^{-1}}$; note here that f_M is negative, so that $(q-p-1)u_{k,\varepsilon}^{q-1} f_M \geq 0$. Consequently, $z_{k,\varepsilon} \geq -\frac{1}{p(t-\tau)}$ on $B_k \times (\tau, \infty)$ for all $\tau > 0$, hence also $z_{k,\varepsilon} \geq -\frac{1}{pt}$ on $B_k \times (0, \infty)$. Taking $\varepsilon \to 0$, we arrive at (2.17). ////

Lemma 2.6 *If $u = \lim_{k\to\infty} u_k$ is radially symmetric and decreasing in $|x|$ for each t then either u exists globally or there is $T < \infty$ and a sequence $t_j \nearrow T$ such that $u(x, t_j) \to \infty$ for all $x \in \mathbb{R}^n$. In other words, for such a solution finite time blow-up occurs either nowhere or everywhere in $\mathbb{R}^n$.*

Proof. If u does not exist globally, there is $T < \infty$ and a sequence $t_j \nearrow T$ such that $u(0, t_j) \nearrow \infty$. For fixed t_j, define a function f on $[0, \infty)$ by $u(x, t_j) =: f(|x|)$. From Lemma 2.5 we infer $\Delta u(t_j) \geq -\frac{1}{pt_j} u^{1-p}(t_j) - u^{q-p}(t_j)$ and thus $f''(r) + \frac{n-1}{r} f'(r) \geq -\frac{1}{f^p(r)}(c_0 f(r) + f^q(r))$ with $c_0 \geq \frac{1}{pt_j}$ independent of j. Writing $a := f(0)$, the claim will follow as soon as we have shown that $f \geq \frac{a}{2}$ on an interval $[0, r_a]$ of length r_a which satisfies $r_a \to \infty$ as $a \to \infty$.
To see this, suppose $f(r_a) = \frac{a}{2}$ for some $r_a < \infty$ – if no such r_a exists, we are done. As $f' \leq 0$ and $f'(0) = 0$, we have

$$\begin{aligned} f(r_a) = \frac{a}{2} &= a + \int_0^{r_a} \int_0^r f''(\rho) d\rho dr \\ &\geq a - \int_0^{r_a} \int_0^r \frac{1}{f^p(\rho)} \Big(c_0 f(\rho) + f^q(\rho) \Big) d\rho dr \\ &\geq a - \Big(\frac{2}{a}\Big)^p (c_0 a + a^q) \frac{r_a^2}{2}, \end{aligned}$$

hence $r_a^2 \geq 2^{-p} \frac{a^{p+1-q}}{1+c_0 a^{1-q}}$ which yields the assertion, for $p+1 > q$. ////

Now we have collected all the tools to be used in

Theorem 2.7 *Suppose $p \geq 1$ and $1 \leq q < p+1$ with $q < \frac{3}{2}$ if $p = 1$. If either $q = 1$ or u_0 satisfies (H1) then the solution $u = \lim_{k\to\infty} u_k$ exists globally.*

Proof. Without loss of generality we may assume $q > 1$ since in the case $q = 1$ it is easily seen by comparison that $u_k(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} e^t$ uniformly in k, so that u clearly exists globally. As u lies below a radially symmetric and (with respect to $|x|$) nonincreasing solution with initial value satisfying (H1), we may furthermore restrict ourselves to the case that u itself has these properties.
Suppose such a u blew up at some time $T < \infty$. We apply Lemma 2.4 with $T_0 := T+1$ and obtain $\alpha > 0$ and $C_0 > 0$ such that

$$\|u_k(t)\|_{L^\alpha(B_k)} \leq C_0 \quad \forall\, t \in (0, T+1). \tag{2.18}$$

Let M be any number larger than $4C_0$ and $\tau := \min\{\frac{T}{2}, p\frac{T}{4} \ln 2, 1\}$. As u blows up in all of $\mathbb{R}^n$ by Lemma 2.6, there is $t_0 \in [T-\tau, T)$ such that $u(x, t_0) > M$ where x is any point on

∂B_r and r is – for convenience – such that $|B_r| = 1$. As $u_k \nearrow u$, we find $k_0 \in \mathbb{N}$ such that $u_k(x, t_0) \geq M$ for all $k \geq k_0$. Now Lemma 2.5 tells us that $\partial_t u_k(x,t) \geq -\frac{1}{p\frac{T}{2}} u_k(x,t)$ and thus $u_k(x,t) \geq Me^{-\frac{2}{pT}(t-t_0)}$ for all $t \geq t_0$ and all $k \geq k_0$. Hence if $t \in (T, T+\tau)$, we have $t - t_0 \leq (T+\tau) - (T-\tau) = 2\tau \leq p\frac{T}{2} \ln 2$ and therefore $u_k(x,t) \geq \frac{M}{2}$. By radial symmetry and monotonicity,

$$u_k(t) \geq \frac{M}{2} \quad \text{on } B_r \text{ for } k \geq k_0 \text{ and } t \in (T, T+\tau).$$

But then we have for such t and $k \geq k_0$

$$\Big(\int_{B_k} u_k^\alpha(t)\Big)^{\frac{1}{\alpha}} \geq \Big(\int_{B_r} (\frac{M}{2})^\alpha\Big)^{\frac{1}{\alpha}} = \frac{M}{2} > 2C_0,$$

which contradicts (2.18). ////

3 The critical case $q = p + 1$

If $q = p+1$, it follows from the results in [Wi1] that for large k (such that the first eigenvalue of $-\Delta$ in B_k with zero Dirichlet boundary data is less than one), u_k blows up in finite time, no matter how small (but positive) $u_{0,k}$ has been chosen. As, by comparison, $u \geq u_k$ in B_k for *any* solution u of (0.3), we can state without further comment

Theorem 3.1 *Suppose $q = p+1$. Then any positive solution of (0.3) blows up in finite time.*

4 The supercritical case $q > p + 1$

Rewriting (0.2) as $u_t = \frac{1}{p+1}\Delta u^{p+1} - pu^{p-1}|\nabla u|^2 + u^q$ and using an equivalent version of (C2) identifying $\beta = \sigma + 1 + \frac{2}{n}$ as critical exponent in (0.1), it is easy to see by a comparison argument that (0.3) has global solutions evolving from sufficiently small initial data, provided $q > p+1+\frac{2}{n}$. However, this reduction to a problem similar to (0.1) does neither – at least not immediately – give us any information about the gap $q \in (p+1, p+1+\frac{2}{n}]$, nor does it clarify whether we can expect blow-up for large data. The L^α-approach, having been successful in the subcritical case yet, seems to fail as well. Alternatively, we shall look for explicit global solutions on the one hand and on the other hand attempt to prove blow-up in the case of large data by an energy-type method as performed e.g. in [FMcL] in a slightly different setting.

Theorem 4.1 *Let $q > p + 1$.*
i) There exists a one-parameter family $(w_a)_{a>0}$ of radially symmetric positive functions $w_a(x)$ vanishing at infinity with $w_a(0) = a$ such that whenever u_0 satisfies (H0) and $u_0 \leq w_a$ in $\mathbb{R}^n$ then the corresponding solution u exists globally and obeys the decay estimate

$$\|u(t)\|_{L^\infty(\mathbb{R}^n)} \leq c(1+t)^{-\frac{1}{q-1}}.$$

ii) For each w satisfying (H0) there is $b > 0$ such that if $u_0 = bw$ then any positive classical solution u evolving from u_0 blows up in finite time.

PROOF. i) Let us look for radially symmetric similarity solutions to (0.3) of the form

$$u(x,t) := (1+t)^{-\alpha} f((1+t)^{-\beta}|x|) \tag{4.1}$$

with positive α and β to be determined. Abbreviating $r := (1+t)^{-\beta}|x|$, we have for such u

$$\begin{aligned}
u_t - u^p\Delta u - u^q &= -\alpha(1+t)^{-\alpha-1}f(r) - \beta(1+t)^{-\alpha-\beta-1}|x|f'(r) \\
&\quad -(1+t)^{-p\alpha}f^p(r)\Big((1+t)^{-\alpha-2\beta}f''(r) - (1+t)^{-\alpha-\beta}\frac{n-1}{|x|}f'(r)\Big) \\
&\quad -(1+t)^{-q\alpha}f^q(r) \\
&= -\alpha(1+t)^{-\alpha-1}f(r) - \beta(1+t)^{-\alpha-1}rf'(r) \\
&\quad -(1+t)^{-p\alpha-\alpha-2\beta}f^p(r)(f''(r) + \frac{n-1}{r}f'(r)) - (1+t)^{-q\alpha}f^q(r).
\end{aligned}$$

If $q\alpha = \alpha+1$ and $p\alpha+\alpha+2\beta = \alpha+1$, i.e. $\alpha = \frac{1}{q-1}$ and $\beta = \frac{q-p-1}{2(q-1)}$, all time exponents are equal, so that u solves the first in (0.3) if and only if f is a positive solution on $(0,\infty)$ of the initial value problem

$$\begin{aligned}
f'' + \frac{n-1}{r}f' + \frac{1}{f^p}(\beta r f' + \alpha f + f^q) &= 0, \quad r\in(0,\infty), \\
f(0) = a, \quad f'(0) &= 0,
\end{aligned} \tag{4.2}$$

with some $a > 0$. Therefore the claim of part i) of the theorem follows if we show that for any $a > 0$, (4.2) has a positive solution $f \in C^2([0,\infty))$. We first prove local solvability of (4.2) near $r = 0$ by rewriting the differential equation as $\frac{1}{r^{n-1}}(r^{n-1}f')' = g(r,f,f')$ with g smooth near the point $(0,a,0)$, and considering the equivalent integral equation

$$f(r) = a + \int_0^r \frac{1}{\rho^{n-1}} \int_0^\rho \sigma^{n-1} g(\sigma, f(\sigma), f'(\sigma))d\sigma d\rho$$

which is solved by standard fixed point arguments in the space $C^1([0,R])$ with sufficiently small $R > 0$. A posteriori, $\tilde g(\sigma) := g(\sigma,f(\sigma),f'(\sigma))$ is continuous at $\sigma = 0$, hence

$$\Big|\frac{1}{r}f'(r) - \frac{\tilde g(0)}{n}\Big| = \Big|\frac{1}{r^n}\int_0^r \sigma^{n-1}(\tilde g(\sigma) - \tilde g(0))d\sigma\Big| \le \frac{1}{n}\max_{\sigma\in[0,r]}|\tilde g(\sigma) - \tilde g(0)| \to 0 \quad \text{as } r\to 0.$$

Thus, f'' is continuous at $r = 0$ and $f \in C^2([0,R])$. Moreover, $f''(0) = \lim_{r\to 0}(-\frac{n-1}{r}f'(r) + \tilde g(r)) = \frac{1}{n}\tilde g(0) < 0$, so that f decreases near $r = 0$ and hence as long as being positive, for (4.2) shows that f cannot have a positive local minimum. Thus, there is a maximal $R \le \infty$ such that f exists and remains strictly positive on $(0,R)$. To see that actually $R = \infty$, suppose on the contrary that $R < \infty$ and consider the case $p > 1$ first. Letting $\varepsilon := \frac{\beta}{p-1}(\frac{R}{2})^n$, we observe that for $r \in (\frac{R}{2}, R)$, the function $\varphi(r) := \varepsilon f^{1-p}(r) - r^{n-1}f'(r)$ satisfies

$$\begin{aligned}
\frac{1}{r^{n-1}}\varphi' &= \Big(\beta r - \frac{\varepsilon(p-1)}{r^{n-1}}\Big)\frac{f'}{f^p} + \alpha f^{1-p} + f^{q-p} \\
&\le f^{1-p}(\alpha + a^{q-1}).
\end{aligned}$$

Thus, writing $\gamma := \frac{R^{n-1}(\alpha+a^{q-1})}{\varepsilon}$ and noting that $\varepsilon f^{1-p} \le \varphi$, we obtain

$$\varphi' \le \gamma\varphi,$$

so that $\varphi(r) \le \varphi(\frac{R}{2}) e^{\gamma(r-\frac{R}{2})}$ for all $r \in (\frac{R}{2}, R)$. In this interval we therefore have

$$|f'| \le c \quad \text{and} \quad c^{-1} \le f \le a,$$

which contradicts the maximality of R.
In the remaining part $p = 1$, we proceed similarly, using that $\varphi(r) := -\varepsilon \ln f(r) - r^{n-1} f'(r)$, with $\varepsilon := \beta(\frac{R}{2})^n$, satisfies $\varphi'(r) \le r^{n-1}(\alpha + f^{q-1}) \le c$ on $(\frac{R}{2}, R)$.
Let us finally show that $f(r) \to 0$ as $r \to \infty$ and thereby complete the proof of part i). Indeed, suppose that we had $f \ge \delta > 0$ on $(0, \infty)$. Then, by (4.2),

$$f'' = -\Big(\frac{n-1}{r} + \frac{\beta}{f^p} r\Big) f' - \frac{\alpha f + f^q}{f^p} \le -2c_1 r f' - c_2 \quad \text{for } r \ge 1$$

with positive numbers c_1 and c_2. An integration of the differential equation $y'(r) = -2c_1 r y(r) - c_2$ leads to

$$f' \le f'(1) e^{-c_1(r^2-1)} - c_2 \int_1^r e^{c_1[\tau^2 - r^2]} d\tau.$$

We estimate the second integral as follows:

$$\int_1^r e^{c_1(\tau - r)(\tau + r)} d\tau \ge \int_1^r e^{2c_1 r(\tau - r)} d\tau = \frac{1}{2c_1 r}(1 - e^{-2c_1 r(r-1)}).$$

Hence, for r_0 sufficiently large and some $c_3 > 0$,

$$f'(r) \le -\frac{c_3}{r} \quad \forall\, r \ge r_0,$$

so that $f(r) \le f(1) - c_3 \ln \frac{r}{r_0}$ for $r > r_0$, implying $f(r) \to -\infty$ as $r \to \infty$ which is again absurd. Thus, $f(r) \to 0$ as $r \to \infty$.

ii) We fix an arbitrary smooth bounded domain $\Omega \subset \mathbb{R}^n$ with principal eigenvalue λ_1 of $-\Delta$ and a corresponding eigenfunction $\Theta \ge 0$ with $\int_\Omega \Theta = 1$. Considering $p > 1$ first, we suppose u exists for $t \le T$ and let $y(t) := \frac{1}{p-1} \int_\Omega u^{1-p}(t) \Theta$, $t \in [0, T]$. Then $y \in C^0([0, T]) \cap C^1((0, T])$ and since $\partial_N \Theta|_{\partial\Omega} < 0$, we have

$$\begin{aligned} y' = -\int_\Omega \frac{u_t}{u^p} \Theta &= -\int_\Omega \Delta u \cdot \Theta - \int_\Omega u^{q-p} \Theta \\ &\le \lambda_1 \int_\Omega u \Theta - \int_\Omega u^{q-p} \Theta. \end{aligned} \tag{4.3}$$

We claim that as long as $y \le \frac{1}{p-1} (\frac{1}{2\lambda_1})^{\frac{p-1}{q-p-1}} =: c_0$, we have

$$y' \le -\lambda_1 (p-1)^{-\frac{1}{p-1}} y^{-\frac{1}{p-1}}, \tag{4.4}$$

from which it will follow that if b is so large that $y_0 := \Big(\frac{1}{p-1} \int_\Omega w^{1-p} \Theta\Big) b^{1-p} \le c_0$ then y decreases and hence remains below c_0 for $t \in [0, T]$; an integration of (4.4) shows that then

$$y(t) \le (y_0^{\frac{p}{p-1}} - c_1 t)^{\frac{p-1}{p}}$$

with $c_1 := \lambda_1 p(p-1)^{-\frac{p}{p-1}}$, and thus T cannot exceed $c_1 y_0^{-\frac{p}{p-1}} < \infty$, that is, u becomes unbounded in finite time.

To prove (4.4), we observe first that due to the Hölder inequality,

$$\int_\Omega u^{1-p}\Theta \geq \Big(\int_\Omega u\Theta\Big)^{1-p} \quad \text{and} \quad \int_\Omega u^{q-p}\Theta \geq \Big(\int_\Omega u\Theta\Big)^{q-p}, \tag{4.5}$$

where we have used $\int_\Omega \Theta = 1$, so that if $y \leq c_0$ then

$$\begin{aligned}\int_\Omega u^{q-p}(t)\Theta &\geq \Big(\int_\Omega u\Theta\Big)^{q-p-1}\int_\Omega u\Theta \\ &\geq \Big(\int_\Omega u^{1-p}\Theta\Big)^{-\frac{q-p-1}{p-1}}\int_\Omega u\Theta \\ &= [(p-1)y]^{-\frac{q-p-1}{p-1}}\int_\Omega u\Theta \\ &\geq 2\lambda_1\int_\Omega u\Theta.\end{aligned}$$

Hence, by (4.3) and (4.5),

$$\begin{aligned} y' &\leq -\lambda_1\int_\Omega u\Theta \leq -\lambda_1\Big(\int_\Omega u^{1-p}\Theta\Big)^{-\frac{1}{p-1}} \\ &= -\lambda_1(p-1)^{-\frac{1}{p-1}}y^{-\frac{1}{p-1}},\end{aligned}$$

as claimed.

If $p = 1$, we let $y(t) := -\int_\Omega \ln u(t)\Theta$ and the proof is similar: Using Hölder's and Jensen's inequalities in estimating $\int_\Omega u^{q-1}\Theta \geq (\int_\Omega u\Theta)^{q-1}$ and $\int_\Omega u\Theta \geq e^{\int_\Omega \ln u\cdot\Theta}$, we see as above that as long as $y \leq \ln(\frac{1}{2\lambda_1})^{\frac{1}{q-2}} =: c_2$, we have $\int_\Omega u\Theta \leq \frac{1}{2\lambda_1}\int_\Omega u^{q-1}\Theta$ and thus

$$y' \leq \lambda_1\int_\Omega u\Theta - \int_\Omega u^{q-1}\Theta \leq -\lambda_1\int_\Omega u\Theta \leq -\lambda_1 e^{\int_\Omega \ln u\cdot\Theta} = -\lambda_1 e^{-y},$$

so that if $y(0) \leq c_2$ – which is true for all sufficiently large b – then $e^{y(t)} \leq e^{y(0)} - \lambda_1 t$ which shows $T \leq \frac{1}{\lambda_1}e^{y(0)}$. ////

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