Refined asymptotics for entire solutions of a biharmonic equation with a supercritical nonlinearity

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Abstract

We consider positive classical radially symmetric solutions of the prototype of a fourth-order nonlinear elliptic equation,

$$\Delta^2 u = u^p, \qquad x \in \mathbb{R}^N, \qquad (\star)$$

where $N \geq 13$ and $p \geq p_c$. Here $p_c > \frac{N+4}{N-4}$ is a critical exponent that is known to mark a borderline in respect of the asymptotic behavior in this problem (Gazzola/Grunau, *Math. Ann.* **334** (2006)): For $p > \frac{N+4}{N-4}$, all positive radial solutions of $(\star)$ approach an explicitly known singular solution u_∞ in the sense that $\frac{u(x)}{u_\infty(x)} \to 1$ as $|x| \to \infty$. If $p < p_c$, this convergence is oscillatory, while for $p \geq p_c$ it is monotone from below.

The present paper studies the precise rate at which this limit is attained, having in mind that it is likely to be expected that quantitative results of this type will be closely linked to the knowledge on domains of attraction of equilibria in the associated parabolic problem. The main goal is to reveal an explicit algebraic convergence rate in the case $p > p_c$, and a rate involving a logarithmic correction when $p = p_c$. These results parallel those known for the second-order counterpart of $(\star)$ (Gui/Ni/Wang, *Comm. Pure Appl. Math.* **45** (1992)), but the technique is completely different.

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Introduction

This work deals with positive radially symmetric classical solutions of

$$\Delta^2 u = u^p, \qquad x \in \mathbb{R}^N, \tag{0.1}$$

where $N \in \mathbb{N}$ and $p > 1$. It is known that concerning the asymptotic behavior of such solutions, explicitly given singular solutions play a key role. Such unbounded solutions exist whenever $N \geq 5$ and $p > \frac{N}{N-4}$, and then are given by

$$u_\infty(x) := L|x|^{-m}, \qquad x \in \mathbb{R}^N \setminus \{0\}, \tag{0.2}$$

where the constant

$$m := \frac{4}{p-1}$$

satisfies $m < N-4$, and where

$$L := \Big\{ m(m+2)(N-2-m)(N-4-m) \Big\}^{\frac{1}{p-1}}. \tag{0.3}$$

More precisely, it was found in [GG1] that if p is even greater than the Sobolev exponent $p_S := \frac{N+4}{N-4}$ then (0.1) possesses radially symmetric positive classical solutions u satisfying

$$\frac{u(x)}{u_\infty(x)} \to 1 \qquad \text{as } |x| \to \infty. \tag{0.4}$$

In order to gain further insight, more recent works studied in more detail how this convergence takes place. In this respect, another exponent $p_c > p_S$ was detected to be critical. This exponent appears when $N \geq 13$ and then can be defined through the relation $m_c = \frac{4}{p_c - 1}$, where m_c is the unique zero of the increasing polynomial function

$$\Big(0, \frac{N-6}{2}\Big) \ni m \mapsto (m+2)(m+4)(N-2-m)(N-4-m) - \frac{N^2(N-4)^2}{16}. \tag{0.5}$$

Namely, it has been proved in [FGK] that if either $N \geq 13$ and $p_S < p < p_c$, or $5 \leq N \leq 12$ and $p > p_S$, then all radial classical solutions of (0.1) oscillate infinitely many times around u_∞ as $|x| \to \infty$; if $N \geq 13$ and $p \geq p_c$, however, the family of regular radial solutions of (0.1) is ordered and the convergence in (0.4) is monotone from below ([K]).

It is the purpose of the present work to refine the latter statement by addressing the question how far positive radial classical solutions of (0.1) differ from u_∞ for large $|x|$, that is, at which rate the convergence in (0.4) occurs. Our main result for supercritical p reads as follows.

Theorem 0.1 *Assume that $N \geq 13$ and $p > p_c$, and let u be a radially symmetric positive classical solution of (0.1). Then there exists $b > 0$ such that*

$$u(x) = L|x|^{-m} - b|x|^{-m-\lambda_2} + o\Big(|x|^{-m-\lambda_2}\Big) \qquad \text{as } |x| \to \infty, \tag{0.6}$$

where L is given by (0.3), $m = \frac{4}{p-1}$, and λ_2 is the positive real number defined by

$$\lambda_2 := \frac{N-4-2m-\sqrt{N^2-4N+8-4\sqrt{M}}}{2} \tag{0.7}$$

with

$$M := (m+2)(m+4)(N-2-m)(N-4-m) + (N-2)^2. \tag{0.8}$$

Let us mention that the number λ_2 arising here is the smallest positive among the four real roots $\lambda_1, \lambda_2, \lambda_3$ and λ_4 of the equation

$$(m+\lambda)(m+\lambda+2)(N-2-m-\lambda)(N-4-m-\lambda) - pL^{p-1} = 0 \qquad (0.9)$$

that is encountered as the characteristic equation when linearizing (0.1) about the singular solution u_∞ ([GG1]). As long as $N \geq 13$ and $p > p_c$, these roots are ordered according to

$$\lambda_1 < 0 < \lambda_2 < \lambda_3 < \lambda_4,$$

whereas λ_2 and λ_3 are both nonreal if $p < p_c$, and $\lambda_2 = \lambda_3 \in \mathbb{R}$ when $p = p_c$ (cf. also Section 1). As a consequence of the latter coincidence, in the critical case $p = p_c$ the asymptotic behavior in (0.1) involves a logarithmic correction:

Theorem 0.2 *Let $N \geq 13$ and $p = p_c$. Then for any radially symmetric positive classical solution u of (0.1) there exists $b > 0$ such that*

$$u(x) = L|x|^{-m} - b|x|^{-m-\lambda_2} \ln|x| + o\Big(|x|^{-m-\lambda_2} \ln|x|\Big) \qquad \text{as } |x| \to \infty,$$

where again $m = \frac{4}{p-1}$ and L and λ_2 are defined through (0.3), (0.7) and (0.8).

In order to put our results in perspective, let us draw some parallels to the second-order analogue,

$$-\Delta u = u^p, \qquad x \in \mathbb{R}^N, \qquad (0.10)$$

and point out how these may give rise to some conjectures on stability and domains of attraction of steady-state solutions in a related parabolic problem:
It is known that the problem (0.10) admits positive classical radial solutions if and only if $N \geq 3$ and $p > \bar{p}_S := \frac{N+2}{N-2}$, and all of these satisfy (0.4), where u_∞ is given by (0.2) with $m = \frac{2}{p-1}$ and $L = \{m(N-2-m)\}^{\frac{1}{p-1}}$. When $N \geq 11$, there exists $\bar{p}_c > \bar{p}_S$ with the property that if $\bar{p}_S < p < \bar{p}_c$ then all regular solutions of (0.10) oscillate infinitely many times around u_∞, whereas if $p \geq \bar{p}_c$ then the regular radial solutions are ordered and approach u_∞ from below. Furthermore, Gui, Ni and Wang ([GNW1]) discovered the asymptotic expansions

$$u(x) = \begin{cases} L|x|^{-m} - b|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1}) & \text{as } |x| \to \infty \quad \text{if } p > \bar{p}_c, \\ L|x|^{-m} - b|x|^{-m-\lambda_1} \ln|x| + o(|x|^{-m-\lambda_1} \ln|x|) & \text{as } |x| \to \infty \quad \text{if } p = \bar{p}_c. \end{cases} \qquad (0.11)$$

Here, $\lambda_1 > 0$ is the smaller among the two roots λ_1, λ_2 of the equation $(m+\lambda)(N-2-m-\lambda) + pL^{p-1} = 0$ which are both real if and only if $p \geq \bar{p}_c$, and which coincide precisely when $p = \bar{p}_c$.
In subsequent research, the expansions in (0.11) played an outstanding role in that they became a source of new insight into the semilinear diffusion equation,

$$U_t = \Delta U + U^p, \qquad x \in \mathbb{R}^N, \ t > 0. \qquad (0.12)$$

When this equation is considered along with nonnegative continuous initial data $u_0 \le u_\infty$ then, for instance, in the case $p > \bar{p}_c$ it turns out that

i) if $u_0(x) \le L|x|^{-m} - b|x|^{-m-\lambda_1+\varepsilon}$ for $|x| > 1$, some $b > 0$ and $0 < \varepsilon < \lambda_1$ then U will tend to zero as $t \to \infty$;

ii) if $u_0(x) = L|x|^{-m} - b|x|^{-m-\lambda_1} + o(|x|^{-m-\lambda_1})$ as $x \to \infty$ then U exists globally and converges to some positive radial classical solution of (0.10);

iii) if $u_0(x) \ge L|x|^{-m} - b|x|^{-m-\lambda_1-\varepsilon}$ for $|x| > 1$, some $b > 0$ and $\varepsilon > 0$ then U exists globally, but U grows up in the sense that $U(0,t) \to \infty$ as $t \to \infty$,

and accordingly modified statements hold for $p = \bar{p}_c$. Details on these mechanisms and the precise time asymptotics can be found in [GNW2] and [FWY2] for i), in [PY] and [FWY1] for ii), and in [GNW2] and [FKWY] for iii).
In light of these results, with Theorem 0.1 and Theorem 0.2 at hand it will not be daring to state a number of corresponding conjectures on global solvability and large time asymptotics in the fourth-order parabolic problem

$$U_t = -\Delta^2 U + |U|^{p-1} U, \qquad x \in \mathbb{R}^N,\ t > 0, \tag{0.13}$$

in the style of i)-iii). Of course, severe technical obstacles are to be overcome in a rigorous analysis, which mainly stem from the lack of appropriate maximum and comparison principles that have served as powerful and essential tools in all of the mentioned works in the second-order framework. Accordingly, the literature on (0.13) and its properties is yet much less complete than that on (0.12); for instance, to the best of our knowledge it is not clear up to now whether initially positive solutions of (0.13) remain nonnegative, or at least eventually become nonnegative; in this respect, only partial results seem to be available ([FGG]). However, it is known that if u_0 is continuous on $\mathbb{R}^N$ and satisfies $|u_0(x)| \le k|x|^{-m}$ for $|x| > 1$ with some sufficiently small $k > 0$, then the (strong) solution of (0.13) with initial data u_0 is global in time and decays to zero as $t \to \infty$ ([GG2]; cf. also [CM] and [GP] for preceding versions). On the other hand, Theorem 2 in [GG2] indicates that blow-up should occur for initial data satisfying $u_0(x) \ge K|x|^{-m}$, $|x| > 1$, whenever $K > 0$ is sufficiently large. Our results strongly suggest that, for instance in the case $p > p_c$, the separatrix between global existence and blow-up be precisely characterized by initial data decaying like $u_0(x) = L|x|^{-m} - b|x|^{-m-\lambda_2} + o(|x|^{-m-\lambda_2})$ with $m = \frac{4}{p-1}$, L as in (0.3) and $b > 0$.

As compared to the methods previously used to prove both the precise asymptotics in the second-order case and the known results on the decay in (0.1), our approach is completely different: It does not directly address the solution in question, for instance, through dynamical systems techniques ([GG1], [K]) or sophisticated monotonicity arguments ([GNW1]). Instead, our approach will be based on an obvious reduction of (0.1) to the two-component second-order Lane-Emden type system

$$\begin{aligned} -\Delta u &= v, \qquad x \in \mathbb{R}^N, \\ -\Delta v &= u^p, \qquad x \in \mathbb{R}^N, \end{aligned} \tag{0.14}$$

and on the construction of (very weak) sub- and supersolutions to (0.14) that are ordered and have the claimed asymptotic decay properties. Since the Lane-Emden system is co-operative, we can rely on appropriate comparison principles for the associated parabolic system. Thus, an argument involving a dynamical procedure in a standard way ([A]) can be applied to obtain the existence of a solution (u, v) of the stationary system (0.14) with the desired asymptotics. By uniqueness, up to scaling, of positive radial solutions to (0.1) ([GG1]), we will conclude.
Each of our comparison functions will be composed out of two simple-structured explicit functions, one of these covering an appropriate inner region near the origin, and the other being useful in a corresponding outer region. Of course, one could alternatively seek for suitably ordered sub- and supersolutions for (0.1) directly, and try to pick a solution in between by means of a suitable fixed point argument. As compared to this, the second-order approach pursued here goes along with a considerable technical advantage: When glueing together inner and outer candidates, in the spirit of a classical second-order procedure ([T]) one may confine oneself with resulting functions that are piecewise smooth but merely continuous as a whole.

Let us remark that as a by-product, independently of [GG1] we obtain a new existence proof for positive solutions of (0.1) when $p \geq p_c$. Moreover, our approach yields that all these solutions lie below u_∞, and thereby also includes the result in [K, Theorem 9 (a)].

The paper is organized as follows. In Section 1 we extract those among the known properties of (0.9) that will be essential for our purpose. In Section 2 we consider the supercritical case $p > p_c$. We explicitly construct couples of continuous functions that are sub- and supersolutions for (0.14) in the distributional sense, and obtain the proof of Theorem 0.1 as a consequence. In Section 3 the same steps, though technically more involved, are carried out for the critical exponent $p = p_c$, with Theorem 0.2 as the main outcome.

In order to simplify notation we shall abbreviate $\Delta u(r) := u_{rr} + \frac{N-1}{r} u_r$ for functions u depending on the real variable $r \geq 0$. Moreover, throughout the rest of the paper we will assume that $N \geq 13$.

1 Preliminaries

Let us first collect some basic facts about the equation (0.9) and its roots, where for convenience we substitute $l = m + \lambda$ and accordingly consider the polynomial

$$P(l) := l(l+2)(N-2-l)(N-4-l) - pL^{p-1}, \qquad l \in \mathbb{R},$$

with L taken from (0.3). This polynomial was analyzed in [GG1], where its zeros were computed as

$$l_1 := \frac{N-4-\sqrt{N^2-4N+8+4\sqrt{M}}}{2}, \quad l_2 := \frac{N-4-\sqrt{N^2-4N+8-4\sqrt{M}}}{2},$$
$$l_3 := \frac{N-4+\sqrt{N^2-4N+8-4\sqrt{M}}}{2}, \quad l_4 := \frac{N-4+\sqrt{N^2-4N+8+4\sqrt{M}}}{2}, \tag{1.1}$$

with M being the constant defined in (0.8). All of these roots are real if and only if $N \geq 13$ and $p \geq p_c$. Moreover, under the assumption $p \geq p_c$ we have

$$l_1 < m < l_2 \leq l_3 < l_4, \tag{1.2}$$

and

$$l_2 < l_3 \qquad \text{if and only if } p > p_c. \tag{1.3}$$

In particular, from (1.2) we see that the number in (0.7) satisfies $\lambda_2 \equiv l_2 - m > 0$. Some further elementary properties of λ_2 are summarized in the following lemma.

Lemma 1.1 *Let $p \geq p_c$. Then the following holds.*
i) $m < l_2 \leq \frac{N-4}{2}$.
ii) $l_2(N-2-l_2) > m(N-2-m) > 0$.
iii) If $p > p_c$ then for all $k \in (l_2, m+\lambda_3)$ we have $P(k) > 0$ and $k(N-2-k) > l_2(N-2-l_2)$.
iv) If $p = p_c$ then $l_2 = \frac{N-4}{2}$.

PROOF. i) This immediately follows from (1.2) and (1.1).
ii) Since $p > p_S$ implies $m < \frac{N-4}{2}$, factorizing

$$I := l_2(N-2-l_2) - m(N-2-m) = (l_2 - m)(N-2-l_2-m)$$

we see using i) that

$$l_2 + m \leq \frac{N-4}{2} + m < N-4,$$

and hence that $I > 0$ because $l_2 > m$.
iii) Similarly, we rewrite

$$J := k(N-2-k) - l_2(N-2-l_2) = (k - l_2)(N-2-k-l_2)$$

and use (1.1) to obtain

$$l_2 + l_3 = N-4 < N-2.$$

Therefore, for all $k \in (l_2, l_3)$ we have $J > 0$, as desired. Moreover, from (1.2) and (1.3) it immediately follows that P changes sign at each of its zeros and hence $P > 0$ in (l_2, l_3) results from the observation that $P(l) \to +\infty$ as $|l| \to \infty$.
iv) This identity directly results from (1.1) upon observing that if $p = p_c$ then $M = \frac{(N^2-4N+8)^2}{16}$ by (0.8) and (0.5). ////

2 The case $p > p_c$

2.1 Construction of a subsolution

Let us start with the technically least involved and most transparent part, namely the construction of a subsolution to (0.14) in the supercritical case. Our plan is to use function couples $(\underline{u}, \underline{v})$ such that $\underline{u}$ and $\underline{v}$ coincide with $\underline{u}_{out}(|\cdot|)$ and $\underline{v}_{out}(|\cdot|)$, respectively, in suitable outer regions, where

$$\begin{aligned} &\underline{u}_{out}(r) := Lr^{-m} - br^{-l} \qquad \text{and} \\ &\underline{v}_{out}(r) := m(N-2-m)Lr^{-m-2} - l(N-2-l)br^{-l-2}, \end{aligned} \tag{2.1}$$

for some $b > 0$, and which are zero near the origin.
The corresponding glueing procedure is prepared by the following lemma.

Lemma 2.1 *Let $p > p_c$ and $l := m + \lambda_2$. Given $b > 0$, we define*

$$\underline{r}_0 := \Big(\frac{b}{L}\Big)^{\frac{1}{l-m}}, \qquad \underline{r}_1 := \Big(\frac{l(N-2-l)b}{m(N-2-m)L}\Big)^{\frac{1}{l-m}} \tag{2.2}$$

for $r > 0$, and let $\underline{u}_{out}$ and $\underline{v}_{out}$ be as in (2.1). Then

$$\underline{r}_0 < \underline{r}_1 \tag{2.3}$$

and

$$\underline{u}_{out}(r) > 0 \quad \textit{if and only if } r > \underline{r}_0 \tag{2.4}$$

as well as

$$\underline{v}_{out}(r) > 0 \quad \textit{if and only if } r > \underline{r}_1. \tag{2.5}$$

PROOF. Since $l(N-2-l) > m(N-2-m) > 0$ by Lemma 1.1, $\underline{r}_1$ is well-defined and larger than $\underline{r}_0$. The equivalences (2.4) and (2.5) are easily checked. ////

The subsolution properties of $\underline{u}_{out}$ and $\underline{v}_{out}$ are easily obtained as follows.

Lemma 2.2 *Assume that $p > p_c, l := m + \lambda_2$ and $b > 0$, and let $\underline{r}_0, \underline{u}_{out}$ and $\underline{v}_{out}$ be as in Lemma 2.1. Then*

$$-\Delta \underline{u}_{out} = \underline{v}_{out} \qquad \textit{for all } r > 0 \tag{2.6}$$

and

$$-\Delta \underline{v}_{out} \le (\underline{u}_{out})^p \qquad \textit{for all } r > \underline{r}_0. \tag{2.7}$$

PROOF. While (2.6) can easily be verified, to see (2.7) we compute

$$\begin{aligned} -\Delta \underline{v}_{out} - (\underline{u}_{out})^p &= m(N-2-m)L \cdot (m+2)(N-4-m)r^{-m-4} \\ &\quad - l(N-2-l)b \cdot (l+2)(N-4-l)r^{-l-4} \\ &\quad - \Big\{Lr^{-m} - br^{-l}\Big\}^p, \qquad r > \underline{r}_0. \end{aligned} \tag{2.8}$$

Here, the convexity of $0 \le s \mapsto s^p$ implies $(1-z)^p \ge 1 - pz$ for all $z \in [0,1]$ and hence

$$\begin{aligned} \Big\{ Lr^{-m} - br^{-l} \Big\}^p &= (Lr^{-m})^p \Big\{ 1 - \frac{b}{L} r^{m-l} \Big\}^p \\ &\ge (Lr^{-m})^p \Big\{ 1 - p\frac{b}{L} r^{m-l} \Big\} \\ &= L^p r^{-m-4} - pL^{p-1} b r^{-l-4} \qquad \text{for } r > \underline{r}_0. \end{aligned} \tag{2.9}$$

Since $L^{p-1} = m(m+2)(N-2-m)(N-4-m)$ by (0.3) and $l(l+2)(N-2-l)(N-4-l) - pL^{p-1} \equiv P(l) = 0$ by choice of l, inserting (2.9) into (2.8) yields (2.7). ////

We can now define a subsolution pair in the announced manner.

Lemma 2.3 *Let $p > p_c, l := m + \lambda_2$ and $b > 0$, and define*

$$\underline{u}(x) := \begin{cases} 0, & 0 \le |x| < \underline{r}_0, \\ \underline{u}_{out}(|x|), & |x| \ge \underline{r}_0, \end{cases} \tag{2.10}$$

and

$$\underline{v}(x) := \begin{cases} 0, & 0 \le |x| < \underline{r}_1, \\ \underline{v}_{out}(|x|), & |x| \ge \underline{r}_1, \end{cases} \tag{2.11}$$

where $\underline{r}_0, \underline{r}_1, \underline{u}_{out}$ and $\underline{v}_{out}$ are as in Lemma 2.2. Then $\underline{u}$ and $\underline{v}$ are nonnegative and Lipschitz continuous on $\mathbb{R}^N$ and satisfy

$$\begin{aligned} -\Delta \underline{u} &\le \underline{v}, \\ -\Delta \underline{v} &\le \underline{u}^p \end{aligned} \tag{2.12}$$

in the sense of distributions on $\mathbb{R}^N$.

Proof. It follows from (2.4) and (2.5) that $\underline{u}$ and $\underline{v}$ are nonnegative and Lipschitz continuous in $\mathbb{R}^N$, and that $(\underline{u}_{out})_r(\underline{r}_0) \ge 0$ and $(\underline{v}_{out})_r(\underline{r}_1) \ge 0$. In order to prove (2.12) it is thus sufficient to show

$$-\Delta \underline{u} \le \underline{v} \qquad \text{for } 0 \le |x| < \underline{r}_0 \text{ and for } |x| > \underline{r}_0 \tag{2.13}$$

and

$$-\Delta \underline{v} \le \underline{u}^p \qquad \text{for } 0 \le |x| < \underline{r}_1 \text{ and for } |x| > \underline{r}_1. \tag{2.14}$$

For $|x| < \underline{r}_0$, both (2.13) and (2.14) are trivial, because $\underline{u} = \underline{v} \equiv 0$ there. When $|x| > \underline{r}_1$, we have $|x| > \underline{r}_0$ by Lemma 2.1 and hence $\underline{u} = \underline{u}_{out}$ and $\underline{v} = \underline{v}_{out}$, so that Lemma 2.2 entails

$$-\Delta \underline{u} = -\Delta \underline{u}_{out} = \underline{v}_{out} = \underline{v}$$

and

$$-\Delta \underline{v} = -\Delta \underline{v}_{out} \le (\underline{u}_{out})^p = \underline{u}^p$$

at such points. If $\underline{r}_0 \le |x| \le \underline{r}_1$, however, we have $\underline{u} = \underline{u}_{out}$ and $\underline{v} = 0$, whence Lemma 2.2 together with (2.4) and (2.5) yields

$$-\Delta \underline{u} = -\Delta \underline{u}_{out} = \underline{v}_{out} \le 0 = \underline{v}$$

and

$$-\Delta \underline{v} = 0 \le \underline{u}^p.$$

This completes the proof. ////

2.2 Construction of a supersolution

In the course of the proof of Lemma 2.2 one might already have expected that when looking for a super- rather than a subsolution one will encounter the problem that $(1-z)^p$ cannot be estimated from above by its first-order expansion. As a consequence, some additional technical expense will be necessary. In particular, our candidates for supersolutions in the outer region will need to contain further terms that can be used to absorb higher order terms in the corresponding differential inequalities (see (2.24) below).
To begin with, for definiteness let us state the following simple calculus inequality.

Lemma 2.4 *Let $p > 1$. Then there exists $C_p > 0$ such that*

$$(1-z)^p \le 1 - pz + C_p z^2 \qquad \textit{for all } z \in [0,1]. \tag{2.15}$$

PROOF. We pick $C_p > 0$ large such that $C_p \ge 2p$ and

$$p(p-1)(1-z)^{p-2} \le 2C_p \qquad \text{for all } z \in \Big(0, \frac{1}{2}\Big). \tag{2.16}$$

Then $\rho(z) := (1-z)^p - 1 + pz - C_p z^2$, $z \in [0,1]$, is concave on $(0, \frac{1}{2})$ and hence nonpositive in this interval, for $\rho(0) = \rho'(0) = 0$. Since for $z \in [\frac{1}{2}, 1]$ we can directly estimate $\rho(z) \le pz - C_p z^2$, using $C_p \ge 2p$ we arrive at (2.15). ////

We can now introduce our outer supersolution.

Lemma 2.5 *Let $p > p_c$ and $l := m + \lambda_2$, and let $k_0 := \min\{m + \lambda_3, 2l - m\}$. Then we have $k_0 > l$, and for all $k \in (l, k_0)$ and each $b > 0$ one can choose $c > 0$ such that*

$$\begin{aligned} &\overline{u}_{out}(r) := Lr^{-m} - br^{-l} + cr^{-k} \qquad \textit{and} \\ &\overline{v}_{out}(r) := m(N-2-m)Lr^{-m-2} - l(N-2-l)br^{-l-2} + k(N-2-k)cr^{-k-2}, \end{aligned} \tag{2.17}$$

satisfy

$$\begin{aligned} -\Delta \overline{u}_{out} &= \overline{v}_{out} \qquad \textit{for all } r > 0 \qquad \textit{and} \\ -\Delta \overline{v}_{out} &\ge (\overline{u}_{out})^p \qquad \textit{for all } r > \overline{r}_0, \end{aligned} \tag{2.18}$$

where

$$\overline{r}_0 := \Big(\frac{c}{b}\Big)^{\frac{1}{k-l}}. \tag{2.19}$$

PROOF. We let $k_0 := \min\{m+\lambda_3, 2l-m\}$. Then Since $l > m$ by Lemma 1.1 i), it easily follows that $k_0 > l$. Now given $k \in (l, k_0)$ and $b > 0$ we know from Lemma 1.1 and the fact that $k < m + \lambda_3$ that $P(k) \equiv k(k+2)(N-2-k)(N-4-k) - pL^{p-1} > 0$. This enables us to pick $c > 0$ large such that

$$P(k)c^{\frac{l-m}{k-l}} \geq C_p L^{p-2} b^{\frac{k-m}{k-l}}, \tag{2.20}$$

where $C_p > 0$ is the constant provided by Lemma 2.4, and such that moreover,

$$\Big(\frac{c}{b}\Big)^{\frac{1}{k-l}} \geq \Big(\frac{b}{L}\Big)^{\frac{1}{l-m}}. \tag{2.21}$$

Then in view of the definitions (2.2) and (2.19), (2.20) and (2.21) are equivalent to saying

$$P(k)c \geq C_p L^{p-2} b^2 \overline{r}_0^{k+m-2l} \tag{2.22}$$

and

$$\overline{r}_0 \geq \underline{r}_0, \tag{2.23}$$

respectively, whence in particular with $\overline{u}_{out}$ and $\overline{v}_{out}$ as defined by (2.17) we have $\overline{u}_{out}(r) > Lr^{-m} - br^{-l} \geq 0$ for all $r \geq \overline{r}_0$. Now by direct computation we easily verify the first relation in (2.18), while

$$\begin{aligned} -\Delta \overline{v}_{out} - (\overline{u}_{out})^p &= m(m+2)(N-2-m)(N-4-m)Lr^{-m-4} \\ &\quad -l(l+2)(N-2-l)(N-4-l)br^{-l-4} \\ &\quad +k(k+2)(N-2-k)(N-4-k)cr^{-k-4} \\ &\quad -\Big\{Lr^{-m} - br^{-l} + cr^{-k}\Big\}^p \end{aligned}$$

for $r > 0$. As $br^{-l} - cr^{-k} > 0$ for all $r > \overline{r}_0$, and since $\overline{u}_{out}(r) \geq 0$ for $r \geq \overline{r}_0$ entails that $z := \frac{b}{L}r^{m-l} - \frac{c}{L}r^{m-k}$ satisfies $z \leq 1$ for such r, by Lemma 2.4 we can estimate

$$\begin{aligned} \Big\{Lr^{-m} - br^{-l} + cr^{-k}\Big\}^p &= (Lr^{-m})^p \cdot \Big\{1 - \Big(\frac{b}{L}r^{m-l} - \frac{c}{L}r^{m-k}\Big)\Big\}^p \\ &\leq (Lr^{-m})^p \cdot \Big\{1 - p\Big(\frac{b}{L}r^{m-l} - \frac{c}{L}r^{m-k}\Big) \\ &\qquad + C_p\Big(\frac{b}{L}r^{m-l} - \frac{c}{L}r^{m-k}\Big)^2\Big\} \\ &= (Lr^{-m})^p - pL^{p-1}br^{-l-4} + pL^{p-1}cr^{-k-4} \\ &\quad + C_p \cdot (Lr^{-m})^p \cdot \Big(\frac{b}{L}r^{m-l} - \frac{c}{L}r^{m-k}\Big)^2 \end{aligned} \tag{2.24}$$

for such r. Estimating $\frac{b}{L}r^{m-l} - \frac{c}{L}r^{m-k} \le \frac{b}{L}r^{m-l}$ here, we arrive at

$$\Big\{Lr^{-m} - br^{-l} + cr^{-k}\Big\}^p \le (Lr^{-m})^p - pL^{p-1}br^{-l-4} + pL^{p-1}cr^{-k-4} + C_pL^{p-2}b^2r^{-4+m-2l}$$

for all $r > \overline{r}_0$. Recalling the definition (0.3) of L we thus infer that

$$\begin{aligned} -\Delta\overline{v}_{out} - (\overline{u}_{out})^p &\ge \Big\{-l(l+2)(N-2-l)(N-4-l) + pL^{p-1}\Big\} \cdot br^{-l-4} \\ &\quad + \Big\{k(k+2)(N-2-k)(N-4-k) - pL^{p-1}\Big\} \cdot cr^{-k-4} \\ &\quad - C_pL^{p-2}b^2r^{-4+m-2l} \\ &= -P(l)\cdot br^{-l-4} + P(k)\cdot cr^{-k-4} - C_pL^{p-2}b^2r^{-4+m-2l} \end{aligned}$$

for all $r > \overline{r}_0$. Since $P(l) = 0$ and $k + m - 2l < 0$ according to our choice of $k < k_0$, we finally obtain

$$\begin{aligned} -\Delta\overline{v}_{out} - (\overline{u}_{out})^p &\ge r^{-k-4}\cdot\Big\{P(k)\cdot c - C_pL^{p-2}b^2r^{k+m-2l}\Big\} \\ &\ge r^{-k-4}\cdot\Big\{P(k)\cdot c - C_pL^{p-2}b^2\overline{r}_0^{k+m-2l}\Big\} \\ &\ge 0 \qquad \text{for all } r > \overline{r}_0 \end{aligned}$$

in view of our largeness assumption (2.22) on c. ////

In the inner part, as a supersolution we shall choose (u, v) as a smooth approximation of the singular solution from below in both of its components.

Lemma 2.6 *Let $p > p_c$. Then for all $\varepsilon > 0$, the couple $(\overline{u}_{in,\varepsilon}, \overline{v}_{in,\varepsilon})$ defined by*

$$\begin{aligned} \overline{u}_{in,\varepsilon}(r) &:= L(r^2+\varepsilon)^{-\frac{m}{2}}, \\ \overline{v}_{in,\varepsilon}(r) &:= m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}}, \end{aligned} \tag{2.25}$$

satisfies

$$\begin{aligned} -\Delta\overline{u}_{in,\varepsilon} &\ge \overline{v}_{in,\varepsilon} \qquad \textit{for all } r \ge 0 \qquad \textit{and} \\ -\Delta\overline{v}_{in,\varepsilon} &\ge (\overline{u}_{in,\varepsilon})^p \qquad \textit{for all } r \ge 0. \end{aligned}$$

PROOF. For $\kappa > 0$ we compute

$$\Big((r^2+\varepsilon)^{-\frac{\kappa}{2}}\Big)_r = -\kappa(r^2+\varepsilon)^{-\frac{\kappa+2}{2}}r$$

and

$$\Big((r^2+\varepsilon)^{-\frac{\kappa}{2}}\Big)_{rr} = -\kappa(r^2+\varepsilon)^{-\frac{\kappa+2}{2}} + \kappa(\gamma+2)(r^2+\varepsilon)^{-\frac{\kappa+4}{2}}r^2,$$

and thus obtain, using that $\Delta\varphi = \varphi_{rr} + \frac{N-1}{r}\varphi_r$ for radial φ,

$$\begin{aligned} -\Delta(r^2+\varepsilon)^{-\frac{\kappa}{2}} &= \kappa N(r^2+\varepsilon)^{-\frac{\kappa+2}{2}} - \kappa(\kappa+2)(r^2+\varepsilon)^{-\frac{\kappa+4}{2}}r^2 \\ &\ge \kappa(N-2-\kappa)(r^2+\varepsilon)^{-\frac{\kappa+2}{2}}, \qquad r \ge 0. \end{aligned}$$

Applying this to $\kappa = m$ we find

$$-\Delta \overline{u}_{in,\varepsilon} \geq m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}} = \overline{v}_{in,\varepsilon},$$

while choosing $\kappa = m+2$ shows that

$$-\Delta \overline{v}_{in,\varepsilon} \geq m(N-2-m)L \cdot (m+2)(N-4-m)(r^2+\varepsilon)^{-\frac{m+4}{2}} = (\overline{u}_{in,\varepsilon})^p$$

for $r \geq 0$ in view of the definitions of $m = \frac{4}{p-1}$ and L. ////

We next make sure that if ε is small enough then inner and outer supersolutions indeed intersect, and that they do so transversally with a favorable direction of the jump in the derivative. The fact that the corresponding intersection points in general are different for both components u and v will force us to assert a certain ordering of inner and outer functions in the intermediate region between both intersections. All this is the objective of the following lemma.

Lemma 2.7 *Let $p > p_c, l = m + \lambda_2, b > 0, c > 0$ and $k \in (l, m+\lambda_3)$. Then there exists $\varepsilon_0 > 0$ with the following property: If, for some $\varepsilon \in (0, \varepsilon_0)$, $\overline{u}_{out}, \overline{v}_{out}, \overline{u}_{in,\varepsilon}$ and $\overline{v}_{in,\varepsilon}$ are as in (2.17) and (2.25) then the numbers $\overline{r}_0(\varepsilon) \in (0,\infty]$ and $\overline{r}_1(\varepsilon) \in (0,\infty]$ defined by*

$$\overline{r}_0(\varepsilon) := \sup \Big\{ r > 0 \ \Big| \ \overline{u}_{in,\varepsilon} < \overline{u}_{out} \ \textit{in } (0,r) \Big\}$$

and

$$\overline{r}_1(\varepsilon) := \sup \Big\{ r > 0 \ \Big| \ \overline{v}_{in,\varepsilon} < \overline{v}_{out} \ \textit{in } (0,r) \Big\}$$

satisfy

$$\overline{r}_0 < \overline{r}_0(\varepsilon) < \overline{r}_1 < \overline{r}_1(\varepsilon) < \overline{r}_1 + 1 \tag{2.26}$$

as well as

$$\overline{u}_{in,\varepsilon} \geq \overline{u}_{out} \qquad \textit{and} \qquad \overline{v}_{in,\varepsilon} \leq \overline{v}_{out} \qquad \textit{in } (\overline{r}_0(\varepsilon), \overline{r}_1(\varepsilon)). \tag{2.27}$$

Here we have set

$$\overline{r}_0 := \Big(\frac{c}{b}\Big)^{\frac{1}{k-l}} \qquad \textit{and} \qquad \overline{r}_1 := \Big(\frac{k(N-2-k)c}{l(N-2-l)b}\Big)^{\frac{1}{k-l}}. \tag{2.28}$$

Proof. For $\varepsilon > 0$ we let

$$\psi_\varepsilon(r) := \overline{u}_{in,\varepsilon}(r) - \overline{u}_{out}(r) = L(r^2+\varepsilon)^{-\frac{m}{2}} - Lr^{-m} + br^{-l} - cr^{-k}, \qquad r > 0,$$

and observe that for all $r > 0$,

$$\psi_\varepsilon(r) \nearrow \psi(r) := br^{-l} - cr^{-k} \qquad \text{as } \varepsilon \searrow 0. \tag{2.29}$$

Since

$$\psi(r) \begin{cases} < 0 & \text{for all } r < \overline{r}_0, \\ = 0 & \text{for } r = \overline{r}_0, \\ > 0 & \text{for all } r > \overline{r}_0, \end{cases} \tag{2.30}$$

we thus have $\overline{r}_0(\varepsilon) > \overline{r}_0$ for all $\varepsilon > 0$. Moreover, for all $\eta > 0$, by (2.29) and (2.30) we can find $\varepsilon_1(\eta) > 0$ such that $\psi_\varepsilon(\overline{r}_0 + \eta) > 0$ whenever $\varepsilon \in (0, \varepsilon_1(\eta))$, so that

$$\overline{r}_0 < \overline{r}_0(\varepsilon) < \overline{r}_0 + \eta \qquad \text{for all } \varepsilon \in (0, \varepsilon_1(\eta)). \tag{2.31}$$

Similarly,

$$\begin{aligned} \chi_\varepsilon(r) &:= \overline{v}_{in,\varepsilon}(r) - \overline{v}_{out}(r) \\ &= m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}} - m(N-2-m)Lr^{-m-2} \\ &\quad + l(N-2-l)br^{-l-2} - k(N-2-k)cr^{-k-2}, \qquad r > 0, \varepsilon > 0, \end{aligned}$$

satisfies

$$\chi_\varepsilon(r) \nearrow \chi(r) := l(N-2-l)br^{-l-2} - k(N-2-k)cr^{-k-2} \qquad \text{as } \varepsilon \searrow 0 \tag{2.32}$$

for all $r > 0$, because $N - 2 - m > 0$ by Lemma 1.1. Now taking $\overline{r}_1$ from (2.28) we have

$$\chi(r) \begin{cases} < 0 & \text{for all } r < \overline{r}_1, \\ = 0 & \text{for } r = \overline{r}_1, \\ > 0 & \text{for all } r > \overline{r}_1, \end{cases}$$

so that arguing as above for each $\eta > 0$ we can fix $\varepsilon_2(\eta) > 0$ such that

$$\overline{r}_1 < \overline{r}_1(\varepsilon) < \overline{r}_1 + \eta \qquad \text{for all } \varepsilon \in (0, \varepsilon_2(\eta)). \tag{2.33}$$

Since due to Lemma 1.1 iii) we know that $\overline{r}_1 > \overline{r}_0$, we can apply (2.31) to $\eta := \frac{\overline{r}_1 - \overline{r}_0}{2}$ and (2.33) to $\eta := 1$ to obtain that (2.26) is valid for all sufficiently small $\varepsilon > 0$.
To verify (2.27), we let $\hat{\psi}_\varepsilon(r) := r^k \psi_\varepsilon(r)$ and $\hat{\chi}_\varepsilon(r) := r^{k+2}\chi_\varepsilon(r)$. Then it is easy to see using (2.29) and (2.32) that $\hat{\psi}_\varepsilon \to \hat{\psi}$ and $\hat{\chi}_\varepsilon \to \hat{\chi}$ in $C^1_{loc}((0,\infty))$ as $\varepsilon \searrow 0$, where $\hat{\psi}(r) = br^{k-l} - c$ and $\hat{\chi}(r) = l(N-2-l)br^{k-l} - k(N-2-k)c$. Since both $\hat{\psi}'$ and $\hat{\chi}'$ are strictly positive on $(0,\infty)$, this implies that also $\hat{\psi}'_\varepsilon > 0$ and $\hat{\chi}'_\varepsilon > 0$ in $[\overline{r}_0, \overline{r}_1 + 1]$ for all appropriately small $\varepsilon > 0$. This, however, clearly entails that for such ε, $\hat{\psi}_\varepsilon(r) > 0$ for all $r \in (\overline{r}_0(\varepsilon), \overline{r}_1 + 1]$ and $\hat{\chi}_\varepsilon(r) < 0$ for all $r \in [\overline{r}_0, \overline{r}_1(\varepsilon))$, which in view of (2.26) implies (2.27). ////

We can now establish the counterpart of Lemma 2.3.

Lemma 2.8 *Let $p > p_c, l = m + \lambda_2$ and $b > 0$. Then there exist $k > l, c > 0$ and $\varepsilon_0 > 0$ such that whenever $\varepsilon \in (0, \varepsilon_0)$,*

$$\overline{u}_\varepsilon(x) := \begin{cases} \overline{u}_{in,\varepsilon}(|x|), & 0 \le |x| < \overline{r}_0(\varepsilon), \\ \overline{u}_{out}(|x|), & |x| \ge \overline{r}_0(\varepsilon), \end{cases} \tag{2.34}$$

and

$$\overline{v}_\varepsilon(x) := \begin{cases} \overline{v}_{in,\varepsilon}(|x|), & 0 \le |x| \le \overline{r}_1(\varepsilon), \\ \overline{v}_{out}(|x|), & |x| > \overline{r}_1(\varepsilon), \end{cases} \tag{2.35}$$

both are nonnegative and Lipschitz continuous on $\mathbb{R}^N$ *and satisfy*

$$\begin{aligned} -\Delta\overline{u}_\varepsilon &\geq \overline{v}_\varepsilon, \\ -\Delta\overline{v}_\varepsilon &\geq \overline{u}_\varepsilon^p \end{aligned} \tag{2.36}$$

in the distributional sense on $\mathbb{R}^N$*. Here,* $\overline{u}_{out}, \overline{v}_{out}, \overline{u}_{in,\varepsilon}, \overline{v}_{in,\varepsilon}, \overline{r}_0(\varepsilon)$ *and* $\overline{r}_1(\varepsilon)$ *are as defined in Lemma 2.5, Lemma 2.6 and Lemma 2.7, respectively.*

PROOF. We choose $k_0 \in (l, m+\lambda_3]$ as in Lemma 2.5 and pick any $k \in (l, k_0)$, and then, given $b > 0$, fix $c > 0$ as provided by the same lemma. We next take $\varepsilon_0 > 0$ as asserted by Lemma 2.7 and claim that upon these choices, for each $\varepsilon \in (0, \varepsilon_0)$ the functions $\overline{u}_\varepsilon$ and $\overline{v}_\varepsilon$ defined by (2.34) and (2.35) have the desired properties. To see this, we proceed as in Lemma 2.3: By the definitions of $\overline{r}_0(\varepsilon)$ and $\overline{r}_1(\varepsilon)$, both $\overline{u}_\varepsilon$ and $\overline{v}_\varepsilon$ are Lipschitz continuous. Moreover, in view of the fact that $(\overline{u}_{in,\varepsilon})_r \geq (\overline{u}_{out})_r$ at $r = \overline{r}_0(\varepsilon)$ as well as $(\overline{v}_{in,\varepsilon})_r \geq (\overline{v}_{out})_r$ at $r = \overline{r}_1(\varepsilon)$, proving (2.36) amounts to verifying that

$$-\Delta\overline{u}_\varepsilon \geq \overline{v}_\varepsilon \qquad \text{for } 0 \leq |x| < \overline{r}_0(\varepsilon) \text{ and for } |x| > \overline{r}_0(\varepsilon) \tag{2.37}$$

and

$$-\Delta\overline{v}_\varepsilon \geq \overline{u}_\varepsilon^p \qquad \text{for } 0 \leq |x| < \overline{r}_1(\varepsilon) \text{ and for } |x| > \overline{r}_1(\varepsilon). \tag{2.38}$$

For $|x| < \overline{r}_0(\varepsilon)$ we have $\overline{u}_\varepsilon = \overline{u}_{in,\varepsilon}(|\cdot|)$ and $\overline{v}_\varepsilon = \overline{v}_{in,\varepsilon}(|\cdot|)$ and thus

$$-\Delta\overline{u}_\varepsilon \geq \overline{v}_\varepsilon \quad \text{and} \quad -\Delta\overline{v}_\varepsilon \geq \overline{u}_\varepsilon^p \qquad \text{for } 0 \leq |x| < \overline{r}_0(\varepsilon) \tag{2.39}$$

are guaranteed by Lemma 2.6. Likewise, Lemma 2.5 ensures

$$-\Delta\overline{u}_\varepsilon \geq \overline{v}_\varepsilon \quad \text{and} \quad -\Delta\overline{v}_\varepsilon \geq \overline{u}_\varepsilon^p \qquad \text{for } |x| > \overline{r}_1(\varepsilon). \tag{2.40}$$

As to the intermediate annular region $\overline{r}_0(\varepsilon) \leq |x| \leq \overline{r}_1(\varepsilon)$, where $\overline{u}_\varepsilon = \overline{u}_{out}(|\cdot|)$ and $\overline{v}_\varepsilon = \overline{v}_{in,\varepsilon}(|\cdot|)$, Lemma 2.5 says that

$$-\Delta\overline{u}_\varepsilon = -\Delta\overline{u}_{out}(|\cdot|) = \overline{v}_{out}(|\cdot|).$$

By Lemma 2.7, however, we can estimate $\overline{v}_{out} \geq \overline{v}_{in,\varepsilon}$ in $(\overline{r}_0(\varepsilon), \overline{r}_1(\varepsilon))$ and hence obtain

$$-\Delta\overline{u}_\varepsilon \geq \overline{v}_\varepsilon \qquad \text{for } \overline{r}_0(\varepsilon) < |x| \leq \overline{r}_1(\varepsilon). \tag{2.41}$$

Similarly, using Lemma 2.6 and again (2.27) we infer that

$$-\Delta\overline{v}_\varepsilon = -\Delta\overline{v}_{in,\varepsilon}(|\cdot|) \geq (\overline{u}_{in,\varepsilon}(|\cdot|))^p \geq (\overline{u}_{out}(|\cdot|))^p = \overline{u}_\varepsilon^p \qquad \text{for } \overline{r}_0(\varepsilon) \leq |x| < \overline{r}_1(\varepsilon).$$

In conjunction with (2.39), (2.40) and (2.41) this establishes (2.37) and (2.38) and thus, showing (2.36), completes the proof. ////

2.3 Construction of a solution and proof of the main result

Performing a dynamical approach, we can now assert the existence of a solution of (0.1) with the desired decay properties. This type of argument is not new, but since we could not find a reference that precisely covers the present situation, we include a proof for the sake of completeness.

Lemma 2.9 *Let $p > p_c$. Then for all $b > 0$, (0.1) possesses a positive radially symmetric classical solution $u \in C^4(\mathbb{R}^N)$ satisfying*

$$u(x) = L|x|^{-m} - b|x|^{-m-\lambda_2} + o(|x|^{-m-\lambda_2}) \qquad \text{as } |x| \to \infty, \tag{2.42}$$

where $\lambda_2 > 0$ is the number defined in (0.7).

PROOF. We write $l := m + \lambda_2$ again and fix $k > l$, $c > 0$ and $\varepsilon_0 > 0$ as provided by Lemma 2.8. As easily checked, it is then possible to pick $\varepsilon \in (0, \varepsilon_0)$ small enough fulfilling

$$L(r^2 + \varepsilon)^{-\frac{m}{2}} \geq Lr^{-m} - br^{-l} \qquad \text{for all } r \in (0, \overline{r}_1 + 1] \tag{2.43}$$

and

$$m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}} \geq m(N-2-m)Lr^{-m-2} - l(N-2-l)br^{-l-2} \text{ for all } r \in (0, \overline{r}_1+1], \tag{2.44}$$

where $\overline{r}_1$ is taken from Lemma 2.7. Note here that the inequality $l > m$ ensures that the right-hand side of (2.43) tends to $-\infty$ as $r \searrow 0$. We claim that then the pairs $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$ introduced in Lemma 2.3 and 2.8, respectively, satisfy

$$\underline{u} \leq \overline{u} \qquad \text{and} \qquad \underline{v} \leq \overline{v} \qquad \text{in } \mathbb{R}^N. \tag{2.45}$$

Indeed, since $\overline{u}$ is nonnegative by Lemma 2.8, the first inequality in (2.45) only needs to be checked at points where $\underline{u}$ is positive, that is, for $|x| > \underline{r}_0 = (\frac{b}{L})^{\frac{1}{l-m}}$. At these points, however, we have $\underline{u}(x) = L|x|^{-m} - b|x|^{-l}$ and hence for large $|x|$

$$\overline{u}(x) = L|x|^{-m} - b|x|^{-l} + c|x|^{-k} \geq L|x|^{-m} - b|x|^{-l} = \underline{u}(x) \qquad \text{if } |x| \geq \overline{r}_0(\varepsilon)$$

and for small $|x|$

$$\overline{u}(x) = L(|x|^2 + \varepsilon)^{-\frac{m}{2}} \geq L|x|^{-m} - b|x|^{-l} = \underline{u}(x) \qquad \text{if } \underline{r}_0 \leq |x| < \overline{r}_0(\varepsilon)$$

due to (2.43). As similar reasoning, involving (2.44), shows the second inequality in (2.45).
Let us now consider the parabolic problem

$$\begin{cases} U_t = \Delta U + V, & x \in \mathbb{R}^N,\ t > 0, \\ V_t = \Delta V + |U|^{p-1}U, & x \in \mathbb{R}^N,\ t > 0, \\ U(x,0) = \underline{u}(x), \quad V(x,0) = \underline{v}(x), & x \in \mathbb{R}^N. \end{cases} \tag{2.46}$$

Since the PDE system in (2.46) is cooperative, the sub- and supersolution properties of $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, as asserted by Lemma 2.3 and Lemma 2.8, along with the ordering (2.45) ensure that in fact (2.46) has a global radially symmetric classical solution satisfying

$$\underline{u}(x) \le U(x,t) \le \overline{u}(x) \quad \text{and} \quad \underline{v}(x) \le V(x,t) \le \overline{v}(x) \qquad \text{for all } x \in \mathbb{R}^N \text{ and } t > 0 \quad (2.47)$$

(see [QS, Propositions 52.22 and 52.14] for a comprehensive demonstration of how to prove corresponding comparison principles for very weak sub- and supersolutions). Moreover, by a straightforward extension of [QS, Proposition 52.20] (cf. [QS, Remark 52.23]), it is possible to show that the subsolution property of the initial data implies monotonicity in time, that is,

$$U_t \ge 0 \quad \text{and} \quad V_t \ge 0 \qquad \text{in } \mathbb{R}^N \times (0, \infty). \quad (2.48)$$

In particular, in view of (2.47) and standard regularity regularity theory this entails that

$$U(\cdot, t) \to u \quad \text{and} \quad V(\cdot, t) \to v \qquad \text{in } C^2_{loc}(\mathbb{R}^N) \quad \text{as } t \to \infty$$

with some smooth limit functions u and v. Moreover, again by (2.48) and parabolic regularity theory, we can pick a sequence of times $t_j \to \infty$ such that both $U_t(\cdot, t_j)$ and $V_t(\cdot, t_j)$ converge to zero in $C^0_{loc}(\mathbb{R}^N)$ as $j \to \infty$, whence upon evaluating (2.46) at $t = t_j$ and letting $j \to \infty$ we conclude that (u, v) is a radially symmetric smooth solution of the stationary problem

$$\begin{cases} 0 = \Delta u + v, & x \in \mathbb{R}^N, \\ 0 = \Delta v + |u|^{p-1} u, & x \in \mathbb{R}^N. \end{cases} \quad (2.49)$$

Observe that (2.47), (2.48) and the strong maximum principle applied to the first equation in (2.46) imply that u must be strictly positive in all of $\mathbb{R}^N$. Hence, (2.49) evidently implies that u is a smooth positive solution of $\Delta^2 u = u^p$ in $\mathbb{R}^N$. Since clearly $\underline{u} \le u \le \overline{u}$ in $\mathbb{R}^N$ by (2.47), we furthermore have

$$L|x|^{-m} - b|x|^{-l} \le u(x) \le L|x|^{-m} - b|x|^{-l} + c|x|^{-k}$$

for all sufficiently large $|x|$ and thereby end up with (2.42) upon recalling that $k > l = m + \lambda_2$. ////

Let us remark that of course one could alternatively construct a solution as a fixed point of $(u, v) \mapsto (-\Delta, -\Delta)^{-1}(v, u^p)$ in a suitable convex set of functions (u, v) with $\underline{u} \le u \le \overline{u}$ and $\underline{v} \le v \le \overline{v}$. This would allow to get along without parabolic theory, but at the cost of some additional compactness considerations.

In any event, the main result in the supercritical case now actually reduces to a corollary.

Proof (of Theorem 0.1). By [GG1], any smooth positive radial solution of (0.1) is uniquely determined by its value at the origin, whence it follows that all such solutions of (0.1) are contained in the one-parameter family $(u_\alpha)_{\alpha > 0}$, where a scaling argument reveals the link

$$u_\beta(x) = \frac{\beta}{\alpha} u_\alpha\Big(\Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{m}} x\Big), \qquad x \in \mathbb{R}^N, \ \alpha > 0, \ \beta > 0,$$

between any two of them. This means that if we let $\tilde{u}$ denote the solution constructed in Lemma 2.9 by choosing $b := 1$ there, and if u is any smooth positive radial solution of (0.1) then $\tilde{u} = u_\alpha$ and $u = u_\beta$ with $\alpha = \tilde{u}(0)$ and $\beta = u(0)$. Hence, by (2.42),

$$\begin{aligned} u(x) &= u_\beta(x) = \frac{\beta}{\alpha} u_\alpha\Big(\Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{m}} x\Big) = \frac{\beta}{\alpha}\tilde{u}\Big(\Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{m}} x\Big) \\ &= \frac{\beta}{\alpha}\Big\{ L\Big|\Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{m}} x\Big|^{-m} - \Big|\Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{m}} x\Big|^{-m-\lambda_2} + o\Big(\Big|\Big(\frac{\beta}{\alpha}\Big)^{\frac{1}{m}} x\Big|^{-m-\lambda_2}\Big)\Big\} \\ &= L|x|^{-m} - \Big(\frac{\beta}{\alpha}\Big)^{-\frac{\lambda_2}{m}} |x|^{-m-\lambda_2} + o(|x|^{-m-\lambda_2}) \qquad \text{as } |x| \to \infty, \end{aligned}$$

which proves (0.6). ////

3 The case $p = p_c$

In the critical case $p = p_c$, we pursue the same strategy as above. As before, near the origin we shall use $(u, v) \equiv (0, 0)$ as a sub- and

$$\begin{aligned} \overline{u}_{in,\varepsilon}(r) &:= L(r^2+\varepsilon)^{-\frac{m}{2}}, \\ \overline{v}_{in,\varepsilon}(r) &:= m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}} \end{aligned} \tag{3.1}$$

as a supersolution with suitably small $\varepsilon > 0$. In an appropriate outer region, however, we this time consider

$$\begin{aligned} &\underline{u}_{out}(r) := Lr^{-m} - br^{-l}\ln\frac{r}{R}, \\ &\underline{v}_{out}(r) := m(N-2-m)Lr^{-m-2} - l(N-2-l)br^{-l-2}\ln\frac{r}{R} + (N-2-2l)br^{-l-2} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \overline{u}_{out}(r) &:= Lr^{-m} - br^{-l}\ln\frac{r}{R} + cr^{-l}\Big(\ln\frac{r}{R}\Big)^\beta \\ \overline{v}_{out}(r) &:= -\Delta\overline{u}_{out}(r) \\ &= m(N-2-m)Lr^{-m-2} - \Big\{l(N-2-l)\ln\frac{r}{R} - (N-2-2l)\Big\}\cdot br^{-l-2} \\ &\quad + \Big\{l(N-2-l)\Big(\ln\frac{r}{R}\Big)^\beta - \beta(N-2-2l)\Big(\ln\frac{r}{R}\Big)^{\beta-1} \\ &\qquad + \beta(1-\beta)\Big(\ln\frac{r}{R}\Big)^{\beta-2}\Big\}\cdot cr^{-l-2} \end{aligned} \tag{3.3}$$

as candidates for a sub- and supersolution, respectively. Here, choosing $l := m+\lambda_2 \equiv \frac{N-4}{2}$ (cf. Lemma 1.1), an arbitrary $b > 0$ and any $\beta < 1$ will be consistent with the desired asymptotics. The remaining parameters $c > 0$ and $R > 0$ are at our disposal and will, along with ε, be adjusted in such a way that sub- and supersolution properties can indeed be proved rigorously within suitable regions, and that inner and outer parts have convenient intersection properties similar to those of the corresponding functions in the previous section. In view of the logarithmic terms appearing in the outer parts, however, some technical modifications will be necessary.

3.1 Construction of a subsolution

Our first goal is to derive an analogue of Lemma 2.1. To this end, given $b > 0$ we let $\underline{u}_{out}$ and $\underline{v}_{out}$ be as above and introduce the numbers

$$\begin{aligned} \underline{r}_0 &:= \sup\Big\{r > 0 \ \Big|\ \underline{u}_{out}(r) \le 0\Big\} \qquad \text{and} \\ \underline{r}_1 &:= \sup\Big\{r > 0 \ \Big|\ \underline{v}_{out}(r) \le 0\Big\}. \end{aligned} \tag{3.4}$$

Although these numbers are in general no longer explicitly computable, we can make sure that upon choosing $R > 0$ suitably, these definitions become meaningful and $\underline{r}_0$ and $\underline{r}_1$ have favorable properties in respect of ordering:

Lemma 3.1 *Let $p = p_c, l = m + \lambda_2 = \frac{N-4}{2}$ and $b > 0$. Then there exists $R > 0$ such that the numbers $\underline{r}_0$ and $\underline{r}_1$ defined in (3.4) are well-defined and satisfy*

$$R < \underline{r}_0 < \underline{r}_1 < \infty, \tag{3.5}$$

and such that moreover

$$0 \le \underline{u}_{out}(r) < Lr^{-m} \qquad \textit{for all } r \ge \underline{r}_0 \tag{3.6}$$

as well as

$$0 \le \underline{v}_{out}(r) < m(N-2-m)Lr^{-m-2} \qquad \textit{for all } r \ge \underline{r}_1 \tag{3.7}$$

hold.

PROOF. Let us pick a number $\mu > 0$ large enough such that

$$\mu > \frac{1}{l-m}, \tag{3.8}$$

$$\mu > \frac{N-2-2l}{(N-2-l-m)(l-m)}, \tag{3.9}$$

$$\mu > \frac{l(N-2-l)}{m(N-2-m)(l-m)} \qquad \text{and} \tag{3.10}$$

$$\mu > \frac{N-2-2l}{l(N-2-l)}. \tag{3.11}$$

We claim that if, given $b > 0$, we let

$$R := e^{-\mu} \cdot \Big(\frac{\mu b}{L}\Big)^{\frac{1}{l-m}} \tag{3.12}$$

and

$$r_0 := \left(\frac{\mu b}{L}\right)^{\frac{1}{l-m}} \tag{3.13}$$

then we have

$$\underline{u}_{out}(r_0) = 0, \tag{3.14}$$

$$\frac{d}{dr}\Big(r^l \underline{u}_{out}(r)\Big) > 0 \qquad \text{for all } r \geq r_0, \tag{3.15}$$

$$\underline{v}_{out}(r_0) < 0 \qquad \text{and} \tag{3.16}$$

$$\frac{d}{dr}\Big(r^{l+2} \underline{v}_{out}(r)\Big) > 0 \qquad \text{for all } r \geq r_0. \tag{3.17}$$

This will imply $\underline{r}_0 = r_0$ and thereby immediately prove (3.5). To see (3.14)-(3.17) we introduce

$$\psi(r) := r^l \underline{u}_{out}(r) = Lr^{l-m} - b \ln \frac{r}{R}$$

and

$$\chi(r) := r^{l+2} \underline{v}_{out}(r) = m(N-2-m)Lr^{l-m} - l(N-2-l)b \ln \frac{r}{R} + (N-2-2l)b$$

for $r > 0$. Using (3.12), it can easily be checked that $\psi(r_0) = 0$, which is (3.14). As to (3.15), we compute ψ' to see that for $r > r_0$ we have

$$\begin{aligned} \psi'(r) &= (l-m)Lr^{l-m-1} - \frac{b}{r} \\ &= \frac{(l-m)b}{r} \cdot \Big\{\frac{L}{b} r^{l-m} - \frac{1}{l-m}\Big\} \\ &> \frac{(l-m)b}{r} \cdot \Big\{\frac{L}{b} \cdot \frac{\mu b}{L} - \frac{1}{l-m}\Big\} \\ &> 0 \end{aligned}$$

by (3.8). Next, by (3.12), (3.16) follows from

$$\begin{aligned} \chi(r_0) &= m(N-2-m)L \cdot \frac{\mu b}{L} - l(N-2-l)b \cdot \mu + (N-2-2l)b \\ &= \Big\{m(N-2-m) - l(N-2-l)\Big\} \cdot \mu b + (N-2-2l)b \\ &= \Big\{-(N-2-l-m)(l-m)\mu + N-2-2l\Big\} \cdot b \\ &< 0 \end{aligned}$$

because of (3.9). Finally, (3.17) is valid since for $r > r_0$,

$$\begin{aligned} \chi'(r) &= m(N-2-m)(l-m)Lr^{l-m-1} - l(N-2-l)b \cdot \frac{1}{r} \\ &= \frac{m(N-2-m)(l-m)b}{r} \cdot \Big\{\frac{L}{b} r^{l-m} - \frac{l(N-2-l)}{m(N-2-m)(l-m)}\Big\} \\ &> \frac{m(N-2-m)(l-m)b}{r} \cdot \Big\{\frac{L}{b} \cdot \frac{\mu b}{L} - \frac{l(N-2-l)}{m(N-2-m)(l-m)}\Big\} \\ &> 0 \end{aligned}$$

is guaranteed by (3.10).
It remains to show the second inequalities in (3.6) and (3.7). Since $\frac{r_0}{R} = e^\mu$, however, (3.6) is an easy consequence of the fact that $\mu > 0$, whereas (3.7) results from (3.11), which in conjunction with (3.5) guarantees that

$$\begin{aligned} m(N-2-m)Lr^{-m-2} - \underline{v}_{out}(r) &= \Big\{ l(N-2-l)\ln\frac{r}{R} - (N-2-2l) \Big\} \cdot br^{-l-2} \\ &> \Big\{ l(N-2-l)\ln\frac{r_0}{R} - (N-2-2l) \Big\} \cdot br^{-l-2} \\ &= \Big\{ l(N-2-l)\cdot\mu - (N-2-2l) \Big\} \cdot br^{-l-2} \\ &> 0 \end{aligned}$$

for all $r > \underline{r}_1$. ////

As in the case $p > p_c$, proving the subsolution property of $(\underline{u}_{out}, \underline{v}_{out})$ will not cause substantial difficulties.

Lemma 3.2 *Let $p = p_c$, $l = m + \lambda_2 = \frac{N-4}{2}$, $b > 0$ and $R > 0$. Then the functions $\underline{u}_{out}$ and $\overline{v}_{out}$ defined in (3.2) satisfy*

$$-\Delta\underline{u}_{out} = \underline{v}_{out} \qquad \textit{for all } r > 0 \tag{3.18}$$

and

$$-\Delta\underline{v}_{out} \le (\underline{u}_{out})^p \qquad \textit{for all } r > \underline{r}_0. \tag{3.19}$$

Proof. By means of the identity

$$\Delta\Big(r^{-\alpha}\ln\frac{r}{R}\Big) = -\alpha(N-2-\alpha)r^{-\alpha-2}\ln\frac{r}{R} + (N-2-2\alpha)r^{-\alpha-2},$$

valid for any $\alpha > 0$ and all $r > 0$, (3.18) can easily be verified, whereas if $r > \underline{r}_0$ then $\underline{u}_{out}(r) > 0$, so that

$$\begin{aligned} -\Delta\underline{v}_{out} - (\underline{u}_{out})^p = {} & m(N-2-m)L\cdot(m+2)(N-4-m)r^{-m-4} \\ & + l(N-2-l)b\cdot\Big\{ -(l+2)(N-4-l)r^{-l-4}\ln\frac{r}{R} + (N-6-2l)r^{-l-4} \Big\} \\ & -(N-2-2l)b\cdot(-l-2)(N-4-l)r^{-l-4} \\ & -\Big\{ Lr^{-m} - br^{-l}\ln\frac{r}{R} \Big\}^p. \end{aligned}$$

Using (0.3) and that at such points we have

$$\Big\{ Lr^{-m} - br^{-l}\ln\frac{r}{R} \Big\}^p \ge (Lr^{-m})^p - pL^{p-1}br^{-l-4}\ln\frac{r}{R}$$

thanks to convexity, after rearranging terms we obtain

$$\begin{aligned} -\Delta\underline{v}_{out} - (\underline{u}_{out})^p \le {} & \Big\{ -l(l+2)(N-2-l)(N-4-l) + pL^{p-1} \Big\} \cdot br^{-l-4}\ln\frac{r}{R} \\ & + \Big\{ l(N-2-l)(N-6-2l) + (l+2)(N-2-2l)(N-4-l) \Big\} \cdot br^{-l-4} \end{aligned} \tag{3.20}$$

wherever $r > \underline{r}_0$. Now by the property of l being a zero of P, the term $-l(l+2)(N-2-l)(N-4-l)+pL^{p-1}$ is zero; since we know that $l = \frac{N-4}{2}$, the simple computation

$$\begin{aligned} & l(N-2-l)(N-6-2l)+(l+2)(N-2-2l)(N-4-l) \\ & = \frac{N-4}{2} \cdot \frac{N}{2} \cdot (-2) + \frac{N}{2} \cdot 2 \cdot \frac{N-4}{2} \\ & = 0 \end{aligned}$$

shows that also the last term in (3.20) vanishes, and that consequently (3.19) holds. ////

3.2 Construction of a supersolution

The proof of the following lemma can be copied word by word from the corresponding Lemma 2.6 in the supercritical case.

Lemma 3.3 *Let $p = p_c$. Then for all $\varepsilon > 0$, the functions $\overline{u}_{in,\varepsilon}$ and $\overline{v}_{in,\varepsilon}$ defined in (3.1) fulfill*

$$\begin{aligned} -\Delta \overline{u}_{in,\varepsilon} \ge \overline{v}_{in,\varepsilon} \qquad & \textit{for all } r \ge 0 \qquad \textit{and} \\ -\Delta \overline{v}_{in,\varepsilon} \ge (\overline{u}_{in,\varepsilon})^p \qquad & \textit{for all } r \ge 0. \end{aligned}$$

As to the outer region, we have the following.

Lemma 3.4 *Let $p = p_c, l = m + \lambda_2 = \frac{N-4}{2}, b > 0$ and $\beta \in (0,1)$. Then there exists $c_0 > 0$ such that whenever $c > c_0$, the pair $(\overline{u}_{out}, \overline{v}_{out})$ as given by (3.3) and the number*

$$\overline{r}_0 := R \exp\Big((\frac{c}{b})^{\frac{1}{1-\beta}} \Big) \tag{3.21}$$

satisfy

$$0 < \overline{u}_{out}(r) < Lr^{-m} \qquad \textit{for all } r > \overline{r}_0 \tag{3.22}$$

and

$$\begin{aligned} -\Delta \overline{u}_{out} = \overline{v}_{out} \qquad & \textit{for all } r > R \qquad \textit{and} \\ -\Delta \overline{v}_{out} \ge (\overline{u}_{out})^p \qquad & \textit{for all } r > \overline{r}_0. \end{aligned} \tag{3.23}$$

Proof. We let $C_p > 0$ denote the constant appearing in Lemma 2.4 and choose some $c_0 > 0$ large fulfilling

$$c_0 > b \cdot \Big(\frac{4(2-\beta)(3-\beta)}{N^2-4N+8} \Big)^{\frac{1-\beta}{2}}, \tag{3.24}$$

$$(l-m) \cdot \Big(\frac{c_0}{b}\Big)^{\frac{1}{1-\beta}} > 4 - \beta \qquad \text{and} \tag{3.25}$$

$$(l-m)L \cdot \Big(R \exp\Big((\frac{c_0}{b})^{\frac{1}{1-\beta}} \Big) \Big)^{l-m} > b, \tag{3.26}$$

and such that

$$c \cdot \left(R \exp\left((\frac{c}{b})^{\frac{1}{1-\beta}} \right) \right)^{l-m} \cdot \left(\frac{c}{b} \right)^{-\frac{4-\beta}{1-\beta}} > \frac{4C_p L^{p-2} b^2}{\beta(1-\beta)(N^2-4N+8)} \qquad \text{for all } c > c_0 \quad (3.27)$$

as well as

$$L \cdot \left(R \exp\left((\frac{c}{b})^{\frac{1}{1-\beta}} \right) \right)^{l-m} > b \cdot \left(\frac{c}{b} \right)^{\frac{1}{1-\beta}} \qquad \text{for all } c > c_0. \qquad (3.28)$$

We now fix $c > c_0$ and let $\overline{u}_{out}, \overline{v}_{out}$ and $\overline{r}_0$ be defined by (3.3) and (3.21). In order to see (3.22), we let

$$f(r) := Lr^{l-m} - b \ln \frac{r}{R}, \qquad r > R,$$

and easily obtain that $f'(r) > 0$ for all $r > \overline{r}_0$ by (3.26), and that $f(\overline{r}_0) > 0$ due to (3.28). This entails that

$$\overline{u}_{out}(r) > r^{-l} \cdot f(r) > 0 \qquad \text{for all } r > \overline{r}_0.$$

As $\overline{u}_{out}(r) < Lr^{-m}$ if and only if $b \ln \frac{r}{R} > c(\ln \frac{r}{R})^\beta$, the definition (3.21) of $\overline{r}_0$ thereby completes the proof of (3.22).

Next, using that for $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}$,

$$\begin{aligned} \Delta\left(r^{-\alpha} \Big(\ln \frac{r}{R} \Big)^\gamma \right) &= -\alpha(N-2-\alpha) r^{-\alpha-2} \Big(\ln \frac{r}{R} \Big)^\gamma \\ &\quad + \gamma(N-2-2\alpha) r^{-\alpha-2} \Big(\ln \frac{r}{R} \Big)^{\gamma-1} \\ &\quad + \gamma(\gamma-1) r^{-\alpha-2} \Big(\ln \frac{r}{R} \Big)^{\gamma-2}, \qquad r > R, \end{aligned}$$

we easily check the first line in (3.23). As to the second, by the same token we compute

$$\begin{aligned} -\Delta \overline{v}_{out} &- (\overline{u}_{out})^p \\ &= -m(N-2-m)L \cdot (-m-2)(N-2-(m+2)) r^{-m-4} \\ &\quad + bl(N-2-l) \cdot \Big\{ -(l+2)(N-2-(l+2)) r^{-l-4} \ln \frac{r}{R} \\ &\qquad\qquad + (N-2-2(l+2)) r^{-l-4} \Big\} \\ &\quad - b(N-2-2l) \cdot (-l-2)(N-2-(l+2)) r^{-l-4} \\ &\quad - cl(N-2-l) \cdot \Big\{ -(l+2)(N-2-(l+2)) r^{-l-4} \Big(\ln \frac{r}{R} \Big)^\beta \\ &\qquad\qquad + \beta(N-2-2(l+2)) r^{-l-4} \Big(\ln \frac{r}{R} \Big)^{\beta-1} \\ &\qquad\qquad + \beta(\beta-1) r^{-l-4} \Big(\ln \frac{r}{R} \Big)^{\beta-2} \Big\} \\ &\quad + c\beta(N-2-2l) \cdot \Big\{ -(l+2)(N-2-(l+2)) r^{-l-4} \Big(\ln \frac{r}{R} \Big)^{\beta-1} \end{aligned}$$

$$
\begin{aligned}
& +(\beta-1)(N-2-2(l+2))r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-2} \\
& +(\beta-1)(\beta-2)r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-3}\Big\} \\
& -c\beta(1-\beta)\cdot\Big\{-(l+2)(N-2-(l+2))r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-2} \\
& +(\beta-2)(N-2-2(l+2))r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-3} \\
& +(\beta-2)(\beta-3)r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-4}\Big\} \\
& -(\overline{u}_{out})^p \\
= \; & +m(m+2)(N-2-m)(N-4-m)Lr^{-m-4} \\
& -l(l+2)(N-2-l)(N-4-l)br^{-l-4}\ln\frac{r}{R} \\
& +\Big\{l(N-2-l)(N-6-2l)+(l+2)(N-2-2l)(N-4-l)\Big\}\cdot br^{-l-4} \\
& +l(l+2)(N-2-l)(N-4-l)cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta} \\
& -\beta\cdot\Big\{l(N-2-l)(N-6-2l)+(l+2)(N-2-2l)(N-4-l)\Big\}\times \\
& \qquad \times c\,r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-1} \\
& +\beta(1-\beta)\cdot\Big\{l(N-2-l)-(N-2-2l)(N-6-2l)+(l+2)(N-4-l)\Big\}\times \\
& \qquad \times c\,r^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-2} \\
& +\beta(1-\beta)(2-\beta)\cdot\Big\{(N-2-2l)+(N-6-2l)\Big\}\cdot cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-3} \\
& -\beta(1-\beta)(2-\beta)(3-\beta)\cdot cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-4} \\
& -(\overline{u}_{out})^p \\
=: \; & I_1+\cdots+I_9 \qquad (3.29)
\end{aligned}
$$

for $r > \overline{r}_0$. Here we use that $l = \frac{N-4}{2}$ implies $I_7 = 0$ and, as before, that $l(N-2-l)(N-6-2l)+(l+2)(N-2-2l)(N-4-l) = 0$, so that $I_3 = I_5 = 0$, and moreover that

$$
\begin{aligned}
l(N-2-l)-(N-2-2l)(N-6-2l) \; & + \; (l+2)(N-4-l) \\
& = \frac{N-4}{2}\cdot\frac{N}{2}-2\cdot(-2)+\frac{N}{2}\cdot\frac{N-4}{2} \\
& = \frac{N^2-4N+8}{2}.
\end{aligned}
$$

Therefore,

$$
\frac{|I_8|}{\frac{1}{2}|I_6|} \;=\; \frac{4(2-\beta)(3-\beta)}{N^2-4N+8}\cdot\Big(\ln\frac{r}{R}\Big)^{-2}
$$

$$
\begin{aligned}
&\le \frac{4(2-\beta)(3-\beta)}{N^2-4N+8}\cdot\Big(\ln\frac{\overline{r}_0}{R}\Big)^{-2}\\
&= \frac{4(2-\beta)(3-\beta)}{N^2-4N+8}\cdot\Big(\frac{c}{b}\Big)^{-\frac{2}{1-\beta}}\\
&< 1
\end{aligned}
$$

for all $r > \overline{r}_0$ in view of (3.24). From (3.29) we thus obtain

$$
\begin{aligned}
-\Delta\overline{v}_{out} - (\overline{u}_{out})^p \ \ge\ & +m(m+2)(N-2-m)(N-4-m)Lr^{-m-4}\\
& -l(l+2)(N-2-l)(N-4-l)\cdot br^{-l-4}\ln\frac{r}{R}\\
& +l(l+2)(N-2-l)(N-4-l)\cdot cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta}\\
& +\frac{\beta(1-\beta)(N^2-4N+8)}{4}\cdot cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-2}\\
& -(\overline{u}_{out})^p \qquad \text{for all } r>\overline{r}_0.
\end{aligned}
\tag{3.30}
$$

Now since $0 < \overline{u}_{out} < Lr^{-m}$ we may invoke Lemma 2.4 to estimate the nonlinearity according to

$$
\begin{aligned}
(\overline{u}_{out})^p &= (Lr^{-m})^p\cdot\bigg\{1-\Big(\frac{b}{L}r^{m-l}\ln\frac{r}{R}-\frac{c}{L}r^{m-l}\Big(\ln\frac{r}{R}\Big)^{\beta}\Big)\bigg\}^p\\
&\le (Lr^{-m})^p\cdot\bigg\{1-p\Big(\frac{b}{L}r^{m-l}\ln\frac{r}{R}-\frac{c}{L}r^{m-l}\Big(\ln\frac{r}{R}\Big)^{\beta}\Big)\\
&\qquad\qquad +C_p\Big(\frac{b}{L}r^{m-l}\ln\frac{r}{R}-\frac{c}{L}r^{m-l}\Big(\ln\frac{r}{R}\Big)^{\beta}\Big)^2\bigg\}\\
&= (Lr^{-m})^p - pL^{p-1}br^{-l-4}\ln\frac{r}{R} + pL^{p-1}cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta}\\
&\quad +C_pL^pr^{-m-4}\Big(\frac{b}{L}r^{m-l}\ln\frac{r}{R}-\frac{c}{L}r^{m-l}\Big(\ln\frac{r}{R}\Big)^{\beta}\Big)^2\\
&\le (Lr^{-m})^p - pL^{p-1}br^{-l-4}\ln\frac{r}{R} + pL^{p-1}cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta}\\
&\quad +C_pL^{p-2}b^2r^{m-2l-4}\Big(\ln\frac{r}{R}\Big)^2 \qquad \text{for } r>\overline{r}_0,
\end{aligned}
$$

because $c > 0$. Inserting this into (3.30) and recalling the definition of L and the fact that $P(l) = 0$ we arrive at the inequality

$$
\begin{aligned}
-\Delta\overline{v}_{out} - (\overline{u}_{out})^p \ \ge\ & \frac{\beta(1-\beta)(N^2-4N+8)}{4}\cdot cr^{-l-4}\Big(\ln\frac{r}{R}\Big)^{\beta-2}\\
& -C_pL^{p-2}b^2r^{m-2l-4}\Big(\ln\frac{r}{R}\Big)^2 \qquad \text{for } r>\overline{r}_0.
\end{aligned}
\tag{3.31}
$$

In order to see that its right-hand side is nonnegative for $r > \overline{r}_0$, we let

$$g(r) := r^{l-m}\Big(\ln\frac{r}{R}\Big)^{\beta-4}, \qquad r > R,$$

and obtain from (3.25) that for $r > \overline{r}_0$,

$$\begin{aligned} g'(r) &= r^{l-m-1}\Big(\ln\frac{r}{R}\Big)^{\beta-5}\cdot\Big\{(l-m)\ln\frac{r}{R}+\beta-4\Big\} \\ &> r^{l-m-1}\Big(\ln\frac{r}{R}\Big)^{\beta-5}\cdot\Big\{(l-m)\cdot\Big(\frac{c}{b}\Big)^{\frac{1}{1-\beta}}+\beta-4\Big\} \\ &> 0. \end{aligned}$$

Hence, (3.27) implies that

$$g(r) > g(\overline{r}_0) > \frac{4C_pL^{p-2}b^2}{c\beta(1-\beta)(N^2-4N+8)} \qquad \text{for all } r > \overline{r}_0,$$

so that according to (3.31),

$$\begin{aligned} -\Delta\overline{v}_{out}-(\overline{u}_{out})^p &\geq \frac{\beta(1-\beta)(N^2-4N+8)c}{4}r^{m-2l-4}\Big(\ln\frac{r}{R}\Big)^2\times \\ &\qquad\qquad \times\Big\{g(r)-\frac{4C_pL^{p-2}b^2}{c\beta(1-\beta)(N^2-4N+8)}\Big\} \\ &> 0 \qquad \text{for all } r > \overline{r}_0, \end{aligned}$$

which proves (3.23). ////

The following analogue of Lemma 2.7 asserts existence and a convenient ordering of transversal intersection points of $\overline{u}_{in,\varepsilon}$ and $\overline{u}_{out}$ on the one hand and of $\overline{v}_{in,\varepsilon}$ and $\overline{v}_{out}$ on the other. Moreover, these intersection points can be put arbitrarily far from the origin by choosing c appropriately large. The latter will be used in Lemma 3.6 in proving that our finally chosen sub- and supersolutions will be ordered in a favorable way (cf. (3.41)).

Lemma 3.5 *Let $p = p_c, l = m + \lambda_2 = \frac{N-4}{2}, b > 0, R > 0$ and $\beta \in (0,1)$ be given. Then there exists $c_1 > 0$ with the following property: Given any $c > c_1$, one can pick $\overline{r}_1 > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$, there exist numbers $\overline{r}_0(\varepsilon)$ and $\overline{r}_1(\varepsilon)$ fulfilling*

$$R\exp\Big(\Big(\frac{c}{b}\Big)^{\frac{1}{1-\beta}}\Big) \equiv \overline{r}_0 < \overline{r}_0(\varepsilon) < \overline{r}_1(\varepsilon) < \overline{r}_1 + 1, \tag{3.32}$$

and such that the functions $\overline{u}_{in,\varepsilon}, \overline{v}_{in,\varepsilon}, \overline{u}_{out}$ and $\overline{v}_{out}$ defined in (3.1) and (3.3) satisfy

$$\overline{u}_{in,\varepsilon} = \overline{u}_{out} \quad \text{and} \quad (\overline{u}_{in,\varepsilon})_r > (\overline{u}_{out})_r \qquad \text{at } r = \overline{r}_0(\varepsilon) \tag{3.33}$$

as well as

$$\overline{v}_{in,\varepsilon} = \overline{v}_{out} \quad \text{and} \quad (\overline{v}_{in,\varepsilon})_r > (\overline{v}_{out})_r \qquad \text{at } r = \overline{r}_1(\varepsilon). \tag{3.34}$$

PROOF. Since $l = \frac{N-4}{2} < N-2$, we can pick $c_1 > 0$ large fulfilling

$$(1-\beta)l(N-2-l)b - \beta(1-\beta)(N-2-2l)b^{\frac{2-\beta}{1-\beta}}c_1^{-\frac{1}{1-\beta}} > 0. \tag{3.35}$$

In order to fix $\varepsilon_0 > 0$ appropriately, for $\varepsilon > 0$ and $r > R$ we introduce

$$\psi_\varepsilon(r) := \overline{u}_{in,\varepsilon}(r) - \overline{u}_{out}(r) = L(r^2+\varepsilon)^{-\frac{m}{2}} - Lr^{-m} + br^{-l}\ln\frac{r}{R} - cr^{-l}\Big(\ln\frac{r}{R}\Big)^{\beta}$$

and

$$\begin{aligned}
\chi_\varepsilon(r) &:= \overline{v}_{in,\varepsilon}(r) - \overline{v}_{out}(r) \\
&= m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}} - m(N-2-m)Lr^{-m-2} \\
&\quad + \Big\{ l(N-2-l)\ln\frac{r}{R} - (N-2-2l)\Big\} \cdot br^{-l-2} \\
&\quad - \Big\{ l(N-2-l)\Big(\ln\frac{r}{R}\Big)^{\beta} - \beta(N-2-2l)\Big(\ln\frac{r}{R}\Big)^{\beta-1} \\
&\qquad + \beta(1-\beta)\Big(\ln\frac{r}{R}\Big)^{\beta-2}\Big\} \cdot cr^{-l-2}.
\end{aligned}$$

Then as $\varepsilon \searrow 0$ we have

$$\psi_\varepsilon(r) \nearrow \psi(r) := br^{-l}\ln\frac{r}{R} - cr^{-l}\Big(\ln\frac{r}{R}\Big)^{\beta} \tag{3.36}$$

and

$$\chi_\varepsilon(r) \nearrow \chi(r) := \Big\{ l(N-2-l)\ln\frac{r}{R} - (N-2-2l)\Big\} \cdot br^{-l-2} \tag{3.37}$$

$$- \Big\{ l(N-2-l)\Big(\ln\frac{r}{R}\Big)^{\beta} - \beta(N-2-2l)\Big(\ln\frac{r}{R}\Big)^{\beta-1} \tag{3.38}$$

$$+ \beta(1-\beta)\Big(\ln\frac{r}{R}\Big)^{\beta-2}\Big\} \cdot cr^{-l-2} \tag{3.39}$$

for all $r > R$. The function ψ has a unique zero at $r = \overline{r}_0$ and is positive for $r > \overline{r}_0$, whereas using $\ln\frac{\overline{r}_0}{R} = (\frac{c}{b})^{\frac{1}{1-\beta}}$ we compute

$$\begin{aligned}
\overline{r}_0^{l+2}\chi(\overline{r}_0) &= \Big\{ l(N-2-l)\cdot\Big(\frac{c}{b}\Big)^{\frac{1}{1-\beta}} - (N-2-2l)\Big\} \cdot b \\
&\quad - \Big\{ l(N-2-l)\cdot\Big(\frac{c}{b}\Big)^{\frac{\beta}{1-\beta}} - \beta(N-2-2l)\cdot\Big(\frac{c}{b}\Big)^{-1} + \beta(1-\beta)\cdot\Big(\frac{c}{b}\Big)^{-\frac{2-\beta}{1-\beta}}\Big\} \cdot c \\
&= l(N-2-l)b^{-\frac{\beta}{1-\beta}}c^{\frac{1}{1-\beta}} - (N-2-2l)b \\
&\quad - l(N-2-l)b^{-\frac{\beta}{1-\beta}}c^{\frac{1}{1-\beta}} + \beta(N-2-2l)b - \beta(1-\beta)b^{\frac{2-\beta}{1-\beta}}c^{-\frac{1}{1-\beta}} \\
&= -(1-\beta)(N-2-2l)b - \beta(1-\beta)b^{\frac{2-\beta}{1-\beta}}c^{-\frac{1}{1-\beta}} \\
&< 0,
\end{aligned}$$

because $0 < \beta < 1$ and $l = \frac{N-4}{2} < \frac{N-2}{2}$. Moreover, according to (3.35), for all $r \ge \overline{r}_0$ we have $\ln \frac{r}{R} \ge (\frac{c}{b})^{\frac{1}{1-\beta}}$ and hence

$$\begin{aligned}
\frac{d}{dr}\Big(r^{l+2}\chi(r)\Big) &= l(N-2-l)\cdot\frac{b}{r} \\
&\quad -\Big\{\beta l(N-2-l)\cdot\Big(\ln\frac{r}{R}\Big)^{\beta-1} + \beta(1-\beta)(N-2-2l)\Big(\ln\frac{r}{R}\Big)^{\beta-2} \\
&\qquad -\beta(1-\beta)(2-\beta)\Big(\ln\frac{r}{R}\Big)^{\beta-3}\Big\}\cdot\frac{c}{r} \\
&\ge \frac{1}{r}\cdot\Big\{l(N-2-l)b - \beta l(N-2-l)\cdot\Big(\frac{c}{b}\Big)^{-1}\cdot c \\
&\quad -\beta(1-\beta)(N-2-2l)\cdot\Big(\frac{c}{b}\Big)^{-\frac{2-\beta}{1-\beta}}\cdot c\Big\} \\
&= \frac{1}{r}\cdot\Big\{(1-\beta)l(N-2-l)b - \beta(1-\beta)(N-2-2l)b^{\frac{2-\beta}{1-\beta}}c^{-\frac{1}{1-\beta}}\Big\} \\
&> 0.
\end{aligned}$$

Since evidently $r^{l+2}\chi(r) \to +\infty$ as $r \to \infty$, it follows that χ has a unique zero $\overline{r}_1$ in $(\overline{r}_0, \infty)$, and that $\chi_r(\overline{r}_1) > 0$.
We now observe that the convergence in (3.36) and (3.37) takes place in $C^1_{loc}((R,\infty))$ and thus in $C^1([\overline{r}_0, \overline{r}_1 + 1])$. Taking into account the monotonicity in (3.36), we easily obtain that if $\varepsilon > 0$ is sufficiently small then also ψ_ε and χ_ε have unique zeros $\overline{r}_0(\varepsilon)$ and $\overline{r}_1(\varepsilon)$, respectively, that enjoy the ordering properties in (3.32) and moreover satisfy $\psi_{\varepsilon r}(\overline{r}_0(\varepsilon)) > 0$ and $\chi_{\varepsilon r}(\overline{r}_1(\varepsilon)) > 0$. This proves (3.33) and (3.34). ////

3.3 Construction of a solution

As a last step towards Theorem 0.2, we can now state the following lemma that is, roughly speaking, a joint version of the above Lemmata 2.3 and 2.8 in the supercritical case.

Lemma 3.6 *Assume that $p = p_c$ and $l = m + \lambda_2 = \frac{N-4}{2}$. Then, given $b > 0$ and $\beta \in (0,1)$, there exist $c > 0$ and $\varepsilon > 0$ with the following property: Let $\underline{r}_0, \underline{r}_1, \overline{r}_0(\varepsilon)$ and $\overline{r}_1(\varepsilon)$ denote the numbers from (3.4) and Lemma 3.5, and let $\overline{u}_{in,\varepsilon}, \overline{v}_{in,\varepsilon}, \overline{u}_{out}, \overline{v}_{out}, \underline{u}_{out}$ and $\underline{v}_{out}$ be defined through (3.1), (3.3) and (3.2), respectively. Then*

$$\underline{r}_0 < \underline{r}_1 < \overline{r}_0(\varepsilon) < \overline{r}_1(\varepsilon), \tag{3.40}$$

and

$$\underline{u}(x) := \begin{cases} 0, & 0 \le |x| \le \underline{r}_0, \\ \underline{u}_{out}(|x|), & |x| > \underline{r}_0, \end{cases}$$

$$\underline{v}(x) := \begin{cases} 0, & 0 \le |x| \le \underline{r}_1, \\ \underline{v}_{out}(|x|), & |x| > \underline{r}_1, \end{cases}$$

$$\overline{u}(x) := \begin{cases} \overline{u}_{in,\varepsilon}(|x|), & 0 \le |x| \le \overline{r}_0(\varepsilon), \\ \overline{u}_{out}(|x|), & |x| > \overline{r}_0(\varepsilon), \end{cases} \qquad \textit{and}$$

$$\overline{v}(x) := \begin{cases} \overline{v}_{in,\varepsilon}(|x|), & 0 \le |x| \le \overline{r}_1(\varepsilon), \\ \overline{v}_{out}(|x|), & |x| > \overline{r}_1(\varepsilon), \end{cases}$$

are nonnegative and satisfy

$$\underline{u} \le \overline{u} \quad \textit{and} \quad \underline{v} \le \overline{v} \qquad \textit{in } \mathbb{R}^N \tag{3.41}$$

as well as

$$\begin{aligned} -\Delta \underline{u} &\le \underline{v}, \\ -\Delta \underline{v} &\le \underline{u}^p \end{aligned} \tag{3.42}$$

and

$$\begin{aligned} -\Delta \overline{u} &\ge \overline{v}, \\ -\Delta \overline{v} &\ge \overline{u}^p \end{aligned} \tag{3.43}$$

in the sense of distributions on $\mathbb{R}^N$.

PROOF. Given $b > 0$, we pick $R > 0$ as in Lemma 3.1 and then obtain that $\underline{r}_0$ and $\underline{r}_1$ as given by (3.4) fulfill (3.5), (3.6) and (3.7). Next, we let $c_0 > 0$ and $c_1 > 0$ denote the constants given by Lemma 3.4 and Lemma 3.5, respectively, and choose some $c > \max\{c_0, c_1\}$ such that

$$\left(\frac{c}{b}\right)^{\frac{1}{1-\beta}} > \ln \frac{\underline{r}_1}{R} \tag{3.44}$$

and

$$\left(\frac{c}{b}\right)^{\frac{1}{1-\beta}} > \frac{\beta(N-2-2l)}{l(N-2-l)}. \tag{3.45}$$

With this value of c being fixed henceforth, we take $\overline{r}_1 > 0$ and $\varepsilon_0 > 0$ as provided by Lemma 3.5. According to (3.6) and (3.7), it is now possible to select $\varepsilon \in (0, \varepsilon_0)$ small enough fulfilling

$$\underline{u}_{out}(r) < L(r^2 + \varepsilon)^{-\frac{m}{2}} \qquad \text{for all } r \in [\underline{r}_0, \overline{r}_1 + 1] \tag{3.46}$$

and

$$\underline{v}_{out}(r) < m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}} \qquad \text{for all } r \in [\underline{r}_1, \overline{r}_1 + 1]. \tag{3.47}$$

Upon these choices, $\overline{r}_0 \equiv R\exp((\frac{c}{b})^{\frac{1}{1-\beta}})$ satisfies $\overline{r}_0 > \underline{r}_1$ by (3.44), and hence from Lemma 3.1 and Lemma 3.5 we obtain

$$\underline{r}_0 < \underline{r}_1 < \overline{r}_0 < \overline{r}_0(\varepsilon) < \overline{r}_1(\varepsilon) < \overline{r}_1 + 1, \tag{3.48}$$

which in particular implies (3.40). Moreover,

$$\overline{u}(x)-\underline{u}(x)=\begin{cases}\overline{u}_{in,\varepsilon}(|x|), & 0\le|x|\le\underline{r}_0,\\ \overline{u}_{in,\varepsilon}(|x|)-\underline{u}_{out}(|x|), & \underline{r}_0<|x|\le\overline{r}_0(\varepsilon),\\ \overline{u}_{out}(|x|)-\underline{u}_{out}(|x|), & |x|>\overline{r}_0(\varepsilon),\end{cases}$$

and

$$\overline{v}(x)-\underline{v}(x)=\begin{cases}\overline{v}_{in,\varepsilon}(|x|), & 0\le|x|\le\underline{r}_1,\\ \overline{v}_{in,\varepsilon}(|x|)-\underline{v}_{out}(|x|), & \underline{r}_1<|x|\le\overline{r}_1(\varepsilon),\\ \overline{v}_{out}(|x|)-\underline{v}_{out}(|x|), & |x|>\overline{r}_1(\varepsilon).\end{cases}$$

Here, clearly, $\overline{u}_{in,\varepsilon}(r)=L(r^2+\varepsilon)^{-\frac{m}{2}}$ and $\overline{v}_{in,\varepsilon}(r)=m(N-2-m)L(r^2+\varepsilon)^{-\frac{m+2}{2}}$ are positive, while (3.46) and (3.47) in conjunction with (3.48) show that $\overline{u}_{in,\varepsilon}-\underline{u}_{out}$ and $\overline{v}_{in,\varepsilon}-\underline{v}_{out}$ are positive on $[\underline{r}_0,\overline{r}_0(\varepsilon)]$ and on $[\underline{r}_1,\overline{r}_1(\varepsilon)]$, respectively. Since

$$\overline{u}_{out}(r)-\underline{u}_{out}(r)=cr^{-l}\Big(\ln\frac{r}{R}\Big)^{\beta}>0 \qquad \text{for all } r>R$$

and

$$\begin{aligned}\overline{v}_{out}(r)-\underline{v}_{out}(r) &= cr^{-l-2}\cdot\Big\{l(N-2-l)\Big(\ln\frac{r}{R}\Big)^{\beta}-\beta(N-2-2l)\Big(\ln\frac{r}{R}\Big)^{\beta-1}\\ &\qquad +\beta(1-\beta)\Big(\ln\frac{r}{R}\Big)^{\beta-2}\Big\}\\ &> cr^{-l-2}\Big(\ln\frac{r}{R}\Big)^{\beta-1}\cdot\Big\{l(N-2-l)\cdot\ln\frac{\overline{r}_0}{R}-\beta(N-2-2l)\Big\}\\ &> 0 \qquad \text{for all } r>\overline{r}_0\end{aligned}$$

due to (3.45), observing that $\overline{r}_0(\varepsilon)>R$ by (3.48) and (3.5) and $\overline{r}_1(\varepsilon)>\overline{r}_0$ by (3.48) we conclude that $\overline{u}-\underline{u}$ and $\overline{v}-\underline{v}$ are positive in $\mathbb{R}^N$. Using (3.40) along with Lemma 3.2, Lemma 3.3, Lemma 3.4 and the intersection properties asserted by Lemma 3.5, one can easily derive the inequalities in (3.42) and (3.43) in the same style as in the proof of Lemma 2.3 and Lemma 2.8. ////

The proof of the following lemma precisely parallels that of Lemma 2.9 and therefore may be omitted.

Lemma 3.7 *If $p=p_c$ then for all $b>0$ there exists a positive radially symmetric classical solution $u\in C^4(\mathbb{R}^N)$ of (0.1) which satisfies*

$$u(x)=L|x|^{-m}-b|x|^{-m-\lambda_2}\ln|x|+o\Big(|x|^{-m-\lambda_2}\ln|x|\Big) \qquad \textit{as } |x|\to\infty,$$

where $\lambda_2>0$ is as given by (0.7).

We can prceed to establish our main result in the critical case.

PROOF (of Theorem 0.2). Theorem 0.2 can now easily be derived from Lemma 3.6 and the asymptotic properties of $\underline{u}$ and $\overline{u}$ by exactly repeating the arguments from Lemma 2.9 and Theorem 0.1. ////

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