Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model

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Abstract

We consider the classical parabolic-parabolic Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n$.
It is proved that in space dimension $n \geq 3$, for each $q > \frac{n}{2}$ and $p > n$ one can find $\varepsilon_0 > 0$ such that if the initial data (u_0, v_0) satisfy $\|u_0\|_{L^q(\Omega)} < \varepsilon$ and $\|\nabla v_0\|_{L^p(\Omega)} < \varepsilon$ then the solution is global in time and bounded and asymptotically behaves like the solution of a discoupled system of linear parabolic equations. In particular, (u, v) approaches the steady state (m, m) as $t \to \infty$, where m is the total mass $m := \int_\Omega u_0$ of the population.
Moreover, we shall show that if Ω is a ball then for arbitrary prescribed $m > 0$ there exist unbounded solutions emanating from initial data (u_0, v_0) having total mass $\int_\Omega u_0 = m$.

Key words: chemotaxis, global existence, boundedness, blow-up
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Introduction

In the mathematical modeling of self-organization of living cells, the Keller-Segel system of partial differential equations,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{0.1}$$

has played an increasingly important role through the last decades. It is used to describe the overall behavior of a collection of cells under the influence of chemotaxis. Under such circumstances, the movement of each individual cell, though still not precisely predictable, follows a favourite direction, namely that towards higher concentrations of a certain chemical signal substance. With $u = u(x, t)$ representing the density of cells and $v = v(x, t)$ the concentration of the chemical, the first equation in (0.1) thus reflects the interplay of undirected diffusive movement on the one hand and 'chemotactical movement' driven by ∇v on the other. The second equation expresses the model assumption that the signal substance, besides diffusing and degrading as most chemicals,

is permanently produced by living cells. Such a coupling is known to occur, for instance, in the paradigm species *Dictyostelium discoideum* ([KS]), but also believed to be present in many more biologically meaningful situations involving chemotaxis ([HP]). A striking feature of (0.1) is that despite its simple mathematical structure it has proved to be able to describe the phenomenon of spatial self-organization of cells: It is known, for instance, that in the spatially two-dimensional setting, (0.1) possesses solutions that undergo a *blow-up* in the sense that the cell density $u(x,t)$ becomes unbounded near some blow-up point in space when t approaches a certain blow-up time $T \le \infty$; since the total mass of cells does not change during the evolution, this means that in such cases the population will essentially aggregate around its blow-up points.
In order to summarize some known results in this direction more precisely, let us turn (0.1) into a full initial-boundary value problem by considering

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \qquad x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, \qquad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \end{cases} \tag{0.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal ν on $\partial\Omega$. The initial functions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$ are assumed to be nonnegative.
Within this framework, well-known results state that

- if $n = 1$ then all solutions of (0.2) are global in time and bounded ([OY]);
- if $n = 2$ then
  - in the case $\int_\Omega u_0 < 4\pi$, the solution will be global and bounded ([NSY], [GZ]), whereas
  - for any $m > 4\pi$ satisfying $m \notin \{4k\pi \mid k \in \mathbb{N}\}$ there exist initial data (u_0, v_0) with $\int_\Omega u_0 = m$ such that the corresponding solution of (0.2) blows up either in finite or infinite time, provided Ω is simply connected ([HWa] and [SeS2]).

In the two-dimensional setting, the outcome in [CC] suggests that in the case $\Omega = \mathbb{R}^2$ not considered here, a similar mass threshold phenomenon should decide between global existence and the possibility of blow-up, but then the conjectured critical mass is 8π.

Some further information on the precise mechanism of blow-up is obtained in [HV], where particular radially symmetric solutions in $\Omega = B_1(0) \subset \mathbb{R}^2$ are constructed that exhibit a finite-time collapse with an essentially Dirac-type blow-up profile. More generally, if $n = 2$ then any solution of (0.2) that is known to blow up in finite time has a finite sum of Dirac deltas plus some $f \in L^1(\Omega)$ as its asymptotic profile near the blow-up time ([NSS]). Apart from that, the large time behavior of *bounded* solutions to (0.2) has been the objective of several studies. For instance, in the two-dimensional setting all bounded solutions stabilize towards some member of the set of equilibria of (0.2) ([FLP]), even though this set may have a complicated structure ([SeS1], [HNSS]). In space dimension one, the dynamical system associated with (0.2) possesses a finite dimensional exponential attractor in $L^2(\Omega) \times W^{1,2}(\Omega)$ ([OY]).

All the above statements concentrate on the cases $n = 1$ and $n = 2$; as to the initial-boundary value problem (0.2) in higher space dimensions, only little appears to be known. In [Bo], it was proved that if $n = 3$ then for each $T > 0$ one can find a smallness condition on u_0 in $W^{1,2}(\Omega) \cap L^\infty(\Omega)$ and on v_0 in $W^{1,2}(\Omega)$ ensuring that (u, v) will exist at least up to time T; on the other hand, numerical evidence ([HP, Section 5.3]) suggests that also blow-up behavior occurs for some initial data in the three-dimensional situation. This conjecture is furthermore confirmed by known results on simplified variants of (0.2), in which after biologically justifiable limit procedures ([JL]), the second equation is replaced with one of the elliptic equations $0 = \Delta v + u - 1$ ([HMV1], [HMV2]), $0 = \Delta v + u$ ([BCKSV]), or $0 = \Delta v - v + u$ ([N1]). In these works, namely, it could be shown that if $n = 3$ (or even higher in some statements) then all these choices allow for solutions blowing up in finite time. Moreover, for some particular initial data even a rather precise description of possible blow-up mechanisms, revealing the occurrence of interesting shock-type blow-up phenomena, is presented in [BCKSV], [HMV1] and [HMV2].

However, to the best of our knowledge, no results are available for the parabolic-parabolic initial-boundary value problem (0.2) that rigorously prove either the existence of bounded solutions, or the occurrence of blow-up. It is not even clear yet whether at all (0.2) possesses any nonstationary global solution if $n \ge 3$. In view of the biological relevance of the particular case $n = 3$, we find it worthwhile to clarify these questions in the present paper. Our main results state that

- if $n \ge 3$, given any $q > \frac{n}{2}$ and $p > n$ one can find a bound for u_0 in $L^q(\Omega)$ and for ∇v_0 in $L^p(\Omega)$ guaranteeing that (u, v) is global in time and bounded (Theorem 2.1); on the other hand,
- if $n \ge 3$ and Ω is a ball then for arbitrarily small mass $m > 0$ there exist u_0 and v_0 having $\int_\Omega u_0 = m$ such that (u, v) blows up either in finite or infinite time (Theorem 3.5).

In other words: Unlike in space dimension $n = 2$, smallness of the population's total mass is definitely not sufficient to prevent chemotactic collapse in higher dimensions. Instead of a smallness condition in $L^1(\Omega)$, we need to require data for which u_0 is small even in some of the smaller spaces $L^{\frac{n}{2}+\varepsilon}(\Omega)$ for some $\varepsilon > 0$. This is fully consistent with related results for the corresponding Cauchy problem in the whole space $\Omega = \mathbb{R}^n$, where a similar feature of the integrability exponent $\frac{n}{2}$ was detected in [CP] for (0.1) and for a parabolic-elliptic simplification thereof in [CPZ].
Moreover, it is possible to characterize the large-time behavior of small-data solutions:

- If both $\|u_0\|_{L^q(\Omega)} < \varepsilon$ and $\|\nabla v_0\|_{L^p(\Omega)} < \varepsilon$ with $\varepsilon > 0$ sufficiently small, then the solution (u, v) of (0.2) satisfies

$$\begin{aligned} &\|u(\cdot,t) - u_H(\cdot,t)\|_{L^\infty(\Omega)} \le C\varepsilon^2 e^{-\lambda_1 t} \qquad \text{and} \\ &\|\nabla(v(\cdot,t) - v_H(\cdot,t))\|_{L^p(\Omega)} \le C\varepsilon^2 e^{-\lambda_1 t} \qquad \text{for all } t > 1, \end{aligned}$$

  with some $C > 0$, where λ_1 denotes the smallest positive eigenvalue of $-\Delta$ in Ω and u_H and v_H are the solutions of $\partial_t u_H = \Delta u_H$ and $\partial_t v_H = \Delta v_H - v_H + u_H$ under the same initial and boundary data (Theorem 2.1).

This means that the solution (u, v) asymptotically behaves like the solution (u_H, v_H) of an actually discoupled system of two linear parabolic equations, one of which is homogeneous and the other one inhomogeneous. In particular, all of our small-data solutions will approach the constant steady state (m, m) at an exponential rate as $t \to \infty$, where $m = \fint_\Omega u_0$. Similar statements on

‘asymptotically linear behavior’ were found in the Cauchy problems in $\Omega = \mathbb{R}^2$ ([N2]) and $\Omega = \mathbb{R}^n$, $n \geq 3$ ([CP]); there, however, the situation is somewhat different from the present one, because the solution of the heat equation in the entire space $\mathbb{R}^n$ converges to zero and not to a positive constant as $t \to \infty$.
Observe that the estimated error, as claimed above, decays in time like $e^{-\lambda_1 t}$ which is precisely the optimal rate of convergence of u_H towards the constant steady state m. Since the perturbation $-\nabla \cdot (u\nabla v)$ in (0.2) should generically affect all modes of u beyond the constant one, this order of decay is the best that can be expected. However, the error is essentially controlled by ε^2, whilst the L^∞ norm of both u and v are of order ε; this illustrates the decreasing influence of the chemotaxis term with shrinking size of the initial data.
We have to leave it as an open question whether or not the integrability exponent $q_0 = \frac{n}{2}$ is indeed critical in respect of blow-up in the sense that smallness of u_0 in $L^{q_0}(\Omega)$ (or in $L^{q_0-\varepsilon}(\Omega)$ for all $\varepsilon > 0$) is *in*sufficient to prevent blow-up. In fact, our Theorem 3.5 below will show that such a role is played by the smaller number $q = \frac{2n}{n+2}$.

1 Preliminaries

It is well-known that (0.2) is well-posed in the sense that it allows for a unique classical solution for any smooth initial data. Moreover, the solution cannot cease to exist unless u becomes unbounded in $L^\infty(\Omega)$. More precisely, we have the following statement that is by far not optimal in respect of regularity of the initial data, but it is sufficient for our purpose. For details and more general assumptions, we refer to [Y], [Bi], [HWi] and the references therein, for instance.

Lemma 1.1 *For any $u_0 \in C^0(\bar\Omega)$ and $v \in C^1(\bar\Omega)$, there exist a maximal existence time $T_{max}(u_0, v_0) \in (0,\infty]$ and a unique pair (u, v) of functions $u, v \in C^0(\bar\Omega\times[0, T_{max}(u_0, v_0)))\cap C^{2,1}(\Omega\times(0, T_{max}(u_0, v_0)))$ such that (u, v) solves (0.2) in the classical sense. Moreover, we have the following alternative:*

$$\textit{Either} \quad T_{max}(u_0, v_0) = \infty, \quad \textit{or} \quad \liminf_{t\nearrow T_{max}(u_0,v_0)} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty \tag{1.1}$$

The following elementary lemma provides some useful information on both the short-time and the large-time behavior of certain integrals that appear in a natural way when standard estimates are applied to variation-of-constants formulae.

Lemma 1.2 *Let $\alpha < 1$, $\beta < 1$ and γ and δ be positive constants such that $\gamma \neq \delta$. Then there exists $C > 0$ such that*

$$\int_0^t \Big(1 + (t-s)^{-\alpha}\Big)e^{-\gamma(t-s)} \cdot (1 + s^{-\beta})e^{-\delta s}ds \leq C\Big(1 + t^{\min\{0,1-\alpha-\beta\}}\Big)e^{-\min\{\gamma,\delta\}\cdot t} \tag{1.2}$$

for all $t > 0$.

Proof. Without loss of generality we may assume that $\gamma < \delta$, since otherwise we can exchange the roles of γ and δ upon substituting $s' = t - s$. Then for some $c > 0$ we have

$$\begin{aligned} I &:= \int_0^t \Big(1 + (t-s)^{-\alpha}\Big)e^{-\gamma(t-s)} \cdot (1 + s^{-\beta})e^{-\delta s}ds \\ &\leq c\int_0^t e^{-\gamma(t-s)}e^{-\delta s}ds + c\int_0^t (t-s)^{-\alpha}s^{-\beta}e^{-\gamma(t-s)}e^{-\delta s}ds \end{aligned}$$

$$= \quad \frac{c}{\delta-\gamma}\Big(e^{-\gamma t}-e^{-\delta t}\Big)+c\,e^{-\gamma t}\cdot\int_0^t (t-s)^{-\alpha}s^{-\beta}e^{-(\delta-\gamma)s}ds.$$

Substituting $s=\sigma t$ in the latter integral, we find

$$I \le \frac{c}{\delta-\gamma}e^{-\gamma t}+ce^{-\gamma t}\cdot t^{1-\alpha-\beta}\cdot\int_0^1 (1-\sigma)^{-\alpha}\sigma^{-\beta}e^{-(\delta-\gamma)\sigma t}d\sigma, \tag{1.3}$$

from which (1.2) follows in the case $\alpha+\beta\ge 1$. If $\alpha+\beta<1$, however, (1.3) proves (1.2) at least for $t\le 1$, whereas for $t>1$ we have

$$\begin{aligned}\int_0^{\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\beta}}}(1-\sigma)^{-\alpha}\sigma^{-\beta}e^{-(\delta-\gamma)\sigma t}d\sigma \quad &\le \quad \Big(\frac{1}{2}\Big)^{-\alpha}\cdot\frac{1}{1-\beta}\cdot\Big(\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\beta}}\Big)^{1-\beta} \\ &= \quad \frac{t^{-(1-\alpha-\beta)}}{2^{1-\alpha-\beta}(1-\beta)}\end{aligned}$$

and

$$\begin{aligned}&\int_{\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\beta}}}^1 (1-\sigma)^{-\alpha}\sigma^{-\beta}e^{-(\delta-\gamma)\sigma t}d\sigma \\ &\qquad \le \quad \Big(\frac{1}{2}t^{-\frac{1-\alpha-\beta}{1-\beta}}\Big)^{-\beta}e^{-(\delta-\gamma)\cdot\frac{1}{2}t^{1-\frac{1-\alpha-\beta}{1-\beta}}}\cdot\int_0^1(1-\sigma)^{-\alpha}d\sigma \\ &\qquad = \quad \frac{2^\beta}{1-\alpha}t^{\frac{\beta(1-\alpha-\beta)}{1-\beta}}e^{-\frac{\delta-\gamma}{2}t^{\frac{\alpha}{1-\beta}}}.\end{aligned}$$

We thus gain from (1.3) that

$$I \le \frac{c}{\delta-\gamma}e^{-\gamma t}+\frac{c}{2^{1-\alpha-\beta}(1-\beta)}e^{-\gamma t}+\frac{2^\beta c}{1-\alpha}\cdot e^{-\gamma t}\cdot t^{\frac{1-\alpha-\beta}{1-\beta}}\cdot e^{-\frac{\delta-\gamma}{2}t^{\frac{\alpha}{1-\beta}}}$$

for all $t>1$. Since $t\mapsto t^{\frac{1-\alpha-\beta}{1-\beta}}\cdot e^{-\frac{\delta-\gamma}{2}t^{\frac{\alpha}{1-\beta}}}$ is bounded for $t>1$ due to the fact that $\delta>\gamma$, we see that (1.2) is valid also for all $t>0$ when $\alpha+\beta<1$. ////

In order to determine the large-time behavior of small-data solutions to (0.2) as precisely as possible, we need another preparation which collects some facts on the asymptotics of the heat semigroup under Neumann boundary conditions. Although most of the statements below are essentially well-known (cf.[QS, Section 48] in case of Dirichlet boundary conditions), we could not find a precise reference in the literature that covers all that is necessary for our purpose; therefore we include a short proof here.

Lemma 1.3 *Let $(e^{t\Delta})_{t\ge0}$ be the Neumann heat semigroup in Ω, and let $\lambda_1>0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist constants $C_1,...,C_4$ depending on Ω only which have the following properties.*

i) If $1\le q\le p\le\infty$ then

$$\|e^{t\Delta}w\|_{L^p(\Omega)} \le C_1(1+t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})})e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \qquad \textit{for all } t>0 \tag{1.4}$$

holds for all $w \in L^q(\Omega)$ *satisfying* $\int_\Omega w = 0$.

ii) If $1 \le q \le p \le \infty$ *then*

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le C_2 \Big(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\Big)\, e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \qquad \text{for all } t>0 \tag{1.5}$$

is true for each $w \in L^q(\Omega)$.

iii) If $2 \le p < \infty$ *then*

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le C_3 e^{-\lambda_1 t}\|\nabla w\|_{L^p(\Omega)} \qquad \text{for all } t>0 \tag{1.6}$$

is valid for all $w \in W^{1,p}(\Omega)$.

iv) Let $1 < q \le p < \infty$. *Then*

$$\|e^{t\Delta} \nabla \cdot w\|_{L^p(\Omega)} \le C_4 \Big(1 + t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\Big)\, e^{-\lambda_1 t}\|w\|_{L^q(\Omega)} \qquad \text{for all } t>0 \tag{1.7}$$

holds for all $w \in (C_0^\infty(\Omega))^n$. *Consequently, for all* $t > 0$ *the operator* $e^{t\Delta}\nabla\cdot$ *possesses a uniquely determined extension to an operator from* $L^q(\Omega)$ *into* $L^p(\Omega)$, *with norm controlled according to (1.7).*

Proof. i) For $t < 2$, (1.4) is a consequence of the well-known smoothing estimate

$$\|e^{t\Delta} z\|_{L^p(\Omega)} \le c_1 t^{-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\|z\|_{L^q(\Omega)} \qquad \text{for all } t<2, \tag{1.8}$$

which can be checked for some c_1 independent of p and q and all $z \in L^q(\Omega)$ using pointwise estimates for Green's function of the Neumann heat semigroup ([M, Theorem 2.2]). As to $t \ge 2$, we first note that upon integrating the heat equation and using the variational definition of λ_1 we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |e^{t\Delta} w|^2 = -\int_\Omega |\nabla e^{t\Delta} w|^2 \le -\lambda_1 \int_\Omega |e^{t\Delta} w|^2$$

for all $t > 0$ and each smooth w satisfying $\int_\Omega w = 0$. Therefore,

$$\|e^{t\Delta} w\|_{L^2(\Omega)} \le e^{-\lambda_1 t}\|w\|_{L^2(\Omega)} \qquad \text{for all } t>0 \tag{1.9}$$

holds for all $w \in L^2(\Omega)$ with $\int_\Omega w = 0$. Now for $p < 2$, using Hölder's inequality and then (1.9) and (1.8) we find

$$\|e^{t\Delta} w\|_{L^p(\Omega)} \le |\Omega|^{\frac{2-p}{2p}}\|e^{t\Delta} w\|_{L^2(\Omega)} \le |\Omega|^{\frac{2-p}{2p}} e^{-\lambda_1(t-1)}\|e^{\Delta} w\|_{L^2(\Omega)} \le |\Omega|^{\frac{2-p}{2p}} c_1 e^{-\lambda_1(t-1)}\|w\|_{L^q(\Omega)}$$

for all $t \ge 2$. By a similar reasoning, for $p \ge 2$ we derive

$$\|e^{t\Delta} w\|_{L^p(\Omega)} \le c_1\|e^{(t-1)\Delta} w\|_{L^2(\Omega)} \le c_1 e^{-\lambda_1(t-2)}\|e^{\Delta} w\|_{L^2(\Omega)} \le c_1^2 e^{-\lambda_1(t-2)}\|w\|_{L^q(\Omega)}$$

for all $t \ge 2$, from which the claim follows.

ii) We first note that for some $c_2 > 0$ independent of p,

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le c_2 t^{-\frac{1}{2}} \|w\|_{L^p(\Omega)} \qquad \text{for all } t \le 1 \tag{1.10}$$

holds for all $w \in L^p(\Omega)$. In fact, for $p = 1$ and for $p = \infty$ this can be seen using pointwise estimates for the spatial gradient of Green's function of $e^{t\Delta}$ ([M, Theorem 2.2]), whereby (1.10) for $1 < p < \infty$ follows from a Marcinkiewicz-type interpolation argument (cf. [GT, Theorem 9.8]). In order to combine this with (1.4), we write $\bar{w} := \frac{1}{|\Omega|}\int_\Omega w$ and thus have $\int_\Omega (w - \bar{w}) = 0$. Since constants are invariant under $e^{t\Delta}$, we thus obtain from (1.10) and (1.4)

$$\begin{aligned} \|\nabla e^{t\Delta} w\|_{L^p(\Omega)} &= \|\nabla e^{\frac{t}{2}\Delta} e^{\frac{t}{2}\Delta}(w - \bar{w})\|_{L^p(\Omega)} \\ &\le c_2 \Big(\frac{t}{2}\Big)^{-\frac{1}{2}} \|e^{\frac{t}{2}\Delta}(w - \bar{w})\|_{L^p(\Omega)} \\ &\le c_2 C_1 \Big(\frac{t}{2}\Big)^{-\frac{1}{2}} \Big(1 + \Big(\frac{t}{2}\Big)^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\Big) e^{-\lambda_1 \frac{t}{2}} \|w - \bar{w}\|_{L^q(\Omega)}, \end{aligned} \tag{1.11}$$

which implies (1.5) for $t < 2$. For $t \ge 2$ we split $e^{t\Delta}$ in a different way to see that

$$\begin{aligned} \|\nabla e^{t\Delta} w\|_{L^p(\Omega)} &= \|\nabla e^{\Delta} e^{(t-1)\Delta}(w - \bar{w})\|_{L^p(\Omega)} \le c_2 \|e^{(t-1)\Delta}(w - \bar{w})\|_{L^p(\Omega)} \\ &\le c_2 C_1 \Big(1 + (t-1)^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})}\Big) e^{-\lambda_1 (t-1)} \|w - \bar{w}\|_{L^q(\Omega)} \\ &\le 4 c_2 C_1 e^{-\lambda_1 (t-1)} \|w\|_{L^q(\Omega)} \end{aligned}$$

for all $t \ge 2$. This together with (1.11) proves (1.5).

iii) Passing to $\hat{w} := w - \frac{1}{|\Omega|}\int_\Omega w$ if necessary, we may assume that $\int_\Omega w = 0$. We first consider the case $t \ge 1$, in which we apply ii), i) and the Poincaré inequality to find

$$\|\nabla e^{t\Delta} w\|_{L^p(\Omega)} \le 2C_2 \|e^{(t-1)\Delta} w\|_{L^p(\Omega)} \le 4C_2 C_1 e^{-(\lambda_1 - 1)t} \|w\|_{L^p(\Omega)} \le 4C_2 C_1 c_P e^{-(\lambda_1 - 1)t} \|\nabla w\|_{L^p(\Omega)},$$

which yields (1.6) for all $t \ge 1$ and any $p \in (1, \infty)$, because, as can easily be verified, the Poincaré constant c_P can be chosen independent of p.
Next, for $p = 2$, multiplying $(e^{t\Delta} w)_t = \Delta e^{t\Delta} w$ by $-\Delta e^{t\Delta} w$ and integrating shows that

$$\|\nabla e^{t\Delta} w\|_{L^2(\Omega)} \le \|\nabla w\|_{L^2(\Omega)} \qquad \text{for all } t \ge 0. \tag{1.12}$$

On the other hand, it is known ([M, formula (2.39)]) that for some $c_3 \ge 1$,

$$\|\nabla e^{t\Delta} w\|_{L^\infty(\Omega)} \le c_3 \|\nabla w\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (0, 1) \tag{1.13}$$

for each $w \in \hat{C}^1(\bar{\Omega}) := \{z \in C^1(\bar{\Omega}) \mid \frac{\partial z}{\partial \nu} = 0 \text{ on } \partial\Omega\}$. A Marcinkiewicz interpolation as in ii) now asserts that (1.6) is valid for each $p \in [2, \infty)$ and $t \in (0, 1)$ and all $w \in \hat{C}^1(\bar{\Omega})$, so that all that remains to be shown is that $\hat{C}^1(\bar{\Omega})$ is dense in $W^{1,p}(\Omega)$. To sketch a possible way to see this, we let $w \in W^{1,p}(\Omega)$ be given and fix $\varepsilon > 0$. Then there exists $w_1 \in C^1(\bar{\Omega})$ such that $\|w - w_1\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2}$. Given $x^0 \in \partial\Omega$, applying a shifting and local flattening procedure if necessary, we may assume that $x^0 = 0$, that $\Omega \subset \{x_n > 0\} := \{x = (x_1, ..., x_n) \in \mathbb{R}^n \mid x_n > 0\}$ and that $\partial\Omega$ is a part of $\{x_n = 0\}$ near x^0. For x near x^0, we define $w_{x^0}(x) := w_1(x_1, ..., x_{n-1}, 0)$

for $x \in \Omega$, so that $w_{x^0} = w_1$ on $\partial\Omega$ and $\frac{\partial w_{x^0}}{\partial\nu} = 0$ on $\partial\Omega$ hold near x^0. The same is thus valid for $w_{x^0,k}(x) := w_{x^0}(x) \cdot (1-\chi(kx_n)) + w_1(x) \cdot \chi(kx_n)$, where $\chi \in C^\infty(\mathbb{R})$ satisfies $\chi_{[2,\infty)} \le \chi \le \chi_{[1,\infty)}$. Since $w_1 \in C^1(\bar\Omega)$, it is easily checked that $w_{x^0,k} \to w_1$ in $W^{1,p}$ in a neighborhood of x^0, so that returning to the original coordinates via a suitable partition of unity will provide some $w_2 \in \hat{C}^1(\bar\Omega)$ such that $\|w_1 - w_2\|_{W^{1,p}(\Omega)} < \frac{\varepsilon}{2}$, which proves the claim.

iv) Given $\varphi \in C_0^\infty(\Omega)$, we use that $e^{t\Delta}$ is self-adjoint in $L^2(\Omega)$ and integrate by parts to see that

$$\begin{aligned}
\Big| \int_\Omega e^{t\Delta}\nabla \cdot w\varphi \Big| &= \Big| - \int_\Omega w \cdot \nabla e^{t\Delta}\varphi \Big| \\
&\le \|w\|_{L^q(\Omega)} \cdot \|\nabla e^{t\Delta}\varphi\|_{L^{q'}(\Omega)} \\
&\le C_2 \Big(1 + t^{-\frac12 - \frac n2 (\frac{1}{p'} - \frac{1}{q'})}\Big) e^{-\lambda_1 t} \|w\|_{L^q(\Omega)} \|\varphi\|_{L^{p'}(\Omega)}
\end{aligned}$$

for all $t > 0$ holds in view of ii), where $\frac1p + \frac{1}{p'} = 1$ and $\frac1q + \frac{1}{q'} = 1$. Since $\frac{1}{p'} - \frac{1}{q'} = \frac1q - \frac1p$, taking the supremum over all such φ satisfying $\|\varphi\|_{L^{p'}(\Omega)} \le 1$ we arrive at (1.7). ////

2 Small-data global solutions and their asymptotics

Having at hand the preliminary material collected above, we are now prepared to prove our main result on global-in-time existence of solutions emanating from suitably small initial data. The proof is organized in such a way that at the same time it yields the desired assertion on the asymptotic behavior of solutions. As compared to the several-step iterative procedure performed in [CP] in the case $\Omega = \mathbb{R}^n$, our proof is based on an essentially one-step contradiction argument and thereby somewhat simpler, but our method seems to be restricted to the case of bounded domains.

Theorem 2.1 *Let $p > n$ and $q > \frac n2$. Then there exist $\varepsilon_0 > 0$ and $C > 0$ with the following property: If $u_0 \in C^0(\bar\Omega)$ and $v_0 \in W^{1,p}(\Omega)$ are nonnegative with*

$$\|u_0\|_{L^q(\Omega)} \le \varepsilon \qquad \text{and} \qquad \|\nabla v_0\|_{L^p(\Omega)} \le \varepsilon \tag{2.1}$$

for some $\varepsilon < \varepsilon_0$ then the solution (u,v) of (0.2) exists globally in time, is bounded and satisfies

$$\Big\| u(\cdot,t) - e^{t\Delta}u_0 \Big\|_{L^\infty(\Omega)} \le C\varepsilon^2 e^{-\lambda_1 t} \qquad \text{for all } t > 1 \tag{2.2}$$

and

$$\Big\| \nabla \Big(v(\cdot,t) - e^{t(\Delta-1)}v_0 - \int_0^t e^{(t-s)(\Delta-1)} e^{s\Delta} u_0 ds \Big) \Big\|_{L^p(\Omega)} \le C\varepsilon^2 e^{-\lambda_1 t} \quad \text{for all } t > 1, \tag{2.3}$$

where $\lambda_1 > 0$ denotes the smallest nonzero eigenvalue of $-\Delta$ in Ω subject to homogeneous Neumann boundary conditions.
In particular, such solutions satisfy

$$\begin{aligned}
&\|u(\cdot,t) - m\|_{L^\infty(\Omega)} \le \bar{C} e^{-\lambda_1 t} \qquad \text{and} \\
&\|v(\cdot,t) - m\|_{L^\infty(\Omega)} \le \bar{C} e^{-\lambda_1 t} \qquad \text{for all } t > 1
\end{aligned} \tag{2.4}$$

(2.5)

with some constant $\bar{C} > 0$.

Remark. At the cost of some technical expense, on the basis of Theorem 2.1 and the ideas in [CP] it is possible to extend the above result to less regular initial data: Upon an approximation argument, namely, one can assert global existence of small-data weak solutions with the asymptotic properties (2.2) and (2.3) for initial data with possibly discontinuous u_0 and ∇v_0, satisfying only (2.1).

PROOF. Since $q > \frac{n}{2}$ and $p > n$, it is possible to fix $q_0 \in (\frac{n}{2}, q)$ and $p_0 \in (n, p]$ such that $q_0 < n$ and $\frac{1}{p_0} > \frac{1}{q_0} - \frac{1}{n}$. With $\varepsilon_0 > 0$ to be specified below, let us assume that (2.1) holds for some $\varepsilon \in (0, \varepsilon_0)$, and let

$$T := \sup \Big\{ \hat{T} > 0 \ \Big| \ \|u(\cdot,t) - e^{t\Delta}u_0\|_{L^\theta(\Omega)} \le \varepsilon\Big(1 + t^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta})}\Big)e^{-\lambda_1 t} \text{ for all } t \in (0, \hat{T}) \text{ and each } \theta \in [p_0, \infty] \Big\} \le \infty.$$

Then T is well-defined and positive with $T \le T_{max}(u_0, v_0)$, because both $u(\cdot,t)$ and $e^{t\Delta}u_0$ are bounded near $t = 0$, while on the other hand, as $1 > t \to 0$ we have $t^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta})} \ge t^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_0})} \to \infty$ uniformly with respect to $\theta \in [p_0, \infty]$.
We first claim that if ε_0 is sufficiently small then actually $T = \infty$ (and hence $T_{max}(u_0, v_0) = \infty$). To this end, we apply ∇ to both sides of the variation-of-constants formula

$$v(\cdot,t) - e^{t(\Delta-1)}v_0 = \int_0^t e^{-(t-s)(\Delta-1)}u(\cdot,s)ds, \qquad t \in (0, T_{max}(u_0, v_0)), \tag{2.6}$$

to obtain from Lemma 1.3 ii) that

$$\Big\|\nabla\Big(v(\cdot,t) - e^{t(\Delta-1)}v_0\Big)\Big\|_{L^{p_0}(\Omega)} \le c_1 \int_0^t \Big(1 + (t-s)^{-\frac{1}{2}}\Big)e^{-(\lambda_1+1)(t-s)}\|u(\cdot,s)\|_{L^{p_0}(\Omega)}ds, \ t \in (0,T), \tag{2.7}$$

with some $c_1 > 0$. By definition of T and Lemma 1.3 i), there exist $c_2 > 0$ and $c_3 > 0$ such that For each $\theta \in [p_0, \infty]$,

$$\begin{aligned} \|u(\cdot,s)\|_{L^\theta(\Omega)} &\le \|u(\cdot,s) - e^{s\Delta}u_0\|_{L^\theta(\Omega)} + \|e^{s\Delta}u_0\|_{L^\theta(\Omega)} \\ &\le \varepsilon\Big(1 + s^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta})}\Big)e^{-\lambda_1 s} + c_2\Big(1 + s^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta})}\Big)e^{-\lambda_1 s} \cdot \|u_0\|_{L^{q_0}(\Omega)} \\ &\le c_3\varepsilon\Big(1 + s^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta})}\Big)e^{-\lambda_1 s} \qquad \text{for all } s \in (0,T), \end{aligned} \tag{2.8}$$

so that in particular, according to Lemma 1.2,

$$\begin{aligned} \Big\|\nabla(v(\cdot,t) - e^{t(\Delta-1)}v_0)\Big\|_{L^{p_0}(\Omega)} &\le c_1c_3\varepsilon \cdot \int_0^t \Big(1 + (t-s)^{-\frac{1}{2}}\Big)e^{-(\lambda_1+1)(t-s)} \times \\ &\qquad \times \Big(1 + s^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_0})}\Big)e^{-\lambda_1 s}\, ds \\ &\le c_4\varepsilon\Big(1 + t^{\min\{0, 1 - \frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_0})\}}\Big)e^{-\min\{\lambda_1+1, \lambda_1\}\cdot t} \\ &= 2c_4\varepsilon e^{-\lambda_1 t} \qquad \text{for all } t \in (0,T) \end{aligned} \tag{2.9}$$

with a certain $c_4 > 0$, because our choices of p_0 and q_0 ensure that $\frac{1}{2} - \frac{n}{2}(\frac{1}{q_0} - \frac{1}{p_0}) > 0$.
Since by Lemma 1.3 iii),

$$\begin{aligned} \|\nabla e^{t(\Delta-1)} v_0\|_{L^{p_0}(\Omega)} &\leq c_5 e^{-(\lambda_1+1)t} \|\nabla v_0\|_{L^{p_0}(\Omega)} \\ &\leq c_5 |\Omega|^{\frac{p-p_0}{pp_0}} \varepsilon e^{-(\lambda_1+1)t} \qquad \text{for all } t \in (0,T) \end{aligned}$$

for some $c_5 > 0$, we thus have

$$\|\nabla v(\cdot,t)\|_{L^{p_0}(\Omega)} \leq c_6 \varepsilon e^{-\lambda_1 t} \qquad \text{for all } t \in (0,T) \tag{2.10}$$

with an appropriate $c_6 > 0$. We now make use of the representation formula for u,

$$u(\cdot,t) - e^{t\Delta} u_0 = -\int_0^t e^{(t-s)\Delta} \nabla \cdot (u(\cdot,s) \nabla v(\cdot,s)) ds, \qquad t \in (0, T_{max}(u_0,v_0)).$$

Invoking Lemma 1.3 iv), (2.8) and (2.10), there exists $c_7 > 0$ such that for arbitrary $\theta \in [p_0, \infty]$ we can estimate

$$\begin{aligned} \|u(\cdot,t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)} &\leq c_7 \int_0^t \Big(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p_0} - \frac{1}{\theta})}\Big) e^{-\lambda_1(t-s)} \times \\ &\qquad \times \|u(\cdot,s) \nabla v(\cdot,s)\|_{L^{p_0}(\Omega)} \, ds \\ &\leq c_7 \int_0^t \Big(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p_0} - \frac{1}{\theta})}\Big) e^{-\lambda_1(t-s)} \times \\ &\qquad \times \|u(\cdot,s)\|_{L^\infty(\Omega)} \|\nabla v(\cdot,s)\|_{L^{p_0}(\Omega)} \, ds \\ &\leq c_7 \int_0^t \Big(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{p_0} - \frac{1}{\theta})}\Big) e^{-\lambda_1(t-s)} \times \\ &\qquad \times c_3 \varepsilon \Big(1 + s^{-\frac{n}{2q_0}}\Big) e^{-\lambda_1 s} \cdot c_6 \varepsilon e^{-(\lambda_1+1)s} \, ds \end{aligned}$$

for all $t \in (0,T)$. Since $2\lambda_1 + 1 > \lambda_1$ and $\frac{1}{2} + \frac{n}{2}(\frac{1}{p_0} - \frac{1}{\theta}) \leq \frac{1}{2} + \frac{n}{2p_0} < 1$ because of $p_0 > n$, we may apply Lemma 1.2 to see that with some $c_8 > 0$,

$$\begin{aligned} \|u(\cdot,t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)} &\leq c_8 \varepsilon^2 \Big(1 + t^{\min\{0, 1 - \frac{1}{2} - \frac{n}{2}(\frac{1}{p_0} - \frac{1}{\theta}) - \frac{n}{2q_0}\}}\Big) e^{-\lambda_1 t} \\ &= c_8 \varepsilon^2 \Big(1 + t^{\min\{0, -\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta}) + \frac{1}{2} - \frac{n}{2p_0}\}}\Big) e^{-\lambda_1 t} \\ &\leq 2c_8 \varepsilon^2 \Big(1 + t^{-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta})}\Big) e^{-\lambda_1 t} \qquad \text{for all } t \in (0,T) \end{aligned} \tag{2.11}$$

holds irrespective of the sign of $-\frac{n}{2}(\frac{1}{q_0} - \frac{1}{\theta}) + \frac{1}{2} - \frac{n}{2p_0}$, where we have used that $\frac{1}{2} - \frac{n}{2p_0}$ is nonnegative. Therefore, if $\varepsilon_0 < \frac{1}{2c_8}$ then the continuity of $t \mapsto \|u(\cdot,t) - e^{t\Delta} u_0\|_{L^\theta(\Omega)}$ excludes the possibility that T be finite. As a consequence, (u,v) exists globally and u satisfies (2.2) in view of (2.11). Moreover, from Lemma 1.3 ii), formula (2.11) (applied to $\theta = \infty$) and Lemma 1.2 we conclude that

$$\Big\|\nabla\Big(v(\cdot,t) - e^{t(\Delta-1)} v_0 - \int_0^t e^{(t-s)(\Delta-1)} e^{s\Delta} u_0 ds\Big)\Big\|_{L^p(\Omega)}$$

$$
\begin{aligned}
&= \Big\| \int_0^t \nabla e^{(t-s)(\Delta-1)} \Big(u(\cdot,s) - e^{s\Delta} u_0 \Big) ds \Big\|_{L^p(\Omega)} \\
&\le c_9 \varepsilon^2 \int_0^t \Big(1 + (t-s)^{-\frac{1}{2}} \Big) e^{-(\lambda_1+1)(t-s)} \Big(1 + s^{-\frac{n}{2q_0}} \Big) e^{-\lambda_1 s}\, ds \\
&\le c_{10} \varepsilon^2 \Big(1 + t^{\min\{0, 1-\frac{1}{2}-\frac{n}{2q_0}\}} \Big) e^{-\lambda_1 t}
\end{aligned}
$$

for all $t > 0$ and certain positive constants c_9 and c_{10}. This proves (2.3).
Now (2.4) is an obvious consequence of this and the identity for the total mass of the chemical, $\int_\Omega v(\cdot,t) = m + (\int_\Omega v_0 - m)e^{-t}$, that can easily be checked upon integrating the second equation in (0.2). ////

3 Aggregation

In parabolic-elliptic simplifications of (0.2), a rather striking method of detecting blow-up in space dimension $n \ge 2$ is based on deriving a favorable ordinary differential inequality for the n-th moment $\int_\Omega |x|^n u(x,t)dx$ ([N2], [N1]). In view of the more complex coupling in the full parabolic-parabolic system (0.2), however, it seems that an adaptation of this approach to the present situation is linked to a number of technical obstacles, and we are not aware of any work in the literature in which blow-up in (0.2) is proved by means of controlling moments.

Proceeding alternatively, we will essentially build our blow-up argument on the use of the functional

$$
F(u,v) := \frac{1}{2} \int_\Omega |\nabla v|^2 + \frac{1}{2} \int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u,
$$

which is known to play the role of an energy in that it satisfies

$$
\int_0^t \int_\Omega v_t^2 + \int_0^t \int_\Omega u \Big| \nabla \ln u - \nabla v \Big|^2 + F(u(\cdot,t), v(\cdot,t)) = F(u_0, v_0) \qquad \text{for } t \in (0,T) \tag{3.1}
$$

whenever (u,v) is a classical solution of (0.2) in $\Omega \times (0,T)$ ([NSY, Lemma 3.3]). This dissipated quantity has frequently been utilized (see also [GZ] or [CC], for instance) in order to *exclude* unboundedness; here we shall employ it to *enforce* blow-up. The strategy we shall pursue is roughly the same as already performed in the study of two-dimensional blow-up phenomena for initial data with large mass (see [SeS2], for instance). The plan is to find a lower bound for the energy of all conceivable steady states and then prove that there exist solutions, having energy below this bound, that cannot be bounded since otherwise they should approach some steady state with a forbidden energy. This approach is quite familiar in the context of scalar parabolic equations (see [L] for a survey), but in the present case of (0.2) it seems that establishing a lower bound for steady-state energies is by far not trivial. We shall therefore restrict ourselves to the setting of radial symmetry, which will essentially be used in the key Lemma 3.4.

We start by stating the following fact which can be proved in quite the same way as its two-dimensional analogue (see [SeS2, Section 2] or also [HWi, Lemma 6.1]); for the sake of completeness, we sketch a possible proof here.

Lemma 3.1 *Suppose that (u,v) is a global bounded solution of (0.2). Then there exist a sequence of times $t_k \to \infty$ and nonnegative functions $u_\infty, v_\infty \in C^2(\bar\Omega)$ such that $u(\cdot,t_k) \to u_\infty$ and $v(\cdot,t_k) \to v_\infty$ in $C^2(\bar\Omega)$ and*

$$\begin{cases} -\Delta v_\infty + v_\infty = u_\infty, & x \in \Omega, \\ \nabla(\ln u_\infty - v_\infty) = 0, & x \in \Omega, \\ \frac{\partial v_\infty}{\partial \nu} = 0, & x \in \partial\Omega, \\ \int_\Omega u_\infty = \int_\Omega v_\infty = m := \int_\Omega u_0 \end{cases} \tag{3.2}$$

as well as

$$F(u_\infty, v_\infty) \le F(u_0, v_0). \tag{3.3}$$

Proof. From the boundedness of (u,v) and parabolic Schauder theory ([LSU]) it follows that both $(u(\cdot,t))_{t>1}$ and $(v(\cdot,t))_{t>1}$ are relatively compact in $C^2(\bar\Omega)$, and that $F(u,v)$ is bounded for $t > 1$. Hence, along a suitable sequence of times $t_k \to \infty$ we obtain $u(\cdot,t_k) \to u_\infty$ and $v(\cdot,t_k) \to v_\infty$ in $C^2(\bar\Omega)$ for some nonnegative $u_\infty, v_\infty \in C^2(\bar\Omega)$ and thus also $F(u(\cdot,t_k), v(\cdot,t_k)) \to F(u_\infty, v_\infty)$ which entails (3.3). In view of (3.1), both integrals $\int_1^\infty \int_\Omega v_t^2$ and $\int_1^\infty \int_\Omega u|\nabla(\ln u - v)|^2$ are finite, whence extracting a subsequence if necessary we may also assume that $\int_\Omega v_t^2(x,t_k)dx \to 0$ and $\int_\Omega u(x,t_k)|\nabla(\ln u(x,t_k) - v(x,t_k))|^2 dx \to 0$ as $k \to \infty$. Here, the former relation yields $-\Delta v_\infty + v_\infty = u_\infty$ in Ω upon evaluating the second equation in (0.2) at $t = t_k$ and letting $k \to \infty$, whereas the second immediately gives

$$u_\infty |\nabla(\ln u_\infty - v_\infty)|^2 = 0 \qquad \text{in } \bar\Omega. \tag{3.4}$$

The last two lines in (3.2) are obvious due to the mass conservation property $\int_\Omega u(x,t)dx \equiv m$ and the first equation in (3.2). Accordingly, we must have $u_\infty \not\equiv 0$, so that the set $\{u_\infty > 0\}$ possesses at least one connected component $\mathcal{C}$ which we claim to coincide with $\bar\Omega$. In fact, if there were $x_0 \in \partial\mathcal{C} \setminus \mathcal{C}$ then there would exist $x_j \in \mathcal{C}$ such that $x_j \to x_0$, By (3.4), we have $\nabla(\ln u_\infty - v_\infty) \equiv 0$ in $\mathcal{C}$ and thus, since $\mathcal{C}$ is connected and relatively open in $\bar\Omega$, $\ln u_\infty - v_\infty \equiv L$ in $\mathcal{C}$ for some constant $L \in \mathbb{R}$. But then

$$u_\infty(x_j) = e^{v_\infty(x_j)+L} \to e^{v_\infty(x_0)+L}$$

as $j \to \infty$, which is absurd since $u_\infty(x_0)$ was assumed to be zero. Having thereby shown that $\mathcal{C} = \bar\Omega$, we have established the second identity in (3.2) and thus completed the proof. ////

We next assert that there exist initial data with arbitrary mass $\int_\Omega u_0$ but having energy below any prescribed bound. In fact, it turns out that it is even possible to bound not only the $L^1(\Omega)$ norm $\int_\Omega u_0$ but also the $L^q(\Omega)$ norm of u_0 for any $q < \frac{2n}{n+2}$.

Lemma 3.2 *Assume $n \ge 3$, and let $q \in (1, \frac{2n}{n+2})$ be given. Then there exists $C = C(\Omega,q) > 0$ such that for all $m > 0$ one can find $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\bar\Omega)$ and $(v_k)_{k\in\mathbb{N}} \subset C^\infty(\bar\Omega)$ with the properties*

$$\int_\Omega u_k = m \qquad \text{and} \qquad \|u_k\|_{L^q(\Omega)} \le Cm \quad \text{for all } k \in \mathbb{N}, \tag{3.5}$$

but which satisfy

$$F(u_k, v_k) \to -\infty \qquad \text{as } k \to \infty. \tag{3.6}$$

PROOF. Without loss of generality we may assume that $0 \in \Omega$, and that $B_{R_0}(0) \subset \Omega \subset B_R(0)$ for some positive R_0 and R. Since $n \ge 3$ and $q < \frac{2n}{n+2}$, we can pick a number $\alpha \in (n - \frac{n}{q}, \frac{n-2}{2})$ and, for $k \in \mathbb{N}$, define

$$u_k(x) := A_k(|x|^2 + \frac{1}{k})^{\frac{-n+\alpha}{2}} \qquad \text{and} \qquad v_k(x) := (|x|^2 + \frac{1}{k})^{-\frac{\alpha}{2}} \qquad \text{for } x \in \bar{\Omega},$$

where

$$A_k := \frac{m}{\int_\Omega (|x|^2 + \frac{1}{k})^{\frac{-n+\alpha}{2}}}.$$

Observe that since $\alpha > 0$, A_k decreases to the positive number $m/(\int_\Omega |x|^{-n+\alpha} dx)$ as $k \to \infty$, and that our choice of A_k asserts that $\int_\Omega u_k = m$ for all k. Moreover,

$$\begin{aligned} \int_\Omega u_k^q &\le \omega_n A_k^q \cdot \int_0^R r^{n-1}(r^2 + \frac{1}{k})^{\frac{-n+\alpha}{2}q} dr \\ &\le \omega_n A_k^q \cdot \int_0^R r^{n-1+(-n+\alpha)q} dr \qquad \text{for all } k \in \mathbb{N}, \end{aligned}$$

where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. Since $\alpha > n - \frac{n}{q}$, this shows that $\|u_k\|_{L^q(\Omega)} \le c_1 m$ and, as a consequence, also

$$\int_\Omega u_k \ln u_k \le c_2 \tag{3.7}$$

hold for all $k \in \mathbb{N}$ with positive constants c_1 and c_2. As to v_k, we estimate

$$\begin{aligned} \int_\Omega |\nabla v_k|^2 &\le \omega_n \cdot \int_0^R r^{n-1} \cdot \Big[\alpha r(r^2 + \frac{1}{k})^{\frac{-\alpha-2}{2}}\Big]^2 dr \\ &\le \omega_n \alpha^2 \cdot \int_0^R r^{n-2\alpha-3} dr. \end{aligned}$$

Using the fact that $\alpha < \frac{n-2}{2}$ and estimating $\int_\Omega v_k^2$ similarly, we conclude that

$$\frac{1}{2}\int_\Omega |\nabla v_k|^2 + \frac{1}{2}\int_\Omega v_k^2 \le c_3 \tag{3.8}$$

is valid for some $c_3 > 0$ and all $k \in \mathbb{N}$.
But

$$\int_\Omega u_k v_k \ge \int_{B_{R_0}(0)} u_k v_k = \omega_n A_k \cdot \int_0^{R_0} r^{n-1}(r^2 + \frac{1}{k})^{-\frac{n}{2}} dr.$$

Since $r^{n-1}(r^2 + \frac{1}{k})^{-\frac{n}{2}}$ increases to $\frac{1}{r}$ as $k \to \infty$ for each $r > 0$, we see that $\int_\Omega u_k v_k \to \infty$ and hence, in view of (3.7) and (3.8), that $F(u_k, v_k) \to -\infty$ as $k \to \infty$. ////

From now on, we restrict ourselves to the case when Ω is a ball in $\mathbb{R}^n$, which allows us to derive some common properties of all conceivable radially symmetric solutions of (3.2), our particular

interest being to find a lower bound for their energies. As a preparation for this, we first assert that the second component v of all such solutions satisfies a universal L^∞ estimate away from the center of the ball.
Throughout the sequel, we shall write $w = w(r)$ when referring to functions w depending on the variable $r = |x|$ only. Also, we abbreviate $B_R := B_R(0)$ for $R > 0$.

Lemma 3.3 *Let $n \geq 3$ and suppose that $\Omega = B_R$ for some $R > 0$. Then for all $m > 0$ and $R_0 \in (0, R)$ there exists $C = C(R, R_0) > 0$ such that for all radially symmetric solutions (u, v) of (3.2) we have*

$$v(r) \leq Cm \qquad \text{for all } r \in [R_0, R]. \tag{3.9}$$

PROOF. Since $\|u\|_{L^1(B_R)} = m$, for each $p \in [1, \frac{n}{n-1})$ elliptic regularity theory applied to the Neumann problem for $-\Delta v + v = u$ ([BS]) provides a constant $c_1 = c_1(R, p)$ such that

$$\|\nabla v\|_{L^p(\Omega)} \leq c_1 m. \tag{3.10}$$

As also $\int_{B_R} v = m$, we trivially have $v(r_0) \leq \frac{m}{|B_R \setminus B_{R_0}|}$ for some $r_0 \in [R_0, R]$ possibly depending on v. For arbitrary $r \in [R_0, R]$ we now estimate

$$\begin{aligned} v(r) &= v(r_0) + \int_{r_0}^{r} v_r(\rho) d\rho \\ &\leq \frac{m}{|B_R \setminus B_{R_0}|} + R_0^{1-n} \cdot \int_{R_0}^{R} \rho^{n-1} |v_r(\rho)| d\rho, \end{aligned}$$

which in conjunction with (3.10) (applied to $p = 1$) yields (3.9). ////

We now pass to the core of our blow-up argument. It consists of finding a lower bound for $F(u, v)$ for all radially symmetric solutions of (3.2). The method used here is based on the use of the Pohozaev multiplier $(x \cdot \nabla v)$ in the elliptic equation $-\Delta v + v = u$. Of course, this has to be combined with the second identity in (3.2) in order to cope with the term involving u. The main advantage of the radial setting here is that the Neumann condition is sufficient to ensure that no boundary terms involving ∇v appear.

Lemma 3.4 *Let $n \geq 3$ and $\Omega = B_R$ for some $R > 0$. Then for all $m > 0$ there exists $C_F = C_F(R, m) > 0$ such that*

$$F(u, v) \geq -C_F \tag{3.11}$$

holds for all radially symmetric solutions (u, v) of (3.2).

PROOF. First, in order to cope with the mixed term $\int_{B_R} uv$ contributing to $F(u, v)$, we test the first equation $-\Delta v + v = u$ in (3.2) by v to obtain $\int_{B_R} uv = \int_{B_R} |\nabla v|^2 + \int_{B_R} v^2$ and hence

$$\begin{aligned} F(u, v) &= -\frac{1}{2}\Big(\int_{B_R} |\nabla v|^2 + \int_{B_R} v^2\Big) + \int_{B_R} u \ln u \\ &\geq -\frac{1}{2}\Big(\int_{B_R} |\nabla v|^2 + \int_{B_R} v^2\Big) - \frac{|B_R|}{e}, \end{aligned} \tag{3.12}$$

so that we only need to concentrate on finding an upper bound for $\int_{B_R} |\nabla v|^2 + \int_{B_R} v^2$. To achieve this, we strongly rely on the radial symmetry, which allows to rewrite the first equation in (3.2) in the form $r^{1-n}(r^{n-1}v_r)_r = v - u$. Multiplying this by $r^{2n-2}v_r$, we obtain

$$\frac{1}{2}\big((r^{n-1}v_r)^2\big)_r = \frac{1}{2}r^{2n-2}(v^2)_r - r^{2n-2}uv_r.$$

Another multiplication by r^{2-n} and an integration by parts over $(0, R)$ yields in view of the boundary condition $v_r(R) = 0$ that

$$\begin{aligned}\frac{n-2}{2}\int_0^R r^{n-1}v_r^2(r)dr &\equiv \frac{n-2}{2}\int_0^R r^{1-n}(r^{n-1}v_r)^2(r)dr \\ &= \frac{1}{2}R^n v^2(R) - \frac{n}{2}\int_0^R r^{n-1}v^2(r)dr - \int_0^R r^n u(r)v_r(r)dr.\end{aligned}$$

Now the second equation in (3.2) provides the identity $\frac{u_r}{u} - v_r = 0$, whereby the last term on the right becomes

$$-\int_0^R r^n u(r)v_r(r)dr = -\int_0^R r^n u_r(r)dr = -R^n u(R) + n\int_0^R r^{n-1}u(r)dr \le n\int_0^R r^{n-1}u(r)dr.$$

Recalling Lemma 3.3 we find

$$\begin{aligned}\frac{n-2}{2}\int_{B_R}|\nabla v|^2 &\le c_1 - \frac{n}{2}\int_{B_R} v^2 + n\int_{B_R} u \\ &= c_1 - \frac{n}{2}\int_{B_R} v^2 + nm\end{aligned}$$

for some $c_1 > 0$ and hence

$$\frac{1}{2}\int_{B_R}|\nabla v|^2 + \frac{1}{2}\int_{B_R} v^2 \le \frac{c_1 + nm}{n-2} - \frac{1}{n-2}\int_{B_R} v^2 \le \frac{c_1+nm}{n-2}. \tag{3.13}$$

Together with (3.12), this yields the desired estimate (3.11) with $C_F(R, m) = \frac{c_1+nm}{n-2} + \frac{|B_R|}{e}$. ////

We can now collect all ingredients to prove the existence of unbounded solutions in the radial setting.

Theorem 3.5 *Let Ω be a ball in $\mathbb{R}^n$ for some $n \ge 3$.*
i) For all $m > 0$ there exist initial data $(u_0, v_0) \in (C^\infty(\bar{\Omega}))^2$ satisfying $\int_\Omega u_0 = m$ such that the corresponding solution $(u.v)$ of (0.2) blows up either in finite or infinite time.
ii) Given any $q \in (1, \frac{2n}{n+2})$ and $\varepsilon > 0$, (0.2) possesses solutions blowing up, either in finite or infinite time, emanating from initial data $(u_0, v_0) \in (C^\infty(\bar{\Omega}))^2$ fulfilling $\|u_0\|_{L^q(\Omega)} < \varepsilon$.

Proof. i) Given $m > 0$, assuming $\Omega = B_R$ we let $C_F = C_F(R, m)$ be as in Lemma 3.4. Then Lemma 3.2 provides a smooth pair (u_0, v_0) of radially symmetric functions satisfying $\int_\Omega u_0 = m$ and $F(u_0, v_0) < -C_F$. Evidently, the corresponding solution (u, v) of (0.2) will inherit the radial symmetry of the data. Thus, if (u, v) were global in time and bounded in $L^\infty(\Omega \times (0,\infty))$ then Lemma 3.1 would ensure the existence of a radially symmetric solution (u_∞, v_∞) of (3.2) such that $F(u_\infty, v_\infty) \le F(u_0, v_0) < C_F$, contradicting the outcome of Lemma 3.4.

ii) This part can be proved similarly: For fixed $\varepsilon > 0$, we pick any $m < \frac{\varepsilon}{C}$, where $C = C(\Omega, q)$ is as given by Lemma 3.2. Then this lemma asserts that the above choice of (u_0, v_0) can be made in such a way that $\|u_0\|_{L^q(\Omega)} \le Cm < \varepsilon$. ////

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