Chemotaxis with logistic source: Very weak global solutions and their boundedness properties

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Abstract

We consider the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + g(u), & x \in \Omega, \ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, where $\chi > 0$ and g generalizes the logistic function $g(u) = Au - bu^\alpha$ with $\alpha > 1$, $A \geq 0$ and $b > 0$.

A concept of very weak solutions is introduced, and global existence of such solutions for any nonnegative initial data $u_0 \in L^1(\Omega)$ is proved under the assumption that $\alpha > 2 - \frac{1}{n}$.

Moreover, boundedness properties of the constructed solutions are studied. Inter alia, it is shown that if b is sufficiently large and $u_0 \in L^\infty(\Omega)$ has small norm in $L^\gamma(\Omega)$ for some $\gamma > \frac{n}{2}$ then the solution is globally bounded.

Finally, in the case that additionally $\alpha > \frac{n}{2}$ holds, a bounded set in $L^\infty(\Omega)$ can be found which eventually attracts very weak solutions emanating from arbitrary L^1 initial data.

The paper closes with numerical experiments that illustrate some of the theoretically established results.

Key words: chemotaxis, global existence, absorbing set
AMS Classification: 35K57 (primary), 35B35, 35B41 (secondary)

Introduction

In any living organism, the communication between individual cells evidently is an indispensible tool for its survival. Accordingly, a large variety of means for cellular communication has been provided during biological evolution. One – rather simple – reaction to an external signal consists of moving either towards or away from the stimulus, and the corresponding behavior is commonly named X-taxis. Here, the template X indicates the particular nature of the stimulus: For instance, *haptotaxis* means oriented movement resulting from a mechanical impulse, *phototaxis, thermotaxis* or *galvanotaxis* are due to stimuli made up by

some source of light, of heat, or of an electric current, respectively.

If a chemical substance is responsible for a change in motion, one is accordingly concerned with *chemotaxis*, and this mechanism appears to be of particular importance also in higher developed organisms, where, for example, it is believed to govern the movements of certain flexible cells such as phagocytes. One distinguishes between *chemoattraction* – aka *positive chemotaxis* – appearing when cells move towards higher concentrations of the substance, and the less frequently observed *chemorepulsion* – the so-called *negative chemotaxis* – meaning that the direction of movement is away from higher and thus towards lower concentrations of the chemical.

In several situations, it is favorable for a cell population to *accumulate* in some region in space; for instance, the slime mold *Dictyostelium Discoideum* forms a fruiting body upon such an aggregation. Chemoattraction can enhance this type of behavior if the individuals themselves secrete the attracting chemical. In 1970, Keller and Segel ([KS]) pursued the problem of finding an appropriate mathematical description of such processes of self-organization. They proposed a model for the time evolution of both the cell density u and the signal substance v, a dimensionless prototype of which reads

$$\begin{cases} u_t = \nabla \cdot (m(u)\nabla u) - \nabla \cdot (f(u,v)\nabla v) + g(u,v), & x \in \Omega, \ t > 0, \\ \Gamma v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases} \tag{0.1}$$

where Ω denotes the considered spatial region and Γ is a positive constant linked to the speed of diffusion of the chemical. The function m measures the ability of cells to diffuse, f represents the sensitivity with respect to chemotaxis, and g models possible production or death of cells.

In the last two decades, considerable progress has been made in the analysis of various particular cases of (0.1), the focus being mainly on the problem whether the respective system of equations is appropriate in the sense that it is able to give a qualitatively correct picture of the phenomenon of accumulation. However, there has non consensus been found yet on the question whether 'accumulation' means that solutions undergo a *blow-up*, that is, become unbounded in either finite or infinite time, or if it is already correctly described by pattern formation of bounded solutions.

As to the 'classical' Keller-Segel model, where $m(u) \equiv 1$, $f(u,v) \equiv \chi > 0$ and $g(u,v) \equiv 0$, it is known, for instance, that some solutions blow up if either the space dimension is $n = 2$ and the total initial population mass is above some threshold level, or if $n \geq 3$; similar results have been asserted for the limit case of this model obtained when $\Gamma = 0$ ([HV], [HMV], [H], [HWa], [N2], [SeS]). Also, questions on pattern formation in bounded domains Ω could be answered in some special cases of (0.1), for instance concerning convergence of all bounded solutions to equilibria (when $f(u) = u$ and $n = 2$, [FLP]), (meta-)stability of steady states (for $f(u,v) \sim u(1-u)$, cf. [PH]), or existence of global attractors (for $f(u) = u$ and $n = 1$, [OY]).

More recently, variants of (0.1) involving non-vanishing sources $g \not\equiv 0$ have

received growing interest. Here, the most commonly considered choices of g exercise a significant dampening effect on the population density u at those points where u itself is large; prototypes are the logistic function

$$g(u,v) = Au - Bu^\alpha, \qquad A > 0, \ B > 0, \ \alpha > 1, \tag{0.2}$$

or modifications thereof, involving further zeros, such as given by the bistable source

$$g(u,v) = u(B-u)(u-A), \qquad 0 < A < B. \tag{0.3}$$

As to the latter, for $\Gamma = 1$ and $\Omega = \mathbb{R}^n$ the behavior along the limiting procedure $\varepsilon \searrow 0$ in $m \equiv \varepsilon^2$ and $f \equiv \varepsilon$ is studied in [MT] and [FMT], where travelling fronts of the corresponding system are investigated by deriving interface equations that are supposed to decribe the dynamics of certain layers.

Logistic sources of the shape (0.2) with the standard choice $\alpha = 2$ have been considered in [OTYM], where global existence of weak solutions in bounded domains Ω along with the existence of a global attractor in an appropriate functional analytical framework has been proved for $f(u,v) \equiv u \cdot \chi(v)$ with smooth bounded functions $\chi(v)$; part of the results can be carried over to the case when χ becomes singular at $v = 0$, cf. [AOTYM].

In the present study we focus on the case $\Gamma = 0$ that is supposed to model the situation when the chemoattractant diffuses very quickly. Moreover, we shall restrict ourselves to the choices $m \equiv 1$, $f(u,v) \equiv \chi v$ and $g(u,v) \equiv g(u)$ and hence subsequently consider the system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + g(u), & x \in \Omega, \ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{0.4}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $\frac{\partial}{\partial \nu}$ denotes the outward normal derivative on $\partial\Omega$ and χ is a given positive constant. The function g is assumed to generalize (0.2) – and (0.3) as well – in the following way: Throughout, g is supposed to belong to $C^1([0,\infty))$ and to satisfy $g(0) \geq 0$. Moreover, with various $\alpha > 1$ we shall suppose that

$(\mathrm{H1}_\alpha) \qquad g(s) \leq a - bs^\alpha \qquad$ for all $s \geq 0$ with some $a \geq 0$ and $b > 0$,

and in some places we will also require a corresponding lower estimate

$(\mathrm{H2}_\alpha) \qquad g(s) \geq -c_0(s + s^\alpha) \qquad$ for all $s \geq 0$ with some $c_0 > 0$.

The system (0.4) – with $g \equiv 0$ – was first introduced in [JL] and later on taken up frequently (see [HV], [N1], [N2], for instance).

Recently, in [TW] the case $\alpha = 2$ in (0.4), $(\mathrm{H1}_\alpha)$ has been considered. Besides some results on steady states concerning regularity, stability, uniqueness and bifurcation, as to the evolution problem the following has been found.

- Assume that g satisfies $(H1_\alpha)$ with $\alpha = 2$ and some $a \geq 0, b > 0$ and $c_0 > 0$, and let $u_0 \in C^0(\bar\Omega)$.

  - If either $n \leq 2$, or $n \geq 3$ and $b > \frac{n-2}{n}\chi$, then (0.4) possesses a unique global bounded classical solution.
  - For arbitrary $n \geq 1$ and $b > 0$, (0.4) admits at least one global weak solution.

In particular, this implies the existence of global bounded solutions for any choice of $b > 0$ in $(H1_\alpha)$ if $\alpha > 2$. It remains open, however, whether in space dimensions $n \geq 3$, a quadratic death rate in (0.4) with small coefficient $b < \frac{n-2}{n}\chi$ might be insufficient to prevent solutions from becoming unbounded.

The purpose of the present work is twofold: Firstly, we would like to investigate whether death rates in (0.4) which are *weaker than quadratic* can enforce a chemotactic collapse in the sense that, for some initial data, no global solution exists in any reasonable space. Secondly, albeit not quite independently, we study the phenomena of of *immediate* and of *eventual regularization* of solutions: Given some unbounded initial data, we ask whether the solution then becomes less singular, possibly even bounded, after some finite time T, and if it may even occur that $T = 0$. Evidently, these considerations are closely related to the possibility of a life after blow-up or, say, a *life beyond collapse* of a chemotactically acting population.

For the heat equation $u_t = \Delta u$, it is well-known that solutions immediately become smooth even when evolving from very irregular initial data such as the dirac distribution; by more sophisticated techiques it has been shown that the same is true also for some finite-time blow-up solutions of the semilinear equation $u_t = \Delta u + u^p$ (with some supercritical $p > 1$) immediately after their blow-up time ([FMP]).

To the best of our knowledge, only little is known about regularization in systems involving nonlinear cross-diffusion such as in (0.1); all available results concentrate on immediately regularizing initial data that are at least sqare integrable. However, since even in the case $g \equiv 0$ any solution of (0.1) formally enjoys the mass conservation property $\int_\Omega u(x,t)dx \equiv \int_\Omega u_0(x)dx$ for all $t > 0$, a more natural requirement on the initial data appears to be $u_0 \in L^1(\Omega)$. All in all, we could not find any result about regularity – not even about existence – of solutions to chemotaxis systems beyond some time at which the solution is merely known to belong to $L^1(\Omega)$.

In light of these premises, our main existence and regularity results may be understood as saying that all $\alpha > 2 - \frac{1}{n}$ are admissible in $(H1_\alpha)$ and $(H2_\alpha)$ (showing inter alia that $\alpha = 2$ should in fact not be a critical number in this respect), and that any $u_0 \in L^1(\Omega)$ is regular enough to allow for a globally defined solution that, though being very weak, immediately becomes less singular than u_0. To be more precise,

- if g satisfies $(H1_\alpha)$ and $(H2_\alpha)$ with some $\alpha > 2 - \frac{1}{n}$ then

– for any nonnegative $u_0 \in L^1(\Omega)$ the problem (0.4) admits a very weak solution (u, v) (Theorem 1.6, cf. also Definitions 1.3 and 1.1-1.2), and

– this solution satisfies $u(\cdot, t) \in L^p(\Omega)$ for a.e. $t > 0$ and any

$$
\begin{cases}
p \leq \infty & \text{if } n = 1, \\
p < \infty & \text{if } n = 2, \\
p \leq \alpha \quad \text{such that } p < \frac{n}{n-2} \cdot \min\{\alpha - 1, 1\} & \text{if } n \geq 3.
\end{cases}
$$

(Corollary 1.7).

If moreover the growth inhibition induced by g is strong enough then some small-data solutions enjoy further boundedness properties in $L^\infty(\Omega)$:

- If (H1$_\alpha$) holds with $\alpha > 1$ and suitably large $b > 0$, and if $\|u_0\|_{L^\infty(\Omega)}$ is sufficiently small then the above solution is bounded (Lemma 2.1).

- If g satisfies (H1$_\alpha$) with $\alpha > 1$ and sufficiently small quotient $\frac{a}{b}$, and if $u_0 \in L^\infty(\Omega)$ has small norm in $L^\gamma(\Omega)$ for some $\gamma > \max\{1, \frac{n}{2}\}$, then the above solution is bounded (Theorem 2.4).

- If (H1$_\alpha$) and (H2$_\alpha$) are valid for some $\alpha > \max\{\frac{n}{2}, 2 - \frac{1}{n}\}$ then for all $\tau > 0$ one can prescribe an upper bound for both $\frac{a}{b}$ and $\|u_0\|_{L^1(\Omega)}$ that ensures boundedness of the above solution for $t > \tau$ (Theorem 2.6).

Finally, in presence of appropriately strong g all of our solutions eventually enter a bounded absorbing set in $L^\infty(\Omega)$:

- If g satisfies (H1$_\alpha$) and (H2$_\alpha$) with some $\alpha > \max\{\frac{n}{2}, 2 - \frac{1}{n}\}$ and sufficiently small ratio $\frac{a}{b}$ then there exists a ball $\mathcal{B}$ in $L^\infty(\Omega)$ such that each of the solutions constructed above eventually enters $\mathcal{B}$ and hence becomes bounded after some finite time (Theorem 2.8).

1 Global solutions for initial data in $L^1(\Omega)$

According to technical difficulties stemming mainly from the cross-diffusion term in (0.4) and the fact that we merely assume $u_0 \in L^1(\Omega)$, our concept of weak solutions differs from the natural notion. We shall deal with solutions that we call *very weak* because as many derivatives concerning u as possible are removed using integration by parts. Moreover, again for technical reasons we shall define a weak solution *not* by requiring its first component u to satisfy *one* integral *identity*, but instead to fulfill *two* integral *inequalities* slightly differing from each other, but in summary indicating that u at the same time is a sub- and a supersolution of the first equation in (0.4).

The first notion that we need is that of a very weak subsolution.

Definition 1.1 *Let $T > 0$. A pair (u, v) of nonnegative functions*

$$u \in L^1(\Omega \times (0, T)), \qquad v \in L^1((0, T); W^{1,1}(\Omega))$$

will be called a very weak subsolution *of (0.4) in $\Omega \times (0, T)$ if*

$$g(u) \quad and \quad u\nabla v \quad belong \ to \ L^1(\Omega \times (0, T)),$$

and if the relations

$$-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0\varphi(\cdot, 0) \le \int_0^T \int_\Omega u\Delta\varphi + \chi \int_0^T \int_\Omega u\nabla v \cdot \nabla\varphi + \int_0^T \int_\Omega g(u)\varphi$$
$$(1.1)$$

and

$$\int_0^T \int_\Omega \nabla v \cdot \nabla\psi + \int_0^T \int_\Omega v\psi = \int_0^T \int_\Omega u\psi \qquad (1.2)$$

hold for all

$$\varphi \in C_0^\infty(\bar\Omega \times [0, T)) \quad with \quad \varphi \ge 0 \quad and \quad \frac{\partial\varphi}{\partial\nu} \ on \ \partial\Omega \times (0, T) \qquad (1.3)$$

and any

$$\psi \in C^\infty(\bar\Omega \times [0, T]). \qquad (1.4)$$

Secondly, we will need some supersolution property. It turns out that the following concept of *entropy subsolution* is suitable for our purpose. In giving names, we follow the notion of The name given here is adapted from the notion of entropy solutions which is commonly used in the context of higher order thin film equations ([DalPGG]).

Definition 1.2 *Let $T > 0$ and $\gamma \in (0, 1)$. Two nonnegative functions*

$$u \in L^{\gamma+1}(\Omega \times (0, T)), \qquad v \in L^1((0, T); W^{1,1}(\Omega)) \cap L^{\gamma+1}(\Omega \times (0, T))$$

form a weak γ-entropy supersolution *of (0.4) in $\Omega \times (0, T)$ if*

$$u^{\gamma-2}|\nabla u|^2, \ u^{\gamma-1}g(u) \quad and \quad u^\gamma\nabla v \quad belong \ to \ L^1(\Omega \times (0, T)),$$

and if

$$-\int_0^T \int_\Omega u^\gamma\varphi_t - \int_\Omega u_0^\gamma\varphi(\cdot, 0) \ge \gamma(1-\gamma)\int_0^T \int_\Omega u^{\gamma-2}|\nabla u|^2\varphi + \int_0^T \int_\Omega u^\gamma\Delta\varphi$$
$$+(1-\gamma)\chi\int_0^T \int_\Omega u^\gamma v\varphi - (1-\gamma)\chi\int_0^T \int_\Omega u^{\gamma+1}\varphi$$
$$+\chi\int_0^T \int_\Omega u^\gamma\nabla v \cdot \nabla\varphi$$
$$+\gamma\int_0^T \int_\Omega u^{\gamma-1}g(u)\varphi \qquad (1.5)$$

as well as (1.2) are valid for all φ and ψ satisfying (1.3) and (1.4).

We finally end up with the following concept which is consistent with that of a classical solution in that if a smooth function is a very weak solution in the sense defined below, then it is a classical solution.

Definition 1.3 *Let $T > 0$. We call a couple (u, v) a* very weak solution *of (0.4) in $\Omega \times (0, T)$ if it is both a very weak subsolution and a weak γ-entropy supersolution of (0.4) in $\Omega \times (0, T)$ for some $\gamma \in (0, 1)$.*
A global *very weak solution of (0.4) is a pair (u, v) of functions defined in $\Omega \times (0, \infty)$ which is a weak solution of (0.4) in $\Omega \times (0, T)$ for all $T > 0$.*

When seeking for weak solutions of (0.4), it appears to be natural that one considers appropriate regularizations of (0.4) which are known to admit global smooth solutions. It turns out that in the present situation this can be done at least in two different ways: The first consists of approximating the chemotactic sensitivity function $f(u) = \chi \cdot u$ in (0.4) by some sequence of functions f_ε (for, say, small $\varepsilon > 0$) with sufficiently small (or even without) growth with respect to u as $u \to \infty$; for instance, it can be shown using the ideas in [HWi] that if $f_\varepsilon(u) \leq C_\varepsilon u^\beta$ with some $\beta < \frac{2}{n}$ then all solutions of the accordingly modified version of (0.4) are global, bounded and hence classical, provided that the initial data are smooth.
For simplicity in presentation, however, we prefer to perform a second variant of regularizing (0.4) which is based on strengthening the death rate in the logistic term rather than weakening the chemoattracting effect. More precisely, throughout the paper we fix a number $\beta > 2$ and, for $\varepsilon \in (0, 1)$, consider the problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) + g(u_\varepsilon) - \varepsilon u_\varepsilon^\beta, & x \in \Omega, \ t > 0, \\ 0 = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega, \ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega, \ t > 0, \\ u_\varepsilon(x, 0) = u_{0\varepsilon}(x), & x \in \Omega, \end{cases} \qquad (1.6)$$

where $(u_{0\varepsilon})_{\varepsilon \in (0,1)} \subset C^0(\bar{\Omega})$ is such that $u_{0\varepsilon} > 0$ in Ω and

$$\|u_{0\varepsilon} - u_0\|_{L^1(\Omega)} \leq \varepsilon. \qquad (1.7)$$

By Theorem 2.5 in [TW], (1.6) has a unique global bounded classical solution $(u_\varepsilon, v_\varepsilon)$. In view of the fact that $g(0) \geq 0$ and the parabolic and elliptic comparison principles applied to in (1.6), $u_\varepsilon \geq 0$ and hence also v_ε is nonnegative. Moreover, we even have $u_\varepsilon > 0$ in $\bar{\Omega} \times [0, \infty)$ by the strong maximum principle.

We proceed to derive ε-independent estimates. The first lemma provides some easily obtained inequalities which nonetheless are crucial for almost everything that follows.

Lemma 1.1 *Suppose g satisfies (H1_α) with some $\alpha > 1$, and let $m := (\frac{a}{b})^{\frac{1}{\alpha}} |\Omega|$. Then for any $t_0 \geq 0$ and each $\varepsilon \in (0, 1)$ the inequalities*

$$\int_\Omega u_\varepsilon(x, t) dx \leq m + e^{-\alpha a^{\frac{\alpha-1}{\alpha}} b^{\frac{1}{\alpha}}(t-t_0)} \cdot \left(\int_\Omega u_\varepsilon(x, t_0) dx - m \right) \quad \text{for } t \geq t_0 \quad (1.8)$$

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and

$$b \int_{t_0}^t \int_\Omega u_\varepsilon^\alpha + \varepsilon \int_{t_0}^t \int_\Omega u_\varepsilon^\beta \le a|\Omega| \cdot (t-t_0) + \int_\Omega u_\varepsilon(x,t_0)dx - \int_\Omega u_\varepsilon(x,t)dx \quad \text{for } t > t_0 \tag{1.9}$$

hold. In particular, writing

$$M_\varepsilon := \max\{m, \|u_{0\varepsilon}\|_{L^1(\Omega)}\}, \tag{1.10}$$

we have the a priori estimates

$$\int_\Omega u_\varepsilon(x,t)dx \le M_\varepsilon \qquad \text{for all } t > 0 \tag{1.11}$$

and

$$\int_0^T \int_\Omega u_\varepsilon^\alpha \le \frac{a|\Omega|T + M_\varepsilon}{b} \qquad \text{for all } T > 0 \tag{1.12}$$

as well as

$$\varepsilon \int_0^T \int_\Omega u_\varepsilon^\beta \le \frac{a|\Omega|T + M_\varepsilon}{b} \qquad \text{for all } T > 0. \tag{1.13}$$

PROOF.   We integrate the first equation in (1.6) over Ω and use (H1$_\alpha$) to see that

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega u_\varepsilon(x,t)dx &= \int_\Omega g(u_\varepsilon) - \varepsilon \int_\Omega u_\varepsilon^\beta \\
&\le a|\Omega| - b \int_\Omega u_\varepsilon^\alpha - \varepsilon \int_\Omega u_\varepsilon^\beta \qquad \text{for } t > 0. \tag{1.14}
\end{aligned}
$$

From the Hölder inequality we obtain $\int_\Omega u_\varepsilon \le |\Omega|^{\frac{\alpha-1}{\alpha}} \cdot (\int_\Omega u_\varepsilon^\alpha)^{\frac{1}{\alpha}}$, and hence $y(t) := \int_\Omega u_\varepsilon(x,t)dx$ satisfies

$$y'(t) \le a|\Omega| - b|\Omega|^{1-\alpha} y^\alpha(t) \qquad \text{for } t > 0.$$

Substituting $z(t) := y(t) - m$, using the convexity of $s \mapsto s^\alpha$ on $(-1,\infty)$ and recalling the definition of m we obtain

$$
\begin{aligned}
z'(t) &\le a|\Omega| - b|\Omega|^{1-\alpha}(M+z)^\alpha \\
&\le a|\Omega| - b|\Omega|^{1-\alpha} m^\alpha \left(1 + \alpha \frac{z(t)}{m}\right) \\
&= -\alpha b|\Omega|^{1-\alpha} m^{\alpha-1} z(t) \\
&= -\alpha a^{\frac{\alpha-1}{\alpha}} b^{\frac{1}{\alpha}} z(t) \qquad \text{for } t > 0.
\end{aligned}
$$

An integration of this differential inequality yields (1.8), whereas (1.9) follows upon integrating (1.14) with respect to time. Now (1.11), (1.12) and (1.13) immediately result from (1.8) and (1.9) and the fact that $\int_\Omega u_\varepsilon(x,t)dx$ is non-negative.   ////

We proceed to derive from the above lemma some bound for that spatial gradient of u_ε.

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Lemma 1.2 *Suppose that g satisfies (H1$_\alpha$) and (H2$_\alpha$) with some $\alpha > 1$. Then for all $\gamma \in (0,1)$ satisfying $\gamma \le \alpha - 1$ there exists $C > 0$ such that for any $\varepsilon \in (0,1)$ and $T > 0$ we have*

$$\int_0^T \int_\Omega u_\varepsilon^{\gamma-2}|\nabla u_\varepsilon|^2 \le C(1+T) \tag{1.15}$$

and

$$g(0) \cdot \int_0^T \int_\Omega u_\varepsilon^{\gamma-1} \le C(1+T). \tag{1.16}$$

PROOF.   We multiply the first equation in (1.6) by $u_\varepsilon^{\gamma-1}$ and integrate by parts over $\Omega \times (0,T)$ to obtain

$$
\begin{aligned}
(1-\gamma)\int_0^T \int_\Omega u_\varepsilon^{\gamma-2}|\nabla u_\varepsilon|^2 \; = \; & \frac{1}{\gamma}\int_\Omega u_\varepsilon^\gamma(x,T)dx - \frac{1}{\gamma}\int_\Omega u_{0\varepsilon}^\gamma(x)dx \\
& + (1-\gamma)\chi \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \\
& - \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}g(u_\varepsilon) + \varepsilon \int_0^T \int_\Omega u_\varepsilon^{\beta+\gamma-1} \tag{1.17}
\end{aligned}
$$

By the Hölder inequality,

$$\frac{1}{\gamma}\int_\Omega u_\varepsilon^\gamma(x,T)dx \le \frac{|\Omega|^{1-\gamma}}{\gamma}\Big(\int_\Omega u_\varepsilon(x,T)dx\Big)^\gamma. \tag{1.18}$$

Once more integrating by parts, again from Hölder's inequality we gain

$$
\begin{aligned}
(1-\gamma)\chi \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \; = \; & -\frac{(1-\gamma)\chi}{\gamma}\int_0^T \int_\Omega u_\varepsilon^\gamma \Delta v_\varepsilon \\
\le \; & \frac{(1-\gamma)\chi}{\gamma}\Big(\int_0^T \int_\Omega u_\varepsilon^\alpha\Big)^{\frac{\gamma}{\alpha}} \cdot \Big(\int_0^T \int_\Omega |\Delta v_\varepsilon|^\alpha\Big)^{\frac{1}{\alpha}} \cdot (|\Omega|T)^{\frac{\alpha-\gamma-1}{\alpha}}. \tag{1.19}
\end{aligned}
$$

In view of the second equation in (1.6) and elliptic L^p theory we know that

$$\int_0^T \int_\Omega |\Delta v_\varepsilon|^\alpha \le c_1 \int_0^T \int_\Omega u_\varepsilon^\alpha$$

holds with some $c_1 > 0$, so that (1.19) implies

$$(1-\gamma)\chi \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}\nabla u_\varepsilon \cdot \nabla v_\varepsilon \le c_2\Big(\int_0^T \int_\Omega u_\varepsilon^\alpha + 1 + T\Big) \tag{1.20}$$

with a certain $c_2 > 0$. Furthermore, since $g(s) \ge g(0) - \hat{c}_0(s + s^\alpha)$ for all $s \ge 0$ and some $\hat{c}_0 \ge c_0$ by (H2$_\alpha$) and the fact that $g \in C^1([0,\infty))$, using Young's

inequality we find

$$-\int_0^T \int_\Omega u_\varepsilon^{\gamma-1} g(u_\varepsilon) \leq -g(0) \cdot \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}$$
$$+\hat{c}_0 \cdot \Big(\int_0^T \int_\Omega u_\varepsilon^{\gamma} + \int_0^T \int_\Omega u_\varepsilon^{\alpha+\gamma-1} \Big)$$
$$\leq -g(0) \cdot \int_0^T \int_\Omega u_\varepsilon^{\gamma-1} + \int_0^T \int_\Omega u_\varepsilon^{\alpha} + c_3 T \quad (1.21)$$

and

$$\varepsilon \int_0^T \int_\Omega u_\varepsilon^{\beta+\gamma-1} \leq \varepsilon \int_0^T \int_\Omega u_\varepsilon^{\beta} + c_4 \varepsilon T \qquad (1.22)$$

with some positive c_3 and c_4.
Collecting (1.17), (1.18) and (1.20)-(1.22), in view of (1.11), (1.12) and (1.13) we arrive at

$$(1-\gamma) \int_0^T \int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 + g(0) \cdot \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}$$
$$\leq \frac{|\Omega|^{1-\gamma}}{\gamma} \cdot M_\varepsilon^{\gamma} + c_2 + (c_2 + c_3 + c_4 \varepsilon) T + (c_2 + 1) \cdot \frac{a|\Omega|T + M_\varepsilon}{b},$$

where M_ε is given by (1.10). Since $M_\varepsilon \leq \max\{(\frac{a}{b})^{\frac{1}{\alpha}} |\Omega|, \|u_0\|_{L^1(\Omega)} + 1\}$ by (1.7), this immediately gives (1.15) and (1.16). ////

The following bound on the time derivative of u_ε involves a very weak norm, but is still sufficient for our purposes.

Lemma 1.3 *Assume that* $(H1_\alpha)$ *and* $(H2_\alpha)$ *hold for some* $\alpha > 1$. *Then for all* $\gamma \in (0,1)$ *satisfying* $\gamma \leq \alpha - 1$ *there exist* $k \in \mathbb{N}$ *and* $C > 0$ *such that for each* $\varepsilon \in (0,1)$ *and* $T > 0$,

$$\|\partial_t (1+u_\varepsilon)^{\frac{\gamma}{2}}\|_{L^1((0,T);(W_0^{k,2}(\Omega))^\star)} \leq C(1+T). \qquad (1.23)$$

PROOF.   We fix $k \in \mathbb{N}$ large such that

$$W_0^{k,2}(\Omega) \hookrightarrow L^\infty(\Omega) \qquad \text{and} \qquad W_0^{k,2}(\Omega) \hookrightarrow W^{1,p}(\Omega) \qquad (1.24)$$

holds for $p := \max\{2, \frac{2\alpha}{2\alpha-\gamma-2}\}$; for instance, we pick any $k > \frac{n+2}{2}$. Given $\psi \in W_0^{k,2}(\Omega)$, multiplying the first equation in (1.6) by $(1+u_\varepsilon)^{\frac{\gamma-2}{2}} \psi$ and integrating by parts, for all $t > 0$ we find

$$\frac{2}{\gamma} \int_\Omega \partial_t (1+u_\varepsilon)^{\frac{\gamma}{2}} \cdot \psi = \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_{\varepsilon t} \cdot \psi$$
$$= \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \Delta u_\varepsilon \cdot \psi - \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) \cdot \psi$$

$$+ \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} g(u_\varepsilon)\psi - \varepsilon \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon^\beta \psi$$

$$= \frac{2-\gamma}{2} \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-4}{2}} |\nabla u_\varepsilon|^2 \psi - \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \nabla u_\varepsilon \cdot \nabla \psi$$

$$- \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \nabla \cdot (u_\varepsilon \nabla v_\varepsilon) \cdot \psi$$

$$+ \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} g(u_\varepsilon)\psi - \varepsilon \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon^\beta \psi. \qquad (1.25)$$

Since $u_\varepsilon \geq 0$ and $\gamma > 0$ implies $\frac{\gamma-4}{2} < \gamma-2$, we have

$$\left| \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-4}{2}} |\nabla u_\varepsilon|^2 \psi \right| \leq \left(\int_\Omega (1+u_\varepsilon)^{\gamma-2} |\nabla u_\varepsilon|^2 \right) \cdot \|\psi\|_{L^\infty(\Omega)}$$

$$\leq \left(\int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 \right) \cdot \|\psi\|_{L^\infty(\Omega)}, \qquad (1.26)$$

and

$$\left| \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \nabla u_\varepsilon \cdot \nabla \psi \right| \leq \left(\int_\Omega (1+u_\varepsilon)^{\gamma-2} |\nabla u_\varepsilon|^2 \right)^{\frac{1}{2}} \cdot \left(\int_\Omega |\nabla \psi|^2 \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{2}\left(1 + \int_\Omega u_\varepsilon^{\gamma-2} \nabla u_\varepsilon|^2 \right) \cdot \|\nabla \psi\|_{L^2(\Omega)}. \qquad (1.27)$$

Another integration by parts in conjunction with the second equation in (1.6) shows that the chemotaxis term can be reaaranged according to

$$- \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \nabla \cdot (u_\varepsilon \nabla v_\varepsilon)\psi$$

$$= - \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon \Delta v_\varepsilon \cdot \psi - \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \cdot \psi$$

$$= - \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon \Delta v_\varepsilon \cdot \psi + \frac{2\chi}{\gamma} \int_\Omega \nabla (1+u_\varepsilon)^{\frac{\gamma}{2}} \cdot \nabla v_\varepsilon \cdot \psi$$

$$= - \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon \Delta v_\varepsilon \cdot \psi - \frac{2\chi}{\gamma} \int_\Omega (1+u_\varepsilon)^{\frac{\gamma}{2}} \Delta v_\varepsilon \cdot \psi$$

$$- \frac{2\chi}{\gamma} \int_\Omega (1+u_\varepsilon)^{\frac{\gamma}{2}} \nabla v_\varepsilon \cdot \nabla \psi$$

$$= - \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon v_\varepsilon \psi + \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon^2 \psi$$

$$- \frac{2\chi}{\gamma} \int_\Omega (1+u_\varepsilon)^{\frac{\gamma}{2}} v_\varepsilon \psi + \frac{2\chi}{\gamma} \int_\Omega (1+u_\varepsilon)^{\frac{\gamma}{2}} u_\varepsilon \psi$$

$$- \frac{2\chi}{\gamma} \int_\Omega (1+u_\varepsilon)^{\frac{\gamma}{2}} \nabla v_\varepsilon \cdot \nabla \psi$$

$$=: \ I_1 + I_2 + I_3 + I_4 + I_5. \qquad (1.28)$$

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Here, applying Hölder's inequality with the three exponents $\frac{2\alpha}{\gamma}$, α and $\frac{2\alpha}{2\alpha-\gamma-2}$ we obtain

$$
\begin{aligned}
|I_1| &\leq \chi \int_\Omega (1+u_\varepsilon)^{\frac{\gamma}{2}} v_\varepsilon |\psi| \\
&\leq \chi \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{\gamma}{2\alpha}} \cdot \Big(\int_\Omega v_\varepsilon^\alpha \Big)^{\frac{1}{\alpha}} \cdot \Big(\int_\Omega |\psi|^{\frac{2\alpha}{2\alpha-\gamma-2}} \Big)^{\frac{2\alpha-\gamma-2}{2\alpha}} . \quad (1.29)
\end{aligned}
$$

Similarly,

$$
|I_3| \leq \frac{2\chi}{\gamma} \cdot \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{\gamma}{2\alpha}} \cdot \Big(\int_\Omega v_\varepsilon^\alpha \Big)^{\frac{1}{\alpha}} \cdot \Big(\int_\Omega |\psi|^{\frac{2\alpha}{2\alpha-\gamma-2}} \Big)^{\frac{2\alpha-\gamma-2}{2\alpha}} \quad (1.30)
$$

and

$$
|I_5| \leq \frac{2\chi}{\gamma} \cdot \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{\gamma}{2\alpha}} \cdot \Big(\int_\Omega |\nabla v_\varepsilon|^\alpha \Big)^{\frac{1}{\alpha}} \cdot \Big(\int_\Omega |\nabla \psi|^{\frac{2\alpha}{2\alpha-\gamma-2}} \Big)^{\frac{2\alpha-\gamma-2}{2\alpha}} , \quad (1.31)
$$

whereas Hölder's inequality with exponents $\frac{2\alpha}{\gamma+2}$ and $\frac{2\alpha}{2\alpha-\gamma-2}$ yields

$$
|I_2| + |I_4| \leq \Big(\chi + \frac{2\chi}{\gamma} \Big) \cdot \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{\gamma+2}{2\alpha}} \cdot \Big(\int_\Omega |\psi|^{\frac{2\alpha}{2\alpha-\gamma-2}} \Big)^{\frac{2\alpha-\gamma-2}{2\alpha}} . \quad (1.32)
$$

Now from the second equation in (1.6) together with standard elliptic L^p estimates we know that

$$
\max \Big\{ \|v_\varepsilon(\cdot,t)\|_{L^\alpha(\Omega)} + \|\nabla v_\varepsilon(\cdot,t)\|_{L^\alpha(\Omega)} \Big\} \leq \|v_\varepsilon(\cdot,t)\|_{W^{2,\alpha}(\Omega)} \leq c_1 \|u_\varepsilon(\cdot,t)\|_{L^\alpha(\Omega)}
$$

for $t > 0$ holds with some constant c_1. Inserting this into (1.29)-(1.32) shows that

$$
\begin{aligned}
|I_1| &+ |I_2| + |I_3| + |I_4| + |I_5| \\
&\leq \Big(\chi c_1 + \frac{2\chi}{\gamma} c_1 + \chi + \frac{2\chi}{\gamma} \Big) \cdot \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{\gamma+2}{2\alpha}} \cdot \|\psi\|_{L^{\frac{2\alpha}{2\alpha-\gamma-2}}(\Omega)} \\
&\quad + \frac{2\chi}{\gamma} c_1 \cdot \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{\gamma+2}{2\alpha}} \cdot \|\nabla\psi\|_{L^{\frac{2\alpha}{2\alpha-\gamma-2}}(\Omega)} \\
&\leq c_2 \Big(1 + \int_\Omega (1+u_\varepsilon)^\alpha \Big) \cdot \|\psi\|_{W^{1,\frac{2\alpha}{2\alpha-\gamma-2}}(\Omega)} \quad (1.33)
\end{aligned}
$$

is valid for some $c_2 > 0$.

As to the logistic term in (1.25), we observe that (H1$_\alpha$) and (H2$_\alpha$) imply that $|g(s)| \leq \tilde{c}_0 (1+s)^\alpha$ holds for all $s \geq 0$ and some $\tilde{c}_0 > 0$, whence

$$
\begin{aligned}
\Big| \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} g(u_\varepsilon) \cdot \psi \Big| &\leq \tilde{c}_0 \cdot \int_\Omega (1+u_\varepsilon)^{\frac{2\alpha+\gamma-2}{2}} |\psi| \\
&\leq \tilde{c}_0 \cdot \Big(\int_\Omega (1+u_\varepsilon)^\alpha \Big)^{\frac{2\alpha+\gamma-2}{2\alpha}} \cdot \Big(\int_\Omega |\psi|^{\frac{2\alpha}{2-\gamma}} \Big)^{\frac{2-\gamma}{2\alpha}} \\
&\leq c_3 \Big(1 + \int_\Omega (1+u_\varepsilon)^\alpha \Big) \cdot \|\psi\|_{L^{\frac{2\alpha}{2-\gamma}}(\Omega)} \quad (1.34)
\end{aligned}
$$

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with suitably large $c_3 > 0$ folows upon applying Hölder's and Young's inequalities. By the same tokens, we find $c_4 > 0$ such that

$$
\begin{aligned}
\left| -\varepsilon \int_\Omega (1+u_\varepsilon)^{\frac{\gamma-2}{2}} u_\varepsilon^\beta \psi \right| &\leq \varepsilon \int_\Omega u_\varepsilon^{\frac{2\beta+\gamma-2}{2}} |\psi| \\
&\leq \varepsilon \left(\int_\Omega u_\varepsilon^\beta \right)^{\frac{2\beta+\gamma-2}{2\beta}} \cdot \left(\int_\Omega |\psi|^{\frac{2\beta}{2-\gamma}} \right)^{\frac{2-\gamma}{2\beta}} \\
&\leq c_4 \varepsilon \left(1 + \int_\Omega u_\varepsilon^\beta \right) \cdot \|\psi\|_{L^{\frac{2\beta}{2-\gamma}}(\Omega)}. \qquad (1.35)
\end{aligned}
$$

Collecting (1.25)-(1.28) and (1.33)-(1.35) and recalling the definition of p, we arrive at the estimate

$$
\begin{aligned}
\left| \frac{2}{\gamma} \int_\Omega \partial_t (1+u_\varepsilon)^{\frac{\gamma}{2}} \psi \right| &\leq c_5 \left(1 + \int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 + \int_\Omega (1+u_\varepsilon)^\alpha + \varepsilon \int_\Omega u_\varepsilon^\beta \right) \times \\
&\quad \times \left(\|\psi\|_{W^{1,p}(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \right) \quad \text{for all } \psi \in W_0^{k,2}(\Omega)
\end{aligned}
$$

with a certain c_5 independent of $\varepsilon \in (0,1), t > 0$ and $\psi \in W_0^{k,2}(\Omega)$.
We now observe that $(1+u_\varepsilon)^\alpha \leq 2^\alpha(1+u_\varepsilon^\alpha)$ and remember (1.24) to obtain

$$
\begin{aligned}
&\left| \frac{2}{\gamma} \int_\Omega \partial_t (1+u_\varepsilon)^{\frac{\gamma}{2}} \psi \right| \\
&\leq c_6 \left(1 + \int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^\alpha + \varepsilon \int_\Omega u_\varepsilon^\beta \right) \|\psi\|_{W_0^{k,2}(\Omega)}
\end{aligned}
$$

for all $\psi \in W_0^{k,2}(\Omega)$ with some $c_6 > 0$. Hence,

$$
\left\| \partial_t \left(1+u_\varepsilon(\cdot,t) \right)^{\frac{\gamma}{2}} \right\|_{(W_0^{k,2}(\Omega))^\star} \leq c_6 \left(1 + \int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 + \int_\Omega u_\varepsilon^\alpha + \varepsilon \int_\Omega u_\varepsilon^\beta \right),
$$

which upon integration over $t \in (0,T)$ yields (1.23) in virtue of the estimates (1.12), (1.13) and (1.15) provided by Lemma 1.1 and Lemma 1.2. ////

As a consequence of the last three lemmata, we obtain the following.

Lemma 1.4 *Let* (H1$_\alpha$) *and* (H2$_\alpha$) *be satisfied with some* $\alpha > 1$. *Then for all* $T > 0$ *and any* $p \in (1,\alpha)$,

$$
(u_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{is strongly precompact in } L^p(\Omega \times (0,T)). \qquad (1.36)
$$

PROOF.   Let $T > 0, p \in (1,\alpha)$ and a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ be given. From (1.12) we know that there exists a nonnegative function u such that

$$
u_\varepsilon \rightharpoonup u \qquad \text{in } L^p(\Omega \times (0,T)) \qquad (1.37)
$$

along a subsequence $\varepsilon = \varepsilon_{j_i}$, $i \to \infty$. On the other hand, Lemma 1.1, Lemma 1.2 and Lemma 1.3 imply that if we pick any $\gamma \in (0,1)$ such that $\gamma \leq \alpha - 1$ then we have

$$
\|(1+u_\varepsilon)^{\frac{\gamma}{2}}\|_{L^2((0,T);W^{1,2}(\Omega))} + \|\partial_t(1+u_\varepsilon)^{\frac{\gamma}{2}}\|_{L^1((0,T);(W_0^{k,2}(\Omega))^\star)} \leq c
$$

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with some $c > 0$ and $k \in \mathbb{N}$. Since $(W_0^{k,2}(\Omega))^\star$ is a Hilbert space, the Aubin-Lions lemma (Theorem 2.3 in [T]) applies to yield strong precompactness of $((1 + u_\varepsilon)^{\frac{\gamma}{2}})_{\varepsilon \in (0,1)}$ in the space $L^2((0,T); L^2(\Omega))$; in particular,

$$u_\varepsilon \to u \qquad \text{a.e. in } \Omega \times (0,T) \tag{1.38}$$

holds along a further subsequence.

Again by Lemma 1.1, $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^q(\Omega \times (0,T))$ with $q = \frac{\alpha}{p}$. Since $q > 1$, this entails that

$$u_\varepsilon^p \rightharpoonup w \qquad \text{in } L^q(\Omega \times (0,T)) \tag{1.39}$$

for another subsequence, where (1.38) asserts the identification $w = u^p$. Choosing $\varphi \equiv 1 \in (L^q(\Omega \times (0,T)))^\star$ as a test functional, we thus find

$$\int_0^T \int_\Omega u_\varepsilon^p \to \int_0^T \int_\Omega u^p.$$

Together with (1.37), this proves the strong convergence $u_\varepsilon \to u$ in the uniformly convex space $L^p(\Omega \times (0,T))$. $////$

One final preparation will provide a compactness property of $(v_\varepsilon)_{\varepsilon \in (0,1)}$.

Lemma 1.5 *Assume* $(H1_\alpha)$ *and* $(H2_\alpha)$ *with some* $\alpha > 1$. *Then for all* $q \in (1, \frac{n\alpha}{n-1})$ *there exists* $C > 0$ *such that*

$$\|\nabla v_\varepsilon\|_{L^q(\Omega \times (0,T))} \leq C(1 + T) \qquad \text{for all } T > 0. \tag{1.40}$$

PROOF. Without loss of generality we may assume that $q \geq \frac{(n+1)\alpha}{n}$, so that $r := \frac{n(q-\alpha)}{\alpha}$ satisfies $r \in [1, \frac{n}{n-1})$. Thus, according to a classical result due to Brézis and Strauss ([BS]), there exists $C_{BS} > 0$ such that for any $w \in C^2(\bar{\Omega})$ satisfying $\frac{\partial w}{\partial \nu} = 0$ on $\partial \Omega$, the estimate

$$\|w\|_{W^{1,r}(\Omega)} \leq C_{BS} \|\Delta w\|_{L^1(\Omega)} \tag{1.41}$$

holds. Since evidently $\int_\Omega v_\varepsilon(\cdot, t) = \int_\Omega u_\varepsilon(\cdot, t)$ for all $t > 0$, from the second equation in (1.6) and Lemma 1.1 we infer that $\|\Delta v_\varepsilon(\cdot, t)\|_{L^1(\Omega)} \leq c_1$ for all $t > 0$ and some $c_1 > 0$. Therefore (1.41) yields

$$\|v_\varepsilon(\cdot, t)\|_{W^{1,r}(\Omega)} \leq c_{BS} \cdot c_1 \qquad \text{for all } t > 0. \tag{1.42}$$

We now invoke the Gagliardo-Nirenberg inequality ([F]) to estimate

$$\|\nabla v_\varepsilon(\cdot, t)\|_{L^q(\Omega)} \leq c_{GN} \|v_\varepsilon(\cdot, t)\|_{W^{2,\alpha}(\Omega)}^\theta \cdot \|v_\varepsilon(\cdot, t)\|_{W^{1,r}(\Omega)}^{1-\theta}$$

for all $t > 0$ with some $c_{GN} > 0$, where

$$1 - \frac{n}{q} = \left(2 - \frac{n}{\alpha}\right)\theta + \left(1 - \frac{n}{r}\right)(1 - \theta),$$

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that is,

$$\theta = \frac{n\alpha(q-r)}{q(\alpha r - nr + n\alpha)} \equiv \frac{\alpha}{q}$$

in view of our definition of r. Since $\|v_\varepsilon(\cdot, t)\|_{W^{2,\alpha}(\Omega)} \leq c_2 \|u_\varepsilon(\cdot, t)\|_{L^\alpha(\Omega)}$ for some constant c_2 by elliptic L^p theory applied to the second equation in (1.6), from (1.42) we obtain

$$\|\nabla v_\varepsilon(\cdot, t)\|_{L^q(\Omega)}^q \leq c_{GN}^q (c_{BS} \cdot c_1)^{q(1-\theta)} \cdot c_2^{q\theta} \cdot \|u_\varepsilon(\cdot, t)\|_{L^\alpha(\Omega)}^\alpha$$

for all $t > 0$ and $\varepsilon \in (0, 1)$. Integrating this with respect to $t \in (0, T)$ and recalling (1.12), we end up with (1.40). ////

We are now in the position to prove our main result concerning existence of very weak solutions.

Theorem 1.6 *Let $\chi > 0$, and suppose that g satisfies* $(H1_\alpha)$ *and* $(H2_\alpha)$ *with some $\alpha > 2 - \frac{1}{n}$. Then for each nonnegative $u_0 \in L^1(\Omega)$, the problem (0.4) possesses at least one global very weak solution (u, v). This solution can be obtained as the limit of an appropriate sequence $\big((u_\varepsilon, v_\varepsilon)\big)_{\varepsilon=\varepsilon_j \searrow 0}$ of global bounded classical solutions of (1.6) in the sense that*

$$u_\varepsilon \to u \qquad a.e. \ in \ \Omega \times (0, \infty), \tag{1.43}$$

$$u_\varepsilon^{\frac{\gamma}{2}} \rightharpoonup u^{\frac{\gamma}{2}} \qquad in \ L^2_{loc}([0, \infty); W^{1,2}(\Omega)), \tag{1.44}$$

$$u_\varepsilon \rightharpoonup u \qquad in \ L^\alpha_{loc}(\bar\Omega \times [0, \infty)) \qquad and \tag{1.45}$$

$$v_\varepsilon \rightharpoonup v \qquad in \ L^\alpha_{loc}([0, \infty); W^{2,\alpha}(\Omega)) \tag{1.46}$$

as $\varepsilon = \varepsilon_j \searrow 0$ for any $\gamma \in (0, 1)$ satisfying $\gamma \leq \alpha - 1$.

PROOF. From (1.12) and elliptic theory applied to the equation for v_ε, we know that $(v_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^\alpha((0, T); W^{2,\alpha}(\Omega))$ for all $T > 0$. In view of Lemma 1.4, Lemma 1.2, (1.12) and Lemma 1.5, we can thus pick a sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ such that (1.43)-(1.46) as well as

$$u_\varepsilon \to u \qquad in \ L^p(\Omega \times (0, T)) \quad for \ all \ p \in [1, \alpha) \tag{1.47}$$

and

$$v_\varepsilon \rightharpoonup v \qquad in \ L^q((0, T); W^{1,q}(\Omega)) \quad for \ all \ q \in \left(1, \frac{n\alpha}{n-1}\right) \tag{1.48}$$

as $\varepsilon = \varepsilon_j \searrow 0$ hold for all $T > 0$ and any $\gamma \in (0, 1)$ satisfying $\gamma \leq \alpha - 1$ with some nonnegative functions u and v.

In order to check that (u, v) is a very weak subsolution of (0.4) in $\Omega \times (0, T)$, let

a test function φ satisfying (1.3) be given. Then multiplying the first equation in (1.6) by φ and integrating by parts, for all $\varepsilon \in (0,1)$ we have

$$- \int_0^T \int_\Omega u_\varepsilon \varphi - \int_\Omega u_{0\varepsilon} \varphi(\cdot,0) \; - \; \int_0^T \int_\Omega u_\varepsilon \Delta\varphi - \chi \int_0^T \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi$$
$$= \; \int_0^T \int_\Omega g(u_\varepsilon)\varphi - \varepsilon \int_0^T \int_\Omega u_\varepsilon^\beta \varphi. \qquad (1.49)$$

By (1.47) and (1.7),

$$- \int_0^T \int_\Omega u_\varepsilon \varphi_t \to - \int_0^T \int_\Omega u\varphi_t, \qquad (1.50)$$

$$- \int_\Omega u_{0\varepsilon} \varphi(\cdot,0) \to - \int_\Omega u_0 \varphi(\cdot,0) \qquad \text{and} \qquad (1.51)$$

$$- \int_0^T \int_\Omega u_\varepsilon \Delta\varphi \to - \int_0^T \int_\Omega u\Delta\varphi \qquad (1.52)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Since $\alpha > 2 - \frac{1}{n} = \frac{2n-1}{n}$, we have $\frac{1}{\alpha} + \frac{1}{\frac{n\alpha}{n-1}} = \frac{2n-1}{n\alpha} < 1$, and hence we can choose $p > 1$ close to α and $q > 1$ close to $\frac{n\alpha}{n-1}$ such that $\frac{1}{p} + \frac{1}{q} \leq 1$. Then (1.47) and (1.48) ensure that

$$- \chi \int_0^T \int_\Omega u_\varepsilon \nabla v_\varepsilon \cdot \nabla\varphi \to - \chi \int_0^T \int_\Omega u\nabla v \cdot \nabla\varphi \qquad (1.53)$$

as $\varepsilon = \varepsilon_j \searrow 0$.
As to the logistic term, we split g according to $g(s) = g_+(s) - g_-(s)$, where $g_+(s) = \max\{0, g(s)\}$ and $g_-(s) = \max\{0, -g(s)\}$ are nonnegative. By (1.47) and the dominated convergence theorem,

$$\int_0^T \int_\Omega g_+(u_\varepsilon)\varphi \to \int_0^T \int_\Omega g_+(u)\varphi,$$

because g_+ evidently is bounded on $[0,\infty)$. Since Fatou's lemma implies

$$\int_0^T \int_\Omega g_-(u)\varphi \leq \liminf_{\varepsilon = \varepsilon_j \searrow 0} \int_0^T \int_\Omega g_-(u_\varepsilon)\varphi,$$

we obtain

$$\int_0^T \int_\Omega g(u)\varphi \geq \limsup_{\varepsilon = \varepsilon_j \searrow 0} \int_0^T \int_\Omega g(u_\varepsilon)\varphi. \qquad (1.54)$$

Altogether, (1.50)-(1.54) and the fact that the last term in (1.49) is nonpositive entail that u satisfies (1.1), whereas (1.46) implies that (1.2) holds for all ψ fulfilling (1.4). Since the regularity requirements made in Definition 1.1 are readily checked to be consequences of (1.44)-(1.46) and (1.48), we conclude that (u,v) in fact is a very weak subsolution of (0.4) in $\Omega \times (0,T)$ for all $T > 0$.

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We next assert that (u, v) is a weak γ-entropy supersolution for any $\gamma \in (0, \alpha-1]$. To this end, we fix φ as required by (1.3) and test the first equation in (1.6) by $u_\varepsilon^{\gamma-1}\varphi$ to obtain

$$-\int_0^T \int_\Omega u_\varepsilon^\gamma \varphi_t - \int_\Omega u_{0\varepsilon}^\gamma \varphi(\cdot, 0)$$

$$= \gamma(1-\gamma) \int_0^T \int_\Omega u_\varepsilon^{\gamma-2} |\nabla u_\varepsilon|^2 \varphi + \int_0^T \int_\Omega u_\varepsilon^\gamma \Delta\varphi$$

$$+ (1-\gamma)\chi \int_0^T \int_\Omega u_\varepsilon^\gamma v_\varepsilon \varphi - (1-\gamma)\chi \int_0^T \int_\Omega u_\varepsilon^{\gamma+1} \varphi$$

$$+ \chi \int_0^T \int_\Omega u_\varepsilon^\gamma \nabla v_\varepsilon \cdot \nabla\varphi$$

$$+ \gamma \int_0^T \int_\Omega u_\varepsilon^{\gamma-1} g(u_\varepsilon)\varphi - \gamma\varepsilon \int_0^T \int_\Omega u_\varepsilon^{\beta+\gamma-1}\varphi. \qquad (1.55)$$

Since $\gamma < 1$, we can again use (1.47) and (1.7) to see that

$$-\int_0^T \int_\Omega u_\varepsilon^\gamma \varphi_t \to -\int_0^T \int_\Omega u^\gamma \varphi_t, \qquad (1.56)$$

$$-\int_\Omega u_{0\varepsilon}^\gamma \varphi(\cdot, 0) \to -\int_\Omega u_0^\gamma \varphi(\cdot, 0) \qquad \text{and} \qquad (1.57)$$

$$\int_0^T \int_\Omega u_\varepsilon^\gamma \Delta\varphi \to \int_0^T \int_\Omega u^\gamma \Delta\varphi, \qquad (1.58)$$

and a simplified variant of the reasoning leading to (1.53) shows that

$$\chi \int_0^T \int_\Omega u_\varepsilon^\gamma \nabla v_\varepsilon \cdot \nabla\varphi \to \chi \int_0^T \int_\Omega u^\gamma \nabla v \cdot \nabla\varphi \qquad (1.59)$$

as well as

$$(1-\gamma)\chi \int_0^T \int_\Omega u_\varepsilon^\gamma v_\varepsilon \varphi \to (1-\gamma)\chi \int_0^T \int_\Omega u^\gamma v\varphi \qquad (1.60)$$

as $\varepsilon = \varepsilon_j \searrow 0$.
Now in order to prove that

$$\gamma \int_0^T \int_\Omega u_\varepsilon^{\gamma-1} g(u_\varepsilon)\varphi \to \gamma \int_0^T \int_\Omega u^{\gamma-1} g(u)\varphi, \qquad (1.61)$$

we first split g via $g(s) = g(0) + h(s)$ with $h \in C^1([0, \infty))$ satisfying $h(0) = 0$ and thus $|h(s)| \le \bar{c}_0(s + s^\alpha)$ for $s \ge 0$ in view of (H1$_\alpha$) and (H2$_\alpha$) . Therefore,

$$\int_0^T \int_\Omega u_\varepsilon^{\gamma-1} g(u_\varepsilon)\varphi = g(0) \cdot \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}\varphi + \int_0^T \int_\Omega u_\varepsilon^{\gamma-1} h(u_\varepsilon)\varphi, \qquad (1.62)$$

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where for $r := \frac{\alpha}{\alpha+\gamma-1} > 1$ we have

$$
\begin{aligned}
\int_0^T \int_\Omega |u_\varepsilon^{\gamma-1} h(u_\varepsilon)|^r &\leq \bar{c}_0^r \int_0^T \int_\Omega |u_\varepsilon^\gamma + u_\varepsilon^{\alpha+\gamma-1}|^r \\
&\leq c_1\Big(1 + \int_0^T \int_\Omega u_\varepsilon^\alpha\Big) \quad\quad (1.63)
\end{aligned}
$$

with some $c_1 > 0$. By (1.12) amd (1.43), we thus infer that $u_\varepsilon^{\gamma-1} h(u_\varepsilon) \rightharpoonup u^{\gamma-1} h(u)$ in $L^r(\Omega \times (0,T))$ and hence

$$
\int_0^T \int_\Omega u_\varepsilon^{\gamma-1} h(u_\varepsilon)\varphi \to \int_0^T \int_\Omega u^{\gamma-1} h(u)\varphi \quad\quad (1.64)
$$

as $\varepsilon = \varepsilon_j \searrow 0$. If $g(0) = 0$, this immediately proves (1.61), while in the case $g(0) > 0$ we apply Lemma 1.2 with γ replaced by any $\gamma_0 \in (0,\gamma)$ to see that $(u_\varepsilon^{\gamma-1})_{\varepsilon\in(0,1)}$ is bounded in $L^s(\Omega \times (0,T))$ with $s = \frac{1-\gamma_0}{1-\gamma} > 1$, so that $u_\varepsilon^{\gamma-1} \rightharpoonup u^{\gamma-1}$ in $L^s(\Omega \times (0,T))$ due to (1.43) and therefore

$$
g(0) \cdot \int_0^T \int_\Omega u_\varepsilon^{\gamma-1}\varphi \to g(0) \cdot \int_0^T \int_\Omega u^{\gamma-1}\varphi
$$

as $\varepsilon = \varepsilon_j \searrow 0$. Together with (1.64), this completes the proof of (1.61).
As to the last term in (1.55), we apply the Hölder inequality to obtain

$$
\Big| -\gamma\varepsilon \int_0^T \int_\Omega u_\varepsilon^{\beta+\gamma-1}\varphi\Big| \leq \gamma \cdot \varepsilon^{\frac{1-\gamma}{\beta}} \cdot \Big(\varepsilon \int_0^T \int_\Omega u_\varepsilon^\beta\Big)^{\frac{\beta+\gamma-1}{\beta}} \cdot \Big(\int_0^T \int_\Omega \varphi^{\frac{\beta}{1-\gamma}}\Big)^{\frac{1-\gamma}{\beta}}
$$

and thus infer from (1.13) that

$$
-\gamma\varepsilon \int_0^T \int_\Omega u_\varepsilon^{\beta+\gamma-1}\varphi \to 0 \quad\quad (1.65)
$$

as $\varepsilon \to 0$.
Finally, the estimate (1.15) guarantees that $(\nabla u_\varepsilon^{\frac{\gamma}{2}})_{\varepsilon\in(0,1)}$ is bounded and hence weaky precompact in $L^2(\Omega \times (0,T))$. Once more due to (1.43), this means that $\nabla u_\varepsilon^{\frac{\gamma}{2}} \rightharpoonup \nabla u^{\frac{\gamma}{2}}$ in $L^2(\Omega \times (0,T))$. Thus, by lower semicontinuity of the seminorm $|||\cdot|||$ on $L^2(\Omega \times (0,T))$ defined by $|||w||| := (\int_0^T \int_\Omega w^2\varphi)^{\frac{1}{2}}$ with respect to weak convergence, we find

$$
\gamma(1-\gamma)\int_0^T \int_\Omega u^{\gamma-2}|\nabla u|^2\varphi \leq \gamma(1-\gamma)\cdot \liminf_{\varepsilon=\varepsilon_j\searrow 0}\int_0^T \int_\Omega u_\varepsilon^{\gamma-2}|\nabla u_\varepsilon|^2\varphi. \quad\quad (1.66)
$$

Collecting (1.55)-(1.61), (1.65) and (1.66), we see that (1.5) in fact is valid. Since the required regularity of (u,v) can be derived from (1.44)-(1.46), (1.48) and (1.63), we thereby see that (u,v) is a γ-entropy supersolution. ////

Combining the regularity properties that u inherits from u_ε via (1.44) and (1.45) with the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^q(\Omega)$ for $n = 1$ and $q = \infty$, or $n \geq 2$ and any $q < \infty$ satisfying $(n-2)q \leq 2n$, we immediately obtain

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Corollary 1.7 *Under the assumptions of Theorem 1.6, we have $u(\cdot, t) \in L^p(\Omega)$ for a.e. $t > 0$ and any*

$$\begin{cases} p \leq \infty & \text{if } n = 1, \\ p < \infty & \text{if } n = 2, \\ p \leq \alpha \quad \text{such that } p < \frac{n}{n-2} \cdot \min\{\alpha - 1, 1\} & \text{if } n \geq 3. \end{cases}$$

2 Boundedness properties

We now turn our attention to the question of boundedness of the very weak solutions constructed above.

2.1 Globally bounded small-data solutions

We start with an observation that is a simple consequence of the parabolic comparison priciple.

Lemma 2.1 *Suppose that g satisfies (H1_α) with some $\alpha > 1$, $a \geq 0$ and sufficiently large $b > 0$ such that there exists a positive number s_0 satisfying*

$$\chi s_0^2 + a - b s_0^\alpha \leq 0. \tag{2.1}$$

Then for all nonnegative $u_0 \in L^\infty(\Omega)$ with $\|u_0\|_{L^\infty(\Omega)} \leq s_0$, (0.4) possesses a global bounded very weak solution (u, v).

PROOF. In (1.6), besides (1.7) we can achieve that $u_{0\varepsilon} \leq s_0$ in Ω. Differentiating the cross-diffusion term in (1.6) and using the equation for v_ε, we find that u_ε satisfies

$$\begin{aligned} u_{\varepsilon t} &= \Delta u_\varepsilon - \chi \nabla u_\varepsilon \cdot \nabla v_\varepsilon - \chi u_\varepsilon v_\varepsilon + \chi u_\varepsilon^2 + g(u_\varepsilon) - \varepsilon u_\varepsilon^\beta \\ &\leq \Delta u_\varepsilon - \chi \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \chi u_\varepsilon^2 + a - b u_\varepsilon^\alpha \end{aligned} \tag{2.2}$$

in $\Omega \times (0, \infty)$ for all $\varepsilon \in (0, 1)$. Since $w(x, t) := s_0$ solves $w_t \geq \Delta w - \chi \nabla w \cdot \nabla v_\varepsilon + \chi w^2 + a - b w^\alpha$ with $\frac{\partial w}{\partial \nu} = 0$ on $\partial \Omega$ and lies above $u_{0\varepsilon}$ initially, the comparison pronciple shows that $u_\varepsilon \leq s_0$ in $\Omega \times (0, \infty)$. Since $\max_{x \in \bar{\Omega}} v_\varepsilon(x, t) \leq \max_{x \in \bar{\Omega}} u_\varepsilon(x, t)$ holds for all $t > 0$ due to an elliptic maximum principle argument, we also have $v_\varepsilon \leq s_0$.

In order to make Theorem 1.6 directly applicable without a re-inspection of its proof, we now manipulate $g(s)$ beyond $s = s_0$ so as to obtain a function $\tilde{g} \in C^1([0, \infty))$ that coincides with g on $[0, s_0]$ and satisfies (H1_α) and (H2_α) with some $\alpha \in (2 - \frac{1}{n}, 2)$. Since $(u_\varepsilon, v_\varepsilon)$ still solves (1.6) with g replaced by $\tilde{g}$, we may conclude from Theorem 1.6 that along an appropriate sequence $\varepsilon = \varepsilon_j \searrow 0$, we obtain a global very weak solution (u, v) of (0.4) satisfying $u \leq s_0$ and $v \leq s_0$ in $\Omega \times (0, \infty)$. ////

The reasoning in the following lemma was partly inspired by that in Theorem 7 in [HR]. Relying on the mass evolution results from Lemma 1.1, it provides an autonomous ordinary differential inequality for u_ε in $L^\gamma(\Omega)$ for arbitrary $\gamma > 1$.

Lemma 2.2 *Let* (H1$_\alpha$) *hold with some* $\alpha > 1$. *For* $t_0 \geq 0$ *and* $\varepsilon \in (0, 1)$, *let*

$$M_\varepsilon(t_0) := \max\{m, \|u_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)}\} \qquad (2.3)$$

with $m = (\frac{a}{b})^{\frac{1}{\alpha}}|\Omega|$ *as in Lemma 1.1.*
Then for all $\gamma > 1$ *satisfying* $\gamma > \frac{n}{2}$ *there exist positive constants* $\kappa > 1, \eta, \mu$
and C *such that for any* $t_0 \geq 0$ *and* $\varepsilon \in (0, 1)$ *we have*

$$\frac{d}{dt} \int_\Omega u_\varepsilon^\gamma(x, t)dx \leq C\Big(\int_\Omega u_\varepsilon^\gamma\Big)^\kappa - \eta \int_\Omega u_\varepsilon^\gamma + C\Big(M_\varepsilon^{\gamma+1}(t_0) + M_\varepsilon^\mu(t_0)\Big) \quad (2.4)$$

for all $t > t_0$.

Remark. Observe that the right-hand side in (2.4) is negative for small positive values of $\int_\Omega u_\varepsilon^\gamma$ whenever $M_\varepsilon(t_0)$ is small. Below, this property will be used in two different situations to achieve boundedness of the norm of $u(\cdot, t)$ in $L^\gamma(\Omega)$ (cf. Theorem 2.4 and lemma 2.5).

PROOF. We multiply the first equation in (1.6) by $u_\varepsilon^{\gamma-1}$, integrate by parts and use the identity $\Delta v_\varepsilon = v_\varepsilon - u_\varepsilon$ as well as (H1$_\alpha$) to see that

$$\frac{1}{\gamma}\frac{d}{dt}\int_\Omega u_\varepsilon^\gamma(x, t)dx \ + \ (\gamma - 1)\int_\Omega u_\varepsilon^{\gamma-2}|\nabla u_\varepsilon|^2$$
$$= \ -\frac{\chi(\gamma - 1)}{\gamma}\int_\Omega u_\varepsilon^\gamma v_\varepsilon + \frac{\chi(\gamma - 1)}{\gamma}\int_\Omega u_\varepsilon^{\gamma+1}$$
$$+ \int_\Omega u_\varepsilon^{\gamma-1}g(u_\varepsilon) - \varepsilon\int_\Omega u_\varepsilon^{\beta+\gamma-1}$$
$$\leq \ \frac{\chi(\gamma - 1)}{\gamma}\int_\Omega u_\varepsilon^{\gamma+1} + a\int_\Omega u_\varepsilon^{\gamma-1} \qquad (2.5)$$

for $t > 0$. Here, in the case $\gamma \leq 2$ we invoke the Hölder inequality to estimate

$$a\int_\Omega u_\varepsilon^{\gamma-1} \leq a|\Omega|^{2-\gamma} \cdot \Big(\int_\Omega u_\varepsilon\Big)^{\gamma-1},$$

while if $\gamma > 2$ then Young's inequality gives

$$a\int_\Omega u_\varepsilon^{\gamma-1} \leq \frac{\chi(\gamma - 1)}{\gamma}\int_\Omega u_\varepsilon^{\gamma+1} + c_1\int_\Omega u_\varepsilon$$

with some $c_1 > 0$. Writing $\mu := \min\{\gamma - 1, 1\}$, we thus have

$$\frac{d}{dt}\int_\Omega u_\varepsilon^\gamma(x, t)dx + \frac{4(\gamma-1)}{\gamma}\int_\Omega |\nabla u_\varepsilon^{\frac{\gamma}{2}}|^2 \leq 2\chi(\gamma-1)\int_\Omega u_\varepsilon^{\gamma+1} + c_2\Big(\int_\Omega u_\varepsilon\Big)^\mu \ (2.6)$$

with $c_2 = \max\{\gamma a|\Omega|^{2-\gamma}, \gamma c_1\}$.
We now use that $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2(\gamma+1)}{\gamma}}(\Omega)$ because $\gamma > \frac{n}{2} > \frac{n}{2} - 1$, and hence

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may apply the Gagliardo-Nirenberg inequality to find $c_{GN} > 0$ such that

$$2\chi(\gamma-1)\int_\Omega u_\varepsilon^{\gamma+1} = 2\chi(\gamma-1)\|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{2(\gamma+1)}{\gamma}}(\Omega)}^{\frac{2(\gamma+1)}{\gamma}}$$

$$\leq c_{GN}\Big(\|\nabla u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma+1)}{\gamma}\theta}\cdot\|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma+1)}{\gamma}(1-\theta)} + \|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{2}{\gamma}}(\Omega)}^{\frac{2(\gamma+1)}{\gamma}}\Big), \qquad (2.7)$$

where

$$-\frac{n\gamma}{2(\gamma+1)} = \Big(1-\frac{n}{2}\Big)\theta - \frac{n}{2}(1-\theta) \equiv \theta - \frac{n}{2},$$

that is,

$$\theta = \frac{n}{2}\Big(1 - \frac{\gamma}{\gamma+1}\Big) = \frac{n}{2(\gamma+1)}.$$

Since $\gamma > \frac{n}{2}$, we may employ Young's inequality with exponents $\frac{2\gamma}{n}$ and $\frac{2\gamma}{2\gamma-n}$ to gain

$$c_{GN}\|\nabla u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma+1)}{\gamma}\theta}\cdot\|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma+1)}{\gamma}(1-\theta)} = c_{GN}\|\nabla u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{n}{\gamma}}\cdot\|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma+1)-n}{\gamma}}$$

$$\leq \frac{2(\gamma-1)}{\gamma}\|\nabla u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^2 + c_3\|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{2\cdot\frac{2(\gamma+1)-n}{2\gamma-n}}$$

with some $c_3 > 0$. Thus, (2.6) and (2.7) imply

$$\frac{d}{dt}\int_\Omega u_\varepsilon^\gamma(x,t)dx + \frac{2(\gamma-1)}{\gamma}\int_\Omega |\nabla u_\varepsilon^{\frac{\gamma}{2}}|^2$$

$$\leq c_3\Big(\int_\Omega u_\varepsilon^\gamma\Big)^{\frac{2(\gamma+1)-n}{2\gamma-n}} + c_{GN}\Big(\int_\Omega u_\varepsilon\Big)^{\gamma+1} + c_2\Big(\int_\Omega u_\varepsilon\Big)^\mu. \qquad (2.8)$$

Finally, the Poincaré inequality provides some $c_P > 0$ such that

$$\int_\Omega u_\varepsilon^\gamma = \|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^2 \leq c_P\Big(\|\nabla u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^2 + \|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{2}{\gamma}}\Omega)}^2\Big)$$

$$= c_P\Big(\int_\Omega |\nabla u_\varepsilon^{\frac{\gamma}{2}}|^2 + \Big(\int_\Omega u_\varepsilon\Big)^\gamma\Big), \qquad (2.9)$$

which inserted into (2.8) yields

$$\frac{d}{dt}\int_\Omega u_\varepsilon^\gamma(x,t)dx \leq c_3\Big(\int_\Omega u_\varepsilon^\gamma\Big)^{\frac{2(\gamma+1)-n}{2\gamma-n}} - \frac{2(\gamma-1)}{\gamma c_P}\int_\Omega u_\varepsilon^\gamma$$

$$+ c_{GN}\Big(\int_\Omega u_\varepsilon\Big)^{\gamma+1} + \frac{2(\gamma-1)}{\gamma c_P}\Big(\int_\Omega u_\varepsilon\Big)^\gamma + c_2\Big(\int_\Omega u_\varepsilon\Big)^\mu.$$

Here, in treating the last three terms we use that the mass evolution estimate (1.8) from Lemma 1.1 implies $\int_\Omega u_\varepsilon(x,t)dx \leq M_\varepsilon(t_0)$ whenever $t > t_0$. Since $\gamma+1 > \gamma > \mu$, a simple interpolation allows us to bound $M_\varepsilon^\gamma(t_0)$ by some multiple of $(M_\varepsilon^{\gamma+1}(t_0) + M_\varepsilon^\mu(t_0))$, so that (2.4) follows. ////

As another preliminary, we shall need the following smoothing property of (1.6).

Lemma 2.3 *Let* (H1$_\alpha$) *be satisfied with some* $\alpha > 1$, *and assume that there exist* $\gamma_0 > \frac{n}{2}$, $C > 0$, $\varepsilon_0 > 0$ *and* $0 \leq t_1 < t_2 \leq \infty$ *such that*

$$\|u_\varepsilon(\cdot, t)\|_{L^{\gamma_0}(\Omega)} \leq C \qquad \text{for all } t \in (t_1, t_2) \tag{2.10}$$

is valid for all $\varepsilon \in (0, \varepsilon_0)$. *Then for any* $\tau > 0$ *we can find* $C(\tau) > 0$ *such that*

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\tau) \qquad \text{for all } t \in (t_1 + \tau, t_2) \tag{2.11}$$

holds whenever $\varepsilon \in (0, \varepsilon_0)$.

PROOF.   The proof closely follows that of Lemma 2.3 and Lemma 2.4 in [TW], and thus we may restrict ourselves to outlining the main steps.

First, we fix any $\gamma > \gamma_0$ and proceed as in deriving (2.6) to obtain

$$\frac{d}{dt} \int_\Omega u_\varepsilon^\gamma(x, t) dx + \frac{4(\gamma - 1)}{\gamma} \int_\Omega |\nabla u_\varepsilon^{\frac{\gamma}{2}}|^2 \leq 2\chi(\gamma - 1) \int_\Omega u_\varepsilon^{\gamma+1} + c_1 \tag{2.12}$$

for $\varepsilon \in (0, \varepsilon_0)$ and some $c_1 > 0$ depending on $\|u_0\|_{L^1(\Omega)}$. By the Gagliardo-Nirenberg inequality,

$$\begin{aligned}
\int_\Omega u_\varepsilon^{\gamma+1} &= \|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{2(\gamma+1)}{\gamma}}(\Omega)}^{\frac{2(\gamma+1)}{\gamma}} \\
&\leq c_{GN}\left(\|\nabla u_\varepsilon^{\frac{\gamma}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma+1)}{\gamma}\theta} \cdot \|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{2\gamma_0}{\gamma}}(\Omega)}^{\frac{2(\gamma+1)}{\gamma}(1-\theta)} + \|u_\varepsilon^{\frac{\gamma}{2}}\|_{L^{\frac{2}{\gamma}}(\Omega)}^{\frac{2(\gamma+1)}{\gamma}}\right)
\end{aligned} \tag{2.13}$$

holds with some $c_{GN} > 0$ and

$$\theta = \frac{n\gamma(\gamma + 1 - \gamma_0)}{(\gamma + 1)(n\gamma - n\gamma_0 + 2\gamma_0)}.$$

Since $\gamma_0 > \frac{n}{2}$, it is easily checked that

$$\frac{2(\gamma + 1)}{\gamma}\theta = \frac{2n(\gamma + 1 - \gamma_0)}{n\gamma - n\gamma_0 + 2\gamma_0} < 2.$$

Hence, from (2.13) we infer upon applying Young's inequality that

$$\left(1 + 2\chi(\gamma - 1)\right) \int_\Omega u_\varepsilon^{\gamma+1}(x, t) dx \leq \frac{4(\gamma - 1)}{\gamma} \int_\Omega |\nabla u_\varepsilon^{\frac{\gamma}{2}}|^2 + c_2 \qquad \text{for all } t \in (t_1, t_2),$$

and thus (2.12) gives

$$\begin{aligned}
\frac{d}{dt} \int_\Omega u_\varepsilon^\gamma(x, t) dx &\leq -\int_\Omega u_\varepsilon^{\gamma+1} + c_3 \\
&\leq -|\Omega|^{-\frac{1}{\gamma}}\left(\int_\Omega u_\varepsilon\gamma\right)^{\frac{\gamma+1}{\gamma}} + c_3 \qquad \text{for } t \in (t_1, t_2),
\end{aligned}$$

where c_2 and c_3 depend on $\|u_0\|_{L^1(\Omega)}$ and C only. Upon integration we obtain, since $\frac{\gamma+1}{\gamma} > 1$, that for all $\varepsilon \in (0, \varepsilon_0)$ and arbitrary $\gamma > \gamma_0$,

$$\|u_\varepsilon(\cdot, t)\|_{L^\gamma(\Omega)} \leq c_4 = c_4(\|u_0\|_{L^1(\Omega)}, C, \gamma, \tau) \qquad \text{for } t \in (t_1 + \tau, t_2). \tag{2.14}$$

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Applying elliptic regularity theory to the second equation in (1.6), we therefore conclude that $(u_\varepsilon \nabla v_\varepsilon)_{\varepsilon \in (0,\varepsilon_0)}$ is bounded in $L^\infty((t_1+\tau, t_2); L^p(\Omega))$ for all $p < \infty$. Now standard arguments relying, for instance, on explicit representation of u_ε involving the semigroup $(e^{t\Delta})_{t\geq 0}$ generated by the Neumann Laplacian in Ω, finally yield the desired uniform bound for u_ε in $L^\infty_{loc}(\bar\Omega \times (t_1, t_2])$ (cf. Lemma 2.4 in [TW] for details on this, or [A] for an alternative reasoning). ////

We now can prove our main result on global bounded small-data solutions.

Theorem 2.4 *Assume that g fulfills* $(\mathrm{H1}_\alpha)$ *with some* $\alpha > 1$. *Then there exists* $\delta > 0$ *with the property that if* $\frac{a}{b} < \delta$ *then for all* $\gamma > \max\{1, \frac{n}{2}\}$ *one can find* $\lambda > 0$ *such that whenever* $u_0 \in L^\infty(\Omega)$ *satisfies* $\|u_0\|_{L^\gamma(\Omega)} < \lambda$, *the problem (0.4) possesses a global bounded very weak solution* (u,v).

PROOF. Given $\gamma > \max\{1, \frac{n}{2}\}$, let κ, η, μ and C be the constants provided by Lemma 2.2. For $M \geq 0$, let

$$\phi_M(\xi) := C\xi^\kappa - \eta\xi + C(M^{\gamma+1} + M^\mu), \qquad \xi \geq 0, \qquad (2.15)$$

and

$$S_M := \left\{ \xi > 0 \;\middle|\; \exists \bar\xi \geq \xi \text{ such that } \phi_M(\bar\xi) = 0 \right\}.$$

Since $\kappa > 1$ and $\eta > 0$, the number $\xi_0 := (\frac{\eta}{C})^{\frac{1}{\kappa-1}}$ belongs to S_0, and thus from a continuous dependence argument it follows that there exists $M_0 > 0$ such that $\frac{\xi_0}{2} \in S_M$ for all $M \leq M_0$. We set

$$\delta := \left(\frac{M_0}{|\Omega|}\right)^\alpha \qquad \text{and} \qquad \lambda := \min\left\{ \left(\frac{\xi_0}{2}\right)^{\frac{1}{\gamma}}, \frac{M_0}{|\Omega|^{\frac{\gamma-1}{\gamma}}} \right\}, \qquad (2.16)$$

and henceforth suppose that $\frac{a}{b} < \delta$ and $\|u_0\|_{L^\gamma(\Omega)} < \lambda$. Then

$$\int_\Omega u_0 \leq |\Omega|^{\frac{\gamma-1}{\gamma}} \cdot \|u_0\|_{L^\gamma(\Omega)} < M_0, \qquad (2.17)$$

and hence, in view of the definition of δ, $M_\varepsilon(0) = \max\{(\frac{a}{b})^{\frac{1}{\alpha}}|\Omega|, \|u_{0\varepsilon}\|_{L^1(\Omega)}\}$ as introduced in Lemma 2.2 satisfies $M_\varepsilon(0) \leq M_0$ for all sufficiently small $\varepsilon > 0$. Since after possibly regularizing $u_{0\varepsilon}$ we may assume that also $\|u_{0\varepsilon}\|_{L^\gamma(\Omega)} < \lambda$ holds for small ε, we obtain

$$\int_\Omega u_{0\varepsilon}^\gamma < \lambda^\gamma \leq \frac{\xi_0}{2} \qquad (2.18)$$

for such ε. Now Lemma 2.2 applies to ensure that $y(t) := \int_\Omega u_\varepsilon^\gamma(x,t)dx$ satisfies $y'(t) \leq \phi_{M_0}(y(t))$ for all $t > 0$, because $M_\varepsilon(0) \leq M_0$ and ϕ_M obviously increases with M. Since $y(0) = \int_\Omega u_{0\varepsilon}^\gamma$ lies below some zero $\bar\xi_0$ of ϕ_{M_0}, it follows from an ODE comparison that $y(t) \leq \bar\xi_0$ for all $t > 0$, and therefore

$$\|u_\varepsilon(\cdot, t)\|_{L^\gamma(\Omega)} \leq \bar\xi_0 \qquad \text{for all } t > 0 \qquad (2.19)$$

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holds for all sufficiently small $\varepsilon > 0$.

In order to be able to apply Lemma 2.3 with an appropriate $\tau > 0$, let us make sure that $(u_\varepsilon)_{\varepsilon \in (0,1)}$ is bounded in $L^\infty(\Omega \times (0,\tau))$ for some $\tau > 0$. Indeed, the fact that $u_0 \in L^\infty(\Omega)$ allows us to assume without loss of generality that $\|u_{0\varepsilon}\|_{L^\infty(\Omega)} \le c_1$ holds for all $\varepsilon \in (0,1)$ and some $c_1 > 0$. Recalling (2.2), we see that

$$u_{\varepsilon t} \le \Delta u_\varepsilon - \chi \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \chi u_\varepsilon^2 + a \qquad \text{in } \Omega \times (0,\infty),$$

which by parabolic comparison implies that

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le z(t) \qquad \text{for all } t \in (0, \tau_z), \tag{2.20}$$

where z denotes the solution of

$$\begin{cases} z' = \chi z^2 + a, & t \in (0, \tau_z), \\ z(0) = c_1, \end{cases}$$

and $\tau_z > 0$ its maximal existence time.

Now due to (2.19), Lemma 2.3 guarantees that for some $\varepsilon_0 > 0$, $(u_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ is bounded in $L^\infty(\Omega \times (\frac{\tau_z}{2}, \infty))$ which together with (2.20) proves boundedness of $(u_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ in $L^\infty(\Omega \times (0, \infty))$. Arguing as in Lemma 2.1, from this we easily conclude that some limit (u, v) of $\big((u_\varepsilon, v_\varepsilon)\big)_{\varepsilon \in (0, \varepsilon_0)}$ as $\varepsilon = \varepsilon_j \searrow 0$ is a globally bounded very weak solution of (0.4). ////

2.2 Eventual boundedness

Our next goal is to show boundedness beyond some prescribed $\tau > 0$. Again, this can be achieved upon imposing a suitable smallness condition on u_0, measured however in $L^1(\Omega)$ rather than in $L^\gamma(\Omega)$ as in Theorem 2.4. Here we once more rely on the differential inequality (2.4).

Lemma 2.5 *Assume that g satisfies* $(H1_\alpha)$ *with some* $\alpha > \frac{n}{2}$. *Then for all $\tau > 0$ there exist positive constants $\delta(\tau)$ and $C(\tau)$ with the following property: If there exist $t_0 \ge 0$ and $\varepsilon_0 > 0$ such that the number*

$$M_\varepsilon(t_0) = \max \left\{ (\frac{a}{b})^{\frac{1}{\alpha}} |\Omega|, \ \|u_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)} \right\}$$

from Lemma 2.2 satisfies

$$M_\varepsilon(t_0) < \delta(\tau)$$

then

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \le C(\tau) \qquad \text{for all } t \ge t_0 + \tau \tag{2.21}$$

is valid whenever $\varepsilon \in (0, \varepsilon_0)$.

PROOF.   As in the proof of Theorem 2.4, for $\xi \geq 0$ and $M \geq 0$ we let $\phi_M(\xi) = C\xi^\kappa - \eta\xi + C(M^{\alpha+1} + M^\mu)$ with κ, η, μ and C taken from Lemma 2.2 upon the choice $\gamma = \alpha > \frac{n}{2}$. Again we find $\xi_0 > 0$ and $M_0 > 0$ such that for all $M \leq M_0$ there exists a zero $\bar{\xi}_0 \geq \frac{\xi_0}{2}$ of ϕ_M. We let

$$\delta(\tau) := \min\left\{\left(\frac{\xi_0}{8}\right)^{\frac{1}{\alpha}} |\Omega|^{\frac{\alpha-1}{\alpha}}, \ \frac{\xi_0 b\tau}{8}, \ M_0\right\} \qquad (2.22)$$

and claim that if $M_\varepsilon(t_0) < \delta(\tau)$ for $\varepsilon < \varepsilon_0$ then (2.21) holds for an appropriately large $C(\tau)$.

To this end, we first employ Lemma 1.1 to obtain

$$\int_\Omega u_\varepsilon(x,t)dx \leq \delta(\tau) \qquad \text{for all } t > t_0 \qquad (2.23)$$

and

$$\int_{t_0}^{t_0+\tau} \int_\Omega u_\varepsilon^\alpha \leq \frac{a|\Omega|\tau + \delta(\tau)}{b}$$

for $\varepsilon < \varepsilon_0$, so that necessarily there must exist some $t_\varepsilon \in (t_0, t_0 + \frac{\tau}{2})$ such that

$$\int_\Omega u_\varepsilon^\alpha(x, t_\varepsilon)dx \leq \frac{2}{\tau} \cdot \frac{a|\Omega|\tau + \delta(\tau)}{b}.$$

Since, by the definition of $M_\varepsilon(t_0)$ and (2.22),

$$\begin{aligned}
\frac{2}{\tau} \cdot \frac{a|\Omega|\tau + \delta(\tau)}{b} &\leq 2|\Omega| \cdot \left(\frac{M_\varepsilon(t_0)}{|\Omega|}\right)^\alpha + \frac{2\delta(\tau)}{b\tau} \\
&\leq 2|\Omega| \cdot \left(\delta(\tau)|\Omega|\right)^\alpha + \frac{2\delta(\tau)}{b\tau} \\
&\leq \frac{\xi_0}{4} + \frac{\xi_0}{4} = \frac{\xi_0}{2},
\end{aligned}$$

we thereby have found $t_\varepsilon \in (t_0, t_0 + \frac{\tau}{2})$ such that

$$\int_\Omega u_\varepsilon^\alpha(x, t_\varepsilon)dx \leq \frac{\xi_0}{2}.$$

As $\|u_\varepsilon(\cdot, t_\varepsilon)\|_{L^1(\Omega)} \leq \delta(\tau) \leq M_0$ by (2.23) and (2.22), the properties of ϕ_{M_0} in conjunction with the differential inequality (2.4) imply that $\|u_\varepsilon(\cdot, t)\|_{L^\alpha(\Omega)}$ is bounded by a constant independent of $t \in (t_0 + \frac{\tau}{2}, \infty)$ and $\varepsilon \in (0, \varepsilon_0)$. Now an application of lemma 2.3 provides the desired L^∞ bound for u_ε in $\Omega \times (t_0 + \tau, \infty)$ and thus also for v_ε, again because of the fact that $\|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq \|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)}$ for all $t > 0$. ////

Now the first of our main results of this section reduces to a corollary that we may state without further comment.

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Theorem 2.6 *Suppose that g fulfills* $(H1_\alpha)$ *and* $(H2_\alpha)$ *with some* $\alpha > \max\{\frac{n}{2}, 2 - \frac{1}{n}\}$. *Then for all* $\tau > 0$ *there exists* $\delta(\tau) > 0$ *such that if*

$$\max\left\{ \left(\frac{a}{b}\right)^{\frac{1}{\alpha}} |\Omega|, \ \|u_0\|_{L^1(\Omega)} \right\} < \delta(\tau)$$

then the weak solution (u, v) *constructed in Theorem 1.6 is bounded in* $\Omega \times (\tau, \infty)$.

Let us finally make sure that *any* of our solutions eventually becomes bounded, regardless of its initial size in $L^1(\Omega)$. In fact, we shall find a bound in $L^\infty(\Omega)$ that is independent of $\|u_0\|_{L^1(\Omega)}$; clearly, however, the time beyond which the corresponding estimate holds will depend on u_0.

Lemma 2.7 *Let* $(H1_\alpha)$ *be satisfied with some* $\alpha > \frac{n}{2}$. *Then there exist positive constants* δ *and* C *with the property that if* $\frac{a}{b} < \delta$ *then for all nonnegative* $u_0 \in L^1(\Omega)$ *one can pick* $T > 0$ *such that for all* $\varepsilon \in (0, 1)$,

$$\|u_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot, t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t \geq T. \qquad (2.24)$$

PROOF.   Let $\delta(1)$ and $C(1)$ be the constants provided by Lemma 2.5 upon the special choice $\tau = 1$. We set

$$\delta := \left(\frac{\delta(1)}{|\Omega|} \right)^\alpha$$

and assume that $\frac{a}{b} < \delta$, so that $m := (\frac{a}{b})^{\frac{1}{\alpha}} |\Omega|$ satisfies $m < \delta(1)$. Then from the inequality (1.8) in Lemma 1.1 and our overall assumption (1.7) we know that

$$\int_\Omega u_\varepsilon(x, t) dx \leq m + \left(\|u_0\|_{L^1(\Omega)} + 1 \right) \cdot e^{-kt}$$

holds for all $t > 0$ and $\varepsilon \in (0, 1)$ with some $k > 0$. In particular, there exists $t_0 > 0$ such that

$$\int_\Omega u_\varepsilon(x, t_0) dx < \delta(1)$$

and hence $M_\varepsilon(t_0) = \max\{m, \|u_\varepsilon(\cdot, t_0)\|_{L^1(\Omega)}\} < 1$ for all $\varepsilon \in (0, 1)$. Accordingly, Lemma 2.5 says that (2.24) must be true for $C := C(1)$ and all $\varepsilon \in (0, 1)$ if we let $T := t_0 + 1$. ////

Taking $\varepsilon = \varepsilon_j \searrow 0$ along an appropriate sequence, we immediately obtain our final result.

Theorem 2.8 *Suppose that g satisfies* $(H1_\alpha)$ *and* $(H2_\alpha)$ *with a fixed number* $\alpha > \max\{\frac{n}{2}, 2 - \frac{1}{n}\}$. *Then there exist* $\delta > 0$ *and a ball* $\mathcal{B}$ *in* $L^\infty(\Omega)$ *such that whenever* $\frac{a}{b} < \delta$ *and* $u_0 \in L^1(\Omega)$ *is nonnegative, there exists* $T > 0$ *with the property that the very weak solution* (u, v) *constructed in Theorem 1.6 satisfies* $u(\cdot, t) \in \mathcal{B}$ *and* $v(\cdot, t) \in \mathcal{B}$ *for all* $t \geq T$.

3 Numerical examples

Let us finally illustrate some of our theoretical results by numerical calculations. In doing so, we restrict ourselves to the case where $\chi = 1$, $\Omega = B_1(0)$ is the unit ball in $\mathbb{R}^n$, and where the initial data u_0 and hence the solution (u, v) are radially symmetric with respect to $x = 0$. The resulting system (0.4) is then actually one-dimensional in space, which considerably reduces the technical expense necessary for our spatial discretization. In particular, we then only need to approximate the radial differential operators $\frac{\partial}{\partial_r}$ and $\frac{\partial^2}{\partial r^2}$ in the standard way by the usual difference operators.

Throughout our numerical experiments, at each time step we first interpret the second equation in (0.4) as a Helmholtz equation for the unknown v with known inhomogeneity u taken from the previous time step. Having thereby found v at the current time, we insert this into the first in (0.4) and then perform an explicit Euler discretization to compute u from this equation, where the time step size can be cotrolled via standard methods familiar from the numerical solution of ODE systems (cf. [S]).

3.1 Smoothing action of the chemotaxis system

A first example refers to problem (0.4) in space dimension $n = 2$, with logistic term given by

$$g(u) = 1 - u^{1.8}, \qquad u \geq 0,$$

and initial data

$$u_0(x) = \frac{0.1}{(|x| + 0.001)^{1.5}}, \qquad 0 \leq |x| \leq 1.$$

Observe that the choice $\alpha = 1.8 < 2$ has not been covered by known results in the literature (for instance by [TW]). The initial data are supposed to be a 'good' approximation of the singular function given by $u_0(x) = 0.1|x|^{-1.5}$ that is not in $L^2(\Omega)$ (the largest previously considered space of admissible initial data in chemotaxis problems, cf. the introduction).

Figure 1 shows the short time behavior of the first component u of the numerical solution, depicted in dependence of the scalar variable $r = |x|$. This illustrates the regularizing effect of the evolution governed by (0.4) even for $\alpha < 2$ and 'bad' initial data, as asserted by Theorem 1.6 and Corollary 1.7.
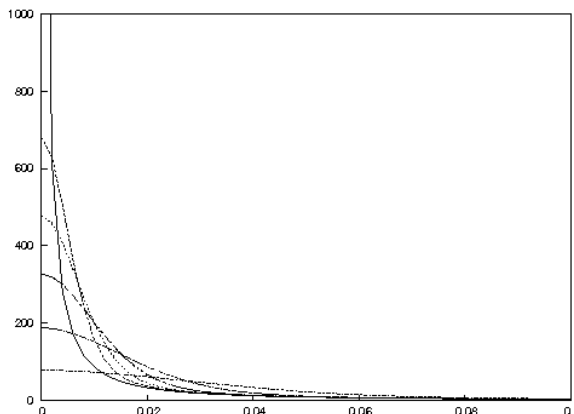
Fig. 1. Abscissa: $r = |x|$; ordinate: First solution component $u = u(r, t)$ at times $t = 0$; $t = 1.22 \cdot 10^{-5}$; $t = 2.44 \cdot 10^{-5}$; $t = 4.88 \cdot 10^{-5}$; $t = 1.22 \cdot 10^{-4}$; $t = 4.46 \cdot 10^{-4}$. Decreasing values of u at $r = 0$ correspond to increasing values of t: The graph with $u(0, t) > 1000$ represents $t = 0$.

3.2 Boundedness of small-data solutions

The motivation for the following example is to demonstrate the assertion on boundedness of solutions emenating from initial data that are sufficiently small in $L^\gamma(\Omega)$ for some $\gamma > \max\{1, \frac{n}{2}\}$, provided that the quotient $\frac{a}{b}$ in $(\mathrm{H1}_\alpha)$ is small enough. For this purpose, we consider the three-dimensional radial version of (0.4) with logistic term

$$g(u) = 1 - 100u^{1.2}, \qquad u \geq 0,$$

and approach the 'small' initial data

$$u_0(x) = 0.0001 \cdot |x|^{-1.2}, \qquad 0 < |x| \leq 1,$$

by the bounded approximates

$$u_0^{(\varepsilon)}(x) = \frac{0.0001}{(|x| + \varepsilon)^{1.2}}, \qquad 0 \leq |x| \leq 1,$$

where ε will attain certain small positive values. Observe that since the integral $\int_0^1 r^2 \cdot r^{-1.2\gamma} dr$ is finite for all $\gamma < 2.5$, the singular data belong to $L^\gamma(\Omega)$ for such γ; we thus may believe that all these data $u_0^{(\varepsilon)}$ are appropriately small in $L^\gamma(\Omega)$ for some $\gamma > \frac{3}{2}$.

Figure 2 shows the time evolution of the spatial L^∞ norm of $u^{(\varepsilon)}$ for some small ε. Though our computational capacity reaches its limit at $\varepsilon = 2.4 \cdot 10^{-8}$, we believe that the corresponding graph can be regarded as a good approximation of the one to be expected for the singular initial function. In any event, Fig. 2 indicates the global boundedness of all approximate solutions. Actually, a closer look at the spatial profile shows that all these numerical solutions approach the constant steady state determined by $u_\infty \equiv (\frac{1}{100})^{\frac{1}{1.2}} \approx 0.0215$ as $t \to \infty$.
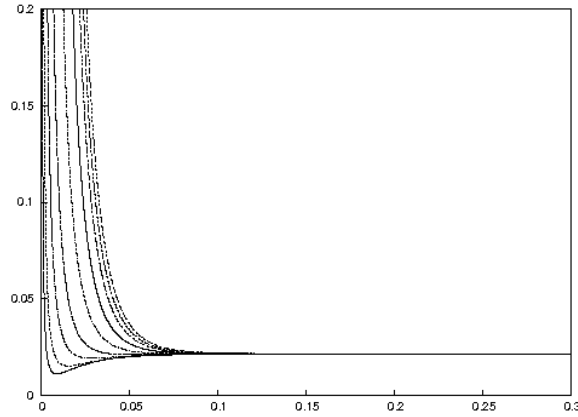
Fig. 2. Abscissa: time t; ordinate: $L^\infty(\Omega)$ norm of the solution $u^{(\varepsilon)}(\cdot, t)$ corresponding to the initial data $u_0^{(\varepsilon)}$ with $\varepsilon = 3 \cdot 10^{-5}$; $\varepsilon = 10^{-5}$; $\varepsilon = 3 \cdot 10^{-6}$; $\varepsilon = 10^{-6}$; $\varepsilon = 3 \cdot 10^{-7}$; $\varepsilon = 10^{-7}$; $\varepsilon = 6 \cdot 10^{-8}$; $\varepsilon = 4 \cdot 10^{-8}$; $\varepsilon = 2.4 \cdot 10^{-8}$. The graphs increase when ε decreases, the largest one belonging to $\varepsilon = 2.4 \cdot 10^{-8}$.

3.3 Unbounded very weak solutions; blow-up

We finally give an example which indicates that in spite of the asserted regularizing effects, very weak solutions need not remain bounded even if they have become smooth at some positive time. To be more precise, we numerically investigate the possibility of finite-time blow-up in (0.4) when $n = 2$ and

$$g(u) = 1 - bu^{1.8}, \qquad u \ge 0,$$

where we consider both $b = 1$ and $b = 0.01$. The initial data are chosen to be

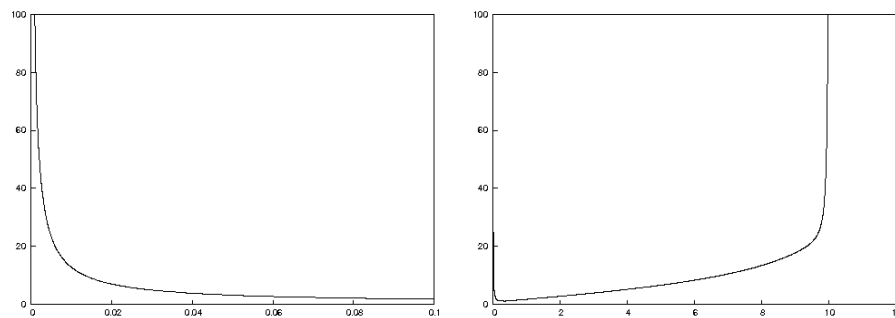$$u_0(x) = \frac{0.1}{(|x| + 0.001)^{1.5}}, \qquad 0 \le |x| \le 1.$$

Fig. 3. Abscissae: t; ordinates: $L^\infty(\Omega)$ norm of the solution $u(\cdot, t)$ in the case $g(u) = 1 - 0.01u^{1.8}$
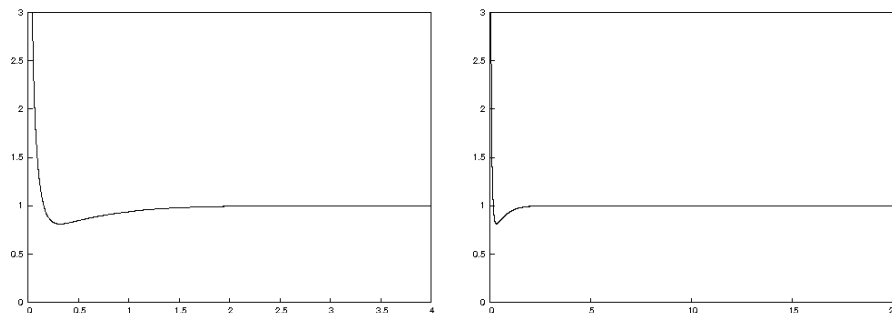
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Fig. 4. Abscissae: t; ordinates: $L^\infty(\Omega)$ norm of the solution $u(\cdot, t)$ in the case $g(u) = 1 - u^{1.8}$

Figure 3 shows that finite-time blow-up occurs when the dampening effect in the logistic term is small ($b = 0.01$), whereas according to Figure 4, the same initial data lead to a globally bounded solution (again stabilizing to the constant equilibrium $(u, v) \equiv 1$) when the growth inhibition is stronger ($b = 1$).
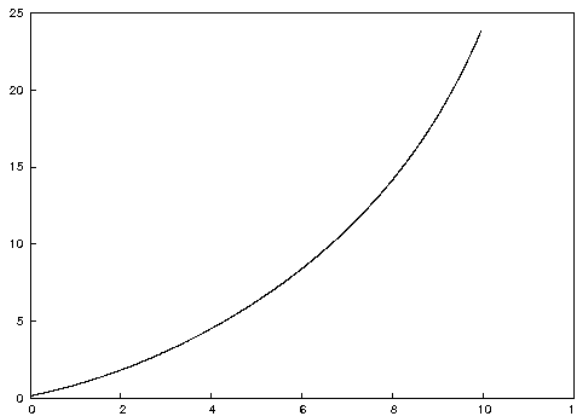

Fig. 5. Abscissa: time T; ordinate: $L^\alpha(\Omega \times (0, T))$ norm of the blow-up solution u in the case $g(u) = 1 - 0.01u^{1.8}$

But our numerical solution, though blowing up in finite time with respect to the norm in $L^\infty(\Omega)$, complies with the space-time summability assertion in Theorem 1.6: As indicated by Figure 5, the integral $\int_0^T \int_\Omega u^\alpha(x, t) dx dt$ remains bounded across – but at least up to – the blow-up time.

3.4 Conclusion

Unfortunately, our algorithm, being essentially of experimental nature and of course lacking any justification by numerical analysis, is not able to compute an unbounded solution beyond its blow-up time. However, in our opinion the above illustrations strongly indicate that logistic growth inhibition gives rise to much a larger variety in the dynamics of (0.4) than one possibly might expect: Besides some mechanisms of regularization and stabilization, we have found numerical evidence suggesting the existence of solutions that model cell aggregation in the

sense of finite-time blow-up – in spite of superlinear logistic dampening. Though the latter needs to be proved in future work, we regard this as a strong advice to rely on (very) weak rather than on classical solutions in the present context.

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