Nonuniqueness in the quenching problem

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Abstract

This paper deals with nonnegative solutions of

$$u_t = \Delta u - u^{-q}\chi_{\{u>0\}} \qquad \text{in } \Omega \times (0,\infty) \qquad \text{(Q)}$$

with $q \in (0,1)$ and prescribed continuous Dirichlet data $B = B(x)$ on $\partial\Omega$. It is proved that for $n \le 6$ there is a critical parameter $q_c \in [0,1)$ with the following property: If $q > q_c$ then there exist at least two continuous weak solutions emanating from some explicitly known stationary solution w: one that coincides with w and another one that satisfies $u \ge w$ but $u \not\equiv w$. For $n \le 6$ and $q \le q_c$ (or $n \ge 7$), however, such a second solution above w is impossible.
Moreover, it is shown that for $n \le 6$, $q > q_c$ and *any* sufficiently small nonnegative boundary data B there exist initial values admitting at least two continuous weak solutions of (Q). The final result asserts that for any n and q nonuniqueness for (Q) holds at least for *some* boundary and initial data.

MSC 2000: 35K57, 35K60, 35R25
Keywords: Singular absorption, nonuniqueness, critical exponent

1 Introduction

This paper deals with the semilinear parabolic boundary value problem with singular absorption,

$$\begin{aligned} u_t &= \Delta u - u^{-q}\chi_{\{u>0\}} \qquad \text{in } \Omega \times (0,\infty), \\ u|_{\partial\Omega} &= B(x), \\ u|_{t=0} &= u_0(x). \end{aligned} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $q \in (0,1)$ is a fixed parameter and $\chi_{\{u>0\}}$ denotes the characteristic function of the set of points where u is positive. The initial data u_0 and the boundary function B are supposed to be nonnegative and continuous and throughout we assume that the compatibility condition

$$u_0|_{\partial\Omega} = B \tag{1.2}$$

holds.
It was shown in [19] that if B is a positive constant then (1.1) admits at least one nonnegative continuous weak solution. Recently, the requirement $B > 0$ could be relaxed in [4], where it has been proved that also in the case $B \equiv 0$ there exists a nonnegative weak solution; however, it was left open there if such solutions are continuous up to $\partial\Omega$.
Several qualitative aspects of solutions to (1.1) with general $q > -1$ have been explored during the last decades. Most of them concerned the possibility or the impossibility of dead cores to occur in finite or in infinite time ([17], [6], [7]), or quenching rates and profiles ([15], [16], [8]). Also, various results on the evolution of the positivity set and on the large time behavior of solutions are available ([5], [2], [1], [9], [12], [10], [3]).
A crucial role in many of these results is played by stationary solutions and especially by the singular steady state $w_{n,q}$ that is explicitly given by

$$w_{n,q}(x) := A_{n,q}|x|^{\frac{2}{q+1}},$$

where

$$A_{n,q} := \Big[\frac{2}{q+1}\Big(\frac{2}{q+1} + n - 2\Big)\Big]^{-\frac{1}{q+1}}.$$

Observe that $w_{n,q} \in C^{1+\theta}(\bar{\Omega})$ with $\theta = \frac{1-q}{q+1} \in (0,1)$, but if $0 \in \Omega$ then $w_{n,q} \notin C^2(\Omega)$, whence $w_{n,q}$ is a continuous weak solution, but not a classical solution of (1.1).
As far as the uniqueness question for (1.1) is concerned, very little is known up to now: For instance, it is easy to see upon a standard manipulation of the nonlinearity in (1.1) that weak solutions are unique (and classical) under the extra assumption $u \ge c > 0$. Similary, uniqueness of weak solutions in the neighboring case $q = -\alpha \in (-1, 0]$ follows from a straightforward argument; then, namely, the nonlinearity $u \mapsto u^\alpha \chi_{\{u>0\}}$ is nondecreasing in u and hence well-behaved with respect to the comparison principle, although it also causes typical effects of strong absorption ([5], [3]). By a more subtle reasoning, Dávila and Montenegro in [4] achieved uniqueness for (1.1) with $B \equiv 0$ within the class of functions satisfying $u(x,t) \ge c(\text{dist}\,(x, \partial\Omega))^\gamma$ for some $\gamma < \frac{2}{q+1}$ and $c > 0$.
The principal goal of the present work is to prove that for a large class of boundary functions B, weak solutions of (1.1) are in general not unique, even if they are assumed to be continuous. Assuming henceforth that

$$0 \in \Omega$$

and writing

$$q_c(n) := \begin{cases} 0 & \text{if } n \le 2, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(10-n)} & \text{if } 3 \le n \le 6, \end{cases} \tag{1.3}$$

we can formulate the first of our main results as follows.

Theorem 1.1 *If $n \le 6$ and $q \in (0,1)$ satisfies $q > q_c(n)$ then for any nonnegative initial data $u_0 \in C^0(\bar\Omega) \cap C^2(\bar\Omega \setminus \{0\})$ fulfilling*

$$u_0 \le w_{n,q} \quad \text{in } \bar\Omega \qquad \text{and} \qquad u_0 \equiv w_{n,q} \quad \text{near } x = 0,$$

there exist at least two distinct continuous weak solutions $u, \tilde u$ of (1.1). These satisfy

$$u(0,t) = 0 \quad \forall\, t > 0 \qquad \text{and} \qquad \tilde u(0,t) > 0 \quad \text{for small } t > 0. \tag{1.4}$$

Particularly, the continuous weak solution $u \equiv w_{n,q}$ with initial value $u_0 = w_{n,q}$ is not unique.

Observe that Theorem 1.1 implies that for any continuous boundary data satisfying $0 \le B \le w_{n,q}|_{\partial\Omega}$, there exist many initial functions for which (1.1) has more than one solution. The exponent $q_c(n)$ indeed appears to be critical with respect to uniqueness. Namely, we have

Lemma 1.2 *Suppose $n \ge 3$ and*

$$\begin{array}{ll} 0 < q \le q_c(n) & \text{if } 3 \le n \le 6, \\ 0 < q < 1 & \text{if } n \ge 7. \end{array} \tag{1.5}$$

Then the solution $u \equiv w_{n,q}$ of (1.1) with $u_0 \equiv w_{n,q}$ is unique within the class

$$\Big\{ v \mid v \text{ is a continuous weak solution of (1.1) with } v \ge w_{n,q} \text{ in } \Omega \times (0,\infty) \Big\};$$

equivalently, $u \equiv w_{n,q}$ is the maximal solution emanating from $w_{n,q}$.

However, if we allow larger zero sets of u_0 (measured e.g. in terms of their Hausdorff dimension), we can prove nonuniqueness for arbitrary n and $q \in (0,1)$ – but then need more technical restrictions on the initial and boundary data:

Theorem 1.3 *Assume $n \ge 1$, $q \in (0,1)$ and*

$$u_0(x) = w_{N,q}(x_1, ..., x_N) \qquad \forall\, x = (x_1, ..., x_n) \in \bar\Omega$$

with some $N \le \min\{n, 6\}$ satisfying $q_c(N) < q$. Then (1.1) has at least two continuous weak solutions with the properties (1.4).

In the course of our analysis we will also prove that the problem (1.1) is at least partially well-posed in the sense that

- for any nonnegative and continuous u_0 and B satisfying (1.2), (1.1) possesses a unique *maximal* continuous weak solution (Theorem 3.5), and that
- if $(u_{0k})_{k\in\mathbb{N}} \subset C^0(\bar{\Omega})$ is a sequence of nonnegative initial data, uniformly convergent to some u_0, then a subsequence of the corresponding sequence of maximal solutions converges locally uniformly in $\bar{\Omega} \times [0,\infty)$ to a continuous weak solution emanating from u_0 (Lemma 3.4).

Unfortunately, the latter limit need not coincide with the maximal solution corresponding to the initial value u_0, as the proof of Theorems 1.1 and 1.3 will show. As a consequence,

- the maximal continuous weak solution does not depend continuously on the initial data

when we measure distances of initial data in $C^0(\bar{\Omega})$ and distances between solutions in any reasonable space that separates continuous functions u and $\tilde{u}$ with the properties (1.4).
Anyhow, this supplements the previously established existence theory for (1.1) in so far as it now provides *continuous* weak solutions for arbitrary – not necessarily positive – boundary data.
We remark that the nonuniqueness result in Theorem 1.1 has an analogue in the study of the Cauchy problem for $u_t = \Delta u + u^p$ with $n \ge 3$ and $p > \frac{n}{n-2}$. For this problem, namely, it is known that there is a critical exponent

$$p_c := \begin{cases} \infty & \text{if } n \le 10, \\ \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \ge 11, \end{cases}$$

which has, apart from a number of other interesting features (cf. [13], [14], [20]), the following property ([11]): If $\frac{n}{n-2} < p < p_c$ then there exist at least two distinct weak solutions evolving from the explicit singular steady state $w(x) = L|x|^{-\frac{2}{p-1}}$, where L is given by $L = (\frac{2}{p-1}(n-2-\frac{2}{p-1}))^{\frac{1}{p-1}}$: One of these is $\tilde{u} \equiv w$ and another one is a selfsimilar solution u that is smooth for $t > 0$, so that particularly $u(0,t) < \infty$ for all $t > 0$. When $n \ge 11$ and $p \ge p_c$, however, such a self-similar solution does not exist, and moreover the solution $u \equiv w$ is the (unique) minimal solution (which is sometimes also called the 'proper' solution; cf. [11, Theorem 10.1]). Observe that if we extend the definition of q_c to $n \ge 11$ then we have $p_c = -q_c$ for such n.
Let us finally mention that for the degenerate counterpart of the PDE in (1.1), $u_t = u^p\Delta u - u^{-q}\chi_{\{u>0\}}$, $p > 0$, it was proved in [23] that nonuniqueness of continuous weak

solutions holds for $p \geq 1$ and $n = 1$ in the whole range $q \in (-1, p-1]$ where continuous solutions exist ([22]). However, nothing seems to be known in the intermediate case $p \in (0,1)$, that is, when the degeneracy is of porous-medium type.

The paper is organzied as follows: In Section 2 we pursue some formal ideas based on a self-similar ansatz. This will on the one hand indicate the criticality of q_c, but on the other hand our approach will also give useful suggestions how to construct nonstationary solutions evolving from steady states. In the rather technical proofs of Section 3 we assert some results on existence and convergence of continuous weak solutions for arbitrary nonnegative initial and boundary data. We also derive a useful estimate from below near $t = 0$ there (see Lemma 3.3) which will enable us to deal with initial data $u_0 \not\equiv w$ in Theorem 1.1. Section 4 is devoted to a rigorous study of the ordinary differential inequality proposed by our formal considerations. This prepares us to prove Theorems 1.1 and 1.3 and Lemma 1.2 in Section 5.

2 Formal analysis

Before going into technical details, let us first demonstrate the basic ideas of our approach. We assume for simplicity that u_0 coincides with $w \equiv w_{n,q}$ in the whole domain and seek for a nonstationary solution u of the form

$$u(x,t) = w(x) + U(x,t),$$

which leads us to the equation

$$U_t = \Delta U + w^{-q} - (w+U)^{-q} \tag{2.1}$$

wherever u is positive, because $\Delta w = w^{-q}$. Let us pursue the self-similar ansatz

$$U(x,t) = t^{\alpha} \cdot f(t^{-\beta}|x|).$$

Such a function would solve (2.1) if and only if

$$\alpha t^{\alpha-1} f(\xi) - \beta t^{\alpha-1} \xi f'(\xi) = t^{\alpha-2\beta}(f''(\xi) + \tfrac{n-1}{\xi} f'(\xi)) + w^{-q}(x) - (w(x) + t^{\alpha} f(\xi))^{-q}, \tag{2.2}$$

where $\xi = t^{-\beta}|x|$, so that the only reasonable choice appears to be $\beta = \frac{1}{2}$. Upon this, we now make use of the fact that

$$w(x) = A_{n,q}|x|^{\frac{2}{q+1}} = A_{n,q}(t^{\frac{1}{2}}\xi)^{\frac{2}{q+1}} = t^{\frac{1}{q+1}} w(\xi).$$

Thus, (2.2) turns into

$$t^{\alpha-1}\Big[f''(\xi)+\frac{n-1}{\xi}f'(\xi)+\frac{\xi}{2}f'(\xi)-\alpha f(\xi)\Big]$$
$$+\,t^{-\frac{q}{q+1}}w^{-q}(\xi)-t^{-\frac{q}{q+1}}\Big(w(\xi)+t^{\alpha-\frac{1}{q+1}}f(\xi)\Big)^{-q}=0,$$

which recommends the choice $\alpha=\frac{1}{q+1}$. In this case, f should satisfy

$$L_0 f := f''+\frac{n-1}{\xi}f'+\frac{\xi}{2}f'-\frac{1}{q+1}f+w^{-q}-(w+f)^{-q}=0, \qquad (2.3)$$

the suppressed argument being ξ now everywhere.
In order to investigate the structure of the operator L_0, let us proceed to transform f via

$$f(\xi)=h(\xi)\cdot w(\xi),$$

the function h supposedly having small values such that u remains nonnegative. Then (2.3) holds if h satisfies

$$\begin{aligned}
0 &= h''w+2h'w'+hw''+\frac{n-1}{\xi}\Big(h'w+hw'\Big)+\frac{\xi}{2}\Big(h'w+hw'\Big)-\frac{1}{q+1}hw \\
&\quad +w^{-q}-\Big(1+h\Big)^{-q}w^{-q} \\
&= w\cdot\Bigg[h''+\frac{n-1}{\xi}h'+2\frac{w'}{w}h'+\frac{\xi}{2}h'-\frac{1}{q+1}h+\frac{\xi}{2}\frac{w'}{w}h \\
&\qquad +\frac{(w''+\frac{n-1}{\xi}w')h+w^{-q}-(1+h)^{-q}w^{-q}}{w}\Bigg].
\end{aligned}$$

Since $w''+\frac{n-1}{\xi}w'=w^{-q}$ and $\frac{w'}{w}=\frac{2}{q+1}\cdot\frac{1}{\xi}$, this amounts to requiring

$$\tilde{L}_0 h := h''+\frac{n-1+\frac{4}{q+1}}{\xi}h'+\frac{\xi}{2}h'+\Big[1+h-(1+h)^{-q}\Big]w^{-q-1}=0. \qquad (2.4)$$

Using the first-order expansion

$$1+h-(1+h)^{-q}\approx(q+1)h, \qquad |h|<<1,$$

and the identity $w^{-q-1}=A_{n,q}^{-q-1}\xi^{-2}$, we obtain that h, if it remains small enough, should satisfy

$$h''+\frac{n-1+\frac{4}{q+1}}{\xi}h'+\frac{(q+1)A_{n,q}^{-q-1}}{\xi^2}h+\frac{\xi}{2}h'\approx 0.$$

Near $\xi = 0$, the last term is small as compared to the second one, whence we end up with the Euler-type ODE

$$h'' + \frac{a_1}{\xi} h' + \frac{a_2}{\xi^2} h \approx 0, \qquad |h| << 1, \quad \xi << 1, \tag{2.5}$$

where $a_1 = n - 1 + \frac{4}{q+1}$ and $a_2 = (q+1) A_{n,q}^{-q-1} \equiv 2 \cdot (\frac{2}{q+1} + n - 2)$. The characteristic roots of this asymptotic equation, that is, the zeros of $\lambda \mapsto \lambda(\lambda - 1) + a_1 \lambda + a_2$, are computed as

$$\lambda_{\pm} = -\frac{a_1 - 1}{2} \pm \frac{1}{2} \sqrt{(a_1 - 1)^2 - 4a_2}.$$

Both these roots have a nonzero imaginary part if and only if $(a_1 - 1)^2 - 4a_2 < 0$ or, equivalently,

$$\begin{aligned} 0 &< (q+1)^2 \cdot [4a_2 - (a_1 - 1)^2] \\ &= 8(q+1)^2 \cdot (\tfrac{2}{q+1} + n - 2) - (q+1)^2 \cdot (n - 2 + \tfrac{4}{q+1})^2 \\ &= 8(n-2)(q+1)^2 + 16(q+1) - (n-2)^2(q+1)^2 \\ &\quad -8(n-2)(q+1) - 16 \\ &= (n-2)(10-n)(q+1)^2 + (32 - 8n)(q+1) - 16 \\ &= (n-2)(10-n)q^2 - 2(n^2 - 8n + 4)q - (n-2)^2. \end{aligned}$$

Solving this inequality with respect to $q \in (0, 1)$, we conclude that the linearized asymptotic equation (2.5) allows oscillating solutions of the form

$$h(\xi) = \xi^{-\frac{a_1 - 1}{2}} \sin\left(\ln \xi^{\frac{1}{2}\sqrt{4a_2 - (a_1-1)^2}} \right) \tag{2.6}$$

(and, of course, their cosine counterparts), if and only if $2 \le n \le 6$ and $q_c(n) < q < 1$. In the case $n = 1$, all solutions are given by

$$\begin{aligned} h(\xi) &= c_1 \xi^{-\frac{2(1-q)}{q+1}} + c_2 \xi^{-1} \qquad \text{if } q \neq \frac{1}{3}, \\ h(\xi) &= c_1 \xi^{-1} + c_2 \xi^{-1} \ln \xi \qquad \text{if } q = \frac{1}{3}. \end{aligned} \tag{2.7}$$

Let us develop from this a strategy on how to construct a solution $u \not\equiv w$ above w but with initial value w. By the approximation procedure performed in Section 3 below, we know that there exists a weak solution (for instance, the maximal solution) $u \ge w$ with initial value w that is the limit of a decreasing sequence of smooth *super*solutions of (1.1) which initially lie strictly above w. Since all nontrivial *solutions* of the exact variant of (2.5) are unbounded near $\xi = 0$ and therefore useless for our puspose, we confine ourselves with constructing *sub*solutions instead. Transformed back to the original coordinates, such functions h, thus satisfying $\tilde{L}_0 h \geq 0$ and required to be small, should give rise to subsolutions $\tilde{u}$ of (1.1) with initial value w. If now we are able to achieve

$$\tilde{u} \leq w \text{ on the lateral boundary of some parabolic domain } Q \text{ of the form } t^{-\frac{1}{2}}|x| < \xi_0 \quad (2.8)$$

and, say,

$$\tilde{u}(0,t) > 0 \qquad \text{for small } t > 0, \tag{2.9}$$

then $\tilde{u} \leq u$ in Q by comparison and particularly $u(0,t) > 0$ for small positive t, whence u cannot coincide with w, though having evolved from w.
Rewriting (2.8) and (2.9) in terms of f, we can formulate as our goal the construction of a function f satisfying

$$L_0 f \geq 0 \qquad \text{in } (0,\xi_0),$$
$$f(0) > 0, \quad f'(0) = 0 \quad \text{and} \quad f(\xi_0) = 0$$

for some $\xi_0 > 0$. As it is easy to see that each positive constant δ satisfies $L_0(\delta) \geq 0$ on $(0,\xi_1)$ for sufficiently small ξ_1 (cf. Corollary 4.3), the main task will thus be to find suitable δ, ξ_1 and a subsolution f for $\xi \geq \xi_1$ with

$$f(\xi_1) = \delta, \qquad f'(\xi_1) = 0,$$

in such a way that

$$f \text{ reaches the value zero at some finite } \xi.$$

It will be a consequence of the oscillatory nature of (2.5) that this in fact is possible under the assumption $2 \leq n \leq 6$ and $q_c(n) < q < 1$. Indeed, in this case we can choose one of the infinitely many decreasing positive branches of the function

$$f(\xi) = \eta \xi^{\frac{2}{q+1} - \frac{a_1 - 1}{2}} \sin\left(\ln \xi^{\frac{1}{2}\sqrt{4a_2 - (a_1-1)^2}} \right) \tag{2.10}$$

induced by (2.6). Here $\eta > 0$ is appropriately small in order to ensure that the smallness conditions imposed above hold on the considered interval.
As to the case $n = 1$, the corresponding candidates take the form

$$\begin{aligned} f(\xi) &= \eta_1 \xi^{\frac{2q}{q+1}} + \eta_2 \xi^{\frac{1-q}{q+1}} && \text{if } q \neq \frac{1}{3}, \\ f(\xi) &= \eta_1 \xi^{\frac{1}{2}} + \eta_2 \xi^{\frac{1}{2}} \ln \xi && \text{if } q = \frac{1}{3}, \end{aligned} \tag{2.11}$$

where all appearing terms are bounded near $\xi = 0$. Thus, if we choose $\eta_1 > 0 > \eta_2$ when $q < \frac{1}{3}$ and $\eta_1 < 0 < \eta_2$ when $q \geq \frac{1}{3}$ then, as desired, these functions attain a positive

maximum and then decrease to $-\infty$.
Let us briefly describe in how far this distinguishes from the case $n \geq 3$ and $q \leq q_c(n)$ in which the roots of (2.5) become real again. Then, namely, we always have $\lambda_- \leq -\frac{a_1-1}{2} = -\frac{n-2}{2} - \frac{2}{q+1} < -\frac{2}{q+1}$ and thus the first term making up

$$f(\xi) = \eta_1 \xi^{\frac{2}{q+1}+\lambda_-} + \eta_2 \xi^{\frac{2}{q+1}+\lambda_+}$$

becomes unbounded near $\xi = 0$. Consequently, the only possibility to enforce a positive maximum of f consists of choosing $\eta_1 < 0$; but then f will have no zero beyond this maximum.

3 Maximal solutions and a convergence result

In the sequel, by a *continuous weak solution* of (1.1) we mean a function $u \in C^0(\bar\Omega \times [0,\infty))$ that satisfies $u|_{t=0} = u_0$ and $u|_{\partial\Omega} = B \equiv u_0|_{\partial\Omega}$ classically and

$$-\int_0^\infty \int_\Omega u\varphi_t - \int_0^\infty \int_\Omega u\Delta\varphi + \int_0^\infty \int_\Omega \chi_{\{u>0\}} u^{-q}\varphi = \int_\Omega u_0\varphi \tag{3.1}$$

for all nonnegative $\varphi \in C_0^\infty(\Omega \times [0,\infty))$. This implicitly requires, of course, that $\chi_{\{u>0\}}u^{-q} \in L^1_{loc}(\Omega \times [0,\infty))$.
A natural method to construct solutions of (1.1) consists of regularizing the problem and consider solutions u_ε of

$$\begin{aligned} u_{\varepsilon t} &= \Delta u_\varepsilon - g_\varepsilon(u_\varepsilon) \qquad \text{on } \Omega \times (0,\infty), \\ u_\varepsilon|_{\partial\Omega} &= B_\varepsilon, \\ u_\varepsilon|_{t=0} &= u_{0\varepsilon}, \end{aligned} \tag{3.2}$$

where B_ε, $u_{0\varepsilon}$ and g_ε are suitably chosen smooth approximations of B, u_0 and $u \mapsto u^{-q}\chi_{\{u>0\}}$, respectively. There are several possible ways to accomplish this. For definiteness, let us define g_ε as follows: We pick a cut-off function $\chi \in C^\infty(\mathbb{R})$ such that $\chi(s) = 0$ for $s \leq 1$, $\chi(s) = 1$ for $s \geq 2$ and $0 \leq \chi' \leq 2$ on $\mathbb{R}$. For $\varepsilon > 0$, we set $\chi_\varepsilon(s) := \chi(\frac{s}{\varepsilon})$ and

$$g_\varepsilon(s) := \begin{cases} \chi_\varepsilon(s) \cdot s^{-q}, & s > 0, \\ 0, & s \leq 0. \end{cases}$$

Moreover, with a fixed sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$, $\varepsilon_j \in (0,1)$, we let $u_{0\varepsilon} \in W^{1,\infty}(\Omega)$ be such that

$$u_0 + \varepsilon \leq u_{0\varepsilon} \leq u_0 + 2\varepsilon \qquad \text{in } \Omega, \tag{3.3}$$

and that $u_{0\varepsilon} \searrow u_0$ as $\varepsilon \to 0$. Here and below, in order to abbreviate notation, we frequently use statements like 'as $\varepsilon \to 0$' or 'for all $\varepsilon \in (0,1)$' which are to be understood as referring to the sequence $\varepsilon = \varepsilon_j$ throughout. Let us also agree upon the convention that if $u_0 \in W^{1,\infty}(\Omega)$ then we always choose $u_{0\varepsilon} := u_0 + \varepsilon$. Finally, we set $B_\varepsilon := u_{0\varepsilon}|_{\partial\Omega}$. Using standard arguments involving parabolic Schauder theory and the comparison principle, one can see that each of the problems (3.2) has a classical solution that satisfies

$$\varepsilon \le u_\varepsilon \le \|u_{0\varepsilon}\|_{L^\infty(\Omega)}.$$

Moreover, the u_ε are ordered and

$$u_\varepsilon \searrow u \qquad \text{as } \varepsilon \to 0$$

holds for some nonnegative $u \in L^\infty(\Omega \times (0,\infty))$.

Lemma 3.1 *i) There exists $c > 0$ such that for each $u_0 \in C^0(\bar{\Omega})$ and all ε we have*

$$|\nabla u_\varepsilon^{\frac{q+1}{2}}(x,t)| \le c \cdot (1 + \|u_0^{\frac{q+1}{2}}\|_{L^\infty(\Omega)}) \cdot \Big(1 + t^{-\frac{1}{2}} + (\text{dist}\,(x,\partial\Omega))^{-1}\Big) \tag{3.4}$$

for all $(x,t) \in \Omega \times (0,\infty)$.
ii) Moreover, if $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ then

$$|\nabla u_\varepsilon^{\frac{q+1}{2}}(x,t)| \le c \cdot (1 + \|u_0^{\frac{q+1}{2}}\|_{W^{1,\infty}(\Omega)}) \cdot \Big(1 + (\text{dist}\,(x,\partial\Omega))^{-1}\Big) \tag{3.5}$$

is valid for any $(x,t) \in \Omega \times (0,\infty)$ and some $c > 0$.

Proof. The proof is a straightforward application of the well-known Berstein technique as demonstrated for some closely related problems in [19], [4] or [18], for instance.
i) For small positive d and τ, fix $\zeta \in C^\infty(\bar{\Omega} \times [0,\infty))$ such that $0 \le \zeta \le 1$, $\zeta(x,t) = 1$ if $\text{dist}\,(x,\partial\Omega) \ge d$ and $t \ge \tau$, $\zeta(x,t) = 0$ if $\text{dist}\,(x,\partial\Omega) \le \frac{d}{2}$ or $t \le \frac{\tau}{2}$, and such that

$$|\zeta_t| \le \frac{c_0}{\tau} \quad \text{and} \quad |\nabla\zeta|^2 + \zeta|\Delta\zeta| \le \frac{c_0}{d^2} \qquad \text{in } \Omega \times (0,\infty)$$

holds with some $c_0 > 0$. The function $v := u_\varepsilon^{\frac{q+1}{2}}$ satisfies

$$v_t = \Delta v + \frac{1-q}{q+1} v^{-1}|\nabla v|^2 - \frac{q+1}{2} v^{-\frac{1-q}{q+1}} g_\varepsilon(v^{\frac{2}{q+1}})$$

and

$$\begin{aligned} \frac{1}{2}(|\nabla v|^2)_t &= \nabla v \cdot \nabla v_t \\ &= \nabla v \cdot \nabla\Delta v - \frac{1-q}{q+1} v^{-2}|\nabla v|^4 + \frac{2(1-q)}{q+1} v^{-1}\langle \nabla v, D^2 v \cdot \nabla v\rangle \\ &\quad -\frac{q+1}{2}\frac{d}{dv}\Big(v^{-\frac{1-q}{q+1}} g_\varepsilon(v^{\frac{2}{q+1}})\Big) \cdot |\nabla v|^2. \end{aligned} \tag{3.6}$$

For any $T > 0$,

$$z(x,t) := \zeta^2(x,t)|\nabla v(x,t)|^2$$

attains its maximum over $\bar{\Omega} \times [0,T]$ at some point $(x_0,t_0) \in \Omega \times (0,T]$; hence at this point, $\nabla z \equiv 2\zeta\nabla\zeta|\nabla v|^2 + 2\zeta^2 D^2 v \cdot \nabla v$ vanishes and $z_t \geq 0$ as well as $\Delta z \leq 0$. Therefore at (x_0,t_0) we find, using (3.6),

$$\begin{aligned}
0 &\leq \frac{1}{2}(z_t - \Delta z) \\
&= \zeta\zeta_t|\nabla v|^2 + \frac{1}{2}\zeta^2 \cdot (|\nabla v|^2)_t - \frac{1}{2}\Delta z \\
&= \zeta\zeta_t|\nabla v|^2 + \zeta^2 \cdot \Big[\nabla v \cdot \nabla\Delta v - \frac{1-q}{q+1}v^{-2}|\nabla v|^4 + \frac{2(1-q)}{q+1}v^{-1}\langle\nabla v, D^2 v \cdot \nabla v\rangle \\
&\qquad\qquad - \frac{q+1}{2}\frac{d}{dv}\Big(v^{-\frac{1-q}{q+1}}g_\varepsilon(v^{\frac{2}{q+1}})\Big) \cdot |\nabla v|^2\Big] \\
&\quad -(|\nabla\zeta|^2 + \zeta\Delta\zeta)|\nabla v|^2 - 4\zeta\langle\nabla\zeta, D^2 v \cdot \nabla v\rangle - \zeta^2|D^2 v|^2 - \zeta^2\nabla v \cdot \nabla\Delta v \\
&= \zeta\zeta_t - \frac{1-q}{q+1}\zeta^2 v^{-2}|\nabla v|^4 - \frac{2(1-q)}{q+1}\zeta v^{-1}|\nabla v|^2\nabla v \cdot \nabla\zeta \\
&\quad -\frac{q+1}{2}\frac{d}{dv}\Big(v^{-\frac{1-q}{q+1}}g_\varepsilon(v^{\frac{2}{q+1}})\Big) \cdot |\nabla v|^2 \\
&\quad -(|\nabla\zeta|^2 + \zeta\Delta\zeta)|\nabla v|^2 + 4|\nabla\zeta|^2|\nabla v|^2 - \zeta^2|D^2 v|^2.
\end{aligned} \tag{3.7}$$

By Young's inequality, we find

$$\Big|\frac{2(1-q)}{q+1}\zeta v^{-1}|\nabla v|^2\nabla v \cdot \nabla\zeta\Big| \leq \frac{1-q}{2(q+1)}\zeta^2 v^{-2}|\nabla v|^4 + \frac{2(1-q)}{q+1}|\nabla\zeta|^2|\nabla v|^2$$

and thus obtain from (3.7) that either $\nabla v(x_0,t_0) = 0$ or

$$\begin{aligned}
\frac{1-q}{2(q+1)}\zeta^2|\nabla v|^2 &\leq \Big[\zeta\zeta_t + \Big(3 + \frac{2(1-q)}{q+1}\Big)|\nabla\zeta|^2 - \zeta\Delta\zeta\Big]v^2 \\
&\quad -\frac{q+1}{2}v^2 \cdot \frac{d}{dv}\Big(v^{-\frac{1-q}{q+1}}g_\varepsilon(v^{\frac{2}{q+1}})\Big).
\end{aligned} \tag{3.8}$$

Since $g_\varepsilon(s) \leq s^{-q}$ and $g'_\varepsilon(s) \geq -qs^{-q-1}$ for all $s > 0$ and $\varepsilon > 0$, we find

$$\begin{aligned}
-\frac{q+1}{2}v^2 \cdot \frac{d}{dv}\Big(v^{-\frac{1-q}{q+1}}g_\varepsilon(v^{\frac{2}{q+1}})\Big) &= \frac{1-q}{2}v^{\frac{2q}{q+1}}g_\varepsilon(v^{\frac{2}{q+1}}) - v^2 g'_\varepsilon(v^{\frac{2}{q+1}}) \\
&\leq \frac{1-q}{2} + q,
\end{aligned}$$

so that (3.8) yields

$$\zeta^2|\nabla v|^2 \le \Big[\frac{c_0}{\tau} + \frac{q+5}{q+1}\cdot\frac{c_0}{d^2} + \frac{c_0}{d^2}\Big]\cdot\|\nabla v\|^2_{L^\infty(\Omega\times(0,\infty))} + \frac{q+1}{2} \tag{3.9}$$

at the pont (x_0,t_0). As $u_\varepsilon \le \|u_{0\varepsilon}\|_{L^\infty(\Omega)}$, this implies (3.4).

ii) The proof in the case $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ (implying $u_0 \in W^{1,\infty}(\Omega)$ and hence $u_{0\varepsilon} = u_0+\varepsilon$) can be run in quite a similar way if we choose ζ independent of t. Then the maximum of z either satisfies an estimate of the form (3.9) without the term containing τ, or it is attained initially. In the latter case, however, we use that $u_{0\varepsilon} = u_0+\varepsilon$ implies $\|\nabla u_{0\varepsilon}^{\frac{2}{q+1}}\|_{L^\infty(\Omega)} \le c\|\nabla u_0^{\frac{2}{q+1}}\|_{L^\infty(\Omega)}$ for some $c>0$ to conclude (3.5). ////

The obtained regularity in space can be used to achieve regularity in time by a standard argument.

Corollary 3.2 *i) For any* $u_0 \in C^0(\bar\Omega)$,

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \textit{is locally equicontinuous in } \Omega\times(0,\infty). \tag{3.10}$$

ii) If even $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ *then*

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \textit{is locally equicontinuous in } \Omega\times[0,\infty). \tag{3.11}$$

Proof. i) Since $\tilde v := u_\varepsilon^{q+1}$ satisfies

$$\tilde v_t = \Delta\tilde v - \frac{q}{q+1}\tilde v^{-1}|\nabla\tilde v|^2 - (q+1)\tilde v^{\frac{q}{q+1}} g_\varepsilon(\tilde v^{\frac{1}{q+1}})$$

with

$$\Big| - (q+1)\tilde v^{\frac{q}{q+1}} g_\varepsilon(\tilde v^{\frac{1}{q+1}})\Big| \le q+1$$

and

$$\frac{q}{q+1}\tilde v^{-1}|\nabla\tilde v|^2 = \frac{4q}{q+1}|\nabla u_\varepsilon^{\frac{q+1}{2}}|^2,$$

the estimate from Lemma 3.1 i) shows that $\tilde v_t - \Delta\tilde v$ is locally bounded in $\Omega\times(0,\infty)$. Thus, standard parabolic theory entails uniform Hölder estimates for $\tilde v$ (and thereby also for u_ε) on arbitrary compact subsets of $\Omega\times(0,\infty)$.

ii) The argument in the case $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ is quite similar. ////

Remark. Involving results on maximal Sobolev regularity and suitable embeddings, for $t \geq \tau > 0$ the method in the above corollary in fact yields uniform Hölder regularity with respect to t for u_ε^{q+1} with *any* Hölder exponent $\gamma < 1$. A more subtle approach shows that away from $t = 0$, in fact $\gamma = 1$ is allowed, that is, u_ε^{q+1} satisfies the corresponding locally uniform Lipschitz estimates in time. The same improvement is possible near $t = 0$, provided that instead of $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$, the slightly stronger condition $u_0 \in C^2(\bar{\Omega})$ is imposed. Both these refinements can be found in [4].

However, in the sequel we shall also need a Lipschitz-type estimate for u_t, but only *from below* and only at $t = 0$. This can be obtained via a simple barrier argument under the weak assumption that Δu_0 be uniformly bounded from below. Since we intend to choose u_0 equal to the singular function w near $x = 0$, it is important for us that we do not require an upper bound for Δu_0.

Lemma 3.3 *Suppose that* $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ *and that*

$$\Delta u_0 \geq -c_0 \qquad \text{in } \Omega \tag{3.12}$$

holds in the sense of distributions with some $c_0 > 0$. *Then for each compact* $K \subset \Omega$ *there exists* $c_K > 0$ *such that*

$$u_\varepsilon^{q+1}(x,t) \geq u_{0\varepsilon}^{q+1}(x) - c_K t \qquad \forall\, (x,t) \in K \times (0,\infty) \tag{3.13}$$

holds.

Proof. Given $K \subset\subset \Omega$, we fix a smooth domain $\Omega' \supset K$ with $\overline{\Omega'} \subset \Omega$ and let $\zeta \in C_0^\infty(\Omega')$ be such that $0 \leq \zeta \leq 1$ in Ω and $\zeta \equiv 1$ in K. Recalling our convention that $u_{0\varepsilon} = u_0 + \varepsilon$ in case of $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$, we calculate

$$\begin{aligned} \Delta u_{0\varepsilon}^{q+1} &= (q+1)(u_0+\varepsilon)^q \Delta u_0 + \frac{4q}{q+1}\Big(\frac{u_0}{u_0+\varepsilon}\Big)^{1-q}\Big|\nabla u_0^{\frac{q+1}{2}}\Big|^2 \\ &\geq -(q+1)(u_0+\varepsilon)^q c_0 \\ &\geq -c_1 \qquad \text{in } \Omega \end{aligned}$$

in the distributional sense, which implies $\Delta(\zeta u_{0\varepsilon}^{q+1}) \geq -c_2 = -c_2(u_0, K)$ in Ω', that is,

$$\int_{\Omega'} \nabla(\zeta u_{0\varepsilon}^{q+1}) \cdot \nabla\varphi \leq c_2 \int_{\Omega'} \varphi \qquad \text{for any nonnegative } \varphi \in W_0^{1,2}(\Omega'). \tag{3.14}$$

Recalling that $v = u_\varepsilon^{q+1}$ satisfies

$$\begin{aligned} v_t &= \Delta v - \frac{q}{q+1} v^{-1}|\nabla v|^2 - (q+1)v^{\frac{q}{q+1}} g_\varepsilon(v^{\frac{1}{q+1}}) \\ &\geq \Delta v - \frac{4q}{q+1}|\nabla u_\varepsilon^{\frac{q+1}{2}}|^2 - (q+1), \end{aligned}$$

we see from Lemma 3.1 ii) that

$$v_t \geq \Delta v - c_3 \qquad \text{in } \Omega' \times (0,\infty) \tag{3.15}$$

with c_3 depending on u_0 and Ω' (hence on K) only. We now let

$$z(x,t) := \zeta(x) \cdot u_{0\varepsilon}^{q+1}(x) - c_K t - v(x,t), \qquad (x,t) \in \Omega' \times (0,\infty),$$

with $c_K := c_2 + c_3$. Then $z(x,0) \leq (\zeta(x)-1)u_{0\varepsilon}^{q+1}(x) \leq 0$ in Ω', so that (3.14) and (3.15) yield

$$\begin{aligned}
\frac{1}{2}\int_{\Omega'} z_+^2(x,t) &= \frac{1}{2}\int_{\Omega'} z_+^2(x,0) + \int_0^t\int_{\Omega'} z_+ \cdot z_{+t} \\
&= \int_0^t\int_{\Omega'} z_+ \cdot (-c_K - v_t) \\
&\leq -c_K \int_0^t\int_{\Omega'} z_+ + \int_0^t\int_{\Omega'} \nabla z_+ \cdot \nabla v + c_3 \int_0^t\int_{\Omega'} z_+ \\
&= -c_K \int_0^t\int_{\Omega'} z_+ + \int_0^t\int_{\Omega'} \nabla z_+ \cdot \nabla(\zeta u_{0\varepsilon}^{q+1}) - \int_0^t\int_{\Omega'} |\nabla z_+|^2 + c_3 \int_0^t\int_{\Omega'} z_+ \\
&\leq (-c_K + c_2 + c_3)\int_0^t\int_{\Omega'} z_+ - \int_0^t\int_{\Omega'} |\nabla z_+|^2 \\
&\leq 0 \qquad \forall\, t > 0.
\end{aligned}$$

This shows that $z_+ \equiv 0$ and thus $v(x,t) \geq \zeta(x)u_{0\varepsilon}^{q+1}(x) - c_K t$ in $\Omega' \times (0,\infty)$, whereby (3.13) follows since $\zeta \equiv 1$ in K. ////

We now turn our attention to the question of existence of continuous weak solutions. In order to avoid a repetition of arguments, we already combine this with the question of convergence of sequences of such weak solutions under the assumption that the corresponding sequences of initial data converge uniformly. This is done in the following lemma.

Lemma 3.4 *Let $u_0 \in C^0(\bar\Omega)$ and suppose $(u_{0k})_{k\in\mathbb{N}} \subset C^0(\bar\Omega)$ is a sequence of nonnegative initial data such that $u_{0k} \to u_0$ uniformly in Ω. Set $u_k := \lim_{\varepsilon\to 0} u_{k\varepsilon}$, where $u_{k\varepsilon}$ denotes the solution of (3.2) with u_0 replaced by u_{0k}. Then the functions u_k are continuous in $\bar\Omega \times [0,\infty)$ and along a subsequence $k_l \to \infty$, we have*

$$u_{k_l} \to \tilde u \qquad \textit{locally uniformly in } \bar\Omega \times [0,\infty)$$

as $k_l \to \infty$, where $\tilde u$ is a continuous weak solution of (1.1) with initial value u_0.

Before giving the somewhat technical proof, let us state two almost immediate but important consequences.

Theorem 3.5 *For any nonnegative $u_0 \in C^0(\bar\Omega)$, (1.1) possesses the unique maximal continuous weak solution $u = \lim_{\varepsilon\to 0} u_\varepsilon$; here, maximality means that if $\tilde u$ is any continuous weak solution of (1.1) then necessarily $\tilde u \le u$.*

PROOF. Choosing $u_{0k} \equiv u_0$ in Lemma 3.4 immediately shows that u in fact is a continuous weak solution. To see its maximality, suppose $\tilde u$ is another continuous weak solution. Due to parabolic regularity theory, $\tilde u$ is smooth in $\{\tilde u > 0\} \cap \{t > 0\}$ and satisfies

$$\begin{aligned} \tilde u_t &= \Delta \tilde u - \tilde u^{-q} \\ &\le \Delta \tilde u - g_\varepsilon(\tilde u) \qquad \text{in } \{\tilde u > 0\} \cap \{t > 0\} \end{aligned}$$

classically. Since $u_\varepsilon \ge \varepsilon$ and $|g'_\varepsilon(s)| \le q\varepsilon^{-q-1}$ for all $s \in \mathbb{R}$, $z := (\tilde u - u_\varepsilon)_+$ satisfies

$$\begin{aligned} \frac{1}{2}\int_\Omega z_+^2(\cdot,t) &= \frac{1}{2}\int_\Omega z_+^2(\cdot,0) + \int_0^t \int_\Omega z_+ \cdot z_{+t} \\ &\le -\int_0^t \int_\Omega |\nabla z_+|^2 + q\varepsilon^{-q-1} \int_0^t \int_\Omega z_+^2 \qquad \forall\, t > 0. \end{aligned}$$

By Gronwall's lemma, $z_+ \equiv 0$ and hence $\tilde u \le u_\varepsilon$ in $\Omega \times (0,\infty)$ for any $\varepsilon > 0$, which proves $\tilde u \le u$. ////

If the u_{0k} are ordered, then it is easy to deduce from (3.3) that the same ordering holds for u_k. Thus, Theorem 3.5 and Lemma 3.4 directly entail

Corollary 3.6 *Assume that u_0 and u_{0k} are nonnegative and continuous in $\bar\Omega$ and such that $u_{0k} \to u_0$ monotonically in $\bar\Omega$. Then as $k \to \infty$, the maximal solutions u_k emanating from u_{0k} converge locally uniformly in $\bar\Omega \times [0,\infty)$ to a continuous weak solution $\tilde u$ of (1.1).*

PROOF (of Lemma 3.4). The proof proceeds in six steps.
Step 1. We first claim that for each $\eta > 0$ there exist $\varepsilon_0 > 0$, $k_0 \in \mathbb{N}$ and $\nu > 0$ such that

$$|u_{k\varepsilon}(x,t) - u_0(x)| \le \frac{\eta}{2} \qquad \forall\,(x,t) \in \Omega_\nu \times (0,\infty),\ \forall \varepsilon < \varepsilon_0,\ \forall k > k_0, \tag{3.16}$$

where $\Omega_\nu := \{x \in \Omega \mid \mathrm{dist}\,(x, \partial\Omega) < \nu\}$.
To this end, let $\eta > 0$ be given and set $\tilde\varepsilon_0 := \frac{\eta}{8}$. Since $u_{0k} \to u_0$ uniformly, $u_{0\tilde\varepsilon_0} > u_0$ and $u_{0k\varepsilon} \le u_{0k} + 2\varepsilon$ in $\bar\Omega$, there exist $k_0 \in \mathbb{N}$ and $\varepsilon_0 < \tilde\varepsilon_0$ such that

$$u_{0k\varepsilon} \le u_{0\tilde\varepsilon_0} \qquad \text{in } \Omega \qquad \forall\, \varepsilon < \varepsilon_0,\ \forall\, k > k_0. \tag{3.17}$$

Since $u_{0\tilde\varepsilon_0}$ beolgs to $W^{1,\infty}(\Omega)$ and $u_{0\tilde\varepsilon_0} \le u_0 + 2\tilde\varepsilon_0$, we can pick some large $c > 0$ (depending possibly on $\tilde\varepsilon_0$) such that

$$u_{0\tilde\varepsilon_0}(x) \le 2\tilde\varepsilon_0 + ce(x) \qquad \text{in } \Omega, \tag{3.18}$$

where $e \in C^0(\bar{\Omega}) \cap C^2(\Omega)$ denotes the solution of $-\Delta e = 1$ in Ω with $e|_{\partial\Omega} = u_0|_{\partial\Omega}$. As $(x,t) \mapsto 2\tilde{\varepsilon}_0 + ce(x)$ is easily seen to be a supersolution of $u_t = \Delta u - g_\varepsilon(u)$ for all $\varepsilon > 0$, the comparison principle yields

$$u_{k\varepsilon} \le 2\tilde{\varepsilon}_0 + ce(x) \qquad \text{in } \Omega \times (0,\infty) \qquad \forall\, \varepsilon < \varepsilon_0,\ \forall\, k > k_0 \tag{3.19}$$

in view of (3.17) and (3.18).
If we now choose $\nu > 0$ small such that $ce(x) \le u_0(x) + \frac{\eta}{4}$ in Ω_ν, we obtain from (3.19) and the choice of $\tilde{\varepsilon}_0$ that

$$u_{k\varepsilon}(x,t) \le u_0(x) + \frac{\eta}{2} \qquad \text{in } \Omega_\nu \times (0,\infty) \qquad \forall\, \varepsilon < \varepsilon_0,\ \forall\, k > k_0. \tag{3.20}$$

In order to prove (3.16), we thus only need to show the corresponding estimate

$$u_{k\varepsilon}(x,t) \ge u_0(x) < \frac{\eta}{2} \qquad \text{in } \Omega_\nu \times (0,\infty) \qquad \forall\, \varepsilon > 0,\ \forall\, k > k_0 \tag{3.21}$$

from below with some possibly smaller ν and larger k_0.
To this end, again by uniform convergence and uniform continuity we can fix $\rho_0 > 0$ and enlarge k_0 if necessary so as to obtain

$$|u_{0k}(x) - u_0(x_0)| \le \frac{\eta}{8} \qquad \forall\, x_0 \in \partial\Omega,\ \forall\, x \in \bar{\Omega} \text{ with } |x - x_0| \le \rho_0, \quad \forall\, k > k_0. \tag{3.22}$$

Moreover, since $\partial\Omega$ is smooth, we find $R < \frac{\rho_0}{4}$ such that for any $x_0 \in \partial\Omega$ there exists $y = y(x_0) \in \mathbb{R}^n \setminus \bar{\Omega}$ such that $\bar{B}_R(y) \cap \bar{\Omega} = \{x_0\}$.
Now given $x_0 \in \partial\Omega$ satisfying $u_0(x_0) \ge \frac{\eta}{4}$, we intend to compare $u_{k\varepsilon}$ from below with the explicit time-independent barrier function $v = v_{x_0}$ defined by

$$v(x,t) := b \cdot \Big[1 - \frac{|x-y| - R}{\rho}\Big]^\kappa \qquad \text{in } Q := \Big\{(x,t) \in \Omega \times (0,\infty)\ \Big|\ R < |x-y| < R + \rho\Big\},$$

where $\kappa := \frac{2}{q+1} > 1$, $y = y(x_0)$, $b := u_0(x_0) - \frac{\eta}{8}$ and

$$\rho := \min\Big\{\frac{\rho_0}{2},\ \frac{(\kappa-1)R}{2(n-1)},\ \sqrt{\kappa(\kappa-1)} \cdot \Big(\frac{\eta}{8}\Big)^{\frac{q+1}{2}}\Big\}.$$

In fact, if $x \in \Omega$ is such that $R < |x-y| < R+\rho$ then $|x-x_0| \le |x-y| + |y-x_0| \le 2R+\rho < \rho_0$ according to the choices of R and ρ, so that (3.22) implies

$$u_{0k\varepsilon}(x) \ge u_0(x_0) - \frac{\eta}{8} = b \ge v(x,0)$$

for such x. In quite a similar fashion one can see that $u_{k\varepsilon} \ge v$ also holds if $x \in \partial\Omega$ and $R < |x-y| < R+\rho$; since v vanishes when $|x-y| = R+\rho$, we therefore find that $u_{k\varepsilon}$

lies above v on the parabolic boundary of Q for all $\varepsilon > 0$ and $k > k_0$. Furthermore, we compute

$$\begin{aligned} v_t - \Delta v + g_\varepsilon(v) &\le v_t - \Delta v + v^{-q} \\ &= \frac{(n-1)\kappa b}{\rho|x-y|} \cdot \Big[1 - \frac{|x-y|-R}{\rho}\Big]^{\kappa-1} - \frac{\kappa(\kappa-1)b}{\rho^2} \cdot \Big[1 - \frac{|x-y|-R}{\rho}\Big]^{\kappa-2} \\ &\quad + b^{-q} \cdot \Big[1 - \frac{|x-y|-R}{\rho}\Big]^{-q\kappa} \\ &=: I_1 - I_2 + I_3 \qquad \text{in } Q \end{aligned}$$

and estimate, using the definition of ρ,

$$\frac{I_1}{\frac{1}{2}I_2} = \frac{2(n-1)\rho}{(\kappa-1)|x-y|} \cdot \Big[1 - \frac{|x-y|-R}{\rho}\Big] \le \frac{2(n-1)\rho}{(\kappa-1)R} \le 1 \qquad \text{in } Q$$

and

$$\frac{I_3}{\frac{1}{2}I_2} = \frac{\rho^2}{\kappa(\kappa-1)b^{q+1}} \le \frac{\rho^2}{\kappa(\kappa-1)\cdot(\frac{\eta}{8})^{q+1}} \le 1 \qquad \text{in } Q,$$

because $b = u_0(x_0) - \frac{\eta}{8} \ge \frac{\eta}{4} - \frac{\eta}{8} = \frac{\eta}{8}$. Hence, $v_t - \Delta v + g_\varepsilon(v) \le 0$ in Q, so that the comparison principle tells us that $u_{k\varepsilon} \ge v$ in Q for all $\varepsilon > 0$ and $k > k_0$.
Now (3.21) follows from this upon a standard argument: Let $\nu < \rho$ be small enough such that $(1-\frac{\nu}{\rho})^\kappa \ge \frac{z-\frac{\eta}{2}}{z-\frac{\eta}{4}}$ for all z satisfying $\frac{\eta}{2} \le z \le \|u_0\|_{L^\infty(\Omega)}$. Further diminishing ν if necessary, we may assume that for all $x \in \Omega_\nu$ there exists a unique $x_0(x) \in \partial\Omega$ with $|x - x_0(x)| = \text{dist}\,(x, \partial\Omega)$. Note that this entails that $x_0(x)$ lies on the line segment spanned by x and $y(x_0(x))$, so that $|x - y(x_0(x))| = \text{dist}\,(x,\partial\Omega) + R$.
Now if $x \in \Omega_n u$ and $u_0(x) < \frac{\eta}{2}$, (3.21) is trivially fulfilled. When $u_0(x) \ge \frac{\eta}{2}$, however, by (3.22) we find $|u_0(x) - u_0(x_0(x))| \le \frac{\eta}{8}$ and particularly $u_0(x_0(x)) > \frac{\eta}{4}$, so that

$$\begin{aligned} u_{k\varepsilon}(x,t) &\ge v_{x_0(x)}(x,t) \\ &= b \cdot \Big[1 - \frac{|x - y(x_0(x))| - R}{\rho}\Big]^\kappa \\ &\ge \Big(u_0(x_0(x)) - \frac{\eta}{8}\Big) \cdot \Big[1 - \frac{\nu}{\rho}\Big]^\kappa \\ &\ge \Big(u_0(x) - \frac{\eta}{4}\Big) \cdot \Big[1 - \frac{\nu}{\rho}\Big]^\kappa \\ &\ge u_0(x) - \frac{\eta}{2} \qquad \forall\, \varepsilon > 0,\ \forall\, k > k_0 \end{aligned}$$

in accordance with our choice of ν. Thus, (3.21) holds for all $x \in \Omega_\nu$.

Step 2. If we additionally require $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ then the convergence $u_\varepsilon \to u$ is uniform in $\bar{\Omega} \times [0,T]$ for any $T > 0$.

In fact, let $\eta > 0$ be given. Applying Step 1 upon the particular choice $u_{0k} \equiv u_0$, we obtain $\varepsilon_0 > 0$ and $\nu > 0$ such that

$$|u_\varepsilon(x,t) - u_0(x)| \le \frac{\eta}{2} \qquad \forall\,(x,t) \in \Omega_\nu \times (0,\infty) \quad \forall\,\varepsilon < \varepsilon_0. \tag{3.23}$$

From Corollary 3.2 ii) and the Arzelà-Ascoli theorem we now gain that $u_\varepsilon \to u$ uniformly in $(\bar{\Omega} \setminus \Omega_\nu) \times [0,T]$, whence there exists $\bar{\varepsilon}_0 > 0$ such that

$$|u_\varepsilon - u_{\varepsilon'}| \le \eta \qquad \text{in } (\bar{\Omega} \setminus \Omega_\nu) \times [0,T] \quad \forall\,\varepsilon, \varepsilon' < \bar{\varepsilon}_0.$$

Combining this with (3.23) completes Step 2.

Step 3. We next claim that for any $\eta > 0$, there exist $\varepsilon_1 > 0$, $k_1 \in \mathbb{N}$ and $\tau_1 > 0$ such that

$$u_0 - \frac{\eta}{2} \le u_{k\varepsilon} \le u_0 + \frac{\eta}{2} \qquad \text{in } \Omega \times (0,\tau_1) \quad \forall\,\varepsilon < \varepsilon_1,\ \forall k > k_1. \tag{3.24}$$

Indeed, given $\eta > 0$ let us choose an auxiliary nonnegative function u_0^- such that $(u_0^-)^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$ and $u_0 - \frac{\eta}{4} \le u_0^- \le (u_0 - \frac{\eta}{8})_+$ in Ω. Then, according to Step 2, the monotonic limit $u^- := \lim_{\varepsilon\to 0} u_\varepsilon^-$ is continuous in $\bar{\Omega}\times[0,\infty)$; here, clearly, u_ε^- denotes the solution to (3.2) emanating from $u_{0\varepsilon}^- = u_0^- + \varepsilon$. Thus, there exists $\tau_1 > 0$ such that $u^-(x,t) \ge u_0^-(x) - \frac{\eta}{4}$ in $\Omega \times (0,\tau_1)$.
Next, due to the uniform convergence $u_{0k} \to u_0$ we can pick $k_1 \in \mathbb{N}$ such that $u_{0k} \ge (u_0 - \frac{\eta}{8})_+$ and hence $u_{0k} \ge u_0^-$ in Ω for all $k \ge k_1$. Since this implies $u_{k\varepsilon} \ge u^-$ for all $\varepsilon > 0$ by a comparison argument, we obtain

$$u_{k\varepsilon} \ge u^- \ge u_0^- - \frac{\eta}{4} \ge u_0 - \frac{\eta}{2} \qquad \text{in } \Omega \times (0,\tau_1) \quad \forall\,\varepsilon > 0,\ \forall\,k > k_1. \tag{3.25}$$

As to the right inequality in (3.24), let us fix $\tilde{\varepsilon}_1 > 0$ such that $u_{0\tilde{\varepsilon}_1} \le u_0 + \frac{\eta}{4}$. Then, thanks to the continuity of $u_{\tilde{\varepsilon}_1}$, we can diminish τ_1 so that

$$u_{\tilde{\varepsilon}_1} \le u_0 + \frac{\eta}{2} \qquad \text{in } \Omega \times (0,\tau_1).$$

We now enlarge k_1 and diminish ε_1 so as to satisfy $\varepsilon_1 < \tilde{\varepsilon}_1$ and $u_{0k\varepsilon} \le u_{0\tilde{\varepsilon}_1}$ in Ω holds for all $k > k_1$ and any $\varepsilon < \varepsilon_1$. Then, again since $g_\varepsilon \ge g_{\tilde{\varepsilon}_1}$ for $\varepsilon < \tilde{\varepsilon}_1$, comparison yields

$$u_{\varepsilon k} \le u_{\tilde{\varepsilon}_1} \le u_0 + \frac{\eta}{2} \qquad \text{in } \Omega \times (0,\tau_1) \quad \forall\,k > k_1,\ \forall\,\varepsilon < \varepsilon_1,$$

which together with (3.25) entails (3.24).

Step 4. We next assert that for any u_0 we have

$$u_\varepsilon \to u \qquad \text{locally uniformly in } \bar\Omega \times [0,\infty),$$

whence particularly u is continuous in $\bar\Omega \times [0,\infty)$.

To see this, we consider the particular sequence $(u_{0k})_{k\in\mathbb{N}}$ given by $u_{0k} \equiv u_0$ for all k. Let $T > 0$ and $\eta > 0$ and fix $\varepsilon_0, \nu, \varepsilon_1$ and τ_1 as in Step 1 and Step 3. From Corollary 3.2 i) and the Arzel/'a-Ascoli theorem we infer that $|u_\varepsilon - u_{\varepsilon'}| \le \eta$ in $(\bar\Omega \setminus \Omega_\nu) \times [\tau_1, T]$ for sufficiently small ε and ε', while Step 1 and Step 3 yield

$$|u_\varepsilon - u_{\varepsilon'}| \le u_\varepsilon \le \eta \quad \text{in } \Omega_\nu \times (0,T) \quad \text{for } 0 < \varepsilon' < \varepsilon < \varepsilon_0 \tag{3.26}$$

and

$$|u_\varepsilon - u_{\varepsilon'}| \le \Big(u_0 + \frac{\eta}{2}\Big) - \Big(u_0 - \frac{\eta}{2}\Big) = \eta \quad \text{in } \bar\Omega \times (0,\tau_1) \quad \text{for } 0 < \varepsilon' < \varepsilon < \varepsilon_1 \tag{3.27}$$

respectively.

Step 5. The functions u_k are continuous in $\bar\Omega \times [0,\infty)$, and a subsequence of $(u_k)_{k\in\mathbb{N}}$ converges locally uniformly in $\bar\Omega \times [0,\infty)$ to some $\tilde u \in C^0(\bar\Omega \times [0,\infty))$.

In fact, applying Step 4 to u_k for fixed k, we see that u_k is continuous. In taking $k \to \infty$, we argue in a similar way as above: We fix $T > 0$ and $\eta > 0$ and take k_0, ν, k_1 and τ_1 as provided by Step 1 and Step 3. By Corollary 3.2 i) and Arzelà-Ascoli, we can extract a subsequence $(k_l)_{l\in\mathbb{N}}$ along which $|u_{k_l} - u_{k_m}| \le \eta$ holds in $(\bar\Omega \setminus \Omega_\nu) \times (\tau_1, T)$ for all large l and m. Letting $\varepsilon \to 0$ in Step 1 and Step 3, we obtain estimates quite similar to (3.26) and (3.27) and conclude that $|u_{k_l} - u_{k_m}| \le \eta$ also holds in $\Omega_\nu \times (0,T)$ and in $\Omega \times (0,\tau_1)$, provided that k_l and k_m are greater than $\max\{k_0, k_1\}$. A standard diagonal extraction procedure now yields the assertion on the whole time interval $[0,\infty)$.

Step 6. We finally claim that $\tilde u$ is a weak solution of (1.1).

To this end, we employ a modified variant of the method used in [19, Theorem 1]: We fix a nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and multiply the equation $u_{k\varepsilon t} = \Delta u_{k\varepsilon} - g_\varepsilon(u_{k\varepsilon})$ by $\chi_\delta(u_{k\varepsilon}) \cdot \varphi$, where χ_δ is the cut-off function introduced above and $\delta > 0$. Integration by parts yields

$$\begin{aligned} -\int_\tau^\infty \int_\Omega \Phi_\delta(u_{k\varepsilon})\varphi_t \;&-\; \int_\tau^\infty \int_\Omega \Phi_\delta(u_{k\varepsilon})\Delta\varphi + \int_\tau^\infty \int_\Omega \chi_\delta(u_{k\varepsilon}) g_\varepsilon(u_{k\varepsilon})\varphi \\ &= \int_\Omega \Phi_\delta(u_{k\varepsilon}(\cdot,\tau))\varphi(\cdot,\tau) - \int_\tau^\infty \int_\Omega \chi_\delta'(u_{k\varepsilon})|\nabla u_{k\varepsilon}|^2\varphi \end{aligned} \tag{3.28}$$

for any $\tau > 0$. Choosing $\tau > 0$ we particularly obtain $\int_0^\infty \int_\Omega \chi_\delta(u_{k\varepsilon}) g_\varepsilon(u_{k\varepsilon})\varphi \le c(\varphi)$ for all k, ε and δ. Here we successively let $\varepsilon \to 0$, then $k = k_l \to \infty$ and finally $\delta > 0$ to achieve with the aid of Fatou's lemma that

$$\tilde{u}^{-q}\chi_{\{\tilde{u}>0\}} \in L^1_{loc}(\Omega \times [0,\infty)). \tag{3.29}$$

This allows us to return to perform appropriate limit procedures in (3.28) along the lines presented in [19], [18] and [4]: Due to Step 4 and Step 5, for any $\delta > 0$ the sets $\{u_{k\varepsilon} \ge \delta\} \cap \operatorname{supp}\varphi$ lie in a compact subset of $\{\tilde{u} > \frac{\delta}{2}\}$ for large k and sufficiently small $\varepsilon < \varepsilon(k)$. Therefore parabolic Schauder theory together with the Arzelà-Ascoli theorem entails

$$\lim_{k=k_l\to\infty} \lim_{\varepsilon\to 0} \int_\tau^\infty \int_\Omega \chi'_\delta(u_{k\varepsilon}) \cdot |\nabla u_{k\varepsilon}|^2 \varphi = \int_\tau^\infty \int_\Omega \chi'_\delta(\tilde{u}) \cdot |\nabla \tilde{u}|^2 \varphi.$$

Thus from (3.28) we easily deduce upon letting $\varepsilon \to 0$ and then $k = k_l \to \infty$ that

$$\begin{aligned} -\int_\tau^\infty \int_\Omega \Phi_\delta(\tilde{u})\varphi_t \;&-\; \int_\tau^\infty \int_\Omega \Phi_\delta(\tilde{u})\Delta\varphi + \int_\tau^\infty \int_\Omega \chi_\delta(\tilde{u}) \cdot \tilde{u}^{-q}\varphi \\ &= \int_\Omega \Phi_\delta(\tilde{u}(\cdot,\tau))\varphi(\cdot,\tau) - \int_\tau^\infty \int_\Omega \chi'_\delta(\tilde{u}) \cdot |\nabla\tilde{u}|^2\varphi. \end{aligned} \tag{3.30}$$

Here we have used that as a consequence of the cut-off property of χ_δ, both convergences $\chi_\delta(u_{k\varepsilon})g_\varepsilon(u_{k\varepsilon}) \to \chi_\delta(u_k) \cdot u_k^{-q}$ as $\varepsilon \to 0$ and $\chi_\delta(u_k) \cdot u_k^{-q} \to \chi_\delta(\tilde{u}) \cdot \tilde{u}^{-q}$ as $k = k_l \to \infty$ hold uniformly in $\operatorname{supp}\varphi$ because of Step 4 and Step 5.
Now by Lemma 3.1 i), we have $|\nabla \tilde{u}^{\frac{q+1}{2}}|^2 \le c(\tau,\varphi)$ in $\operatorname{supp}\varphi \cap \{t \ge \tau\}$ and hence the second term on the right of (3.30) can be estimated according to

$$\begin{aligned} \Big| \int_\tau^\infty \int_\Omega \chi'_\delta(\tilde{u}) \cdot |\nabla\tilde{u}|^2\varphi \Big| \;&\le\; \frac{4}{(q+1)^2} \cdot c(\tau,\varphi) \cdot \frac{2}{\delta} \cdot \int_\tau^\infty \int_{\{\delta \le \tilde{u}(\cdot,t) \le 2\delta\}} \tilde{u}^{1-q}\varphi \\ &\le\; \frac{8}{(q+1)^2} \cdot c(\tau,\varphi) \cdot \int_\tau^\infty \int_{\{\delta \le \tilde{u}(\cdot,t) \le 2\delta\}} \tilde{u}^{-q}\varphi \\ &\to\; 0 \qquad \text{as } \delta \to 0 \end{aligned}$$

in view of (3.29) and the dominated convergence theorem.
Since $\chi_\delta \nearrow \chi_{\{0,\infty\}}$ and $\Phi_\delta(s) \nearrow s$ as $\delta \to 0$, we once again invoke the dominated convergence theroem and (3.29) to obtain from (3.30) in the limit $\delta \to 0$

$$-\int_\tau^\infty \int_\Omega \tilde{u}\varphi_t - \int_\tau^\infty \int_\Omega \tilde{u}\Delta\varphi + \int_\tau^\infty \int_\Omega \tilde{u}^{-q}\chi_{\{\tilde{u}>0\}}\varphi = \int_\Omega \tilde{u}(\cdot,\tau)\varphi(\cdot,\tau).$$

Finally, from (3.29) and the fact that $\tilde{u}$ is continuous down to $t = 0$ it is clear that we can take $\tau \to 0$ in the latter identity to conclude that $\tilde{u}$ in fact is a weak solution of (1.1). ////

4 ODE analysis

The aim of the present section is to construct explicit subsolutions of the operator L_0 introduced in Section 2. More generally, for any $\Lambda \ge 0$ and all supercritical q we shall find $f \in W^{2,\infty}((0,\infty))$ such that

$$L_\Lambda f := f'' + \frac{n-1}{\xi} f' + \frac{\xi}{2} f' - \Big(\frac{1}{q+1} + \Lambda\Big) f + w^{-q} - \Big(w+f\Big)^{-q} \ge 0 \qquad \text{a.e. in } (0,\infty).$$

Here and throughout this section, we may write w instead of $w_{n,q}$ since there is no danger of confusion. The parameter Λ will finally be set zero in the proof of Theorem 1.1, but we need to choose $\Lambda > 0$ in the proof of Theorem 1.3 in order to compensate some effects that are due to a possibly positive difference $n - N$ in that theorem. This role of positive Λ will become clear in the proof of Lemma 5.3.

The construction of f basically consists of three steps. The first and third of these provide a constant subsolution near $\xi = 0$ and a negative subsolution near $\xi = \infty$, starting from any prescribed zero with arbitrary negative initial slope. The main part is done in the second step which essentially uses the ideas from Section 2 in constructing subsolutions in an intermediate region which suitably match the constant inner subsolution and reach zero at some finite ξ_0.

For later reference, let us start with the following lemma, the elementary proof of which may be omitted. It quantifies the error made in formal expansions of the form $(1 \pm s)^\alpha \approx 1 \pm \alpha s$.

Lemma 4.1 *Let* $q > 0$.
i) For any $s_0 > 0$,

$$(1+s)^{-q} \le 1 - (1-\mu(s_0))qs \qquad \forall\, s \in [0, s_0]$$

holds with $\mu(s_0) := 1 - (1+s_0)^{-q-1}$.
ii) Given $s_1 \in (0,1)$,

$$(1-s)^{-q} \le 1 + (1+\nu(s_1))qs \qquad \forall\, s \in [0, s_1]$$

is valid with $\nu(s_1) := (1-s_1)^{-q-1} - 1$.
iii) For $s_2 \in (0,1)$ *we can estimate*

$$(1-s)^{q+1} \le 1 - (1-\theta(s_2)) \cdot (q+1)s \qquad \forall\, s \in [0, s_2]$$

with $\theta(s_2) = 1 - (1-s_2)^q$.

4.1 Subsolutions near $\xi = 0$

We first intend to detect up to which maximal value of ξ the constant function $\xi \mapsto \delta$ remains a subsolution. This is prepared by

Lemma 4.2 *Given $\Lambda \geq 0$, let*

$$P_\delta(\xi) := -\Big(\frac{1}{q+1} + \Lambda\Big)\delta + w^{-q}(\xi) - \Big(w(\xi) + \delta\Big)^{-q} \qquad \text{for } \xi > 0 \text{ and } \delta \in (0,1).$$

Then $P'_\delta(\xi) < 0$ for all $\xi \in (0,\infty)$, and there exists a unique $\bar{\xi}_\delta \in (0,\infty)$ such that $P_\delta(\bar{\xi}_\delta) = 0$. Moreover, we have $\liminf_{\delta\to 0} \bar{\xi}_\delta \geq \bar{\xi}$, where

$$\bar{\xi} := \sqrt{\frac{qA_{n,q}^{-q-1}}{\frac{1}{q+1} + \Lambda}}. \tag{4.1}$$

Proof. Since $w' > 0$, we have

$$\begin{aligned} P'_\delta(\xi) &= -qw^{-q-1}w' + q(w+\delta)^{-q-1}w' \\ &< 0 \qquad \forall\, \xi > 0 \end{aligned}$$

due to the fact that $\delta > 0$. Thus P_δ has a unique positive zero $\bar{\xi}_\delta$, because $P_\delta(\xi) \to +\infty$ as $\xi \to 0$ and $P_\delta(\xi) \to -(\frac{1}{q+1} + \Lambda) < 0$ as $\xi \to \infty$. To see that $\liminf_{\delta\to 0} \bar{\xi}_\delta \geq \bar{\xi}$ we only need to show that for any fixed $\xi < \bar{\xi}$ we have $P_\delta(\xi) > 0$ for sufficiently small δ. To this end we fix $\mu > 0$ small such that

$$-\frac{1}{q+1} - \Lambda + (1-\mu)qA_{n,q}^{-q-1}\xi^{-2} > 0, \tag{4.2}$$

which is possible for any $\xi < \bar{\xi}$. Next, we choose $s_0 > 0$ small such that $\mu(s_0) \equiv 1 - (1 + s_0)^{-q-1} < \mu$. Then for all $\delta < s_0 \cdot w(\xi)$ we infer from Lemma 4.1 that

$$\begin{aligned} P_\delta(\xi) &= -\Big(\frac{1}{q+1} + \Lambda\Big)\delta + w^{-q}(\xi) - w^{-q}(\xi) \cdot \Big(1 + \frac{\delta}{w(\xi)}\Big)^{-q} \\ &\geq -\Big(\frac{1}{q+1} + \Lambda\Big)\delta + (1 - \mu(s_0))qw^{-q-1}(\xi)\delta \\ &> \Big[-\frac{1}{q+1} - \Lambda + (1-\mu)qA_{n,q}^{-q-1}\xi^{-2}\Big]\delta \\ &> 0 \end{aligned}$$

by (4.2), whence the proof is complete. ////

Remark. It is easy to see that P_δ is negative on $[\bar{\xi}, \infty)$ for all δ, whence it actually follows that $\bar{\xi}_\delta \to \bar{\xi}$ as $\delta \to 0$, but we do not need this in the sequel.

As an obvious consequence of Lemma 4.2 we state

Corollary 4.3 *Let $\Lambda \geq 0$. Then there exists $\delta_0 > 0$ such that for all $\delta \leq \delta_0$,*

$$L_\Lambda(\delta) \equiv -\Big(\frac{1}{q+1} + \Lambda\Big)\delta + w^{-q}(\xi) - (w(\xi) + \delta)^{-q} \geq 0 \qquad \forall\, \xi < \frac{\bar{\xi}}{2}.$$

4.2 Subsolutions in the intermediate region

The shape of our candidates for subsolutions in a region away from $\xi = 0$ is strongly aligned with the functions formally found in (2.10) and (2.11); accordingly, the criticality of $q_c(n)$ will now become evident. Since we are interested in qualitative behavior only, we do not aim at rediscovering the exact exponents appearing proposed in these equations. This significantly eases the proof and particularly enables us to avoid a distinction between the cases $q \neq \frac{1}{3}$ and $q = \frac{1}{3}$ in the one-dimensional setting, as suggested by (2.11).

Lemma 4.4 *Let $n \leq 6$ and $q > q_c(n)$. Then for any $\Lambda \geq 0$, each $\bar{\xi}_0 > 0$ and all sufficiently small $\delta > 0$ there exist positive numbers ξ_0 and ξ_1 with $\xi_1 < \xi_0 < \bar{\xi}_0$ and a function $f_{im} \in C^2([\xi_1, \xi_0])$ such that*

$$f_{im} > 0 \quad \text{on } (\xi_1, \xi_0), \qquad f_{im}(\xi_0) = 0, \qquad f'_{im}(\xi_0) < 0, \tag{4.3}$$

$$f_{im}(\xi_1) = \delta \qquad \text{and} \qquad f'_{im}(\xi_1) = 0 \tag{4.4}$$

hold as well as

$$L_\Lambda f_{im} \geq 0 \qquad \text{on } (\xi_1, \xi_0).$$

PROOF. According to the different formal expansions obtained in (2.10) and (2.11), we split the proof into the cases $n \geq 2$ and $n = 1$.
i) For $2 \leq n \leq 6$, we shall take f_{im} as a decreasing positive branch of the rapidly oscillating function

$$f(\xi) := \eta \xi^{-\gamma} \sin(\ln \xi^{\kappa\gamma}), \tag{4.5}$$

where γ is close to $\frac{n-2}{2}$ and η and κ are sufficiently small positive numbers.
To be more precise, observe that due to $q > q_c(n)$, we have $qA_{n,q}^{-q-1} > \frac{n-2}{2}$, whence it is possible to choose $\tilde{q} \in (0, q)$ still fulfilling $\tilde{q}A_{n,q}^{-q-1} > \frac{n-2}{2}$, and then $s_0 > 0$ small such that the constant $\mu(s_0) \equiv 1 - (1 + s_0)^{-q-1}$ from Lemma 4.1 satisfies $(1 - \mu(s_0))q \geq \tilde{q}$. We next pick $\gamma \in (\frac{n-2}{2}, \tilde{q}A_{n,q}^{-q-1})$, which enables us to find a positive κ with $\kappa^2 < \frac{\tilde{q}A_{n,q}^{-q-1}}{\gamma^2} - 1$.

Then for any $\eta > 0$, f as defined through (4.5) is positive in (ξ_m^-, ξ_m^+) and has a zeros in $\xi_m^\pm$, where $\ln(\xi_m^-)^{\kappa\gamma} = -2m\pi$ and $\ln(\xi_m^+)^{\kappa\gamma} = -(2m-1)\pi$, that is,

$$\xi_m^- = \exp\Big(-\frac{2m\pi}{\kappa\gamma}\Big) \qquad \text{and} \qquad \xi_m^+ = \exp\Big(-\frac{(2m-1)\pi}{\kappa\gamma}\Big)$$

for arbitrary integers m. Since $\xi_m^+ \to 0$ as $m \to \infty$, it is clear that we can fix a large positive m such that $\xi_0 := \xi_m^+$ lies below $\bar{\xi}_0$ and satisfies

$$\xi_0 \le \sqrt{\frac{\tilde{q}A_{n,q}^{-q-1} - (1+\kappa^2)\gamma^2}{\frac{\gamma}{2} + \frac{1}{q+1} + \Lambda}} \qquad \text{and} \qquad \xi_0 \le \sqrt{2(2\gamma + 2 - n)}. \tag{4.6}$$

We now let ξ_1 denote the unique point in (ξ_m^-, ξ_m^+) where f attains its maximum over this interval, i.e. we set

$$\xi_1 := \exp\Big(\frac{\arctan\kappa - 2m\pi}{\kappa\gamma}\Big).$$

In order to complete the definition of f, we finally let

$$\begin{array}{rcl} \eta_0 & := & s_0 \cdot A_{n,q} \cdot \xi_1^{\gamma + \frac{2}{q+1}} \qquad \text{and} \\ \delta_0 & := & \eta_0 \xi_1^{-\gamma} \sin(\ln \xi_1^{\kappa\gamma}) \end{array}$$

and, given any $\delta < \delta_0$, set $\eta := \frac{\delta}{\xi_1^{-\gamma}\sin(\ln\xi_1^{\kappa\gamma})} \in (0, \eta_0)$. This guarantees that $f(\xi_1) = \delta$, $f'(\xi_1) = 0$ and, moreover, that f satisfies

$$\frac{f(\xi)}{w(\xi)} \le \frac{\eta}{A_{n,q}} \xi^{-\gamma - \frac{2}{q+1}} \le s_0 \qquad \forall\, \xi \in (\xi_1, \xi_0).$$

As a consequence, Lemma 4.1 tells us that

$$\begin{array}{rcl} w^{-q} - (w+f)^{-q} & = & w^{-q} - w^{-q}\Big(1 + \dfrac{f}{w}\Big)^{-q} \\ & \ge & w^{-q} - w^{-q}\Big[1 - (1+\mu(s_0))q\dfrac{f}{w}\Big] \\ & = & (1+\mu(s_0))qw^{-q-1}f \\ & \ge & \tilde{q}A_{n,q}^{-q-1}\xi^{-2}f \qquad \forall \xi \in (\xi_1, \xi_0). \end{array}$$

Thus, computing

$$\begin{array}{rcl} f'(\xi) & = & -\eta\gamma\xi^{-\gamma-1}\sin\Big(\ln(\xi^{\kappa\gamma})\Big) + \eta\kappa\gamma\xi^{-\gamma-1}\cos\Big(\ln(\xi^{\kappa\gamma})\Big), \\ f''(\xi) & = & \eta[\gamma(\gamma+1) - \kappa^2\gamma^2]\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big) - \eta\kappa\gamma(2\gamma+1)\xi^{-\gamma-2}\cos\Big(\ln(\xi^{\kappa\gamma})\Big), \end{array}$$

we obtain

$$\begin{aligned}
L_\Lambda f &\geq \eta[\gamma(\gamma+1)-\kappa^2\gamma^2-(n-1)\gamma]\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\quad -\eta[\kappa\gamma(2\gamma+1)-(n-1)\kappa\gamma]\xi^{-\gamma-2}\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\quad -\frac{\eta\gamma}{2}\xi^{-\gamma}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)+\frac{\eta\kappa\gamma}{2}\xi^{-\gamma}\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\quad -\eta\Big(\frac{1}{q+1}+\Lambda\Big)\xi^{-\gamma}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)+\eta\cdot\tilde{q}A_{n,q}^{-q-1}\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&= \eta\cdot\Big\{[(1-\kappa^2)\gamma^2-(n-2)\gamma+\tilde{q}A_{n,q}^{-q-1}]\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\qquad -\kappa\gamma(2\gamma+2-n)\xi^{-\gamma-2}\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\qquad -\Big(\frac{\gamma}{2}+\frac{1}{q+1}+\Lambda\Big)\xi^{-\gamma}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\qquad +\frac{\kappa\gamma}{2}\xi^{-\gamma}\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\Big\}\qquad \forall\,\xi\in(\xi_1,\xi_0).
\end{aligned}$$

Let S denote the set of points in (ξ_1,ξ_0) where $\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\geq 0$. For $\xi\in S$, we observe that $\xi>\xi_1$ implies $\tan\xi\geq\kappa$, so that $\kappa\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\leq\sin\Big(\ln(\xi^{\kappa\gamma})\Big)$. Hence

$$\begin{aligned}
\frac{1}{\eta}\cdot L_\Lambda f &\geq \Big[(1-\kappa^2)\gamma^2-(n-2)\gamma)+\tilde{q}A_{n,q}^{-q-1}-\gamma(2\gamma+2-n)\Big]\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\quad -\Big(\frac{\gamma}{2}+\frac{1}{q+1}+\Lambda\Big)\xi_0^2\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&= \Big[-(1+\kappa^2)\gamma^2+\tilde{q}A_{n,q}^{-q-1}-\Big(\frac{\gamma}{2}+\frac{1}{q+1}+\Lambda\Big)\xi_0^2\Big]\xi^{-\gamma-2}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\geq 0\qquad\forall\,\xi\in S
\end{aligned}$$

in view of the first condition in (4.6). If $\xi\in(\xi_1,\xi_0)\setminus S$, however, we have

$$\begin{aligned}
&-\kappa\gamma(2\gamma+2-n)\xi^{-\gamma-2}\cos\Big(\ln(\xi^{\kappa\gamma})\Big)+\frac{\kappa\gamma}{2}\xi^{-\gamma}\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\\
&\qquad =\kappa\gamma\xi^{-\gamma-2}\cdot\Big[-\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\Big]\cdot\Big[2\gamma+2-n-\frac{1}{2}\xi^2\Big]\\
&\qquad =\kappa\gamma\xi^{-\gamma-2}\cdot\Big[-\cos\Big(\ln(\xi^{\kappa\gamma})\Big)\Big]\cdot\Big[2\gamma+2-n-\frac{1}{2}\xi_0^2\Big]\\
&\qquad =0
\end{aligned}$$

by the second restriction in (4.6). Therefore

$$\frac{1}{\eta}\cdot L_\Lambda f \geq \Big[(1-\kappa^2)\gamma^2-(n-2)\gamma+\tilde{q}A_{n,q}^{-q-1}-\Big(\frac{\gamma}{2}+\frac{1}{q+1}+\Lambda\Big)\xi_0^2\Big]\xi^{-\gamma}\sin\Big(\ln(\xi^{\kappa\gamma})\Big)$$

$$\begin{aligned} &\geq \Big[(1-\kappa^2)\gamma^2 - (n-2)\gamma + (1+\kappa^2)\gamma^2\Big]\xi^{-\gamma}\sin\Big(\ln(\xi^{\kappa\gamma})\Big) \\ &= [2\gamma^2 - (n-2)\gamma]\xi^{-\gamma}\sin\Big(\ln(\xi^{\kappa\gamma})\Big) \\ &\geq 0 \qquad \forall\,\xi \in (\xi_1,\xi_0)\setminus S \end{aligned}$$

due to (4.6) and the fact that $\gamma > \frac{n-2}{2}$. Altogether, we conclude that $L_\Lambda f \geq 0$ in (ξ_1, ξ_0), so that $f_{im} := f|_{[\xi_1,\xi_0]}$ has all the desired properties.

ii) The case $n = 1$ runs similarly, involving

$$f(\xi) := \eta\Big(\xi^\gamma - \kappa\xi^{\frac{1}{2}}\Big)$$

instead, where η and $\gamma < \frac{1}{2}$ are small positive numbers and κ will be large.
In contrast to the former case, we now fix any $\tilde{q} \in (0, q)$ (not necessarily close to q) and then, as before, $s_0 > 0$ such that $(1 - \mu(s_0))q \geq \tilde{q}$. First choosing γ small and then $\kappa > 0$ large, we can achieve that

$$\gamma(\gamma-1) + \tilde{q}A_{1,q}^{-q-1} - \Big(\frac{1}{4} + \frac{1}{q+1} + \Lambda\Big)\kappa^{-\frac{2}{\frac{1}{2}-\gamma}} > 0 \tag{4.7}$$

holds as well as

$$\xi_0 := \Big(\frac{1}{\kappa}\Big)^{\frac{1}{\frac{1}{2}-\gamma}} < \bar{\xi}_0.$$

Introducing furthermore

$$\xi_1 := \Big(\frac{2\gamma}{\kappa}\Big)^{\frac{1}{\frac{1}{2}-\gamma}},$$

we then see that for each positive η, f will be positive and decreasing on (ξ_1, ξ_0) with $f(\xi_0) = f'(\xi_1) = 0$. Set $\eta_0 := s_0 A_{1,q}\xi_1^{\gamma-\frac{2}{q+1}}$ and $\delta_0 := \eta_0 \cdot [\xi_1^\gamma - \kappa\xi_1^{\frac{1}{2}}]$. Then, given $\delta < \delta_0$, we choose $\eta := \frac{\delta}{\xi_1^\gamma - \kappa\xi_1^{\frac{1}{2}}}$, so that $\eta < \eta_0$ and hence for all $\xi \in (\xi_1, \xi_0)$ we have

$$\frac{f(\xi)}{w(\xi)} \leq \frac{f(\xi_1)}{w(\xi_1)} \leq \frac{\eta\xi_1^\gamma}{A_{1,q}\cdot\xi_1^{\frac{2}{q+1}}} \leq s_0.$$

Therefore again $w^{-q} - (w+f)^{-q} \geq \tilde{q}A_{1,q}^{-q-1}\xi^{-2}f$ in (ξ_1, ξ_0) and, consequently,

$$\frac{1}{\eta}\cdot L_\Lambda f \;\; \geq \;\; \gamma(\gamma-1)\xi^{\gamma-2} + \frac{\kappa}{4}\xi^{-\frac{3}{2}} + \frac{\gamma}{2}\xi^\gamma - \frac{\kappa}{4}\xi^{\frac{1}{2}}$$

$$
\begin{aligned}
& -\Big(\frac{1}{q+1}+\Lambda\Big)\xi^{\gamma}+\kappa\Big(\frac{1}{q+1}+\Lambda\Big)\xi^{\frac{1}{2}} \\
& +\tilde{q}A_{1,q}^{-q-1}\xi^{\gamma-2}-\kappa\tilde{q}A_{1,q}^{-q-1}\xi^{-\frac{3}{2}} \\
\geq \;& \Big[\gamma(\gamma-1)+\tilde{q}A_{1,q}^{-q-1}-\frac{\kappa}{4}\xi^{\frac{5}{4}-\gamma}-\Big(\frac{1}{q+1}+\Lambda\Big)\Big]\xi^{\gamma-2} \\
& +\kappa\Big[\frac{1}{4}-\tilde{q}A_{1,q}^{-q-1}\Big]\xi^{-\frac{3}{2}}.
\end{aligned}
$$

Since $\tilde{q}<q$, we find

$$
\frac{1}{4}-\tilde{q}A_{1,q}^{-q-1}>\frac{1}{4}-qA_{1,q}^{-q-1}=\frac{1}{4}-\frac{2q(1-q)}{(q+1)^2}=\frac{(3q-1)^2}{4(q+1)^2}\geq 0.
$$

Furthermore, as long as $\xi<\xi_0$,

$$
\begin{aligned}
\frac{\kappa}{4}\xi^{\frac{5}{4}-\gamma}+\Big(\frac{1}{q+1}+\Lambda\Big)\xi^2 \;&\leq\; \frac{\kappa}{4}\cdot\Big(\frac{1}{\kappa}\Big)^{\frac{\frac{5}{2}-\gamma}{\frac{1}{2}-\gamma}}+\Big(\frac{1}{q+1}+\Lambda\Big)\cdot\Big(\frac{1}{\kappa}\Big)^{\frac{2}{\frac{1}{2}-\gamma}} \\
&=\; \Big(\frac{1}{4}-\frac{1}{q+1}+\Lambda\Big)\kappa^{-\frac{2}{\frac{1}{2}-\gamma}}.
\end{aligned}
$$

Hence, using (4.7) we obtain $L_\Lambda f\geq 0$ in (ξ_1,ξ_0). ////

4.3 Subsolutions in the outer region

We now make sure that it is possible to construct a negative subsolution starting from a given positive zero with prescribed negative initial slope and certain asymptotics near infinity. This behavior for large ξ ensures that on the one hand $|f|$ remains small as compared to w, so that one more linearization will be justifyable (cf. the proof of Lemma 5.2). On the other hand, $|f|$ decays so slowly at infinity that the induced parabolic subsolution remains below u_ε on some line $|x|=\xi_2 t^{\frac{1}{2}}$, even if the initial data are above w *only near* $x=0$ (which will also be important in the proof of Lemma 5.2).

Lemma 4.5 *Let* $\Lambda\geq 0$ *and* $q\in(0,1)$. *Then, given* $\xi_0>0$ *and* $c_0>0$, *there exists* $f_{out}\in W^{2,\infty}((\xi_0,\infty))$ *such that*

$$
\begin{aligned}
& f_{out}(\xi_0)=0, \quad f'_{out}(\xi_0)=-c_0, \quad -w<f_{out}<0 \text{ on } (\xi_0,\infty) \qquad \text{and} \\
& \frac{f_{out}(\xi)}{w(\xi)}\to 0 \quad \text{as well as} \quad w^q(\xi)f_{out}(\xi)\to-\infty \qquad \text{as } \xi\to\infty
\end{aligned}
\tag{4.8}
$$

hold, and such that

$$
L_\Lambda f\geq 0 \qquad a.e. \text{ in } (\xi_0,\infty) \tag{4.9}
$$

is valid.

PROOF. For small positive $\delta < 1$ to be specified below, let

$$f_2(\xi) := c_\delta \cdot (\xi_\delta - \xi)^2 - \delta \qquad \text{for } \xi \in \Big[\xi_0, \xi_0 + \frac{4\delta}{c_0}\Big],$$

where c_δ and ξ_δ are adjusted in such a way that $f_2(\xi_0) = 0$ and $f_2'(\xi_0) = -c_0$, that is,

$$c_\delta = \frac{c_0^2}{4\delta} \qquad \text{and} \qquad \xi_\delta = \xi_0 + \frac{2\delta}{c_0}.$$

Then $f_2 < 0$ on $(\xi_0, \xi_0 + \frac{4\delta}{c_0})$ and if we restrict δ so as to satisfy $\delta \le \frac{1}{2} w(\xi_0)$ then we have the lower estimate $\frac{f_2(\xi)}{w(\xi)} \ge \frac{-\delta}{w(\xi_0)} \ge -\frac{1}{2}$. Therefore from Lemma 4.1 we find $(1 + \frac{f_2}{w})^{-q} \le 1 - q \cdot 2^{q+1} \frac{f_2}{w}$ and hence

$$\begin{aligned} w^{-q} - (w + f_2)^{-q} &= w^{-q}\Big[1 - \Big(1 + \frac{f_2}{w}\Big)^{-q}\Big] \\ &\ge q \cdot 2^{q+1} w^{-q-1} f_2 \\ &\ge -\frac{a}{\xi_2}\delta \qquad \text{in } \Big(\xi_0, \xi_0 + \frac{4\delta}{c_0}\Big) \end{aligned}$$

with $a := q \cdot 2^{q+1} A_{n,q}^{-q-1}$. Thus, calculating the derivatives of f_2 and using $f_2 \le 0$ we can estimate

$$\begin{aligned} Lf_2 &\ge 2c_\delta - \frac{n-1}{\xi} \cdot 2c_\delta(\xi_\delta - \xi) - c_\delta \xi(\xi_\delta - \xi) - \Big(\frac{1}{q+1} + \Lambda\Big) f_2 - \frac{a}{\xi^2}\delta \\ &\ge 2c_\delta - \frac{n-1}{\xi_0} \cdot 2c_\delta(\xi_\delta - \xi_0) - c_\delta \xi_\delta(\xi_\delta - \xi_0) - \frac{a}{\xi_0^2}\delta \\ &= \frac{c_0^2}{2\delta} - \frac{n-1}{\xi_0} \cdot \frac{c_0^2}{2\delta} \cdot \frac{2\delta}{c_0} - \frac{c_0^2}{4\delta} \cdot \Big(\xi_0 + \frac{2\delta}{c_0}\Big) \cdot \frac{2\delta}{c_0} - \frac{a}{\xi_0^2}\delta \qquad \text{in } \Big(\xi_0, \xi_0 + \frac{4\delta}{c_0}\Big). \end{aligned}$$

Choosing now δ small enough we can achieve $Lf_2 \ge 0$ on $(\xi_0, \xi_0 + \frac{4\delta}{c_0})$. With this value of δ and any $\gamma \in (0, \frac{2q}{q+1})$ fixed henceforth, we observe that since f_2 attains a negative minimum at $\xi = \xi_\delta$, it is possible to pick some $\xi_1 > \xi_\delta$ close to ξ_δ such that $c_1 := -f_2(\xi_1) \in (0, \frac{1}{2} w(\xi_0))$ and $c_2 := f_2'(\xi_1)$ is a positive number small enough such that

$$\frac{\gamma^2 \xi_0 c_1}{2c_2} \ge \gamma(\gamma + 1) + a. \tag{4.10}$$

We then define

$$f_3(\xi) := -d(\xi - \xi_2)^{-\gamma} \qquad \text{for } \xi \in [\xi_1, \infty),$$

where d and ξ_2 are such that $f_3(\xi_1) = f_2(\xi_1) = -c_1$ and $f_3'(\xi_1) = f_2'(\xi_1) = c_2$, i.e.

$$d = \frac{\gamma^\gamma c_1^{\gamma+1}}{c_2^\gamma} \qquad \text{and} \qquad \xi_2 = \xi_1 - \frac{\gamma c_1}{c_2}.$$

Then f_3 increases on (ξ_1, ∞), so that particularly $\frac{f_3(\xi)}{w(\xi)} \geq \frac{f_3(\xi_1)}{w(\xi)} \geq \frac{f_3(\xi_1)}{w(\xi_0)} > -\frac{1}{2}$ and thus again

$$w^{-q} - (w + f_3)^{-q} \geq \frac{a}{\xi^2} f_3 \qquad \text{on } (\xi_1, \infty).$$

Consequently, noting $f_3 \leq 0$ and $f_3' \geq 0$ we obtain

$$\begin{aligned}
Lf_3 &\geq -d\gamma(\gamma+1)(\xi - \xi_2)^{-\gamma-2} + \frac{\xi}{2} \cdot \gamma d(\xi - \xi_2)^{-\gamma-1} - \frac{a}{\xi^2} d(\xi - \xi_2)^{-\gamma} \\
&= d(\xi - \xi_2)^{-\gamma-2} \cdot \Big[-\gamma(\gamma+1) + \frac{\gamma\xi}{2}(\xi - \xi_2) - a\Big(\frac{\xi - \xi_2}{\xi}\Big)^2 \Big] \\
&\geq d(\xi - \xi_2)^{-\gamma-2} \cdot \Big[-\gamma(\gamma+1) + \frac{\gamma\xi_0}{2}(\xi_1 - \xi_2) - a \Big] \\
&= d(\xi - \xi_2)^{-\gamma-2} \cdot \Big[-\gamma(\gamma+1) + \frac{\gamma\xi_0}{2} \cdot \frac{\gamma c_1}{c_2} - a \Big] \\
&\geq 0 \qquad \text{in } (\xi_1, \infty)
\end{aligned}$$

in view of (4.10). As we chose γ positive but smaller than $\frac{2q}{q+1}$, we furthermore have $f_3(\xi) \to 0$ (whence also $\frac{f_3(\xi)}{w(\xi)} \to 0$) and

$$w^q(\xi) f_3(\xi) = -dA_{n,q}^q \xi^{\frac{2q}{q+1} - \gamma} \to -\infty$$

as $\xi \to \infty$.
It is now evident that

$$f_{out}(\xi) := \begin{cases} f_2(\xi), & \xi \in [\xi_0, \xi_1], \\ f_3(\xi), & \xi \in (\xi_1, \infty), \end{cases}$$

belongs to $W^{2,\infty}((\xi_0, \infty))$ and fulfills $Lf \geq 0$ a.e. in (ξ_0, ∞). ////

4.4 Subsolutions on the whole interval $[0, \infty)$

All that remains is to glue the three subsolutions together appropriately to obtain

Lemma 4.6 *Suppose $n \le 6$ and $q > q_c(n)$. Then for each $\Lambda \ge 0$ there exists a function $f \in W^{2,\infty}((0,\infty))$ with the properties*

$$f(0) > 0, \quad f'(0) = 0,$$
$$\frac{f(\xi)}{w(\xi)} \to 0 \quad \textit{and} \quad w^q(\xi) f(\xi) \to -\infty \quad \textit{as } \xi \to \infty$$

as well as

$$L_\Lambda f \equiv f'' + \frac{n-1}{\xi} f' + \frac{\xi}{2} f' - \Big(\frac{1}{q+1} + \Lambda\Big) f + w^{-q} - (w+f)^{-q} \ge 0 \qquad \textit{a.e. in } (0,\infty).$$

Proof. Our plan is to set

$$f(\xi) := \begin{cases} \delta, & \xi \le \xi_1, \\ f_{im}(\xi), & \xi_1 < \xi \le \xi_0, \\ f_{out}(\xi), & \xi > \xi_0, \end{cases} \tag{4.11}$$

with suitably small $\delta > 0$, $0 < \xi_1 < \xi_0$ and the functions f_{im} and f_{out} being taken from Lemma 4.4 and Lemma 4.5, respectively.
For this purpose, let δ_0 and $\bar{\xi}$ be as in Corollary 4.3. Then from Lemma 4.4 we know there exist numbers ξ_0 and ξ_1 with $0 < \xi_1 < \xi_0 < \frac{\bar{\xi}}{2}$ and a function $f_{im} \in C^2([\xi_1, \xi_0])$ such that

$$\delta := f_{im}(\xi_1) \le \delta_0, \qquad f'_{im}(\xi_1) = 0,$$
$$f_{im}(\xi_0) = 0, \qquad f'_{im}(\xi_0) < 0$$

and

$$L_\Lambda f_{im} \ge 0 \qquad \text{in } (\xi_1, \xi_0).$$

Furthermore, since $\delta \le \delta_0$ and $\xi_1 < \frac{\bar{\xi}}{2}$, it results from Corollary 4.3 that $L_\Lambda(\delta) \ge 0$ on $(0, \xi_1)$.
Let now f_{out} be as provided by Lemma 4.5 with prescribed derivative $f'_{out}(\xi_0) = f'_{im}(\xi_0)$. It is then an immediate consequence of our construction that f defined by (4.11) has all the desired properties. ////

5 Nonuniqueness: The main results

5.1 Solutions remaining below steady states

In the first part of this section we shall utilize our convergence lemma 3.4 (resp. Corollary 3.6) to show that if the initial data u_0 are bounded above by some of the mentioned explicit

steady states then the same will be true for *some* continuous weak solution evolving from u_0. This may be little surprising and one might suspect it to be a trivial consequence of some comparison argument. However, as we shall see below in Theorems 1.1 and 1.3, this implication does in general *not* hold for the maximal solution.
For the construction of suitable candidates for comparison, let us introduce for any $n \in \mathbb{N}$, $q \in (0,1)$ and $\delta > 0$ the radially symmetric regular steady states $w_\delta = w_{n,q,\delta}$ defined through the initial-value problem

$$\begin{aligned} w_{\delta rr} + \tfrac{n-1}{r} w_{\delta r} &= w_\delta^{-q} \quad \text{in } (0,\infty), \\ w_\delta(0) = \delta, \qquad & w_{\delta r}(0) = 0. \end{aligned} \tag{5.1}$$

Here and below we will identify w_δ with the induced radial function $\tilde{w}_\delta(x) := w_\delta(|x|)$ defined on $\mathbb{R}^n$. It is known (see [6] and [24, Lemma 6.2.2], for instance) that

$$w_{n,q,\delta} \to w_{n,q} \quad \text{locally uniformly in } \mathbb{R}^n. \tag{5.2}$$

Lemma 5.1 *If there exists $N \le n$ such that*

$$u_0(x) \le w_{N,q}(x_1, ..., x_N) \equiv A_{N,q} \cdot \Big|(x_1, ..., x_N)\Big|^{\frac{2}{q+1}} \qquad \forall\, x = (x_1, ..., x_N, ..., x_n) \in \bar{\Omega}$$

then the exists a continuous weak solution $\tilde{u}$ of (1.1) satisfying

$$\tilde{u}(x,t) \le w_{N,q}(x_1, ..., x_N) \qquad \forall\, (x,t) \in \Omega \times (0,\infty). \tag{5.3}$$

PROOF. For brevity let us write $w = w_{N,q}$ and $x = (x', x'')$ with $x' = (x_1, ..., x_N)$ and $x'' = (x_{N+1}, ..., x_n)$.
For $\delta > 0$, let w_δ be the regular steady state in $\mathbb{R}^N$ given by (5.1). Then, since

$$w_\delta(x') \to w(x') \qquad \text{as } \delta \to 0 \quad \text{uniformly in bounded subsets of } \mathbb{R}^N \tag{5.4}$$

by (5.2), we can choose a nondecreasing sequence of nonnegative functions $u_{0k} \in C^0(\bar{\Omega})$ satisfying

$$u_{0k}(x) < w_{\frac{1}{k}}(x') \qquad \text{in } \bar{\Omega} \qquad \text{and} \tag{5.5}$$

$$u_{0k} \nearrow u_0 \qquad \text{in } \bar{\Omega}. \tag{5.6}$$

Then Corollary 3.6 ensures that the maximal solutions u_k of (1.1) emanating from u_{0k} increase to a continuous weak solution $\tilde{u}$ with initial data u_0. In order to estimate $\tilde{u}$ from above, we consider the solutions $u_{k\varepsilon}$ of (3.2) with suitable initial values $u_{0k\varepsilon}$ lying between $u_{0k}+\varepsilon$ and $u_{0k}+2\varepsilon$. Due to our particular choice of g_ε, we have $g_\varepsilon(s) = s^{-q}$ for all $s \ge 2\varepsilon$, whence for any $\delta > 0$,

$$w_{\delta t} - \Delta w_\delta - g_\varepsilon(w_\delta) = -\Delta w_\delta - w_\delta^{-q} = 0 \qquad \text{in } \Omega \times (0,\infty)$$

holds for all $\varepsilon \le \frac{\delta}{2}$, because $w_\delta \ge \delta$. Therefore, from (5.5) and a comparison argument we find $u_{k\varepsilon}(x,t) \le w_{\frac{1}{k}}(x')$ in $\Omega \times (0,\infty)$ for sufficiently small ε, so that $u_k = \lim_{\varepsilon\to 0} u_{k\varepsilon}$ satisfies

$$u_k(x,t) \le w_{\frac{1}{k}}(x') \qquad \forall\,(x,t) \in \Omega \times (0,\infty). \tag{5.7}$$

In view of (5.4), (5.3) immediately follows from (5.7). ////

5.2 Solutions instantaneously lifting off steady states

On the other hand, the subsolutions supplied by Section 4 give rise to maximal solutions which immediately lift off at $x = 0$, provided that q is supercritical and u_0 does not fall below $w_{n,q}$ near the origin. A combination of the lower Lipschitz estimate in time (Lemma 3.3) with the behavior of f for large ξ (Lemma 4.6) makes it possible to abstain from any further condition on u_0 such as, for instance, a further estimate from below on the whole domain.

Lemma 5.2 *Suppose $n \le 6$, $q_c(n) < q < 1$, and that u_0 satisfies $u_0^{\frac{q+1}{2}} \in W^{1,\infty}(\Omega)$, $\Delta u_0 \ge -c_0$ in the sense of distributions on Ω and*

$$u_0(x) \ge w(x) \equiv A_{n,q}|x|^{\frac{2}{q+1}} \qquad \forall\, x \in \bar{B}_R(0) \subset \Omega \tag{5.8}$$

with some $c_0 > 0$ and $R > 0$. Then there exist $t_0 > 0$ and $\delta > 0$ such that the maximal continuous weak solution u of (1.1) satisfies

$$u(0,t) \ge \delta t^{\frac{1}{q+1}} \qquad \forall\, t \in (0,t_0). \tag{5.9}$$

PROOF. Since $\Delta u_0 \ge -c_0$ in Ω and $u_0 \ge w$ in $\bar{B}_R(0)$, we can apply Lemma 3.3 with $K := \bar{B}_R(0)$ to obtain

$$\begin{aligned} u_\varepsilon^{q+1}(x,t) &\ge u_{0\varepsilon}^{q+1}(x) - c_K t \\ &\ge w^{q+1}(x) - c_K t \qquad \forall\,(x,t) \in \bar{B}_R(0) \times (0,\infty). \end{aligned} \tag{5.10}$$

Let us take f from Lemma 4.6 (with $\Lambda := 0$) and $\theta := \theta(\frac{1}{2})$ as in Lemma 4.1 iii). Then There exists some large ξ_2 such that

$$\frac{f(\xi_2)}{w(\xi_2)} \ge -\frac{1}{2} \tag{5.11}$$

and

$$w^q(\xi_2) \cdot f(\xi_2) \le \frac{-c_K}{(1-\theta)(q+1)}. \tag{5.12}$$

We set $t_0 := (\frac{R}{\xi_2})^2$ and define the comparison function

$$u^-(x,t) := w(x) + t^{\frac{1}{q+1}} f(t^{-\frac{1}{2}}|x|), \qquad (x,t) \in Q,$$

on the parabolic domain

$$Q := \Big\{(x,t) \in \mathbb{R}^{n+1} \mid t \in (0,t_0) \text{ and } |x| < \xi_2 t^{\frac{1}{2}}\Big\}.$$

If $|x| = \xi_2 t^{\frac{1}{2}}$ then, by the identity $w(x) = t^{\frac{1}{q+1}} w(\xi_2)$, we obtain from (5.11) that

$$\frac{t^{\frac{1}{q+1}} f(t^{-\frac{1}{2}}|x|)}{w(x)} = \frac{f(\xi_2)}{w(\xi_2)} \geq -\frac{1}{2}.$$

Therefore Lemma 4.1 iii) and (5.12) entail that if $|x| = \xi_2 t^{\frac{1}{2}}$ then

$$\begin{aligned}
(u^-(x,t))^{q+1} &= w^{q+1}(x) \cdot \Big(1 + \frac{t^{\frac{1}{q+1}} f(t^{-\frac{1}{2}}|x|)}{w(x)}\Big)^{q+1} \\
&\leq w^{q+1}(x) \cdot \Big(1 + (1-\theta)(q+1) \cdot \frac{t^{\frac{1}{q+1}} f(\xi_2)}{w(x)}\Big) \\
&\leq w^{q+1}(x) + (1-\theta)(q+1) \cdot w^q(x) \cdot t^{\frac{1}{q+1}} f(\xi_2) \\
&= w^{q+1}(x) + (1-\theta)(q+1) \cdot w^q(\xi_2) \cdot f(\xi_2) \cdot t \\
&\leq w^{q+1}(x) - c_K t \\
&\leq u_\varepsilon^{q+1}(x,t) \qquad \forall\, t \in (0,t_0)
\end{aligned}$$

in virtue of (5.10), because our choice of t_0 guarantees that such x satisfy $|x| \leq \xi_2 t_0^{\frac{1}{2}} = R$ for all $t < t_0$. Thus, $z := u^- - u_\varepsilon$ fulfills

$$z \leq 0 \quad \text{on } \partial\Omega(t) \qquad \forall\, t \in (0,t_0), \tag{5.13}$$

where $\Omega(t) := \{x \in \mathbb{R}^n \mid |x| < \xi_2 t^{\frac{1}{2}}\}$. Furthermore, from the differential inequality satisfied by f we derive, writing $\xi = t^{-\frac{1}{2}}|x|$,

$$\begin{aligned}
u_t^- &- \Delta u^- + g_\varepsilon(u^-) \leq u_t^- - \Delta u^- + (u^-)^{-q} \\
&= \frac{1}{q+1} t^{-\frac{-q}{q+1}} f(\xi) - \frac{1}{2} t^{-\frac{q}{q+1}} \xi f'(\xi) - \Delta w(x) - t^{-\frac{q}{q+1}} \Big(f''(\xi) + \frac{n-1}{\xi} f'(\xi)\Big) \\
&\quad + \Big(w(x) + t^{\frac{1}{q+1}} f(\xi)\Big)^{-q} \\
&= -t^{-\frac{q}{q+1}} \cdot \Big[f''(\xi) + \frac{n-1}{\xi} f'(\xi) + \frac{\xi}{2} f'(\xi) - \frac{1}{q+1} f(\xi) + w^{-q}(\xi) - \Big(w(\xi) + f(\xi)\Big)^{-q}\Big] \\
&\leq 0 \qquad \text{a.e. in } Q,
\end{aligned}$$

where we again have used the scaling property $w(x) = t^{\frac{1}{q+1}} w(\xi)$. Consequently,

$$\begin{aligned} z_t - \Delta z &\leq -g_\varepsilon(u^-) + g_\varepsilon(u_\varepsilon) \\ &\leq c_\varepsilon z_+ \qquad \text{a.e. in } Q \end{aligned}$$

holds with $c_\varepsilon := \|g'_\varepsilon\|_{L^\infty((0,\infty))}$. In view of (5.13), z_+ vanishes on $\partial\Omega(t)$ for each $t \in (0, t_0)$, which we apply twice in deducing

$$\begin{aligned} \frac{1}{2}\int_{\Omega(t)} z_+^2(\cdot,t) &= \frac{1}{2}\int_{\Omega(\tau)} z_+^2(\cdot,\tau) \int_\tau^t \int_{\Omega(s)} z_+ \cdot z_t \\ &\leq -\int_\tau^t \int_{\Omega(s)} |\nabla z_+|^2 + c_\varepsilon \int_\tau^t \int_{\Omega(s)} z_+^2(s) \qquad \forall\, 0 < \tau < t < t_0. \end{aligned} \tag{5.14}$$

Since furthermore the assumption $u_{0\varepsilon} \geq u_0 + \varepsilon$ in $\bar\Omega$ together with the continuity of u_ε and u^- implies that $u^- \leq u_\varepsilon$ for small t, Gronwall's lemma turns (5.14) into the inequality $z_+ \leq 0$ in Q. Thus, $u_\varepsilon \geq u^-$ in Q and, particularly,

$$u_\varepsilon(0,t) \geq u^-(0,t) = f(0) t^{\frac{1}{q+1}},$$

which results in (5.9) with $\delta := f(0) > 0$. ////

Proof (of Theorem 1.1). We only need to combine Lemma 5.1 with Lemma 5.2; in applying the latter, we observe that the lower estimate $\Delta u_0 \geq -c_0$ holds for any u_0 from the hypothesis of Theorem 1.1, because $\Delta u_0 \equiv \Delta w \geq 0$ within some ball $B_R(0)$ and $\Delta u_0 \geq -\|u_0\|_{C^2(\bar\Omega\setminus B_R(0))}$ outside. ////

The above method can be modified to construct lift-off solutions also for arbitrary n and q. The idea here is to choose any $N \leq n$ such that q is supercritical with respect to N and then work with $w_{N,q}$, adding 'dummy' variables $x_{N+1}, ..., x_n$. This, however, means that we are no longer free to choose the boundary values at will. The proof of the following lemma is the only place in which we need to choose $\Lambda > 0$.

Lemma 5.3 *Suppose $n \geq 1$, $q \in (0,1)$, and that*

$$u_0(x) \geq w_{N,q}(x_1, ..., x_N) \equiv A_{N,q}|(x_1, ..., x_N)|^{\frac{2}{q+1}} \qquad \forall\, x = (x_1, ..., x_N, ..., x_n) \in \bar\Omega \tag{5.15}$$

with some $N \leq \min\{n, 6\}$ fulfilling $q_c(N) < q$. Then there exist $t_0 > 0$ and $\delta > 0$ such that the maximal continuous weak solution u of (1.1) satisfies

$$u(0,t) \geq \delta t^{\frac{1}{q+1}} \qquad \forall\, t \in (0, t_0). \tag{5.16}$$

Proof. Throughout the proof we abbreviate $w = w_{N,q}$ and write $x = (x', x'')$ with $x' = (x_1, ..., x_N)$ and $x'' = (x_{N+1}, ..., x_n)$. Here we may assume that $N < n$, for the case $N = n$ has already been covered by Theorem 1.1.
First observe that (5.15) implies that for all ε we have

$$u_\varepsilon(x,t) \geq w(x') \qquad \text{in } \Omega \times (0,\infty) \tag{5.17}$$

by classical comparison, because u_ε is positive and w is smooth and satisfies $w_t - \Delta w + w^{-q} = 0$ classically wherever it is positive.
Since $0 \in \Omega$, there exists $R > 0$ such that the set of points $x = (x', x'')$ satisfying both $|x'| \leq R$ and $|x''| \leq R$ is contained in Ω. Let $\Lambda > 0$ denote the principal eigenvalue of the $(n-N)$-dimensional Laplacian in the ball $\{x'' \in \mathbb{R}^{n-N} \mid |x''| < R\}$ and Θ be the corresponding eigenfunction with $\max\Theta \equiv \Theta(0) = 1$. According to Lemma 4.6, we can pick $\xi_0 > 0$ and a function $f \in W^{2,\infty}((0,\xi_0))$ satisfying $f'(0) = 0$, $f(\xi) > 0$ for $\xi \in [0,\xi_0)$, $f(\xi_0) = 0$ and

$$f''(\xi) + \frac{N-1}{\xi}f'(\xi) - \Big(\frac{1}{q+1} + \Lambda\Big)f(\xi) + w^{-q}(\xi) - \Big(w(\xi) + f(\xi)\Big)^{-q} \geq 0 \quad \text{for a.e. } \xi \in (0,\xi_0). \tag{5.18}$$

We define

$$u^-(x,t) := w(x') + t^{\frac{1}{q+1}} f(t^{-\frac{1}{2}}|x'|) \cdot \Theta(x'')$$

for $(x,t) \in Q$, where

$$Q := \Big\{(x,t) \in \mathbb{R}^{n+1} \ \Big|\ t \in (0,t_0),\ |x''| < R \text{ and } |x'| < \xi_0 t^{\frac{1}{2}}\Big\}$$

lies in $\Omega \times (0,t_0)$ if we let $t_0 := \min\{1, (\frac{R}{\xi_0})^2\}$. For each cross-section $\Omega(t) := \{x \in \mathbb{R}^n | \, ||x''| < R \text{ and } |x'| < \xi_0 t^{\frac{1}{2}}\}$, $0 < t < t_0$, in view of (5.17) we find

$$u^- \leq u_\varepsilon \qquad \text{on } \partial\Omega(t), \tag{5.19}$$

because if $|x''| = R$ then $\Theta(x'') = 0$, while $f(t^{-\frac{1}{2}}|x'|) = 0$ whenever $|x'| = \xi_0 t^{\frac{1}{2}}$. In order to derive a parabolic inequality for u^-, we calculate, writing $\xi = t^{-\frac{1}{2}}|x'|$,

$$\begin{aligned} u^-_t - \Delta u^- + g_\varepsilon(u^-) &\leq u^-_t - \Delta u^- + (u^-)^{-q} \\ &= \frac{1}{q+1} t^{\frac{1}{q+1}-1} f(\xi)\Theta(x'') - \frac{1}{2} t^{\frac{1}{q+1}-\frac{3}{2}} |x'| f'(\xi)\Theta(x'') \\ &\quad -\Delta_{x'} w(x') - t^{\frac{1}{q+1}-1}\Big(f''(\xi) + \frac{N-1}{\xi}f'(\xi)\Big)\cdot\Theta(x'') \\ &\quad -t^{\frac{1}{q+1}} f(\xi)\cdot\Delta_{x''}\Theta(x'') + \Big(w(x') + t^{\frac{1}{q+1}}f(\xi)\Theta(x'')\Big)^{-q} \qquad \text{a.e. in } Q. \end{aligned}$$

Using the equations defining w and Θ and the identity $w(x') = t^{\frac{1}{q+1}} w(\xi)$, we derive from this

$$\begin{aligned}
u_t^- - \Delta u^- + g_\varepsilon(u^-) &\le \frac{1}{q+1} t^{-\frac{q}{q+1}} f\Theta - \frac{1}{2} t^{-\frac{q}{q+1}} \xi f'\Theta - t^{-\frac{q}{q+1}} w^{-q}(\xi) \\
&\quad - t^{-\frac{q}{q+1}} \Big(f'' + \frac{N-1}{\xi} f' \Big) \Theta + \Lambda t^{\frac{1}{q+1}} f\Theta + t^{-\frac{q}{q+1}} \Big(w(\xi) + f\Theta \Big)^{-q} \\
&= -t^{-\frac{q}{q+1}} \Theta \cdot \Big[f'' + \frac{N-1}{\xi} f' + \frac{\xi}{2} f' - \Big(\frac{1}{q+1} + \Lambda t \Big) f \\
&\qquad + \frac{w^{-q}(\xi) - (w(\xi) + f\Theta)^{-q}}{\Theta} \Big] \quad \text{a.e. in } Q. \qquad (5.20)
\end{aligned}$$

Fortunately, for each $\xi > 0$ the function

$$\varphi(s) := \frac{w^{-q}(\xi) - (w(\xi) + f(\xi) \cdot s)^{-q}}{s}, \qquad s > 0,$$

satisfies

$$\begin{aligned}
\varphi'(s) &= \frac{q\Big(w(\xi) + f(\xi)s\Big)^{-q-1} \cdot f(\xi)s - w^{-q}(\xi) + \Big(w(\xi) + f(\xi)s\Big)^{-q}}{s^2} \\
&\le 0 \qquad \forall\, s > 0,
\end{aligned}$$

because the convexity of $\sigma \mapsto (1-\sigma)^{-q}$ on $(0,1)$ implies that

$$w^{-q}(\xi) = \Big(w(\xi) + f(\xi)s - f(\xi)s\Big)^{-q} \ge \Big(w(\xi) + f(\xi)s\Big)^{-q} - q\Big(w(\xi) + f(\xi)s\Big)^{-q-1} \cdot f(\xi)s$$

for all $s > 0$. Therefore, since $\Theta(x'') \in (0,1]$ for all $|x''| < R$, we have $\varphi(\Theta(x'')) \ge \varphi(1)$ and thus (5.20) gives

$$\begin{aligned}
u_t^- - \Delta u^- + g_\varepsilon(u^-) &\le -t^{-\frac{q}{q+1}} \Theta \cdot \Big[f'' + \frac{N-1}{\xi} f' + \frac{\xi}{2} f' - \Big(\frac{1}{q+1} + \Lambda t \Big) f \\
&\qquad + w^{-q} - (w(\xi) + f\Theta)^{-q} \Big] \quad \text{a.e. in } Q.
\end{aligned}$$

Recalling (5.18) and the restriction $t < t_0 \le 1$ in Q, we end up with

$$u_t^- - \Delta u^- + g_\varepsilon(u^-) \le 0 \qquad \text{a.e. in } Q.$$

Now we may use the same comparison argument as in the proof of Theorem 1.1 to achieve $u^- \le u_\varepsilon$ and thereby verify (5.16) with $\delta := f(0)$. ////

Proof (of Theorem 1.3). This is an immediate consequence of Lemma 5.1 and Lemma 5.3. ////

5.3 A uniqueness result for $q \le q_c(n)$

We now prove that for $n \ge 3$ and $q \le q_c(n)$ (any $q \in (0,1)$ if $n \ge 7$), the stationary solution $u \equiv w_{n,q}$ is unique 'from above' in the sense that there is no continuous weak solution other than u which remains bounded from below by $w_{n,q}$ for all times. The argument we use essentially relies on the knowledge of the optimal constant $\frac{4}{(n-2)^2}$ in the Hardy inequality

$$\int_\Omega |x|^{-2}\varphi^2(x)dx \le \frac{4}{(n-2)^2}\int_\Omega |\nabla\varphi|^2 \qquad \forall\, \varphi \in W_0^{1,2}(\Omega) \tag{5.21}$$

which can easily be verified for $n \ge 3$ and any open set $\Omega \subset \mathbb{R}^n$ ([21]). Quite a similar reasoning was used in [20, Proposition 3.5] to show L^2 stability of steady states of $u_t = \Delta u + u^p$ for $n \ge 11$ and $p \ge p_c = \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$.

PROOF (of Lemma 1.2). Assuming that u is a solution with $u \ge w = w_{n,q}$ in $\Omega \times (0,\infty)$, we know from the continuity of u and parabolic regularity theory that wherever u is positive, u is smooth and satisfies $u_t = \Delta u - u^{-q}$. Thus, at the points where $z := u - w > 0$, we have that

$$z_t - \Delta z = -u^{-q} + w^{-q} = q\psi^{-q-1}(x,t)\cdot z$$

with some ψ fulfilling $\psi(x,t) \ge w(x) = A_{n,q}|x|^{\frac{2}{q+1}}$, because $u \ge w$. Accordingly,

$$z_t - \Delta z \le qA_{n,q}^{-q-1}|x|^{-2}z$$

at such points, which upon multiplication by $(z-\delta)_+, \delta > 0$, gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (z-\delta)_+^2 + \int_\Omega |\nabla(z-\delta)_+|^2 \le qA_{n,q}^{-q-1}\int_\Omega |x|^{-2}(z-\delta)_+^2 + \delta qA_{n,q}^{-q-1}\int_\Omega |x|^{-2}(z-\delta)_+ \tag{5.22}$$

for all $t > 0$. Using the Hardy inequality (5.21) we obtain, since $z|_{\partial\Omega} = 0$,

$$\begin{aligned} qA_{n,q}^{-q-1}\int_\Omega |x|^{-2}(z-\delta)_+^2 &\le qA_{n,q}^{-q-1}\cdot\frac{4}{(n-2)^2}\int_\Omega |\nabla(z-\delta)_+|^2 \\ &\le \int_\Omega |\nabla(z-\delta)_+|^2, \end{aligned} \tag{5.23}$$

provided that

$$\varphi(q) := qA_{n,q}^{-q-1} \equiv \frac{2q}{q+1}\Big(\frac{2}{q+1} + n - 2\Big) \le \frac{(n-2)^2}{4}. \tag{5.24}$$

In the case $n \ge 7$, we have $n - 1 \le \frac{(n-2)^2}{4}$. Hence we can use the easily verified fact that $\varphi'(q) > 0$ for $q \in (0,1)$ to obtain

$$\varphi(q) \le \varphi(1) = n - 1 \le \frac{(n-2)^2}{4} \qquad \forall\, q \in (0,1). \tag{5.25}$$

For $n \leq 6$, (5.24) is equivalent to the quadratic inequality

$$Q(q) := (10-n)(n-2)q^2 - 2(n^2-8n+4)q - (n-2)^2 \leq 0, \tag{5.26}$$

the larger root of which is precisely $q_c(n)$, while the smaller one is nonpositive, for $Q(0) \leq 0$. Therefore, if $0 < q \leq q_c(n)$ then (5.26) holds. From (5.22)-(5.26) we conclude that if (1.5) holds then

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (z-\delta)_+^2 \leq \delta q A_{n,q}^{-q-1} \int_\Omega |x|^{-2}(z-\delta)_+ \qquad \forall\, t > 0$$

for each $\delta > 0$. In the limit $\delta \to 0$ this means that $z_+ \equiv 0$ and thus $u \leq w$. ////

Acknowledgment The author would like to thank the anonymous referees for their useful remarks.

References

[1] Bandle, C., Sperb, R.P., Stakgold, I.: Diffusion and reaction with monotone kinetics. Nonlinear Anal., Theory Methods Appl. **8**, 321-333 (1984)

[2] Bandle, C., Stakgold, I.: The formation of the dead core in parabolic reaction-diffusion problems. Trans. Am. Math. Soc. **286**, 275-293 (1984)

[3] Chen, X.-Y., Matano, H., Mimura, M.: Finite-point extinction and continuity of interfaces in a nonlinear diffusion equation with strong absorption. J. reine angew. Math. **459**, 1-36 (1995)

[4] Dávila, J., Montenegro, M.: Existence and asymptotic behavior for a singular parabolic equation. Trans. Amer. Math. Soc. **357** (5), 1801-1828 (2005)

[5] Evans, L.C., Knerr, B.F.: Instantaneous shrinking of the support of nonnegative solutions to certain nonlinear parabolic equations and variational inequalities. Illinios J. of Math. **23** (1), 153-166 (1979)

[6] Fila, M., Hulshof, J., Quittner, P.: The Quenching Problem on the N-dimensional Ball. Nonlinear diffusion equations and their equilibrium states, Proc. 3rd Conf., Gregynog/UK 1989, Prog. Nonlinear Differ. Equ. Appl. **7**, 183-196 (1992)

[7] Fila, M., Kawohl, B.: Is quenching in infinite time possible? Q. Appl. Math. **48** (3), 531-534 (1990)

[8] Fila, M., Kawohl, B.: Asymptotic analysis of quenching problems. Rocky Mt. J. Math. **22** (2), 563-577 (1992)

[9] Fila, M., Levine, H.A., Vázquez, J.L.: Stabilization of solutions of weakly singular quenching problems. Proc. Am. Math. Soc. **119** (2), 555-559 (1993)

[10] Friedman, A., Herrero, M.A.: Extinction properties of semilinear heat equations with strong absorption. J. Math. Anal. Appl. **124**, 530-546 (1987)

[11] Galaktionov, V.A., Vázquez, J.L.: Continuation of Blowup Solutions of Nonlinear Heat Equations in Several Space Dimensions. Comm. Pure Appl. Math. **50**, 1-67 (1997)

[12] Gilding, B.H., Kersner, R.: Instantaneous extinction, step discontinuities and blow-up. Nonlinearity **16**, 843-854 (2003)

[13] Gui, C., Ni, W.-M., Wang, X.: On the stability and instability of positive steady states of a semilinear heat equation in $\mathbb{R}^n$. Comm. Pure Appl. Math. **45**, 1153-1181 (1992)

[14] Gui, C., Ni, W.-M., Wang, X.: Further study on a nonlinear heat equation. J. Differential Equations **169**, 588-613 (2001)

[15] Guo, J.-S.: On the quenching behavior of the solution of a semilinear parabolic equation. J. Math. Anal. Appl. **151**, 58-79 (1990)

[16] Guo, J.-S., Souplet, Ph.: Fast rate of formation of dead-core for the heat equation with strong absorption and applications to fast blow-up. Math. Annalen **331**, 651-667 (2005)

[17] Kawarada, H.: On solutions of initial -boundary problem for $u_t = u_{xx} + 1/(1-u)$. Publ. Res. Inst. Math. Sci., Kyoto Univ. **10**, 729-736 (1975)

[18] Kawohl, B., Kersner, R.: On Degenerate Diffusion with very Strong Absorption. Math. Meth. Appl. Sci. **15**, 469-477 (1992)

[19] Phillips, D.: Existence of Solutions of Quenching Problems. Applicable Anal. **24**, 253-264 (1987)

[20] Poláčik, P., Yanagida, E.: On bounded and unbounded global solutions of a supercritical semilinear heat equation. Math. Annalen **327**, 745-771 (2003)

[21] Vázquez, J. L., Zuazua, E.: The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential. J. Funct. Anal. **173**, 103-153 (2000)

[22] Winkler, M.:A strongly degenerate diffusion equation with strong absorption. Math. Nachrichten **277**, 83-101 (2004)

[23] Winkler, M.: Nonuniqueness in strongly degenerate diffusion equations with strong absorption. Proc. Conf. Equadiff 2003, Hasselt/Belgium (F. Dumortier et al., Eds.), 313-316 (2004)

[24] Winkler, M.: On strongly degenerate diffusion with strong absorption. Habilitation thesis, Aachen (2004)