Oscillating solutions and large ω-limit sets in a degenerate parabolic equation

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Abstract

The paper deals with positive solutions of the initial-boundary value problem for

$$u_t = f(u)(\Delta u + \lambda_1 u) \qquad (\star)$$

with zero Dirichlet data in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$. Here $f \in C^0([0,\infty)) \cap C^1((0,\infty))$ is positive on $(0,\infty)$ with $f(0) = 0$, and λ_1 is exactly the first Dirichlet eigenvalue of $-\Delta$ in Ω. In this setting, $(\star)$ may possess oscillating solutions in presence of a sufficiently strong degeneracy. More precisely, writing $H(s) := \int_1^s \frac{d\sigma}{f(\sigma)}$, it is shown that if $\int_0 sH(s)ds = -\infty$ then there exist global classical solutions of $(\star)$ satisfying $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ and $\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = 0$.

Under the additional structural assumption $\frac{sf'(s)}{f(s)} \geq \kappa > 0$, $s > 0$, this result can be sharpened: If $\int_0 sH(s)ds = -\infty$ then $(\star)$ has a global solution with its ω-limit set being the ordered arc that consists of all nonnegative multiples of the principal Laplacian eigenfunction. On the other hand, under the above additional assumption the opposite condition $\int_0 sH(s)ds > -\infty$ ensures that all solutions of $(\star)$ will stabilize to a single equilibrium.

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1 Introduction

We consider nonnegative classical solutions of the problem

$$\begin{cases} u_t = f(u)(\Delta u + \lambda_1 u) & \text{in } \Omega \times (0,T), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$, where $\lambda_1 > 0$ denotes the first Dirichlet eigenvalue of $-\Delta$ in Ω. The function $f \in C^0([0,\infty)) \cap C^1((0,\infty))$ is assumed to be positive in $(0,\infty)$ with $f(0) = 0$, whereby (1.1) degenerates near points where u is small.

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Equations of this type arise, for instance, in simplified models in electromagnetism, in differential geometry and in population dynamics ([Lo], [An], [Al]), where usually $f(u) = u^p$ with some $p \geq 1$. When $0 < p < 1$, (1.1) can be transformed into the well-understood porous medium equation $v_t = \Delta v^m + \lambda_1 v^m$ with $m = \frac{1}{1-p}$.

The main goal of this work is to show that (1.1), despite of its seemingly simple structure, may exhibit an unexpectedly complex dynamics: Namely, we shall see that if the degeneracy is strong enough in a certain sense then (1.1) possesses global unbounded solutions which oscillate in time with amplitude growing to infinity. It will furthermore turn out that these solutions even may have an *unbounded but ordered* ω-limit set

$$\omega(u_0) := \{w \in L^2(\Omega) \mid u(\cdot, t_k) \to w \text{ in } L^2(\Omega) \text{ for some } t_k \to \infty\}.$$

Before making these statements more precise, let us first recall some basically well-known results on asymptotic behavior in second-order diffusion equations. It is in accordance with the dissipative structure of such equations that, generically, their global-in-time solutions prefer to stabilize, as time approaches inifinity, to either a single stationary profile, or to '∞' in a suitable topology. Moreover, in most reasonable situations such a regular profile must belong to the set $\mathcal{E}$ of stationary solutions of (1.1), so that, generically, one has that $\omega(u_0)$ consists of at most one element of $\mathcal{E}$.

A well-studied example is the corresponding Dirichlet problem for the semilinear equation $u_t = \Delta u + g(u)$ with sufficiently smooth g. As to this, it is known that all bounded solutions stabilize if either $n = 1$ ([Ze], [Ma]), g is analytic ([Je]), or if Ω is a ball and u is nonnegative ([HP]). Other sufficient criteria involve some a priori knowledge on $\mathcal{E}$: For instance, if $\mathcal{E}$ is a totally ordered set of functions then the Hopf boundary point lemma enforces all bounded solutions to settle down to one of them ([Li]). More information on this and related equations, including further references, can be found in [PS] and [Po].

In the more general equation $u_t = f(u)(\Delta u + g(u))$ with $f \in C^1(\mathbb{R})$, most of these results continue to be valid under the extra assumption that f be bounded from below by a positive constant. Even in the case when f is allowed to touch the value zero in the sense described in the beginning, we have stabilization of all solutions of the resulting degenerate problem, provided the initial data belong to $W_0^{1,2}(\Omega)$, are positive in Ω and $\mathcal{E}$ is known to be a discrete set ([Win5]). For degenerate equations of this type, however, some technical devices, frequently serving as useful support in the treatment of semilinear problems, are no longer available, for instance the Hopf boundary point lemma ([Win2]). A posteriori, our results will show that these tools in fact appear to be inevitable for the proof of stabilization.

In order to explain this and state our main results, let us note that the particular choice of λ_1 in (1.1) entails that the set $\mathcal{E}$ of nonnegative steady states of (1.1) is precisely given by $\mathcal{E} = \{\alpha\Theta \mid \alpha \geq 0\}$, where Θ denotes any positive principal Dirichlet eigenfunction of $-\Delta$ in Ω. Therefore $\mathcal{E}$ forms a totally ordered set, but if f is sufficiently small near zero then unlike the case $f \equiv 1$ this ordering property need not enforce solutions to select one of the equilibria as a stabilization point. To be more precise, let us write

$$H(s) := \int_1^s \frac{d\sigma}{f(\sigma)}, \qquad s > 0,$$

and observe that H is bounded above on bounded subsets of $(0, \infty)$, while the strength of the degeneracy in (1.1) is reflected in the possibly singular behavior of H near $s = 0$:

The equation is weakly degenerate if $H(0) > -\infty$, whereas the smaller f becomes near zero, the stronger will be the tendency of $H(s)$ towards $-\infty$ as $s \to 0$. In the present context, the condition

$$\int_0 sH(s)ds = -\infty \tag{1.2}$$

will mark a crucial borderline in respect of stabilization. Namely, the first of our main results reads as follows.

- If (1.2) holds then (1.1) possesses a global positive classical solution u that oscillates in the sense that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \qquad \text{and}$$
$$\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = 0$$

  (Lemma 3.8).

Under an additional structural assumption on f we can refine this statement and also show that (1.2) is sharp:

- If (1.2) holds and $\frac{sf'(s)}{f(s)} \ge \kappa > 0$ for $s > 0$ then there exists $u_0 \in C^0(\bar{\Omega})$ admitting a global positive classical solution u of (1.1) that has the ω-limit set

$$\omega(u_0) = \{\alpha\Theta \mid \alpha \ge 0\}$$

  (Theorem 4.4).

- Conversely, if $\int_0 sH(s)ds > -\infty$ but still $\frac{sf'(s)}{f(s)} \ge \kappa > 0$ then all positive classical solutions of (1.1) are global and bounded. Moreover, for any such solution there exists a unique $\alpha \ge 0$ such that

$$u(\cdot,t) \to \alpha\Theta \qquad \text{in } L^2(\Omega) \qquad \text{as } t \to \infty$$

  (Theorem 4.5).

In the literature we could find only few results on the dynamics of (1.1) so far, basically dealing with the case when $f(u) = u^p$ with $p > 0$, and all of them concentrating on the proof of statements which are *positive* in respect of stabilization: Summarizing these, namely, we know that for such f all solutions are global in time and approach their ω-limit set as $t \to \infty$; moreover, $\omega(u_0) \subset \{\alpha\Theta \mid \alpha \ge 0\}$ and if $p < 3$ then $\omega(u_0)$ actually is a singleton (see [Wie] and [Win3], for instance). On the other hand, if λ_1 is substituted by any number *larger* than λ_1 then it is well-known that all positive solutions blow up in finite time (cf. [FMcL] for the special case $f(u) = u^2$), while the replacement by any constant *smaller* than λ_1 enforces all solutions to stabilize to zero asymptotically ([Wie]).

The core of our detection of oscillating solutions will be formed by an iterative construction of suitable initial data. The idea of this procedure goes back to [PY], where nonstabilizing solutions to the Cauchy problem in $\mathbb{R}^n$ for $u_t = \Delta u + u^p$ were constructed in the parameter

regime $n \geq 11$, $p \geq \frac{(n-2)^2-4n+8\sqrt{n-1}}{(n-2)(n-10)}$; a similar approach was used in the related work [Win6] to find oscillating solutions for the linear degenerate equation which is obtained when $f(u)$ is replaced with $a(x)$ in (1.1), where a is positive in Ω but vanishes in some point on the boundary.
In the present framework this procedure will essentially rely on two ingredients: firstly, appropriate global attractivity properties of the steady state $w \equiv 0$ and the 'steady state' ∞, and secondly some particular result on continuous dependence of solutions to (1.1) with respect to perturbations of the initial data. Both these ingredients will be provided by Section 3.2 (for $w \equiv 0$) and Section 3.3 (for the attractivity of ∞), respectively, after some basic preparatory material and a more convenient reformulation of (1.2) have been given in Section 2 and Section 3.1. The main statements will be demonstrated in Section 3.4 and, for $\frac{sf'(s)}{f(s)} \geq \kappa > 0$, in Section 4.

2 Preliminaries

For the sake of definiteness, throughout the sequel we shall assume that the principal Laplacian eigenfunction Θ is normalized in such a way that $\max_{\Omega} \Theta = 1$. Then it is a well-known consequence of the smoothness of $\partial\Omega$ that there exist positive constants $\theta_\pm$ such that

$$\theta_- \mathrm{dist}\,(x, \partial\Omega) \leq \Theta(x) \leq \theta_+ \mathrm{dist}\,(x, \partial\Omega) \qquad \text{for all } x \in \Omega. \tag{2.1}$$

The following lemma can be proved by standard methods in the context of degenerate equations of the present type (cf. [Win5], [Wie], [Win3] and the references therein, for instance).

Lemma 2.1 *Suppose $u_0 \in C^0(\bar{\Omega})$ is positive in Ω with $u_0|_{\partial\Omega} = 0$. Then there exists a maximal existence time $T_{max} = T_{max}(u) \leq \infty$ and a unique classical solution u of (1.1). This solution can be obtained as the limit in $C^0_{loc}(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\Omega \times (0, T_{max}))$ of the decreasing net of solutions u_ε, $\varepsilon \searrow 0$, of the non-degenerate problems*

$$\begin{cases} u_{\varepsilon t} = f(u_\varepsilon)(\Delta u_\varepsilon + \lambda_1 u_\varepsilon) & \text{in } \Omega \times (0, T_{max}(u_\varepsilon)), \\ u_\varepsilon|_{\partial\Omega} = \varepsilon, \\ u_\varepsilon|_{t=0} = u_0 + \varepsilon. \end{cases} \tag{2.2}$$

Moreover, if $T_{max}(u) < \infty$ then $\|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \nearrow T_{max}(u)$.

Besides an approximation from above, it will be useful to know that u can as well be approximated from below by 'solutions' with compactly supported initial data:

Lemma 2.2 *Assuming that u_0 is as in Lemma 2.1, let $u_\eta^- := \lim_{\varepsilon \to 0} u_{\eta\varepsilon}^-$ denote the limit of the solutions of (2.2) corresponding to the initial data $u_{0\eta}^- := (u_0 - \eta)_+$. Then $u_\eta^- \in C^0(\{u_0 > \eta\} \times [0, T_{max}(u))) \cap C^{2,1}(\{u_0 > \eta\} \times (0, T_{max}(u)))$, and we have $u_\eta^- \nearrow u$ as $\eta \searrow 0$, where u is the solution of (1.1).*

Remark. Since their initial data are not positive everywhere, the functions u_η^- need not be continuous outside $\{u_0 > \eta\}$ (cf. [BDalPU]); particularly, they might not be classical solutions of (1.1). However, in view of the nonuniqueness results for weak solutions given in [LDalP], for instance, we avoid introducing any weak solution concept here, because it will actually not be necessary.

Proof. By comparison, $u_{\eta\varepsilon}^- \le u_\varepsilon$ for all ε and hence $T_{max}(u_\eta^-) \ge T_{max}(u)$ and $u_\eta^- \le u$. Clearly, the u_η^- are ordered and thus $u_\eta^- \nearrow u^-$ in $\Omega \times [0, T_{max}(u))$ holds for some $u^- \le u$. For any ball $B \subset\subset \{u_0 > \eta\}$, we have $(u_0 - \eta)_+ \ge c_0 \Theta_B$ in B for some $c_0 = c_0(B) > 0$, where Θ_B denotes the principal Laplacian eigenfunction in B with $\max_{x\in B} \Theta_B(x) = 1$, corresponding to the eigenvalue $\lambda_1(B) > \lambda_1$. Writing $c_1 := \|f\|_{L^\infty((0,c_0))}$, $y(t) := c_0 \exp(-(\lambda_1(B) - \lambda_1)c_1 t)$ and $v(x,t) := y(t)\Theta_B(x)$, we then calculate

$$\begin{aligned}
v_t - f(v)(\Delta v + \lambda_1 v) &= y'\Theta_B - f(y\Theta_B)(-y \cdot \lambda_1(B)\Theta_B + \lambda_1 y \Theta_B) \\
&= \Big[y' + (\lambda_1(B) - \lambda_1) f(y\Theta_B) \cdot y\Big] \Theta_B \\
&\le \Big[y' + (\lambda_1(B) - \lambda_1) c_1 y\Big] \Theta_B \\
&= 0 \qquad \text{in } B \times (0, \infty).
\end{aligned}$$

Therefore, $u_{\eta\varepsilon}^- \ge v$ in $B \times (0,\infty)$ for all ε by the comparison principle, which together with parabolic Schauder estimates shows that u_η^- is positive and continuous in $\{u_0 > \eta\} \times [0, T_{max}(u))$ and contained in $C^{2,1}(\{u_0 > \eta\} \times (0, T_{max}(u)))$. Since u_η^- increases as η decreases, we consequently can once more invoke Schauder theory and a standard barrier argument near $t = 0$ to conclude that $u_\eta^- \to u^-$ in $C^0_{loc}(\Omega \times [0, T_{max}(u))) \cap C^{2,1}_{loc}(\Omega \times (0, T_{max}(u)))$. Now the barrier-type estimate $0 \le u_\eta^- \le u$ ensures that u^- is continuous even up to $\partial\Omega$, and that accordingly u^- is a positive classical solution of (1.1). Thus, the uniqueness statement in Lemma 2.1 asserts that $u^- \equiv u$. ∎

An elementary but nonetheless essential conservation property is provided by the following lemma. As compared to the case $f \equiv 1$, it can be interpreted as a substitute for the constancy of the first Fourier coefficient $\int_\Omega u(\cdot,t)\Theta$ in this linear equation.

Lemma 2.3 *We have*

$$\int_\Omega H(u(x,t))\Theta(x)dx = \int_\Omega H(u_0(x))\Theta(x)dx \qquad \textit{for all } t \in (0, T_{max}(u)), \tag{2.3}$$

which is to be understood as an identity in $\mathbb{R} \cup \{-\infty\}$.

Proof. We multiply (2.2) by $\frac{\Theta}{f(u_\varepsilon)}$ and integrate by parts to obtain

$$\begin{aligned}
\int_\Omega H(u_\varepsilon(x,t))\Theta(x)dx &= \int_\Omega H(u_0(x)+\varepsilon)\Theta(x)dx + \int_0^t \int_\Omega (\Delta u_\varepsilon + \lambda_1 u_\varepsilon)\Theta \\
&= \int_\Omega H(u_0(x)+\varepsilon)\Theta(x)dx - \varepsilon \int_0^t \int_{\partial\Omega} \partial_\nu \Theta \\
&= \int_\Omega H(u_0(x)+\varepsilon)\Theta(x)dx - \varepsilon \int_0^t \int_\Omega \Delta\Theta \\
&= \int_\Omega H(u_0(x)+\varepsilon)\Theta(x)dx + \varepsilon\lambda_1 \int_\Omega \Theta(x)dx \cdot t.
\end{aligned}$$

Sicne $u_\varepsilon \searrow u$ in $\Omega \times [0, T_{max}(u))$ and H is increasing, the monotone convergence theorem thus proves (2.3). ∎

3 The case $\int_0 sH(s)ds = -\infty$

3.1 A reformulation of $\int_0 sH(s)ds = -\infty$

In order to establish a connection between the (non-)integrability of $sH(s)$ and the quantities involved in Lemma 2.3, it will be convenient to have a statement equivalent to $\int_0 sH(s)ds = -\infty$, but made up of integrals over Ω and involving Θ appropriately. In view of (2.1), it is plausible to guess that such a statement might be that $\int_\Omega H(\Theta)\Theta dx = -\infty$. That this in fact is true will become clear upon the following elementary lemma, which we formulate in a way slightly more general than actually required here.

Lemma 3.1 *Suppose $G \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary and $\Psi : (0,\infty) \to \mathbb{R}$ is continuous and nonnegative. Then we have*

$$\int_G \Psi(\mathrm{dist}\,(x,\partial G))dx < \infty \quad \textit{if and only if} \quad \int_0 \Psi(s)ds < \infty. \tag{3.1}$$

Remark. At the cost of slightly more technical expense, results in a similar spirit can be derived for arbitrary – not necessarily smooth – domains ([Dj]).

Proof. Utilizing suitable local representations of ∂G, one can see that there exist positive constants δ, c_0 and c_1 such that

$$c_0(d_2 - d_1) \le \Big|\{d_1 < \mathrm{dist}\,(x,\partial G) \le d_2\}\Big| \le c_1(d_2 - d_1) \quad \text{whenever } 0 < d_1 < d_2 \le \delta, \tag{3.2}$$

where we abbreviate $\{d_1 < \mathrm{dist}\,(x,\partial G) \le d_2\} := \{x \in G \mid d_1 < \mathrm{dist}\,(x,\partial G) \le d_2\}$. Now for any $\eta > 0$, the interval $[\eta, \delta]$ can be equidistantly decomposed by setting $d_k := \eta + k \cdot \frac{\delta-\eta}{N}$, $k = 0, ..., N$, where $N \in \mathbb{N}$. Choosing $\xi_k \in [d_{k-1}, d_k]$ such that $\Psi(\xi_k) = \max\limits_{s\in[d_{k-1},d_k]} \Psi(s)$, we then have, using (3.2),

$$\begin{aligned}
\int_{\{\eta<\mathrm{dist}\,(x,\partial G)\le\delta\}} \Psi(\mathrm{dist}\,(x,\partial G))dx &= \sum_{k=1}^N \int_{\{d_{k-1}<\mathrm{dist}\,(x,\partial G)\le d_k\}} \Psi(\mathrm{dist}\,(x,\partial G))dx \\
&\le \sum_{k=1}^N \Big|\{d_{k-1} < \mathrm{dist}\,(x,\partial G) \le d_k\}\Big| \cdot \Psi(\xi_k) \\
&\le c_1 \sum_{k=1}^N (d_k - d_{k-1}) \cdot \Psi(\xi_k).
\end{aligned}$$

As $N \to \infty$, the latter sum tends to $\int_\eta^\delta \Psi(s)ds$. Thus, taking $\eta \to 0$ shows that

$$\int_{\{\mathrm{dist}\,(x,\partial G)\le\delta\}} \Psi(\mathrm{dist}\,(x,\partial G))dx \le c_1 \int_0^\delta \Psi(s)ds. \tag{3.3}$$

Similarly one can use the left estimate in (3.2) to derive

$$\int_{\{\mathrm{dist}\,(x,\partial G)\le\delta\}} \Psi(\mathrm{dist}\,(x,\partial G))dx \ge c_0 \int_0^\delta \Psi(s)ds. \tag{3.4}$$

Since $x \mapsto \Psi(\mathrm{dist}\,(x,\partial G))$ is continuous in $\{\delta \le \mathrm{dist}\,(x,\partial G) \le \mathrm{diam}\, G\}$, (3.3) and (3.4) prove (3.1). ∎

Now we immediately have

Corollary 3.2 *We have*

$$\int_0 sH(s)ds > -\infty \qquad \textit{if and only if} \qquad \int_\Omega \Theta \cdot H(\alpha\Theta)dx > -\infty \quad \textit{for all } \alpha > 0. \tag{3.5}$$

Proof. Assume that $\int_0 sH(s)ds > -\infty$. Since H increases, we obtain from (2.1) that

$$\int_\Omega \Theta \cdot H(\alpha\Theta)dx \ge \alpha\theta_- \int_\Omega dist(x,\partial\Omega) \cdot H(\alpha\theta_- \mathrm{dist}\,(x,\partial\Omega))dx,$$

so that $\int_\Omega \Theta \cdot H(\alpha\Theta)dx > -\infty$ follows upon applying Lemma 3.1 to $\Psi(s) := s \cdot H_-(\alpha\theta_- s) \equiv s \cdot \max\{-H(\alpha\theta_- s), 0\}$. The opposite implication can be seen in quite the same way. ∎

3.2 Compactly supported initial data

The main purpose of this section is to assert that solutions of (1.1) become appropriately small after some time if their initial data are sufficiently close to some compactly supported function. The important point here is that this holds true irrespective of the actual size of the data (measured e.g. in $L^\infty(\Omega)$).
Before going into the details, let us point out some arguments which let this surprising property appear more plausible. Assume that u_0 is smooth and has compact support in Ω. Then earlier work indicates that if the degeneracy in (1.1) is strong enough (such that $H(0) = -\infty$, which is clearly implied by $\int_0 sH(s)ds = -\infty$), then the support of u will be constant in time (see [LDalP], [Win4]). Now, secondly, u must be globally bounded (because a suitably large multiple of Θ is a supersolution lying above u_0), but the constancy of the support entails that the only equilibrium that u can converge to is $w \equiv 0$. Therefore u must decay asymptotically and, by continuous dependence, the same should hold for positive data slightly differing from u_0. Of course, these arguments are based on a heuristic reasoning, so that the actual proof, though essentially pursuing the same ideas, becomes more involved.
The following lemma on an auxiliary one-dimensional problem shall allow us to prove an appropriate version of the mentioned support property in a self-contained way.

Lemma 3.3 *Suppose $H(0) = -\infty$, $d > 0$ and $v_0 \in C^0([0,d])$ is nonnegative and has compact support in $(0,d)$, and let $A > 0$. Then for each $\varepsilon \in (0,1)$, the problem*

$$\begin{cases} v_{\varepsilon t} = f(v_\varepsilon)(v_{\varepsilon xx} + A), & x \in (0,d),\ t > 0, \\ v_\varepsilon(0,t) = v_\varepsilon(d,t) = \varepsilon, & t > 0, \\ v_\varepsilon(x,0) = v_0(x) + \varepsilon, & x \in (0,d), \end{cases} \tag{3.6}$$

has a global bounded classical solution v_ε. Moreover, there exists $\xi > 0$ such that for all $T > 0$ we have

$$v_\varepsilon \to 0 \qquad \textit{uniformly in } [0,\xi]\times[0,T] \textit{ as } \varepsilon \to 0. \tag{3.7}$$

Proof. Let $e \in C^2([0,d])$ satisfy $-e_{xx} = 1$ in $(0,d)$ and $e(0) = e(d) = 0$ (that is, we set $e(x) = \frac{d^2}{8} - \frac{1}{2}(x - \frac{d}{2})^2)$ and choose $B \geq A$ such that $v_0 \leq B \cdot e$ in $(0,d)$ which is possible since v_0 has compact support. It can then be seen upon straightforward comparison of v_ε with $\underline{v}(x,t) := \varepsilon$ and $\bar{v}(x,t) := \varepsilon + Be(x)$ that

$$\varepsilon \leq v(x,t) \leq \varepsilon + Be(x) \qquad \text{for all } x \in (0,d)$$

holds as long as v_ε exists. Particularly, v_ε is global in time and bounded.
To see (3.7), we first note that in view of the comparison principle it is sufficient to prove (3.7) in the case that v_0 (and hence v_ε) be symmetric with respect to the center $x = \frac{d}{2}$ of the interval $(0,d)$, and that v_0 is nondecreasing for $x \leq \frac{d}{2}$, vanishing for $x \leq 2\xi$ with some $\xi \in (0, \frac{d}{4})$.
Once more from the comparison principle we gain that the v_ε are ordered and

$$v_\varepsilon \searrow v \qquad \text{in } [0,d]\times[0,\infty) \ \text{ as } \varepsilon \searrow 0 \tag{3.8}$$

holds with a nonnegative limit function v. Clearly, for all $t > 0$, $v(\cdot,t)$ is nondecreasing for $x \leq \frac{d}{2}$. Our goal is to show that

$$v(\xi,t) = 0 \qquad \text{for all } t > 0; \tag{3.9}$$

once this has been proved, we will know that $v \equiv 0$ in $[0,\xi]\times[0,\infty)$, whereupon Dini's theorem will turn the monotone convergence in (3.8) into locally uniform convergence as claimed by (3.7).
Assuming (3.9) is false, we can find $t_0 > 0$ and $\delta > 0$ such that

$$v(x,t_0) \geq \delta \qquad \text{for all } x \in (\xi, 2\xi). \tag{3.10}$$

We then fix a nonnegative nontrivial $\varphi \in C_0^\infty((\xi,2\xi))$ and multiply (3.6) by $\frac{\varphi(x)}{f(v_\varepsilon)}$. We then obtain after integrating by parts

$$\begin{aligned}
\int_0^d \Big[H(v_\varepsilon(x,t_0)) - H(v_0(x)+\varepsilon)\Big]\cdot\varphi(x)dx &= \int_0^{t_0}\int_0^d (v_{\varepsilon xx} + A)\varphi dxdt \\
&= \int_0^{t_0}\int_0^d (v_\varepsilon \varphi_{xx} + A\varphi)dxdt \\
&\leq c_1 \qquad \text{for all } \varepsilon \in (0,1)
\end{aligned} \tag{3.11}$$

with a constant $c_1 = c_1(t_0,\varphi)$ independent of ε. On the other hand, since $v_0 \equiv 0$ on $\operatorname{supp}\varphi$ and H is increasing, we infer from (3.10) that

$$\begin{aligned}
\int_0^d \Big[H(v_\varepsilon(x,t_0)) - H(v_0(x)+\varepsilon)\Big]\cdot\varphi(x)dx &\geq \int_0^d \Big[H(\delta) - H(\varepsilon)\Big]\cdot\varphi(x)dx \\
&\to +\infty \qquad \text{as } \varepsilon \to 0,
\end{aligned}$$

because $H(\varepsilon) \to -\infty$ as $\varepsilon \to 0$ and φ was nontrivial. This contradiction with (3.11) establishes (3.9) and thereby completes the proof. ∎

As a consequence, if $\operatorname{supp} u_0 \subset\subset \Omega$ then the limit of the solutions u_ε of (2.2) has its support contained in some subset different from all of $\bar\Omega$. We believe that this subset actually coincides with $\operatorname{supp} u_0$, but we do not need this here.

Lemma 3.4 *Assume that $H(0) = -\infty$. Suppose $u_0 \in C^0(\bar\Omega)$ is nonnegative and has compact support in Ω, and let u_ε denote the solution of (2.2). Then we have $T_{max}(u_\varepsilon) \nearrow \infty$ as $\varepsilon \searrow 0$, and there exists a nonempty open subset G of Ω such that for all $T > 0$ we have*

$$u_\varepsilon \to 0 \qquad \textit{uniformly in } \bar G \times [0,T] \textit{ as } \varepsilon \to 0. \tag{3.12}$$

Proof. The hypothesis particularly entails that $u_0 \le \tilde u_0 := c\Theta$ holds in Ω with some $c > 0$. By comparison, the solution $\tilde u$ emanating from $\tilde u_0$ lies below $c\Theta$ as long as it exists, whence $\tilde u$ must be global. Therefore $T_{max}(\tilde u_\varepsilon) \nearrow \infty$ as $\varepsilon \searrow 0$, which implies that the same is true for $T_{max}(u_\varepsilon)$, because $u_\varepsilon \le \tilde u_\varepsilon$ by the maximum principle.
As a consequence, for all $T > 0$ there exist $M_T > 0$ and $\varepsilon_T > 0$ such that

$$u_\varepsilon \le M_T \qquad \text{in } \Omega \times (0,T) \ \text{ for all } \varepsilon < \varepsilon_T. \tag{3.13}$$

Since $\bar\Omega$ is compact, after an appropriate affine change of variables we may assume that $0 \in \partial\Omega$ and that Ω is contained in the half-space $\{x = (x_1, x') \in \mathbb{R}^n \mid x_1 \in (0,d)\}$, where $d := \operatorname{diam}\Omega$. Due to the support property of u_0 it is then possible to pick some nonnegative $v_0 \in C_0^\infty((0,d))$ such that

$$u_0(x) \le v_0(x_1) \qquad \text{for all } x = (x_1, x') \in \Omega. \tag{3.14}$$

For $\varepsilon \in (0,1)$, we now let $v_\varepsilon = v_\varepsilon(x_1,t)$ denote the solution of the one-dimensional problem

$$\begin{cases} v_{\varepsilon t} = f(v_\varepsilon)(v_{\varepsilon x_1 x_1} + \lambda_1 M_T), & x_1 \in (0,d),\ t > 0, \\ v_\varepsilon(0,t) = v_\varepsilon(d,t) = \varepsilon, & t > 0, \\ v_\varepsilon(x_1, 0) = v_0(x_1) + \varepsilon, & x_1 \in (0,d). \end{cases}$$

According to Lemma 3.3, such a solution exists globally and satisfies

$$v_\varepsilon \to 0 \qquad \text{uniformly in } [0,\xi] \times [0,T] \ \text{ as } \varepsilon \to 0 \tag{3.15}$$

with some $\xi > 0$. But in view of (3.13) and (3.14), the comparison principle ensures that

$$u_\varepsilon(x,t) \le v_\varepsilon(x_1,t) \qquad \text{for all } (x_1,x') \in \bar\Omega,\ t \in [0,T].$$

Combined with (3.15) this yields (3.12) upon the selection $G := \{(x_1, x') \in \Omega \mid x_1 \in (0,\xi)\}$. ∎

Using the support property in the above formulation, we are now able to take the final two steps of our heuristic approach at the same time, namely convergence to zero for compactly supported initial data, and the continuous dependence argument.

Lemma 3.5 *Suppose $H(0) = -\infty$ and $u_0 \in C^0(\bar{\Omega})$ is nonnegative and has compact support in Ω. Then for all $\delta > 0$ there exist $T > 0$ and $\nu > 0$ such that if $\tilde{u}_0 \in C^0(\bar{\Omega})$ is positive in Ω and vanishes on $\partial\Omega$ then*

$$\|\tilde{u}_0 - u_0\|_{L^\infty(\Omega)} \leq \nu \qquad \textit{implies} \qquad T_{max}(\tilde{u}_0) > T \textit{ and } \|\tilde{u}(\cdot, T)\|_{L^\infty(\Omega)} < \delta \tag{3.16}$$

for the solution $\tilde{u}$ of (1.1) with $\tilde{u}|_{t=0} = \tilde{u}_0$.

Proof. In order to prove the lemma it will be sufficient to show that, given $\delta > 0$, there exist $T > 0$ and $\varepsilon_0 > 0$ such that the solution u_{ε_0} of (2.2) exists at least up to T and satisfies

$$\|u_{\varepsilon_0}(\cdot, T)\|_{L^\infty(\Omega)} < \delta. \tag{3.17}$$

In fact, if this is true then for all $\tilde{u}_0 \in C^0(\bar{\Omega})$ such that $\tilde{u}_0|_{\partial\Omega} = 0$ and $0 < \tilde{u}_0 \leq u_0 + \frac{\varepsilon_0}{2}$, we can use comparison to conclude $\tilde{u}_\varepsilon \leq u_{\varepsilon_0}$ in $\Omega \times [0, T]$ for all $\varepsilon < \frac{\varepsilon_0}{2}$ and therefore (3.16) holds with $\nu = \frac{\varepsilon_0}{2}$.
Since u_0 has compact support, Lemma 3.4 says that $T_{max}(u_\varepsilon) \to \infty$ as $\varepsilon \to 0$, and provides an open set $G \subset \Omega$ such that (3.12) holds. Let us pick some smooth subdomain $\tilde{\Omega} \subset \Omega$ such that $\tilde{\Omega} \neq \Omega$ and $\Omega \setminus G$ is contained in $\tilde{\Omega}$. Then (3.12) entails that for all $T > 0$,

$$u_\varepsilon \to 0 \qquad \text{uniformly in } (\Omega \setminus \tilde{\Omega}) \times [0, T] \quad \text{as } \varepsilon \to 0 \tag{3.18}$$

and moreover, as $\partial\tilde{\Omega} \subset \partial\Omega \cup G$ and $u_\varepsilon \equiv \varepsilon$ on $\partial\Omega$, that

$$u_\varepsilon \to 0 \qquad \text{uniformly in } \partial\tilde{\Omega} \times [0, T] \quad \text{as } \varepsilon \to 0 \tag{3.19}$$

holds for any fixed $T > 0$ as well.
Consider the elliptic problem

$$\begin{cases} -\Delta w - \lambda_1 w = 1 & \text{in } \tilde{\Omega}, \\ w|_{\partial\tilde{\Omega}} = 1. \end{cases} \tag{3.20}$$

From the strict monotonicity of the principal Laplacian eigenvalue with respect to the domain it follows that the first eigenvalue $\tilde{\lambda}_1$ of $-\Delta$ in $\tilde{\Omega}$ is larger than λ_1 and hence (3.20) has a unique solution w that satisfies $1 \leq w(x) \leq M$ for all $x \in \tilde{\Omega}$ with some constant M. Therefore the number

$$y_0 := \sup_{\varepsilon \in (0,1)} \left\| \frac{u_0 + \varepsilon}{w} \right\|_{L^\infty(\Omega)} \tag{3.21}$$

is finite. Thus, introducing a nondecreasing locally Lipschitz continuous minorant f_0 of f on $[0, y_0]$ by

$$f_0(s) := \min_{\sigma \in [s, y_0]} f(\sigma), \qquad s \in [0, y_0],$$

the initial-value problem

$$\begin{cases} y'(t) = -\frac{1}{M} f_0(y) y, & t > 0, \\ y(0) = y_0, \end{cases}$$

has a unique decreasing solution. Since f_0 is positive on $(0, y_0]$, this solution converges to zero as $t \to \infty$, so that, given $\delta > 0$, we can find $T > 0$ such that $y(T) \le \frac{\delta}{4M}$. By continuous dependence, there hence exists $\eta \in (0, \frac{\delta}{2})$ such that the solution y_η of

$$\begin{cases} y_\eta'(t) = -\frac{1}{M} f_0(y_\eta)(y_\eta - \lambda_1 \eta), \qquad t > 0, \\ y_\eta(0) = y_0, \end{cases} \tag{3.22}$$

decreases to $\lambda_1 \eta$ as $t \to \infty$ and satisfies

$$y_\eta(T) \le \frac{\delta}{2M}. \tag{3.23}$$

Now according to (3.18) and (3.19) let us fix $\varepsilon_0 \in (0,1)$ small such that

$$u_{\varepsilon_0}(\cdot, T) \le \delta \quad \text{in } \Omega \setminus \tilde{\Omega} \qquad \text{and} \qquad u_{\varepsilon_0} \le \eta \quad \text{on } \partial\tilde{\Omega} \times [0, T]. \tag{3.24}$$

Then the function

$$\bar{u}(x,t) := \eta + y_\eta(t) \cdot w(x), \qquad x \in \tilde{\Omega},\ t \in [0, T],$$

majorizes u_{ε_0} initially because of (3.21), while on $\partial\tilde{\Omega} \times [0,T]$ we have $\bar{u} \ge \eta \ge u_{\varepsilon_0}$ by (3.24). From (3.20) and (3.22) we furthermore obtain

$$\begin{aligned} \bar{u}_t - f(\bar{u})(\Delta \bar{u} + \lambda_1 \bar{u}) &= y_\eta' w - f(\bar{u}) \Big[y_\eta (\Delta w + \lambda_1 w) + \lambda_1 \eta \Big] \\ &= y_\eta' w + f(\bar{u})(y_\eta - \lambda_1 \eta) \\ &\ge y_\eta' M + f_0(y_\eta)(y_\eta - \lambda_1 \eta) \\ &= 0 \qquad \text{in } \tilde{\Omega} \times (0, T), \end{aligned}$$

where we have used the monotonicity properties of y_η and f_0 and the fact that $\bar{u} \ge y_\eta$. Altogether, the comparison principle entails $\bar{u} \ge u_{\varepsilon_0}$ in $\tilde{\Omega} \times [0,T]$ and particularly

$$u_{\varepsilon_0}(x, T) \le \bar{u}(x, T) \le \eta + \frac{\delta}{2M} \cdot M \le \delta \qquad \text{for all } x \in \tilde{\Omega}.$$

Recalling the first inequality in (3.24), we end up with (3.17). ∎

3.3 Initial data that are large near $\partial\Omega$

We proceed to demonstrate, by quite a different method, a similar attractivity property of ∞. As in the previous section, it is essentially the behavior of u_0 near the boundary that makes up the following sufficient condition for u to be unbounded.

Lemma 3.6 *Suppose that $\int_0 sH(s)ds = -\infty$. Then for all $u_0 \in C^0(\bar{\Omega})$ that are positive in Ω, vanish on $\partial\Omega$ and satisfy*

$$\int_\Omega H(u_0) \cdot \Theta dx > -\infty, \tag{3.25}$$

the corresponding solution u of (1.1) is unbounded; that is,

$$\limsup_{t \to T_{max}(u)} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{3.26}$$

Proof. Let us assume on the contrary that u be bounded (and hence global), say, $u \le M$ in $\Omega \times (0,\infty)$. We then may modify the source term in (1.1) as follows: Let $g \in C^1(\mathbb{R})$ be nondecreasing and such that

$$\lambda_1 \cdot \min\{s, M\} \le g(s) \le \lambda_1 \cdot \min\{s, 2M\} \qquad \text{for all } s \in \mathbb{R}.$$

Namely, u actually staisfies $u_t = f(u)(\Delta u + g(u))$ in $\Omega \times (0,\infty)$ and hence

$$u \le \tilde{u}_\varepsilon \qquad \text{in } \Omega \times (0,\infty) \tag{3.27}$$

holds for all the solutions $\tilde{u}_\varepsilon$, $\varepsilon \in (0,1)$, of the problems

$$\begin{cases} \tilde{u}_{\varepsilon t} = f(\tilde{u}_\varepsilon)(\Delta \tilde{u}_\varepsilon + g(\tilde{u}_\varepsilon)) & \text{in } \Omega \times (0,\infty), \\ \tilde{u}_\varepsilon|_{\partial\Omega} = \varepsilon, \\ \tilde{u}_\varepsilon|_{t=0} = u_0 + \varepsilon. \end{cases} \tag{3.28}$$

Clearly, since g is bounded from above, $\tilde{u}_\varepsilon$ is global in time and satisfies

$$\varepsilon \le \tilde{u}_\varepsilon(x,t) \le \|u_0 + \varepsilon\|_{L^\infty(\Omega)} + 2Me(x) \qquad \text{in } \Omega \times (0,\infty) \tag{3.29}$$

by comparison, where $-\Delta e = 1$ in Ω with $e|_{\partial\Omega} = 0$. Indeed, the time-independent functions on the left and on the right of (3.29) can easily be checked to be a sub- and a supersolution of (3.28), respectively.
Hence, each $\tilde{u}_\varepsilon$ is a bounded solution of the quasilinear uniformly parabolic problem (3.28). Therefore, standard energy arguments (cf. [Win5] for a version adapted to problems of this type) together with parabolic Schauder theory ensure that $\{\tilde{u}_\varepsilon(\cdot,t) \mid t \ge 1\}$ is relatively compact in $C^2(\bar{\Omega})$, and that the ω-limit set of $\tilde{u}_\varepsilon$ is contained in the set of steady states of (3.28). Particularly, this means that for all $\varepsilon \in (0,1)$ we can find some $t_\varepsilon > 0$ such that

$$\|\tilde{u}_\varepsilon(\cdot, t_\varepsilon) - w_\varepsilon\|_{L^\infty(\Omega)} \le \varepsilon, \tag{3.30}$$

where $w_\varepsilon \in C^2(\bar{\Omega})$ is a positive classical solution of

$$\begin{cases} \Delta w_\varepsilon + g(w_\varepsilon) = 0 & \text{in } \Omega, \\ w_\varepsilon|_{\partial\Omega} = \varepsilon. \end{cases}$$

Again due to the boundedness of g from above (or also by (3.29)), these w_ε are bounded in $L^\infty(\Omega)$, uniformly with respect to ε. Therefore elliptic Schauder theory guarantees that $w_\varepsilon \to w$ holds in $C^2(\bar{\Omega})$. In particular, w is in $C^1(\bar{\Omega})$ and thus satisfies

$$w(x) \le c\Theta(x) \qquad \text{for all } x \in \Omega \tag{3.31}$$

with some $c > 0$. Now from Lemma 2.3, (3.27) and (3.30) we infer that

$$\begin{aligned} \int_\Omega H(u_0) \cdot \Theta dx &= \int_\Omega H(u(x,t_\varepsilon)) \cdot \Theta(x) dx \\ &\le \int_\Omega H(\tilde{u}_\varepsilon(x,t_\varepsilon)) \cdot \Theta(x) dx \\ &\le \int_\Omega H(w_\varepsilon(x) + \varepsilon) \cdot \Theta(x) dx. \end{aligned}$$

Hence, an application of Fatou's lemma shows that

$$\int_\Omega H(w) \cdot \Theta dx \ge \int_\Omega H(u_0) \cdot \Theta dx.$$

In view of (3.31), this leads to the conclusion

$$\int_\Omega H(c\Theta) \cdot \Theta dx \ge \int_\Omega H(u_0) \cdot \Theta dx,$$

which is absurd, however, because of (3.25) and the assumption on H. Therefore u must be unbounded. ∎

Combining this with the continuous dependence statements implicitly contained in Lemma 2.1 and Lemma 2.2, we obtain the following analogue of Lemma 3.5.

Corollary 3.7 *Assume that $\int_0 sH(s)ds = -\infty$, and that $u_0 \in C^0(\bar\Omega)$ is positive in Ω with $u_0|_{\partial\Omega} = 0$ and*

$$\int_\Omega H(u_0)\Theta dx > -\infty.$$

Then for all $M > 0$ there exist $\nu > 0$ and $T > 0$ such that if $\tilde u_0 \in C^0(\bar\Omega)$ is positive in Ω and vanishes on $\partial\Omega$ then

$$\|\tilde u_0 - u_0\|_{L^\infty(\Omega)} \le \nu \qquad \textit{implies} \qquad T_{max}(\tilde u) > T \textit{ and } \|\tilde u(\cdot, T)\|_{L^\infty(\Omega)} \ge M, \tag{3.32}$$

where $\tilde u$ denotes the solution of (1.1) emanating from $\tilde u_0$.

Proof. Given $M > 0$, according to Lemma 3.6 we can pick $T > 0$ and $x_0 \in \Omega$ such that $u(x_0, T) = \|u(\cdot, T)\|_{L^\infty(\Omega)} \ge M + 1$. In view of Lemma 2.1 and Lemma 2.2, there exist $\varepsilon_0 > 0$ and $\eta_0 > 0$ such that $T_{max}(u_{\varepsilon_0}) > T$ and $u^-_{\eta_0}(x_0, T) \ge u(x_0, T) - 1$. Thus, whenever $\|u_0 - \tilde u_0\|_{L^\infty(\Omega)} \le \nu := \min\{\varepsilon_0, \eta_0\}$, we have $u^-_{\eta_0} \le \tilde u_0 \le u_0 + \varepsilon_0$ in Ω and hence the comparison principle entails $T_{max}(\tilde u) > T$ and

$$\|\tilde u(\cdot, T)\|_{L^\infty(\Omega)} \ge \tilde u(x_0, T) \ge u^-_{\eta_0}(x_0, T) \ge u(x_0, T) - 1 \ge M,$$

as claimed. ∎

3.4 Oscillating solutions

We are now ready to construct oscillating solutions in the case $\int_0 sH(s)ds = -\infty$. The proof of the following lemma parallels that of Lemma 4.2 in [Win6] in some parts, but since essential adaptations are necessary, we find it convenient to repeat all steps here.

Lemma 3.8 *Suppose $\int_0 sH(s)ds = -\infty$. Then there is a function $u_0 \in C^0(\bar\Omega)$ which is positive in Ω and vanishes on $\partial\Omega$ such that the solution u of (1.1) is global in time and has the following property: There exists a sequence $(t_k)_{k\in\mathbb{N}} \subset (0, \infty)$ with $t_k \to \infty$ as $k \to \infty$ and*

$$\begin{cases} \|u(\cdot, t_k)\|_{L^\infty(\Omega)} \ge k & \textit{if } k \textit{ is odd,} \\ \|u(\cdot, t_k)\|_{L^\infty(\Omega)} \le \frac{1}{k} & \textit{if } k \textit{ is even.} \end{cases} \tag{3.33}$$

Proof. As a technical preliminary, let us fix a nonincreasing $\zeta \in C^\infty([0,\infty))$ such that $\zeta \equiv 1$ in $[0,\frac{1}{2}]$ and $\zeta \equiv 0$ on $[1,\infty)$. For $\delta > 0$ and $x \in \bar{\Omega}$ we set $\zeta_\delta(x) := \zeta(\frac{\mathrm{dist}\,(x,\partial\Omega)}{\delta})$. Then $\zeta_\delta \in W^{1,\infty}(\Omega)$ and $\mathrm{supp}\,\zeta_\delta \subset \bar{\Omega} \setminus \Omega_\delta$, where $\Omega_\delta := \{x \in \Omega \mid \mathrm{dist}\,(x,\partial\Omega) > \delta\}$.
These cut-off functions will be used in the following iteration procedure in which a sequence of initial data u_{0k} is constructed, the limit of which will be u_0. The basic idea in each step is to modify $u_{0,k-1}$ in a small neighborhood of $\partial\Omega$ in such a way that the corresponding solutions eventually either become unbounded or small.
More precisely, we claim that there exist sequences of positive numbers δ_k, ν_k and t_k and initial data $u_{0k}, k \in \mathbb{N}$, such that

$$\delta_k < \frac{\delta_{k-1}}{2}, \qquad k \geq 2, \tag{3.34}$$

$$\nu_k < \frac{\nu_{k-1}}{2^{k-1}}, \qquad k \geq 2, \tag{3.35}$$

$$u_{0k} \in C^0(\bar{\Omega}), \quad u_{0k} > 0 \text{ in } \Omega, \quad u_{0k}|_{\partial\Omega} = 0, \qquad k \geq 1, \tag{3.36}$$

$$u_{0k} \equiv u_{0,k-1} \quad \text{in } \Omega_{\delta_k}, \qquad k \geq 2, \tag{3.37}$$

$$\|u_{0k} - u_{0,k-1}\|_{L^\infty(\Omega)} \leq \frac{\nu_{k-1}}{2}, \qquad k \geq 2, \tag{3.38}$$

and such that for all $k \geq 1$,

$$\|\tilde{u}_0 - u_{0k}\|_{L^\infty(\Omega)} \leq \nu_k \qquad \text{implies}$$
$$T_{max}(\tilde{u}) > t_k \quad \text{and} \quad \begin{cases} \|\tilde{u}(\cdot,t_k)\|_{L^\infty(\Omega)} \geq k & \text{if } k \text{ is odd,} \\ \|\tilde{u}(\cdot,t_k)\|_{L^\infty(\Omega)} \leq \frac{1}{k} & \text{if } k \text{ is even,} \end{cases} \tag{3.39}$$

whenever $\tilde{u}_0 \in C^0(\bar{\Omega})$ is positive in Ω, vanishing on $\partial\Omega$, and $\tilde{u}$ denotes the corresponding solution of (1.1) in $\Omega \times (0, T_{max}(\tilde{u}))$.
To initiate the recursive definition of these objects, we fix any $\varphi \in C^0(\bar{\Omega})$ such that $\varphi > 0$ in Ω, $\varphi|_{\partial\Omega} = 0$ and φ is large near $\partial\Omega$ in the sense that

$$\int_\Omega H(\varphi) \cdot \Theta > -\infty.$$

Setting

$$u_{01} := \varphi,$$

we then infer from Corollary 3.7 that there exist $\nu_1 > 0$ and $t_1 > 0$ such that

$$\|\tilde{u}_0 - u_{01}\|_{L^\infty(\Omega)} \leq \nu_1 \qquad \text{implies} \qquad T_{max}(\tilde{u}) > t_1 \quad \text{and} \quad \|\tilde{u}(\cdot,t_1)\|_{L^\infty(\Omega)} \geq 1$$

for $\tilde{u}_0$ and $\tilde{u}$ as above. Thus, (3.36) and (3.39) hold for $k = 1$. The first step is completed by fixing any positive number δ_1.
Next, assume that δ_j, ν_j, t_j and u_{0j} have already been defined for $1 \leq j < k$ and some $k \geq 2$, and that (3.34)-(3.39) hold up to $j = k-1$.
If k is even, we continue as follows: Since $u_{0,k-1}|_{\partial\Omega} = 0$ and $k \geq 2$, we can pick a positive $\delta_k < \frac{\delta_{k-1}}{2}$ such that

$$\|u_{0,k-1}\|_{L^\infty(\Omega \setminus \Omega_{\delta_k})} \leq \Big(\frac{1}{2} - \frac{1}{2^k}\Big)\nu_{k-1} \tag{3.40}$$

and define an auxiliary function $\hat{u}_{0k}$ by

$$\hat{u}_{0k} := (1 - \zeta_{\delta_k}) u_{0,k-1}. \tag{3.41}$$

Then $\hat{u}_{0k} \in C^0(\bar{\Omega})$ has compact support in Ω, so that Lemma 3.5 provides some positive t_k and

$$\nu_k < \frac{\nu_{k-1}}{2^{k-1}} \tag{3.42}$$

such that whenever $\tilde{u}_0 \in C^0(\bar{\Omega})$ is positive in Ω with $\tilde{u}_0|_{\partial\Omega} = 0$,

$$\|\tilde{u}_0 - \hat{u}_{0k}\|_{L^\infty(\Omega)} \le 2\nu_k \qquad \text{implies} \quad T_{max}(\tilde{u}) > t_k \quad \text{and} \quad \|\tilde{u}(\cdot, t_k)\|_{L^\infty(\Omega)} \le \frac{1}{k} \tag{3.43}$$

for the corresponding solution $\tilde{u}$ of (1.1).
In order to achieve interior positivity of u_{0k}, we finally let

$$u_{0k} := \hat{u}_{0k} + \frac{\nu_k}{2} \zeta_{\delta_k} \Theta. \tag{3.44}$$

Then (3.34)-(3.37) are clearly satisfied, whereas (3.40), (3.41), (3.44) and (3.42) imply

$$\begin{aligned}
\|u_{0k} - u_{0,k-1}\|_{L^\infty(\Omega)} &\le \|u_{0k} - \hat{u}_{0k}\|_{L^\infty(\Omega)} + \|\hat{u}_{0k} - u_{0,k-1}\|_{L^\infty(\Omega)} \\
&\le \frac{\nu_k}{2} + \|\zeta_{\delta_k} u_{0,k-1}\|_{L^\infty(\Omega \setminus \Omega_{\delta_k})} \\
&\le \frac{\nu_k}{2} + \Big(\frac{1}{2} - \frac{1}{2^k}\Big)\nu_{k-1} \\
&\le \frac{\nu_{k-1}}{2},
\end{aligned}$$

which gives (3.38). Moreover, (3.39) results from (3.43), because if $\|\tilde{u}_0 - u_{0k}\|_{L^\infty(\Omega)} \le \nu_k$ then $\|\tilde{u}_0 - \hat{u}_{0k}\|_{L^\infty(\Omega)} \le \frac{3}{2}\nu_k < 2\nu_k$ by (3.44).
On the other hand, if k is odd then we choose $\delta_k < \frac{\delta_{k-1}}{2}$ so as to satisfy

$$\|\varphi\|_{L^\infty(\Omega \setminus \Omega_{\delta_k})} + \|u_{0,k-1}\|_{L^\infty(\Omega \setminus \Omega_{\delta_k})} \le \frac{\nu_{k-1}}{2} \tag{3.45}$$

and directly set

$$u_{0k} := (1 - \zeta_{\delta_k}) u_{0,k-1} + \zeta_{\delta_k} \varphi.$$

Then $u_{0k} > 0$ in Ω and $u_{0k} \equiv \varphi$ near $\partial\Omega$, so that $\int_\Omega H(u_{0k}) \cdot \Theta > -\infty$ according to our choice of φ. Thus, Corollary 3.7 applies to yield $t_k > 0$ and $\nu_k < \frac{\nu_{k-1}}{2^{k-1}}$ such that (3.39) holds. While (3.34)-(3.37) are obviously fulfilled now, (3.38) again follows from (3.45) and the support properties of ζ used in

$$\begin{aligned}
\|u_{0k} - u_{0,k-1}\|_{L^\infty(\Omega)} &\le \|\zeta_{\delta_k}(\varphi - u_{0,k-1})\|_{L^\infty(\Omega \setminus \Omega_{\delta_k})} \\
&\le \frac{\nu_{k-1}}{2}.
\end{aligned}$$

Having thereby completed the construction of the u_{0k}, we next combine (3.38) with (3.35) to obtain for $1 \le k < k'$

$$
\begin{aligned}
\|u_{0k} - u_{0k'}\|_{L^\infty(\Omega)} &\le \sum_{i=k+1}^{k'} \frac{\nu_{i-1}}{2} \\
&= \frac{1}{2}\Big(\nu_k + \sum_{i=k+2}^{k'} \nu_{i-1}\Big) \\
&\le \frac{1}{2}\Big(\nu_k + \sum_{i=k+2}^{k'} \frac{\nu_{i-2}}{2^{i-2}}\Big) \\
&\le \frac{1}{2}\Big(\nu_k + \nu_k \cdot \sum_{i=k+2}^{k'} \frac{1}{2^{i-2}}\Big) \\
&= \nu_k.
\end{aligned}
\tag{3.46}
$$

Consequently, as $k \to \infty$ we have $u_{0k} \to u_0$ in $C^0(\bar{\Omega})$ for some u_0 that hence vanishes on $\partial\Omega$. From (3.36) and (3.37) it is clear that $u_0 > 0$ in Ω, and taking $k' \to \infty$ in (3.46) shows that $\|u_0 - u_{0k}\|_{L^\infty(\Omega)} \le \nu_k$ for all $k \in \mathbb{N}$. Therefore (3.34) results from (3.39), and since the solution u of (1.1) cannot undergo a finite time extinction (see Lemma 2.1), u is global in time and, necessarily, t_k must tend to $+\infty$ as $k \to \infty$. ∎

4 The case $\frac{sf'(s)}{f(s)} > \kappa > 0$

In this section we assume that f satisfies the one-sided estimate

$$\frac{sf'(s)}{f(s)} \ge \kappa > 0 \qquad \text{for all } s > 0 \tag{4.1}$$

with some constant κ. This structural assumption is satisfied, for instance, by $f(s) = s^p$ for any $p > 0$ (with $\kappa = p$), but also by suitable extensions of $f(s) = e^{-s^{-p}}$, $p > 0$, $0 < s \le s_0 < \infty$, to all of $(0, \infty)$. Thus, (4.1) allows for very strong degeneracies, but excludes oscillating f.

4.1 Preliminary conclusions from (4.1)

Our starting point is a technically very useful consequence of (4.1), namely the following *semi-convexity* estimate for solutions of (1.1) (cf. [Ha]).

Lemma 4.1 *If (4.1) holds then the solution u of (1.1) satisfies*

$$\frac{u_t}{u} \ge -\frac{1}{\kappa t} \qquad in\ \Omega \times (0, T_{max}). \tag{4.2}$$

Proof. For $\varepsilon > 0$, let $z_\varepsilon := \frac{u_{\varepsilon t}}{u_\varepsilon}$. By a straightforward computation, we see that $z_t = \frac{u_\varepsilon f'(u_\varepsilon)}{f(u_\varepsilon)} \cdot z^2 + \frac{f(u_\varepsilon)}{u_\varepsilon} \cdot (2\nabla u_\varepsilon \cdot \nabla z + u_\varepsilon \Delta z)$ in $\Omega \times (0, T_{max}(u_\varepsilon))$. By comparison, for all

$0 < \tau < T < T_{max}(u_\varepsilon)$, in the region $\Omega \times (\tau, T)$ the function $z(x,t)$ therefore lies above the solution $y = y(t)$ of $y'(t) = \kappa y^2$ satisfying $y(\tau) = -M$, that is,

$$z(x,t) \geq -\frac{1}{\frac{1}{M} + \kappa(t-\tau)} \qquad \text{for } x \in \Omega,\ t \in (\tau, T),$$

for all sufficiently large $M > 0$ depending on ε. Here we let $M \to \infty$, then $\varepsilon \to 0$ and $\tau \to 0$ and finally $T \to T_{max}(u)$ to arrive at (4.2). ∎

Our main tool in this section, Lemma 4.3, requires one further preliminary lemma on monotone elliptic problems.

Lemma 4.2 *Let $g \in C^1((0,\infty))$ be nonincreasing and nonnegative.*
i) For all $\eta > 0$, the operator $A_\eta := -\Delta - \eta g(\cdot)$ satisfies the following comparison principle: If $w_1, w_2 \in C^0(\bar{\Omega}) \cap W^{1,2}_{loc}(\Omega)$ are positive in Ω and such that

$$-\Delta w_1 - \eta g(w_1) \leq -\Delta w_2 - \eta g(w_2) \quad \text{in } \Omega, \qquad \text{and} \quad w_1 \leq w_2 \quad \text{on } \partial\Omega,$$

then $w_1 \leq w_2$ in Ω.
ii) Suppose that $\eta_j \searrow 0$ as $j \to \infty$, and that $(h_j)_{j\in\mathbb{N}} \subset L^\infty(\Omega)$ is such that $\sup_{j\in\mathbb{N}} \|h_j\|_{L^\infty(\Omega)} < \infty$ and $h_j \to 0$ in $L^2(\Omega)$ as $j \to \infty$. Then the problems

$$\begin{cases} -\Delta W_j = \eta_j g(W_j) + h_j & \text{in } \Omega, \\ W_j|_{\partial\Omega} = 0, \end{cases} \tag{4.3}$$

have unique positive weak solutions $W_j \in C^0(\bar{\Omega}) \cap C^1(\Omega)$, and we have

$$W_j \to 0 \qquad \text{in } C^0(\bar{\Omega}) \qquad \text{as } j \to \infty, \tag{4.4}$$

Proof. i) This part can easily be seen upon multiplying the inequality $-\Delta(w_1 - w_2) \leq \eta(g(w_1) - g(w_2))$ by $(w_1 - w_2 - \mu)_+$, $\mu > 0$, integrating over Ω, using the monotonicity of g, and then letting $\mu \to 0$.
ii) By standard arguments involving elliptic Schauder estimates ([GT, Chapter 8]) and part i), we obtain existence and uniqueness of a positive solution of (4.3); details of similar constructions can be found in [CRT] or [Win1], for instance. Moreover, this W_j satisfies

$$W_j \leq W_{1j} + W_{2j} \qquad \text{in } \Omega, \tag{4.5}$$

where

$$-\Delta W_{1j} = \eta_j g(W_{1j}) \quad \text{in } \Omega, \qquad W_{1j}|_{\partial\Omega} = 0,$$

and

$$-\Delta W_{2j} = h_j \quad \text{in } \Omega, \qquad W_{2j}|_{\partial\Omega} = 0.$$

Indeed, this follows from the above comparison principle, because the positivity of W_{2j} and the monotonicity of g imply that

$$A_{\eta_j}(W_{1j} + W_{2j}) = \eta_j g(W_{1j}) + h_j - \eta_j g(W_{1j} + W_{2j}) \geq h_j = A_{\eta_j} W_j \qquad \text{in } \Omega.$$

Since the h_j are uniformly bounded in $L^\infty(\Omega)$ and converge to zero in $L^2(\Omega)$, elliptic theory ([GT, Theorem 8.33]) says that $W_{2j} \to 0$ in $C^1(\bar{\Omega})$. Hence, (4.4) will follow if we can show that $W_{1j} \to 0$ uniformly in Ω. To this end, we may assume after a coordinate translation that $\Omega \subset \{(x_1, x') \in \mathbb{R}^n \mid 0 < x_1 < d\}$ for some $d \geq \operatorname{diam}\Omega$ and consider the auxiliary problems

$$\begin{cases} -v_{j\xi\xi} = \eta_j g(v_j) & \text{in } (0,d), \\ v_j(0) = v_j(d) = 0. \end{cases} \tag{4.6}$$

By comparison, we then have $W_{1j}(x) \leq v_j(x_1)$ for all $x = (x_1, x') \in \Omega$, so that it is sufficient to show that $v_j \to 0$ in $C^0([0,d])$ as $j \to \infty$. We multiply (4.6) by $(v_j - \frac{1}{i})_+$, $i \in \mathbb{N}$, and integrate, using the monotonicity of g and Hölder's inequality, to obtain

$$\begin{aligned} \int_0^d \Big|\partial_x (v_j - \tfrac{1}{i})_+\Big|^2 &\leq \eta_j \int_0^d g(v_j)(v_j - \tfrac{1}{i})_+ \\ &\leq \eta_j g(\tfrac{1}{i}) d^{\frac{1}{2}} \cdot \Big(\int_0^d (v_j - \tfrac{1}{i})_+^2\Big)^{\frac{1}{2}} \\ &\leq \eta_j g(\tfrac{1}{i}) d^{\frac{1}{2}} c_p \cdot \Big(\int_0^d \Big|\partial_x (v_j - \tfrac{1}{i})_+\Big|^2\Big)^{\frac{1}{2}}, \end{aligned}$$

where c_p is a Poincaré constant on $(0,d)$. Therefore for all $i \in \mathbb{N}$ we can pick a large $j_i \in \mathbb{N}$ such that

$$\int_0^d \Big|\partial_x (v_{j_i} - \tfrac{1}{i})_+\Big|^2 \leq \frac{1}{i},$$

which implies $(v_{j_i} - \frac{1}{i})_+ \to 0$ in $W_0^{1,2}((0,d)) \hookrightarrow C^0([0,d])$. Since a straightforward comparison argument shows that v_j decreases with j, this entails that $v_j \to 0$ in $C^0([0,d])$ holds along the whole sequence. ∎

The main preparation for the results in Theorems 4.5 and 4.4 below is done in the following lemma. Its proof contains some ideas already used in [Win3, Theorem 2.1], where a similar statement was used in ruling out finite time blow-up in the special case $f(s) = s^p$ for any $p > 0$.

Lemma 4.3 *Suppose (4.1) holds and $(t_k)_{k\in\mathbb{N}} \subset (0, T_{max}(u))$ is such that*

$$\mu := \liminf_{k\to\infty} \|u(\cdot, t_k)\|_{L^\infty(\Omega)} > 0, \tag{4.7}$$

and that

either $t_k \to \infty$ *or* $\|u(\cdot,t_k)\|_{L^\infty(\Omega)} \to \infty$ *as* $k \to \infty$.

Then there exist $\alpha > 0$ and a subsequence of indices $k_j \to \infty$ such that

$$\frac{u(\cdot, t_{k_j})}{\|u(\cdot, t_{k_j})\|_{L^\infty(\Omega)}} \to \alpha\Theta \qquad \text{in } L^2(\Omega) \qquad \text{as } j \to \infty. \tag{4.8}$$

Proof. We set $m_k := \max\{1, \|u(\cdot,t_k)\|_{L^\infty(\Omega)}\}$ and $q_k(x) := \frac{u(x,t_k)}{m_k}$ for $k \in \mathbb{N}$. By Lemma 4.1,

$$-\Delta q_k \le \lambda_1 q_k + \frac{1}{\kappa t_k m_k} \frac{u(\cdot,t_k)}{f(u(\cdot,t_k))} \qquad \text{in } \Omega,$$

where we may assume $\kappa \in (0,1)$ for later convenience. In order to cope with the second term on the right, we observe that (4.1) implies that $g(s) := \frac{s^\kappa}{f(s)}$ is nonincreasing for $s > 0$. Thus, since $q_k \le u(\cdot,t_k) \le m_k$ and $\kappa < 1$,

$$\begin{aligned} -\Delta q_k &\le \lambda_1 q_k + \frac{1}{\kappa t_k m_k} \cdot u^{1-\kappa}(\cdot,t_k) \cdot g(u(\cdot,t_k)) \\ &\le \lambda_1 q_k + \frac{1}{\kappa t_k m_k^\kappa} \cdot g(q_k) \\ &= \lambda_1 q_k + \eta_k g(q_k) \qquad \text{in } \Omega, \end{aligned} \tag{4.9}$$

where $\eta_k := \frac{1}{\kappa t_k m_k^\kappa} \to 0$ as $k \to \infty$ by hypothesis. For $\delta > 0$, we multiply (4.9) by $(q_k - \delta)_+$ and integrate to obtain

$$\begin{aligned} \int_\Omega |\nabla (q_k - \delta)_+|^2 &\le \lambda_1 \int_\Omega (q_k - \delta)_+^2 + \lambda_1 \delta \int_\Omega (q_k - \delta)_+ + \eta_k \int_\Omega g(q_k) \cdot (q_k - \delta)_+ \\ &\le \lambda_1 \int_\Omega (q_k - \delta)_+^2 + \lambda_1 \delta |\Omega| + \eta_k g(\delta) |\Omega|, \end{aligned} \tag{4.10}$$

because $q_k \le 1$ and g is nonincreasing. Hence, for all $j \in \mathbb{N}$ we can find some large $k_j \in \mathbb{N}$ such that

$$\int_\Omega |\nabla (q_{k_j} - \tfrac{1}{j})_+|^2 \le \lambda_1 \int_\Omega (q_{k_j} - \tfrac{1}{j})_+^2 + \frac{2\lambda_1 |\Omega|}{j}$$

Thus, along a subsequence we have, keeping indices unchanged, that $(q_{k_j} - \frac{1}{j})_+ \rightharpoonup Q$ in $W_0^{1,2}(\Omega)$ as well as $q_{k_j} \to Q$ in $L^2(\Omega)$ and a.e. in Ω, where $\int_\Omega |\nabla Q|^2 \le \lambda_1 \int_\Omega Q^2$. By simplicity of the principal Laplacian eigenfunction, however, this means that $Q = \alpha \Theta$ for some $\alpha \in [0,1]$, so that it remains to be shown that α must be positive. In fact, suppose on the contrary that $\alpha = 0$, that is, $q_{k_j} \to 0$ in $L^2(\Omega)$. Recalling (4.9) and applying Lemma 4.2, we obtain that

$$q_{k_j} \le W_j \qquad \text{in } \Omega,$$

where W_j solves (4.3) with $h_j := \lambda_1 q_{k_j}$ (and η_j replaced by η_{k_j}, of course). But (4.4) says that $W_j \to 0$ uniformly in Ω, whereas by (4.7) we have $\|q_{k_j}\|_{L^\infty(\Omega)} \ge \mu > 0$ for all j. This contradiction completes the proof. ∎

4.2 An unbounded ordered ω-limit set in the case $\int_0 H(s)ds = -\infty$

With the tools just provided we can now easily improve the result of Lemma 3.8: Namely, if $\int_0 sH(s)ds = -\infty$ and, additionally, (4.1) holds then the oscillating solution constructed in Lemma 3.8 in fact has an unbounded ω-limit set consisting of all nonnegative steady states of (1.1).

Theorem 4.4 *Assume f is such that (4.1) holds as well as $\int_0 sH(s) = -\infty$. Then there exists $u_0 \in C^0(\bar{\Omega})$ with $u_0 > 0$ in Ω and $u_0|_{\partial\Omega} = 0$, such that the solution u of (1.1) is global and unbounded and satisfies*

$$\omega(u_0) = \{\alpha\Theta \mid \alpha \in [0,\infty)\}. \tag{4.11}$$

Proof. From Lemma 3.8 we know that there exists a function u_0 in the indicated class such that u is global and satisfies $\liminf_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = 0$ and $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$. But the latter entails that also $\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^2(\Omega)} = \infty$, because if $t_k \to \infty$ is such that $\|u(\cdot,t_k)\|_{L^\infty(\Omega)} \to \infty$ as $k \to \infty$ then Lemma 4.3 asserts that $\limsup_{k\to\infty} \frac{\|u(\cdot,t_k)\|_{L^2(\Omega)}}{\|u(\cdot,t_k)\|_{L^\infty(\Omega)}}$ must be positive.
From this and the continuity of $t \mapsto \|u(\cdot,t)\|_{L^2(\Omega)}$ we infer that for any $\alpha > 0$ there exists a sequence $t_k \to \infty$ such that

$$\|u(\cdot,t_k)\|_{L^2(\Omega)} = \alpha\|\Theta\|_{L^2(\Omega)} \qquad \text{for all } k \in \mathbb{N}. \tag{4.12}$$

Again by Lemma 4.3, $\|u(\cdot,t_k)\|_{L^\infty(\Omega)}$ cannot be unbounded, whence for a subsequence we have $\|u(\cdot,t_{k_j})\|_{L^\infty(\Omega)} \to \mu$, where μ must be positive due to (4.12). One more application of Lemma 4.3 shows that after another extraction process we gain

$$u(\cdot,t_{k_{j_i}}) \to \mu\tilde{\alpha}\Theta \qquad \text{in } L^2(\Omega) \qquad \text{as } i \to \infty$$

with some $\tilde{\alpha} > 0$. Combining this with (4.12) yields $\mu\tilde{\alpha} = \alpha$ and thus $u(\cdot,t_{k_{j_i}}) \to \alpha\Theta$ in $L^2(\Omega)$ as $i \to \infty$. This shows that each positive multiple of Θ lies in $\omega(u_0)$, and hence $\omega(u_0) \supset \{\alpha\Theta \mid \alpha \geq 0\}$, because $\omega(u_0)$ is closed. The opposite inclusion is an immediate consequence of Lemma 4.3. ∎

4.3 Stabilization in case of $\int_0 sH(s)ds > -\infty$

If $\int_0 sH(s)ds$ is finite then the following theorem states that all solutions stabilize. In particular, this indicates that the above condition $\int_0 sH(s)ds = -\infty$ in fact was sharp in respect of the occurrence of oscillating solutions.

Theorem 4.5 *Suppose (4.1) holds as well as $\int_0 sH(s)ds > -\infty$. Then the solution u of (1.1) is global and bounded. Moreover, we have*

$$u(\cdot,t) \to \alpha\Theta \qquad \text{in } L^2(\Omega) \qquad \text{as } t \to \infty, \tag{4.13}$$

where $\alpha \geq 0$ is the uniquely determined number satisfying

$$\begin{cases} \alpha = 0 & \text{if } \int_\Omega H(u_0)\Theta = -\infty, \\ \int_\Omega H(\alpha\Theta)\Theta = \int_\Omega H(u_0)\Theta & \text{if } \int_\Omega H(u_0)\Theta > -\infty. \end{cases} \tag{4.14}$$

Proof. In order to prove that u is global and bounded, by the comparison principle it is sufficient to show this for u_0 so large that $u_0 \geq \Theta$ in Ω. Under this additional assumption, we clearly know that $u(\cdot,t) \geq \Theta$ holds in Ω for all $t \in (0, T_{max}(u))$. We fix any $M > \|u_0\|_{L^\infty(\Omega)}$ and define $H_M(s) := \min\{H(s), H(M)\}$ for $s > 0$. Then we have

$$H(\Theta)\Theta \leq H_M(u(\cdot,t))\Theta \leq H(M)\Theta \qquad \text{in } \Omega \qquad \text{for all } t \in (0, T_{max}(u)), \tag{4.15}$$

whence $H_M(u(\cdot,t))$ is integrable by Corollary 3.2 and

$$\int_\Omega H_M(u(\cdot,t))\Theta \leq \int_\Omega H(u(\cdot,t))\Theta = \int_\Omega H(u_0)\Theta < \int_\Omega H(M)\Theta \qquad \text{for all } t \in (0, T_{max}(u)) \tag{4.16}$$

as a consequence of Lemma 2.3. Now if u were unbounded then $\|u(\cdot,t_k)\|_{L^\infty(\Omega)} \to \infty$ for some $t_k \to T_{max}(u) \leq \infty$ and thus Lemma 4.3 allows to pass to a subsequence along which $u(\cdot,t_k) \to \infty$ a.e. in Ω holds. Therefore $H_M(u(\cdot,t_k)) \to H(M)$ a.e. in Ω, which together with the two-sided estimate (4.15) implies

$$\int_\Omega H_M(u(\cdot,t_k))\Theta \to \int_\Omega H(M)\Theta \qquad \text{as } k \to \infty$$

by the dominated convergence theorem. This, however, contradicts (4.16).

The proof of stabilization to $\alpha\Theta$ as $t \to \infty$ follows the basic idea introduced in [Win3, Theorem 2.4]: In light of Lemma 4.3 we only need to make sure that if $u(\cdot,t_k) \to \beta\Theta$ in $L^2(\Omega)$ and a.e. in Ω for some $t_k \to \infty$, then $\beta = \alpha$. To see this, let us first assume that $u_0 \geq c_0\Theta$ in Ω with some $c_0 > 0$. Then $u(\cdot,t) \geq c_0\Theta$ for all $t > 0$ by comparison and hence $H(c_0\Theta)\Theta \leq H(u(\cdot,t_k))\Theta \leq H(N)\Theta$ provides a uniform two-sided $L^1(\Omega)$-bound, where $N := \|u\|_{L^\infty(\Omega\times(0,\infty))}$ is known to be finite now. Therefore we achieve $\int_\Omega H(u_0)\Theta \equiv \int_\Omega H(u(\cdot,t_k)\Theta \to \int_\Omega H(\beta\Theta)\Theta > -\infty$ as $k \to \infty$ by the dominated convergence theorem, which identifies $\beta = \alpha$.
For general u_0, we consider the approximate solutions $u^{(\delta)}$ of (1.1) emanating from $u_0^{(\delta)} := \max\{u_0, \delta\Theta\}$. According to what we have just shown, these satisfy $u^{(\delta)}(\cdot,t) \to \alpha_\delta\Theta$ as $t \to \infty$, where $\alpha_\delta > 0$ is given by $\int_\Omega H(\alpha_\delta\Theta)\Theta = \int_\Omega H(u_0^{(\delta)})\Theta$. Since $H' > 0$ and $u^{(\delta)} \searrow u_0$ as $\delta \searrow 0$, the monotone convergence theorem implies that $\alpha_\delta \searrow \alpha \geq 0$ as $\delta \searrow 0$. But $u^{(\delta)}$ lies above u by comparison, so that $\beta \leq \alpha_\delta$ for all $\delta > 0$ and thus $\beta \leq \alpha$. In the case $\alpha = 0$ this already proves $\beta = \alpha$. If $\alpha > 0$, the opposite inequality $\beta \geq \alpha$ can be obtained upon combining the pointwise convergence $H(u(\cdot,t_k))\Theta \to H(\beta\Theta)\Theta$ a.e. in Ω with the upper estimate $H(u(\cdot,t_k))\Theta \leq H(N)\Theta$: Applying Fatou's lemma shows that $\int_\Omega H(\beta\Theta)\Theta \geq \int_\Omega H(u_0)\Theta$ and hence in fact $\beta \geq \alpha$. ∎

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