Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect

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Abstract

We consider the elliptic-parabolic PDE system

$$\begin{cases} u_t &= \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\psi(u)\nabla v)\ , \quad x\in\Omega\ ,\ t>0\ ,\\ 0 &= \Delta v - M + u\ , \qquad\qquad\qquad\qquad x\in\Omega\ ,\ t>0\ ,\end{cases}$$

with nonnegative initial data u_0 having mean value M, under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega\subset\mathbb{R}^n$. The nonlinearities ϕ and ψ are supposed to generalize the prototypes

$$\phi(u) = (u+1)^{-p}\ , \qquad \psi(u) = u(u+1)^{q-1}$$

with $p\geq 0$ and $q\in\mathbb{R}$. Problems of this type arise as simplified models in the theoretical description of chemotaxis phenomena under the influence of the *volume-filling effect* as introduced by Painter and Hillen (*Can. Appl. Math. Q.* **10**, 501-543 (2002)).
It is proved that if $p+q<\frac{2}{n}$ then all solutions are global in time and bounded, whereas if $p+q>\frac{2}{n}$, $q>0$, and Ω is a ball then there exist solutions that become unbounded in finite time. The former result is consistent with the aggregation-inhibiting effect of the volume-filling mechanism; the latter, however, is shown to imply that if the space dimension is at least three then chemotactic collapse may occur even despite the presence of some nonlinearities that supposedly model a volume-filling effect in the sense of Painter and Hillen.

Key words: chemotaxis, global existence, boundedness, blow-up
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Introduction

The theoretical description of patterning phenomena in living organisms has become an increasingly active and important field of mathematical biology through the last decades. One particular principle that is known to govern such a process of cellular self-organization is the mechanism of *chemotaxis*, by which one means the movement of cells that is directed in the sense that the migrating cells preferably follow the concentration gradient of a certain chemical signal substance (cf. [HP] for a recent survey providing numerous biological examples).
According to the emergence of spatial structures, Keller and Segel ([KS]) introduced a model where the distribution of cells is continuous in space and thus determined by a function $u=u(x,t)$ that satisfies a PDE

$$u_t = \nabla\cdot(\phi(u)\nabla u) - \nabla\cdot(\chi(u,v)\nabla v)\ , \tag{0.1}$$

where $v=v(x,t)$ stands for the concentration of the chemical. Besides capturing undirected, purely diffusive behavior measured by the (self-)diffusivity $\phi(u)$, (0.1) also accounts for cellular movement towards (or away from) higher chemical concentrations by incorporating the mechanism of *cross-diffusion* which enters through any nontrivial choice of the so-called chemotactic sensitivity $\chi(u,v)$. In the last 15 years, a large variety of particular problems based on (0.1) has been studied, and it has been shown that the simple element of cross-diffusion is indeed able to describe the experimentally observed phenomenon of cell aggregation. Here the most commonly underlying, albeit rather extreme, mathematical translation of the statement that (0.1) models aggregation consists of requiring that (0.1) possesses solutions which become unbounded somewhere in space either in finite or infinite time. This especially seems to be appropriate when (0.1) enjoys the mass conservation property

$$\int_\Omega u(x,t)\mathrm{d}x \equiv \text{const.}\ , \qquad t>0\ , \tag{0.2}$$

which is, for instance, guaranteed if (0.1) is posed in the whole physical space, or in a bounded region with no-flux boundary conditions; then, namely, the mass of unbounded solutions should essentially concentrate near points where u is large.

In the original Keller-Segel model, (0.1) is supplemented by a parabolic equation for the unknown v that reflects that the chemical diffuses and degrades, and that it is produced by the cells themselves; a dimensionless prototype of such a diffusion equation is

$$v_t = \Delta v - v + u. \tag{0.3}$$

However, the analysis of the full parabolic-parabolic system (0.1), (0.3) turned out to be quite involved: Even the simplest reasonable choices $\phi(u) \equiv 1$ and $\chi(u,v) = u$, leading to the 'classical' Keller-Segel model, bring about severe difficulties. Although it is known that the corresponding Neumann boundary value problem in bounded domains in $\mathbb{R}^n$ has only bounded solutions if $n = 1$ ([OY]), or $n = 2$ and small total mass of cells ([NSY], [GZ]), and that aggregation may occur for $n = 2$ and large mass ([HWa]), there is only one result available in the literature that asserts (radially symmetric) aggregation in *finite time* when $n = 2$ ([HV]), and the latter actually refers to one single unbounded solution only, leaving open the possibility that finite-time aggregation might be a non-generic, unstable phenomenon.

On the basis of the fact that in many relevant applications the chemical diffuses much faster than the cells move, in [JL] the stationary equation

$$0 = \Delta v - M + u$$

is derived as a biologically still meaningful asymptotic limit, where M denotes the spatial mean of the cell density. Upon this simplification and the further restriction to bounded domains and to chemotactic sensitivities of the form $\chi(u,v) = \psi(u)$, one is led to studying the elliptic-parabolic initial-boundary value problem

$$\left\{ \begin{array}{ll} u_t = \nabla\cdot\big(\phi(u)\nabla u\big) - \nabla\cdot\big(\psi(u)\nabla v\big)\ , & x\in\Omega,\ t>0\ , \\ 0 = \Delta v - M + u\ , & x\in\Omega\ ,\ t>0\ , \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0\ , & x\in\partial\Omega\ ,\ t>0\ , \\ u(x,0) = u_0(x)\ , & x\in\Omega\ , \\ \int_\Omega v(x,t)\mathrm{d}x = 0\ , & t>0\ , \end{array} \right. \tag{0.4}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, with $\frac{\partial}{\partial\nu}$ denoting outward normal derivatives on $\partial\Omega$, and $M>0$. Representing a cell density, the initial function u_0 is assumed to be nonnegative and to satisfy $\frac{1}{|\Omega|}\int_\Omega u_0 = M$. Unlike in the previous lines, here and throughout the sequel v denotes the deviation of the signal concentration from its spatial mean, rather than this concentration itself; for convenience, however, we omit a relabelling.

Several functional forms of ϕ and ψ have been studied as ingredients to (0.4) or closely related variants thereof in the literature. For instance, it is known that the classical choice $\phi(u)\equiv 1$ and $\psi(u) = u$ in the case $n = 2$ leads to global boundedness if $\int_\Omega u_0$ is small but admits blow-up solutions if $\int_\Omega u_0$ is large ([JL]). Whereas in space dimension $n = 1$, all solutions are global and

bounded, blow-up occurs for some initial data if $n \geq 3$ ([CW]).
As to nonlinear diffusion, the methods in [K] apply to $n = 2$, $\phi(u) = D(u+1)^{-p}$, $D > 0$, $p \in \mathbb{R}$, and $\psi(u) = u$ and show that if $p < -1$ then all solutions are global and bounded. In [CC], this result was carried over to the case of any $p < 0$ and even that of $p = 0$ under the additional assumption that D is sufficiently large. Recently, a further generalization was obtained in [CW], where the same problem was investigated in arbitrary space dimension $n \geq 1$. The boundedness result was extended there to any $p < \frac{2}{n} - 1$, whereas for each $p > \frac{2}{n} - 1$ radially symmetric solutions were constructed that blow up in finite time.
A similar critical relationship between power-type diffusion and cross-diffusion in (0.1) appears in the case of positive powers in the corresponding Cauchy problem in $\Omega = \mathbb{R}^n$: Namely, it is known that when $\phi(u) = u^{-p}$ and $\psi(u) = u^q$ with some $p \leq 0$ and $q > \frac{1}{2}$ the condition $q < p + \frac{2}{n}$ excludes blow-up, whereas if $q > p + \frac{2}{n}$ then blow-up solutions exist (cf. [SK], [LS] and the references therein).
In the present study we concentrate on nonlinearities ϕ and ψ that vanish asymptotically as $u \to \infty$. The motivation for this stems from [PH], where via a random walk approach the crucial Keller-Segel equation (0.1) was re-invented in such a way that it incorporates the fact that the ability of cells to move becomes small when the cells are densely packed. The precise derivation suggests to choose ϕ and ψ according to

$$\phi(u) = Q(u) - uQ'(u) \qquad \text{and} \qquad \psi(u) = \chi u Q(u) \tag{0.5}$$

with some constant $\chi > 0$, where $Q(u)$ is proportional to the probability that a cell, currently at some position with density u, will move away from this position. In [HP1], it was proved that if Q is chosen according to $Q(u) = (A - u)_+$ for some $A > 0$ then no blow-up occurs in the parabolic-parabolic system (0.1), (0.3) when posed on a compact Riemannian manifold without boundary. It appears to be fairly open, however, if less restrictive decay conditions on $Q(u)$ as $u \to \infty$ will also prevent blow-up (cf. [CC] for a discussion of some mathematical difficulties).
We shall restrict our analysis on the situation when $Q(u)$ decays algebraically; more generally, we will consider the case where ϕ and ψ, independently and not necessarily linked through (0.5), asymptotically behave according to

$$\phi(u) \simeq u^{-p} , \qquad \psi(u) \simeq u^q , \qquad u \simeq \infty , \tag{0.6}$$

with some $p \geq 0$ and $q \in \mathbb{R}$. Within this context, we shall obtain the following results.

- If $p + q < \frac{2}{n}$ then all solutions of (0.4) are global and bounded (Corollary 2.3).
- If $p+q > \frac{2}{n}$ and Ω is a ball then for any initial data having their mass concentrated sufficiently close to the center of Ω, the corresponding solution will undergo a blow-up in finite time, provided that $q > 0$ (Theorem 3.5).

For example, if the probability in (0.5) has the form $Q(u) = (u+1)^{-\alpha}$ with some $\alpha \geq 0$ (as suggested in [CC], for instance) then ϕ and ψ satisfy (0.6) with $p = \alpha$ and $q = 1 - \alpha$ and hence $p + q = 1$. Consequently, *any* rate of algebraic decay of $Q(u)$ completely excludes blow-up in space dimension one, whereas in the case $n \geq 3$ none of these Q is sufficient to prevent aggregation. On the one hand, this generalizes the results in [CC] that assert the same but only in space dimension $n = 2$, and on the other it improves the outcome of [CM-R], where absence of any collapse was proved when $n = 3$ under the assumption $\alpha > 2$.

Note that our results, in particular those concerning blow-up, apply to any space dimension $n \geq 1$. In particular, if $\phi(u) \simeq u^{-p}$ and $\psi(u) \simeq u^{2-p+\varepsilon}$ with some $p \geq 0$ and $\varepsilon > 0$ satisfying $2 - p + \varepsilon > 0$ then blow-up occurs in the one-dimensional version of (0.4), even though elliptic theory and (0.2) imply that ∇v is uniformly bounded.

1 Local existence and uniqueness

The following local existence and uniqueness result is rather standard; a similar reasoning can be found in [CW], for instance. Since we could not find a precise reference in the literature that exactly matches to our situation, however, we include a short proof for the sake of completeness.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and ϕ and ψ belong to $C^{1+\theta}_{\rm loc}([0,\infty))$ for some $\theta > 0$ and satisfy $\phi > 0$ and $\psi \ge 0$ in $[0,\infty)$. Furthermore, assume that the nonnegative function u_0 belongs to $C^\alpha(\bar\Omega)$ for some $\alpha > 0$, and that $\frac{1}{|\Omega|}\int_\Omega u_0 = M$. Then there exists a unique classical solution (u,v) of (0.4) that can be extended up to its maximal existence time $T_{\max} \in (0,\infty]$. Here,*

$$\textit{either} \quad T_{\max} = \infty \qquad \textit{or} \qquad \lim_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty \ . \tag{1.1}$$

Proof. The proof is carried out in three steps.
Step 1. Let us first assert uniqueness of classical solutions.
To this end, assume that both (u_1,v_1) and (u_2,v_2) solve (0.4) in $\Omega\times(0,T)$ with some $T > 0$. Since $-\Delta(v_1 - v_2) = u_1 - u_2$ and $(u_1 - u_2)_t$ belongs to $C^0(\bar\Omega \times (0,T))$ by standard parabolic regularity theory, elliptic estimates show that $(v_1 + v_2)_t$ exists and belongs to $C^0((0,T); C^{1+\beta}(\bar\Omega))$ for all $\beta \in (0,1)$. It follows that $|\nabla(v_1 - v_2)|^2$ has a continuous time derivative in $\bar\Omega \times (0,T)$ with

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla(v_1-v_2)|^2 &= -\int_\Omega \Delta(v_1-v_2)_t \cdot (v_1-v_2) \\
&= \int_\Omega (u_1-u_2)_t \cdot (v_1-v_2) \\
&= -\int_\Omega \nabla(\Phi(u_1)-\Phi(u_2))\cdot\nabla(v_1-v_2) \\
&\quad + \int_\Omega \Big(\psi(u_1)\nabla v_1 - \psi(u_2)\nabla v_2\Big)\cdot\nabla(v_1-v_2)
\end{aligned} \tag{1.2}$$

for all $t \in (0,T)$, with $\Phi(s) := \int_0^s \phi(\sigma)\mathrm{d}\sigma$. Using the mean value theorem, for all $T_0 < T$ we obtain

$$\begin{aligned}
-\int_\Omega \nabla(\Phi(u_1)-\Phi(u_2))\cdot\nabla(v_1-v_2) &= \int_\Omega (\Phi(u_1)-\Phi(u_2))\cdot\Delta(v_1-v_2) \\
&= -\int_\Omega (\Phi(u_1)-\Phi(u_2))\cdot(u_1-u_2) \\
&\le -c_1\int_\Omega |u_1-u_2|^2 \qquad \text{for all } t\in(0,T_0)\ ,
\end{aligned} \tag{1.3}$$

where abbreviating $K \equiv K(T_0) := \max\{\|u_1\|_{L^\infty(\Omega\times(0,T_0))}, \|u_2\|_{L^\infty(\Omega\times(0,T_0))}\}$ we have set $c_1 := \min_{s\in[0,K]} \phi(s) > 0$. Next, by the Cauchy-Schwarz inequality,

$$\int_\Omega \Big(\psi(u_1)\nabla v_1 - \psi(u_2)\nabla v_2\Big)\cdot\nabla(v_1-v_2) \le \Big(\int_\Omega \big|\psi(u_1)\nabla v_1 - \psi(u_2)\nabla v_2\big|^2\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\nabla(v_1-v_2)|^2\Big)^{\frac{1}{2}}.$$

Here, in view of the Lipschitz regularity of ψ on $[0,K(T_0)]$ and elliptic theory,

$$|\psi(u_1)-\psi(u_2)| \le c_2|u_1-u_2|, \qquad \psi(u_2)\le c_3 \qquad \text{and} \qquad |\nabla v_1| \le c_4 \qquad \text{in } \Omega\times(0,T_0)$$

hold with certain positive constants c_2, c_3 and c_4 depending on $K(T_0)$ only, so that

$$\begin{aligned}
\int_\Omega \big|\psi(u_1)\nabla v_1 - \psi(u_2)\nabla v_2\big|^2 &\le 2\int_\Omega |\psi(u_1)-\psi(u_2)|^2|\nabla v_1|^2 + 2\int_\Omega \psi^2(u_2)\cdot|\nabla(v_1-v_2)|^2 \\
&\le 2c_2^2c_4^2\int_\Omega |u_1-u_2|^2 + 2c_3^2\int_\Omega |\nabla(v_1-v_2)|^2 \text{ for all } t\in(0,T_0)\ .
\end{aligned}$$

Hence, Young's inequality entails

$$
\begin{aligned}
\int_\Omega \Big(\psi(u_1)\nabla v_1 - \psi(u_2)\nabla v_2\Big)\cdot\nabla(v_1-v_2) &\le \sqrt{2}c_2c_4\Big(\int_\Omega |u_1-u_2|^2\Big)^{\frac12}\cdot\Big(\int_\Omega |\nabla(v_1-v_2)|^2\Big)^{\frac12} \\
&\quad +\sqrt{2}c_3\int_\Omega |\nabla(v_1-v_2)|^2 \\
&\le \frac{c_1}{2}\int_\Omega |u_1-u_2|^2 \\
&\quad +\Big(\frac{c_2^2c_4^2}{c_1}+\sqrt{2}c_3\Big)\cdot\int_\Omega |\nabla(v_1-v_2)|^2 \qquad (1.4)
\end{aligned}
$$

for all $t\in(0,T_0)$. Inserting (1.3) and (1.4) into (1.2) and applying Grönwall's lemma, we end up with the inequality

$$
\frac12\int_\Omega |\nabla(v_1-v_2)|^2(\cdot,t)+\frac{c_1}{2}\int_\tau^t\int_\Omega |u_1-u_2|^2 \le \Big(\frac12\int_\Omega|\nabla(v_1-v_2)|^2(\cdot,\tau)\Big)\cdot e^{c_5(t-\tau)},\ 0<\tau<t<T_0,
$$

with some $c_5>0$ depending on T_0 only. Since $\nabla(v_1-v_2)$ is continuous in $\bar\Omega\times[0,T_0]$ by continuity of u_1 and u_2 and elliptic estimates, we may let $\tau\searrow 0$ and then $T_0\nearrow T$ to conclude that $(u_1,v_1)\equiv(u_2,v_2)$ in $\Omega\times(0,T)$.

Step 2. We next claim that for all $K>\frac{M}{2}$ one can find a positive number $T(K)$ such that whenever $\bar u_0$ belongs to $C^{\bar\alpha}(\bar\Omega)$ with some $\bar\alpha>0$ and satisfies $\frac{1}{|\Omega|}\int_\Omega \bar u_0=M$ as well as $0\le\bar u_0\le K$ in Ω then (0.4) has a classical solution (u,v) in $\Omega\times(0,T(K))$ with initial data $u(\cdot,0)=\bar u_0$.
To see this, we let $y=y(t)$ denote the solution of the initial-value problem

$$
\left\{\begin{array}{rcll} y' &=& (2K-M)\psi(y)\ , & t\in(0,T_y)\ , \\ y(0) &=& K\ , & \end{array}\right. \qquad (1.5)
$$

defined up to its maximal existence time $T_y\le\infty$, and let

$$
T(K) := \min\Big\{1,\ \sup\{T\in(0,T_y]\ |\ y\le 2K \text{ on } (0,T)\}\Big\}\ . \qquad (1.6)
$$

We now pick some functions ϕ_K and ψ_K belonging to $C^{1+\theta}(\mathbb{R})$ and satisfying $\phi_K\equiv\phi$ and $\psi_K\equiv\psi$ on $[0,2K]$ as well as

$$
\inf_{s\in\mathbb{R}}\phi_K(s)>0\ ,\quad \sup_{s\in\mathbb{R}}\phi_K(s)<\infty\ ,\quad \psi_K(s)\ge 0\ \text{ for all } s\ge 0 \quad\text{and}\quad \sup_{s\in\mathbb{R}}\psi_K(s)<\infty\ . \qquad (1.7)
$$

We consider the fixed point problem $Fu=u$ for the operator F, where $F\bar u$ is defined to be the first component u of the solution (u,v) to the discoupled system

$$
\left\{\begin{array}{ll} u_t=\nabla\cdot(\phi_K(u)\nabla u)-\nabla\cdot(\psi_K(u)\nabla v)\ , & x\in\Omega\ ,\ t>0\ , \\ 0=\Delta v-M+u\ , & x\in\Omega\ ,\ t>0\ , \\ \frac{\partial u}{\partial\nu}=\frac{\partial v}{\partial\nu}=0\ , & x\in\partial\Omega\ ,\ t>0\ , \\ u(x,0)=\bar u_0(x)\ , & x\in\Omega\ , \\ \int_\Omega v(x,t)\mathrm{d}x=0\ , & t>0\ . \end{array}\right. \qquad (1.8)
$$

Here, $\bar u$ is taken from the closed bounded convex set

$$
S:=\Big\{\bar u\in X\ \Big|\ 0\le\bar u\le 2K \text{ in } \bar\Omega\times[0,T(K)] \text{ and } \frac{1}{|\Omega|}\int_\Omega \bar u(x,t)\mathrm{d}x=M \text{ for all } t\in[0,T(K)]\Big\}
$$

in the space $X:=C^{\beta,\frac{\beta}{2}}(\bar\Omega\times[0,T(K)])$ with $\beta\in(0,1)$ small to be fixed below. According to (1.7) and the regularity properties of ϕ_K and ψ_K, standard elliptic and parabolic theory imply that

(1.8) possesses a unique classical solution (u,v) in $\Omega \times (0,T_0)$ with some $T_0 \in (0,T(K))$. By (1.5) and the definition of $T(K)$, we have $0 \le y(t) \le 2K$ for all $t \in [0,T(K))$, so that $\hat{u}(x,t) := y(t)$ satisfies

$$\hat{u}_t - \nabla \cdot (\phi_K(\hat{u})\nabla \hat{u}) + \nabla \cdot (\psi_K(\hat{u})\nabla v) = y' - (\hat{u} - M)\psi_K(y) \ge y' - (2K-M)\psi(y) = 0$$

in $\Omega \times (0,T_0)$. Hence, the comparison principle ensures that $0 \le u(x,t) \le \hat{u}(x,t) = y(t) \le 2K$ for all $x \in \Omega$ and $t \in (0,T_0)$, which implies that (u,v) can be extended so as to exist in all of $\Omega \times (0,T(K))$. Since by elliptic theory and the Sobolev embedding theorem,

$$\|\nabla v\|_{L^\infty(\Omega\times(0,T))} \le c_6 \|\Delta v\|_{L^\infty((0,T);W^{2,n+1}(\Omega))} \le c_6 c_7 \cdot (2K+M)$$

holds with some positive c_6 and c_7, parabolic theory ([LSU, Theorem V.1.1]) asserts a uniform bound for u in $C^{\hat{\alpha},\frac{\hat{\alpha}}{2}}(\bar{\Omega}\times[0,T(K)])$ for some $\hat{\alpha} \in (0,\bar{\alpha})$. Choosing $\beta < \hat{\alpha}$ now, we see that $F(S)$ is a relatively compact subset of X. As $0 \le u \le 2K$ and, evidently, $\int_\Omega u(x,t)\mathrm{d}x \equiv \int_\Omega u_0$, we also have $F(S) \subset S$. Moreover, since the solution of (1.8) is unique, it is easy to see using elliptic and parabolic theory and compactness arguments that F is continuous with respect to the topology of X. Therefore, the Schauder fixed point theorem states that $Fu = u$ holds for some $u \in S$ which evidently solves (0.4) classically in $\Omega \times (0,T)$.

Step 3. We are now in the position to conclude the proof: Applying Steps 1 and 2 to $\bar{u}_0 := u_0$, we obtain a local solution (u,v) of (0.4) in $\Omega \times (0,T)$ with some $T > 0$. Since $u(\cdot,t)$ is Hölder continuous in $\bar{\Omega}$ for each $t \in (0,T)$ by parabolic regularity theory, and since the constant $T(K)$ in Step 2 depends on $\|\bar{u}_0\|_{L^\infty(\Omega)}$ only, it follows from a standard argument that (u,v) can be extended up to some maximal $T_{\max} \le \infty$ and that (1.1) holds. ////

2 Boundedness for $p+q<\frac{2}{n}$

In our boundedness proof below, we shall use the following version of the Gagliardo-Nirenberg inequality in which special attention is paid to the dependence of the appearing constants on the integrability powers.

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let* $2^\star := \begin{cases} \frac{2n}{n-2}, & n>2, \\ \infty, & n \le 2. \end{cases}$ *Then for all $p^\star \in (2,\infty)$ satisfying $p^\star \le 2^\star$ and any $\theta \in (0,2)$ there exists a positive constant C_{GN} such that whenever $s \in [\theta, p^\star]$,*

$$\|w\|_{L^s(\Omega)} \le C_{GN}\big(\|\nabla w\|^a_{L^2(\Omega)} \cdot \|w\|^{1-a}_{L^\theta(\Omega)} + \|w\|_{L^\theta(\Omega)}\big) \qquad \textit{for all } w \in W^{1,2}(\Omega) \tag{2.1}$$

is valid with

$$a = \frac{\frac{n}{\theta} - \frac{n}{s}}{1 - \frac{n}{2} + \frac{n}{\theta}} . \tag{2.2}$$

Proof. By the Hölder inequality,

$$\|w\|_{L^s(\Omega)} \le \|w\|^b_{L^{p^\star}(\Omega)} \|w\|^{1-b}_{L^\theta(\Omega)} ,$$

where $b = \frac{\frac{1}{\theta}-\frac{1}{s}}{\frac{1}{\theta}-\frac{1}{p^\star}}$. Since $p^\star \le 2^\star$, the standard Gagliardo-Nirenberg inequality ([F]) says that

$$\|w\|_{L^{p^\star}(\Omega)} \le \bar{C}_{GN} \|w\|^d_{W^{1,2}(\Omega)} \|w\|^{1-d}_{L^\theta(\Omega)}$$

with some $\bar{C}_{GN} > 0$ and $d = \frac{\frac{n}{\theta}-\frac{n}{p^\star}}{1-\frac{n}{2}+\frac{n}{\theta}}$. Since $bd = a$, these estimates together with the Poincaré inequality in the form

$$\|w\|_{W^{1,2}(\Omega)} \le C_P(\|\nabla w\|_{L^2(\Omega)} + \|w\|_{L^\theta(\Omega)})$$

and the fact that $(A+B)^a \le 2^a(A^a+B^a)$ for $A \ge 0$ and $B \ge 0$ imply that (2.1) holds with $C_{GN} = (2C_P)^a \cdot (\bar{C}_{GN})^b$. ////

We can now prove the following key lemma towards global boundedness of solutions under the assumption that ψ asymptotically is weak enough as compared to ϕ. The method we perform is strongly inspired by Alikakos' iteration technique ([A]), but the original idea needs some adaptation to the present situation. This is mainly due to the fact that the diffusive term may be significantly weakened when u becomes large.

Lemma 2.2 *Assume that ϕ and ψ are continuous on $[0,\infty)$ and such that ϕ is positive, ψ is nonnegative and*

$$\phi(s) \ge c_\phi s^{-p} \qquad \text{and} \qquad \psi(s) \le c_\psi s^q \qquad \text{for all } s \ge 1 \tag{2.3}$$

with constants $c_\phi > 0, c_\psi > 0$ and $p \ge 0$ and $q \in \mathbb{R}$ satisfying

$$p + q < \frac{2}{n} \ . \tag{2.4}$$

Then for all $L > 0$ there exists $C(L) > 0$ such that whenever (u,v) is a classical solution of (0.4) in $\Omega \times (0,T)$ with some $T \in (0,\infty]$ fulfilling $0 \le u_0 \le L$ in Ω, we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C(L) \qquad \text{for all } t \in (0,T) \ . \tag{2.5}$$

PROOF. Passing to $\bar{q} := \max\{q, -p\}$ if necessary, we may assume without loss of generality that (2.3) holds with $p \ge 0$ and some q satisfying both (2.4) and the lower estimate $p + q \ge 0$. Writing $\omega := \max\{1, \frac{1}{|\Omega|}\}$, from (2.3) we infer the existence of positive constants $\hat{c}_\phi$ and $\hat{c}_\psi$ such that

$$\phi(s) \ge \hat{c}_\phi (s+\omega)^{-p} \qquad \text{and} \qquad \psi(s) \le \hat{c}_\psi (s+\omega)^q \qquad \text{for all } s \ge 0 \ . \tag{2.6}$$

Let us fix $\theta \in (0,1]$ small such that

$$(n-2)\theta < \frac{4}{p} \qquad \text{and} \qquad (1-p-q)\theta < 2 \ , \tag{2.7}$$

and set

$$\gamma_k := \Gamma \cdot Z^k - \Gamma + 1 \ , \qquad k \in \mathbb{N}_0 \ ,$$

with

$$Z := \frac{2}{\theta} \qquad \text{and} \qquad \Gamma := 1 + \frac{p}{\frac{2}{\theta} - 1} \ .$$

Then $\gamma_0 = 1$, γ_k increases with k and

$$Z^k \le \gamma_k \le \Gamma \cdot Z^k \qquad \text{for all } k \in \mathbb{N}_0 \ . \tag{2.8}$$

Moreover, for $k \in \mathbb{N}_0$ we define

$$K_k := \sup_{t \in (0,T)} \int_\Omega (u+\omega)^{\gamma_k}(x,t)\mathrm{d}x \ \le \infty \ ,$$

so that $K_0 = (M+\omega)|\Omega|$ due to the mass conservation property in (0.4). In the case that $K_{k_j} \le \int_\Omega (u_0+\omega)^{\gamma_{k_j}}$ occurs along some sequence $k_j \to \infty$, we may take the $\frac{1}{\gamma_{k_j}}$-th power on both sides here and let $j \to \infty$ to easily end up with (2.5) upon the choice $C(L) := L + \omega$. We thus only need to consider the case when $S := \{k \in \mathbb{N}_0 \mid K_j > \int_\Omega (u_0+\omega)^{\gamma_j} \text{for all } j > k\}$ is not empty, in which $k_\star := \min S$ is well-defined and

$$K_k > \int_\Omega (u_0+\omega)^{\gamma_k} \qquad \text{for all } k > k_\star \tag{2.9}$$

holds as well as

$$K_{k_\star} \le \int_\Omega (u_0+\omega)^{\gamma_{k_\star}} \ . \tag{2.10}$$

We now multiply the first equation in (0.4) by $\gamma_k(u+\omega)^{\gamma_k-1}$ for $k \ge 1$ and integrate by parts to obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\omega)^{\gamma_k} &+ \gamma_k(\gamma_k-1)\int_\Omega (u+\omega)^{\gamma_k-2}\phi(u)|\nabla u|^2 \\ &= \gamma_k(\gamma_k-1)\int_\Omega (u+\omega)^{\gamma_k-2}\psi(u)\nabla u\cdot\nabla v \\ &= \gamma_k(\gamma_k-1)\int_\Omega \nabla\chi(u)\cdot\nabla v \qquad \text{for } t\in(0,T)\ ,\end{aligned}$$

where we have set

$$\chi(s) := \int_0^s (\sigma+\omega)^{\gamma_k-2}\cdot\psi(\sigma)\mathrm{d}\sigma \qquad \text{for } s\ge 0\ .$$

By (2.6) and the fact that $\gamma_k+q-1 \ge \gamma_1+q-1 = \frac{2}{\theta}+p+q-1 > 0$ for all $k\ge 1$ due to (2.7),

$$\chi(s) \le \hat{c}_\psi\cdot\int_0^s (\sigma+\omega)^{\gamma_k+q-2}\mathrm{d}\sigma \le \frac{\hat{c}_\psi}{\gamma_k+q-1}(s+\omega)^{\gamma_k+q-1} \qquad \text{for all } s\ge 0\ .$$

Hence, once more integrating by parts and taking into account the equation $\Delta v = M-u = (M+\omega)-(u+\omega)$, we find

$$\begin{aligned}\gamma_k(\gamma_k-1)\int_\Omega \nabla\chi(u)\cdot\nabla v &= -\gamma_k(\gamma_k-1)\int_\Omega \chi(u)\Delta v \\ &= -\gamma_k(\gamma_k-1)(M+\omega)\chi(u) + \gamma_k(\gamma_k-1)\int_\Omega (u+\omega)\chi(u) \\ &\le \gamma_k(\gamma_k-1)\int_\Omega (u+\omega)\chi(u) \\ &\le \frac{\gamma_k(\gamma_k-1)\hat{c}_\psi}{\gamma_k+q-1}\int_\Omega (u+\omega)^{\gamma_k+q}\ ,\end{aligned}$$

because $\chi(u)\ge 0$. Using the lower estimate for ϕ in (2.6), we therefore deduce the inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\omega)^{\gamma_k} + \frac{4\gamma_k(\gamma_k-1)\hat{c}_\phi}{(\gamma_k-p)^2}\int_\Omega \left|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right|^2 \le \frac{\gamma_k(\gamma_k-1)\hat{c}_\psi}{\gamma_k+q-1}\int_\Omega (u+\omega)^{\gamma_k+q} \tag{2.11}$$

for all $t\in(0,T)$. Since for all $k\in\mathbb{N}$ we have

$$\frac{4\gamma_k(\gamma_k-1)\hat{c}_\phi}{(\gamma_k-p)^2} \ge \frac{4(\gamma_k-1)\hat{c}_\phi}{\gamma_k} \ge \frac{4(\gamma_1-1)\hat{c}_\phi}{\gamma_1} =: c_1$$

and

$$\frac{\gamma_k(\gamma_k-1)\hat{c}_\psi}{\gamma_k+q-1} \le \frac{\gamma_k}{\gamma_k+q-1}\hat{c}_\psi\gamma_k \le \max\left\{1,\ \frac{\gamma_1}{\gamma_1+q-1}\right\}\cdot\hat{c}_\psi\gamma_k = c_2\gamma_k$$

with an obvious choice of c_2, (2.11) implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\omega)^{\gamma_k} + c_1\int_\Omega \left|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right|^2 \le c_2\gamma_k\int_\Omega (u+\omega)^{\gamma_k+q} \qquad \text{for } t\in(0,T)\ . \tag{2.12}$$

To estimate the term on the right, we observe that Lemma 2.1 provides a constant C_{GN}, independent of k, such that

$$\|w\|_{L^{\frac{2(\gamma_k+q)}{\gamma_k-p}}(\Omega)} \le C_{GN}\Big(\|\nabla w\|^a_{L^2(\Omega)}\cdot\|w\|^{1-a}_{L^\theta(\Omega)} + \|w\|_{L^\theta(\Omega)}\Big) \qquad \text{for all } w\in W^{1,2}(\Omega)$$

with

$$a = \frac{\frac{n}{\theta} - \frac{n(\gamma_k - p)}{2(\gamma_k+q)}}{1 - \frac{n}{2} + \frac{n}{\theta}} ,$$

because, in view of our choice $\theta \le 1$,

$$\theta \le 1 \le \frac{2(\gamma_k+q)}{\gamma_k - p} = 2 + \frac{2(p+q)}{\gamma_k - p} \le 2 + \frac{2(p+q)}{\gamma_1 - p} = 2 + \theta(p+q) < 2 + \frac{2}{n} \quad \text{for all } k \in \mathbb{N} \tag{2.13}$$

and $(n-2)\cdot(2+\frac{2}{n}) < 2n$ for $n \ge 1$. Applying this to $w := (u+\omega)^{\frac{\gamma_k - p}{2}}$ and using the inequality $(A+B)^\alpha \le 2^\alpha(A^\alpha + B^\alpha)$, valid for all positive A, B and α, we have

$$\begin{aligned}
c_2\gamma_k \int_\Omega (u+\omega)^{\gamma_k+q} &= c_2\gamma_k \|w\|^{\frac{2(\gamma_k+q)}{\gamma_k-p}}_{L^{\frac{2(\gamma_k+q)}{\gamma_k-p}}(\Omega)} \\
&\le c_2\gamma_k \cdot C_{GN}^{\frac{2(\gamma_k+q)}{\gamma_k-p}} \left(\left\|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^a_{L^2(\Omega)} \cdot \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{1-a}_{L^\theta(\Omega)} + \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|_{L^\theta(\Omega)} \right)^{\frac{2(\gamma_k+q)}{\gamma_k-p}} \\
&\le c_2\gamma_k \cdot (2C_{GN})^{\frac{2(\gamma_k+q)}{\gamma_k-p}} \Bigg(\left\|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)a}{\gamma_k-p}}_{L^2(\Omega)} \cdot \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)(1-a)}{\gamma_k-p}}_{L^\theta(\Omega)} \\
&\qquad\qquad + \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)}{\gamma_k-p}}_{L^\theta(\Omega)} \Bigg) \\
&\le c_3\gamma_k \left\|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)a}{\gamma_k-p}}_{L^2(\Omega)} \cdot \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)(1-a)}{\gamma_k-p}}_{L^\theta(\Omega)} + c_3\gamma_k \cdot \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)}{\gamma_k-p}}_{L^\theta(\Omega)}
\end{aligned} \tag{2.14}$$

for all $t \in (0,T)$ with some $c_3 > 0$ which, again by (2.13), does not depend on $k \in \mathbb{N}$. Now the smallness condition (2.4) becomes crucial in that it guarantees that for any $k \in \mathbb{N}$ we have

$$\begin{aligned}
\frac{2(\gamma_k+q)a}{\gamma_k-p} &= \frac{(\frac{2}{\theta}-1)n}{1-\frac{n}{2}+\frac{n}{\theta}} + \frac{2(p+q)\frac{n}{\theta}}{(1-\frac{n}{2}+\frac{n}{\theta})(\gamma_k-p)} \\
&\le \frac{(\frac{2}{\theta}-1)n}{1-\frac{n}{2}+\frac{n}{\theta}} + \frac{2(p+q)\frac{n}{\theta}}{(1-\frac{n}{2}+\frac{n}{\theta})(\gamma_1-p)} \\
&= \frac{(\frac{2}{\theta}-1+p+q)n}{1-\frac{n}{2}+\frac{n}{\theta}} \\
&< \frac{(\frac{2}{\theta}-1)n}{1-\frac{n}{2}+\frac{n}{\theta}} + \frac{2}{1-\frac{n}{2}+\frac{n}{\theta}} \\
&= 2 .
\end{aligned} \tag{2.15}$$

It is therefore possible to apply Young's inequality,

$$AB \le \eta A^s + (s-1)s^{-\frac{s}{s-1}} \cdot \eta^{-\frac{1}{s-1}} \cdot B^{\frac{s}{s-1}} \qquad \forall A \ge 0 ,\ B \ge 0 ,\ \eta > 0 , \ \text{ and } s > 1$$

with $s = \frac{2}{\frac{2(\gamma_k+q)a}{\gamma_k-p}} = \frac{\gamma_k-p}{(\gamma_k+q)a}$ to estimate

$$\begin{aligned}
&c_3\gamma_k \left\|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)a}{\gamma_k-p}}_{L^2(\Omega)} \cdot \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)(1-a)}{\gamma_k-p}}_{L^\theta(\Omega)} \\
&\le \frac{c_1}{2}\int_\Omega \left|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\right|^2 + (s-1)s^{-\frac{s}{s-1}} \cdot \left(\frac{2}{c_1}\right)^{\frac{1}{s-1}} \cdot (c_3\gamma_k)^{\frac{s}{s-1}} \cdot \left\|(u+\omega)^{\frac{\gamma_k-p}{2}}\right\|^{\frac{2(\gamma_k+q)(1-a)s}{(\gamma_k-p)(s-1)}}_{L^\theta(\Omega)}
\end{aligned} \tag{2.16}$$

Here, (2.15) implies $s \ge s_\star := 2 \cdot \frac{1-\frac{n}{2}+\frac{n}{\theta}}{(\frac{2}{\theta}-1+p+q)n} > 1$ for all $k \ge 1$, whereas evidently

$$s \equiv \frac{1-\frac{n}{2}+\frac{n}{\theta}}{\frac{n}{\theta}-\frac{n}{2}+\frac{n}{\theta}\cdot\frac{p+q}{\gamma_k-p}} \le \frac{1-\frac{n}{2}+\frac{n}{\theta}}{\frac{n}{\theta}-\frac{n}{2}} =: s^\star ,$$

because $\gamma_k > p$. Therefore,

$$(s-1)s^{-\frac{s}{s-1}} \cdot \Big(\frac{2}{c_1}\Big)^{\frac{1}{s-1}} \cdot (c_3\gamma_k)^{\frac{s}{s-1}} \le c_4{\gamma_k}^b \tag{2.17}$$

with some $c_4 > 0$ independent of k and $b := \frac{s^\star}{s_\star - 1} > 1$. The power appearing in the last term in (2.16) can explicitly be computed according to

$$\begin{aligned}
\frac{2(\gamma_k+q)(1-a)s}{(\gamma_k-p)(s-1)} &= 2\cdot\frac{(\gamma_k+q)(1-a)}{(\gamma_k-p)(1-\frac{(\gamma_k+q)a}{\gamma_k-p})} = 2\cdot\frac{\gamma_k+q-(\gamma_k+q)a}{\gamma_k-p-(\gamma_k+q)a}\\
&= 2\cdot\Big(1+\frac{p+q}{\gamma_k-p-(\gamma_k+q)a}\Big) = 2\cdot\Bigg(1+\frac{p+q}{\gamma_k-p-\frac{(\gamma_k-p)(\frac{n}{\theta}-\frac{n}{2})+(p+q)\frac{n}{\theta}}{1-\frac{n}{2}+\frac{n}{\theta}}}\Bigg)\\
&= 2+\frac{2(p+q)(1-\frac{n}{2}+\frac{n}{\theta})}{\gamma_k-p-(p+q)\frac{n}{\theta}}\ .
\end{aligned}$$

Altogether, from (2.12), (2.14), (2.16) and (2.17) we thus infer that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\omega)^{\gamma_k} + \frac{c_1}{2}\int_\Omega \Big|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\Big|^2 &\le c_4{\gamma_k}^b \Big\|(u+\omega)^{\frac{\gamma_k-p}{2}}\Big\|_{L^\theta(\Omega)}^{2+\frac{2(p+q)(1-\frac{n}{2}+\frac{n}{\theta})}{\gamma_k-p-(p+q)\frac{n}{\theta}}}\\
&\quad + c_3\gamma_k\Big\|(u+\omega)^{\frac{\gamma_k-p}{2}}\Big\|_{L^\theta(\Omega)}^{2+\frac{2(p+q)}{\gamma_k-p}}
\end{aligned}\tag{2.18}$$

for all $t\in(0,T)$, where we have rewritten $\frac{2(\gamma_k+q)}{\gamma_k-p}$ in a convenient way.
Next, we invoke the Poincaré inequality in the version

$$c_5\|w\|^2_{L^{p\theta+2}(\Omega)} \le \int_\Omega |\nabla w|^2 + \|w\|^2_{L^\theta(\Omega)} \qquad \text{for all } w\in W^{1,2}(\Omega)\ ,$$

which is valid with some $c_5 > 0$ because of the fact that (2.7) implies $(n-2)(p\theta+2) < 2n$. Combining this with the Hölder inequality, we see that

$$\begin{aligned}
\int_\Omega \Big|\nabla(u+\omega)^{\frac{\gamma_k-p}{2}}\Big|^2 + \Big\|(u+\omega)^{\frac{\gamma_k-p}{2}}\Big\|^2_{L^\theta(\Omega)} &\ge c_5\Big(\int_\Omega (u+\omega)^{\frac{(\gamma_k-p)(p\theta+2)}{2}}\Big)^{\frac{2}{p\theta+2}}\\
&\ge c_5|\Omega|^{-(\frac{p\theta}{p\theta+2}-\frac{p}{\gamma_k})}\cdot\Big(\int_\Omega (u+\omega)^{\gamma_k}\Big)^{\frac{\gamma_k-p}{\gamma_k}}\\
&\ge c_6\Big(\int_\Omega (u+\omega)^{\gamma_k}\Big)^{\frac{\gamma_k-p}{\gamma_k}} \qquad \text{for all } t\in(0,T)
\end{aligned}$$

is true with $c_6 := c_5\min\{1,|\Omega|^{-\frac{p\theta}{p\theta+2}}\}$. Inserted into (2.18) this yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\omega)^{\gamma_k} &\le -\frac{c_1c_6}{2}\Big(\int_\Omega (u+\omega)^{\gamma_k}\Big)^{\frac{\gamma_k-p}{\gamma_k}}\\
&\quad +\frac{c_1}{2}\Big\|(u+\omega)^{\frac{\gamma_k-p}{2}}\Big\|^2_{L^\theta(\Omega)}\\
&\quad +c_4{\gamma_k}^b\Big\|(u+\omega)^{\frac{\gamma_k-p}{2}}\Big\|_{L^\theta(\Omega)}^{2+\frac{2(p+q)(1-\frac{n}{2}+\frac{n}{\theta})}{\gamma_k-p-(p+q)\frac{n}{\theta}}}\\
&\quad +c_3\gamma_k\Big\|(u+\omega)^{\frac{\gamma_k-p}{2}}\Big\|_{L^\theta(\Omega)}^{2+\frac{2(p+q)}{\gamma_k-p}} \qquad \text{for all } t\in(0,T)\ .
\end{aligned}$$

We now recall the definitions of K_{k-1} and Z and note that $\frac{\gamma_k - p}{2}\theta = \gamma_{k-1}$ in further estimating

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\omega)^{\gamma_k} \le -\frac{c_1 c_6}{2}\Big(\int_\Omega (u+\omega)^{\gamma_k}\Big)^{\frac{\gamma_k - p}{\gamma_k}} + c_7 {\gamma_k}^b K_{k-1}^{Z(1+\bar{\varepsilon}_k)} \qquad \text{for all } t\in(0,T) \;, \tag{2.19}$$

where $c_7 := \frac{c_1}{2} + c_4 + c_3$ and

$$\bar{\varepsilon}_k := \max\left\{ \frac{p+q}{\gamma_k - p}, \frac{(p+q)(1-\frac{n}{2}+\frac{n}{\theta})}{\gamma_k - p - (p+q)\frac{n}{\theta}} \right\} \;. \tag{2.20}$$

Here we have used that $1 \le \gamma_k \le {\gamma_k}^b$ for all k because of $b>1$, and that our choice of ω made in the beginning of the proof ensures that $K_{k-1} \ge 1$.
Integrating the differential inequality (2.19), we find

$$\int_\Omega (u+\omega)^{\gamma_k}(\cdot,t) \le \max\left\{ \int_\Omega (u_0+\omega)^{\gamma_k},\; \Big[\frac{2c_7}{c_1c_6}{\gamma_k}^b (K_{k-1})^{Z(1+\bar{\varepsilon}_k)}\Big]^{\frac{\gamma_k}{\gamma_k - p}} \right\} \qquad \text{for all } t\in(0,T) \;.$$

Since $\frac{\gamma_k}{\gamma_k - p} = 1 + \frac{p}{\gamma_k - p} \le 1 + \frac{p\theta}{2}$ for all $k \ge 1$, in view of (2.8) this implies

$$\int_\Omega (u+\omega)^{\gamma_k}(\cdot,t) \le \max\left\{ \int_\Omega (u_0+\omega)^{\gamma_k},\, c_8 \cdot Z^{dk}(K_{k-1})^{Z(1+\varepsilon_k)} \right\} \qquad \text{for all } t\in(0,T) \tag{2.21}$$

with $d := (1+\frac{p\theta}{2})b, c_8 := (\frac{2c_7\Gamma^b}{c_1c_6})^{1+\frac{p\theta}{2}}$, and

$$\varepsilon_k := \frac{\gamma_k \bar{\varepsilon}_k + p}{\gamma_k - p} \;. \tag{2.22}$$

Taking $\sup_{t\in(0,T)}$ on both sides of (2.21) and noting (2.9), we thus arrive at the recursive estimate

$$K_k \le c_8 \cdot Z^{dk}(K_{k-1})^{Z(1+\varepsilon_k)} \qquad \text{for all } k > k_\star + 1 \;,$$

whereas

$$\begin{aligned} K_{k_\star+1} &\le c_8 \cdot Z^{d(k_\star+1)} \max\{[\frac{1}{c_8 Z^{d(k_\star+1)}}\int_\Omega (u_0+\omega)^{\gamma_{k_\star+1}}]^{\frac{1}{Z(1+\varepsilon_{k_\star+1})}} K_{k_\star}\}^{Z(1+\varepsilon_{k_\star+1})} \\ &=: c_8 \cdot Z^{d(k_\star+1)}(\widetilde{K}_{k_\star})^{Z(1+\varepsilon_{k_\star+1})} \;. \end{aligned} \tag{2.23}$$

(2.24)

A straightforward induction shows that

$$K_{k_\star+j} \le c_8^{1+\sum_{i=1}^{j-1} Z^i\cdot\prod_{l=0}^{i-1}(1+\varepsilon_{k_\star+j-l})} \cdot Z^{d\left(k_\star+j+\sum_{i=1}^{j-1}(k_\star+j-i)\cdot Z^i\cdot\prod_{l=0}^{i-1}(1+\varepsilon_{k_\star+j-l})\right)} \cdot \widetilde{K}_{k_\star}^{Z^j\cdot\prod_{l=1}^{j}(1+\varepsilon_{k_\star+l})} \tag{2.25}$$

for all $j \ge 1$. We now observe that by (2.20), (2.22) and (2.8), we have

$$\varepsilon_k \le c_9 \cdot Z^{-k} \qquad \text{for all } k \ge 1$$

with some $c_9 > 0$, so that we can estimate, using that $\ln(1+\xi) \le \xi$ for all $\xi \ge 0$,

$$\prod_{l=1}^{j}(1+\varepsilon_{k_\star+l}) = e^{\sum_{l=1}^{j}\ln(1+\varepsilon_{k_\star+l})} \le e^{\sum_{l=1}^{j}\varepsilon_{k_\star+l}} \le e^{\frac{c_9}{Z-1}}$$

and

$$\begin{aligned} 1+\sum_{i=1}^{j-1} Z^i \cdot \prod_{l=0}^{i-1}(1+\varepsilon_{k_\star+j-l}) &\le 1+\sum_{i=1}^{j-1} Z^i \cdot e^{c_9\cdot\sum_{l=0}^{i-1} Z^{-(k_\star+j-l)}} \\ &\le 1+\sum_{i=1}^{j-1} Z^i \cdot e^{\frac{c_9}{Z-1}} \\ &\le 1+\frac{Z^j - Z}{Z-1}\cdot e^{\frac{c_9}{Z-1}} \\ &\le Z^j \cdot e^{\frac{c_9}{Z-1}} \end{aligned}$$

for all $j \ge 1$. Similarly,

$$\begin{aligned}
k_\star + j + \sum_{i=1}^{j-1} (k_\star + j - i) \cdot Z^i \cdot \prod_{l=0}^{i-1} (1 + \varepsilon_{k_\star + j - l}) &\le k_\star + j + \sum_{i=1}^{j-1} (k_\star + j - i) \cdot Z^i \cdot e^{\frac{c_9}{Z-1}} \\
&\le e^{\frac{c_9}{Z-1}} \cdot \sum_{i=0}^{j-1} (k_\star + j - i) \cdot Z^i \\
&= e^{\frac{c_9}{Z-1}} \cdot \frac{1}{(Z-1)^2} [k_\star \cdot (Z^{j+1} - Z^j - Z + 1) \\
&\qquad - j(Z-1) + Z^{j+1} - Z] \\
&\le e^{\frac{c_9}{Z-1}} \cdot \frac{1}{(Z-1)^2} [k_\star \cdot (Z^{j+1} + 1) + Z^{j+1}]
\end{aligned}$$

for all $j \ge 1$, whereupon taking the $\gamma_{k_\star+j}$-th root on both sides of (2.25) in view of (2.8) leads to

$$K_{k_\star+j}^{\frac{1}{\gamma_{k_\star+j}}} \le K_{k_\star+j}^{\frac{1}{Z^{k_\star+j}}} \le c_8^{Z^{-k_\star} e^{\frac{c_9}{Z-1}}} \cdot Z^{de^{\frac{c_9}{Z-1}} \cdot \frac{1}{(Z-1)^2} [k_\star \cdot (Z^{1-k_\star} + Z^{-k_\star - j}) + Z^{1-k_\star}]} \cdot \widetilde{K}_{k_\star}^{Z^{-k_\star} e^{\frac{c_9}{Z-1}}}$$

for all $j \ge 1$. Recalling (2.10) and (2.23), we infer that the right-hand side is bounded, uniformly in j and irrespective of the particular value of $k_\star$, by a constant that can be estimated from above by $c_{10} \|u_0 + \omega\|_{L^\infty(\Omega)}^{c_{11}}$ with certain positive c_{10} and c_{11}. After letting $j \to \infty$ we easily arrive at (2.5), as asserted. ////

Combining the above lemma with Theorem 1.1, we immediately obtain the following.

Corollary 2.3 *Suppose that ϕ and ψ are in $C^{1+\theta}_{\mathrm{loc}}([0,\infty))$ for some $\theta > 0$, that $\phi > 0$ and $\psi \ge 0$ on $[0,\infty)$, and that*

$$\phi(s) \ge c_\phi s^{-p} \qquad \text{and} \qquad \psi(s) \le c_\psi s^q \qquad \text{for all } s \ge 1$$

with $c_\phi > 0, c_\psi > 0$, $p \ge 0$ and $q \in \mathbb{R}$ such that

$$p + q < \frac{2}{n} .$$

Then for any nonnegative and uniformly Hölder continuous u_0, (0.4) possesses a unique global bounded solution.

3 Finite-time blow-up for $p + q > \frac{2}{n}$

In the sequel we set $\Omega = B_R := B(0,R)$ and assume that the nonnegative data are radially symmetric and satisfy $u_0 \in C^\vartheta(\overline{\Omega})$ for some $\vartheta > 0$ and

$$\phi, \psi \in C^{1+\theta}_{\mathrm{loc}}([0,\infty)) \ , \ \phi > 0 \ , \ \psi \ge 0$$

for some $\theta > 0$. By Theorem 1.1 there exists a classical solution (u,v) up to a maximal existence time $T_{\max} \in (0,\infty]$, where (u,v) is also radially symmetric because of the uniqueness result. Without any danger of confusion we shall write $u = u(r,t)$ for $r = |x|$ throughout.

We will use some transformations in order to achieve a differential equation which allows for some kind of comparison principle. Multiplying r^{n-1} and integrating the radial differential equation for u in

$$\left\{ \begin{array}{rcll}
u_t &=& \frac{1}{r^{n-1}} (r^{n-1} \phi(u) u_r)_r - \frac{1}{r^{n-1}} (r^{n-1} \psi(u) v_r)_r \ , & r \in (0,R) \ , \ t \in (0,T_{\max}) \ , \\
0 &=& \frac{1}{r^{n-1}} (r^{n-1} v_r)_r + u - M \ , & r \in (0,R) \ , \ t \in (0,T_{\max}) \ , \\
u_r &=& 0 \ , & r = R \ , \ t \in [0,T_{\max}) \ , \\
v_r &=& 0 \ , & r = R \ , \ t \in [0,T_{\max}) \ , \\
u(r,0) &=& u_0 \ , & r \in [0,R) \ ,
\end{array} \right.$$

over $(0,r)$ implies

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\int_0^r \rho^{n-1}u\mathrm{d}\rho &=& \int_0^r(\rho^{n-1}\phi(u)u_r)_r\mathrm{d}\rho - \int_0^r(\rho^{n-1}\psi(u)v_r)_r\mathrm{d}\rho \\ &=& r^{n-1}\phi(u)u_r - [r^{n-1}v_r]\psi(u) \\ &=& r^{n-1}\phi(u)u_r + \psi(u)\int_0^r \rho^{n-1}u\mathrm{d}\rho - \frac{M}{n}r^n\psi(u)\ . \end{array}$$

We set

$$U(r,t) := \int_0^r \rho^{n-1}u(\rho,t)\mathrm{d}\rho\ .$$

Then we have

$$u = \frac{U_r}{r^{n-1}}\ ,\ u_r = \frac{U_{rr}}{r^{n-1}} - (n-1)\frac{U_r}{r^n}\ ,$$

which implies the following differential equation for U

$$U_t = \phi\left(\frac{U_r}{r^{n-1}}\right)\cdot\left(U_{rr} - \frac{n-1}{r}U_r\right) + \psi\left(\frac{U_r}{r^{n-1}}\right)U - \frac{M}{n}r^n\psi\left(\frac{U_r}{r^{n-1}}\right)\ .$$

By

$$W(s,t) := n\cdot U\left(s^{\frac{1}{n}},t\right)$$

one obtains, substituting with $r = s^{\frac{1}{n}}$,

$$U = \frac{1}{n}W\ ,\ U_t = \frac{1}{n}W_t\ ,\ U_r = W_s s^{\frac{n-1}{n}}\ ,\ U_{rr} = ns^{2-\frac{2}{n}}W_{ss} + (n-1)s^{1-\frac{2}{n}}W_s\ .$$

Defining the differential operator $\mathcal{P}$ by

$$\mathcal{P}\widetilde{W} := \widetilde{W}_t - F(s,\widetilde{W},\widetilde{W}_s,\widetilde{W}_{ss}) \tag{3.1}$$

with

$$F(x,u,y,z) := n^2x^{2-\frac{2}{n}}\phi(y)z + (u-Mx)\psi(y) \tag{3.2}$$

we arrive at the following problem for W:

$$\left\{\begin{array}{rcll} \mathcal{P}W &=& 0\ , & s\in(0,R^n)\ ,\ t\in(0,T_{\max})\ , \\ W(0,t) &=& 0\ , & t\in[0,T_{\max})\ , \\ W(R^n,t) &=& MR^n\ , & t\in[0,T_{\max})\ , \\ W(s,0) &=& W_0(s) := n\int_0^{s^{\frac{1}{n}}}\rho^{n-1}u_0(\rho)\mathrm{d}\rho\ , & s\in[0,R^n]\ . \end{array}\right.$$

We want to apply a comparison principle to this differential equation. Since the subsolutions we have in mind (see below) do not possess second order derivatives with respect to s at *all* points, we need a weak comparison principle.

Theorem 3.1 (Weak comparison principle) *Let $\overline{W},\underline{W} : [0,R^n]\times[0,T]\to[0,\infty)$ be two non-negative functions with*

$$\underline{W}(s,0) \le \overline{W}(s,0) \text{ for all } s\in[0,R^n]$$

and

$$\underline{W}(0,t) = \overline{W}(0,t) = 0 \text{ and } \underline{W}(R^n,t) \le \overline{W}(R^n,t) \text{ for all } t\in[0,T]\ .$$

We also assume $\overline{W},\underline{W}\in C^1([0,R^n]\times[0,T])\cap L^1([0,T];W^{2,\infty}([0,R^n]))$ with $\overline{W}_s,\underline{W}_s\ge 0$, and suppose that $\overline{W},\underline{W}$ possess a second order derivatives with respect to s almost everywhere, and that $\mathcal{P}\underline{W}\le\mathcal{P}\overline{W}$ almost everywhere with $\mathcal{P}$ taken from (3.1) and (3.2). Then it follows that

$$\underline{W}\le\overline{W} \text{ on } [0,R^n]\times[0,T]\ .$$

PROOF. From

$$\overline{W}_t - n^2 s^{2-\frac{2}{n}}\phi(\overline{W}_s)\overline{W}_{ss} - (\overline{W}-Ms)\psi(\overline{W}_s) \geq \underline{W}_t - n^2 s^{2-\frac{2}{n}}\phi(\underline{W}_s)\underline{W}_{ss} - (\underline{W}-Ms)\psi(\underline{W}_s)$$

we obtain, writing $z(s,t) := (\underline{W}-\overline{W})(s,t)$, after dividing by $s^{2-\frac{2}{n}}$ that almost everywhere

$$s^{-2+\frac{2}{n}} z_t \leq n^2(\phi(\underline{W}_s)\underline{W}_{ss} - \phi(\overline{W}_s)\overline{W}_{ss}) + s^{-2+\frac{2}{n}}[(\underline{W}-Ms)\psi(\underline{W}_s) - (\overline{W}-Ms)\psi(\overline{W}_s)]\ .$$

Now observe $z_+ z_t = z_+(z_+)_t = \frac{1}{2}(z_+^2)_t$ and $z_+(\cdot,0)=0$. Multiplying this by z_+ and integrating over $[0,R^n]\times[0,\tau]$ for $\tau\in(0,T]$ we find

$$\begin{aligned}
\tfrac{1}{2}\int_0^{R^n} s^{-2+\frac{2}{n}} z_+^2(s,\tau) &= \tfrac{1}{2}\int_0^{R^n}\int_0^\tau s^{-2+\frac{2}{n}}(z_+^2)_t = \int_0^{R^n}\int_0^\tau s^{-2+\frac{2}{n}} z_+ z_t \\
&\leq n^2\int_0^\tau\int_0^{R^n}(\phi(\underline{W}_s)\underline{W}_{ss} - \phi(\overline{W}_s)\overline{W}_{ss})z_+ \\
&\quad + \int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}[(\underline{W}-Ms)\psi(\underline{W}_s) - (\overline{W}-Ms)\psi(\overline{W}_s)]z_+\ .
\end{aligned}$$

We set $\Phi(x) := \int_0^x \phi(s)\mathrm{d}s$. Then because of $z_+(0,\tau) = z_+(R^n,\tau) = 0$ for all $\tau\in[0,T]$, we have

$$\begin{aligned}
\tfrac{1}{2}\int_0^{R^n} s^{-2+\frac{2}{n}} z_+^2(s,\tau) &\leq n^2\int_0^\tau\int_0^{R^n}(\Phi(\underline{W}_s) - \Phi(\overline{W}_s))_s z_+ \\
&\quad + \int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}[(\underline{W}-Ms)\psi(\underline{W}_s) - (\overline{W}-Ms)\psi(\overline{W}_s)]z_+ \\
&= -n^2\int_0^\tau\int_0^{R^n}(\Phi(\underline{W}_s) - \Phi(\overline{W}_s))(z_+)_s \\
&\quad + \int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}[(\underline{W}-Ms)\psi(\underline{W}_s) - (\overline{W}-Ms)\psi(\overline{W}_s)]z_+\ .
\end{aligned}$$

Now, we define

$$C_1 := \max\{\|\overline{W}_s\|_{C([0,R^n]\times[0,T])}, \|\underline{W}_s\|_{C([0,R^n]\times[0,T])}\} < \infty$$

and

$$C_2 := \min\{\phi(x) \mid x\in[0,C_1]\} > 0\ .$$

Φ is nondecreasing since $\phi>0$. Thus one obtains by the mean value theorem

$$\begin{aligned}
(\Phi(\underline{W}_s) - \Phi(\overline{W}_s))(z_+)_s &= |\Phi(\underline{W}_s) - \Phi(\overline{W}_s)||\underline{W}_s - \overline{W}_s|\chi_{\{z\geq 0\}} \\
&\geq C_2(\underline{W}_s - \overline{W}_s)^2\chi_{\{z\geq 0\}} \\
&= C_2((z_+)_s)^2\ .
\end{aligned}$$

This implies

$$\begin{aligned}
\int_0^{R^n} s^{-2+\frac{2}{n}} z_+^2(s,\tau) &\leq -2C_2 n^2\int_0^\tau\int_0^{R^n}((z_+)_s)^2 \\
&\quad +2\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}[(\underline{W}-Ms)\psi(\underline{W}_s) - (\overline{W}-Ms)\psi(\overline{W}_s)]z_+ \\
&= -2C_2 n^2\int_0^\tau\int_0^{R^n}((z_+)_s)^2 \\
&\quad +2\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}(\overline{W}-Ms)(\psi(\underline{W}_s) - \psi(\overline{W}_s))\cdot z_+ \\
&\quad +2\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}\psi(\underline{W}_s)\cdot(z_+)^2\ .
\end{aligned}$$

With the aid of Young's inequality and

$$C_3 := \sup\{s^{-2+\frac{2}{n}}(\overline{W}-Ms)^2 \mid s\in(0,R^n]\ ,\ \tau\in[0,T]\} < \infty$$

and

$$C_4 := \max\{\psi(x) \mid x\in[0,C_1]\} < \infty\ ,$$

in the case $C_3C_4>0$ for $\varepsilon := \frac{C_2 n^2}{4C_4^2 C_3} > 0$ one sees that

$$\begin{aligned}
&\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}(\overline{W}-Ms)(\psi(\underline{W}_s) - \psi(\overline{W}_s))\cdot z_+ \\
\leq\ &\varepsilon\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}(\overline{W}-Ms)^2(\psi(\underline{W}_s) - \psi(\overline{W}_s))^2 + \tfrac{1}{4\varepsilon}\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}(z_+)^2 \\
\leq\ &\varepsilon 4C_4^2 C_3\int_0^\tau\int_0^{R^n}(\psi(\underline{W}_s) - \psi(\overline{W}_s))^2 + \tfrac{1}{4\varepsilon}\int_0^\tau\int_0^{R^n} s^{-2+\frac{2}{n}}(z_+)^2\ ,
\end{aligned}$$

hence

$$\int_0^{R^n} s^{-2+\frac{2}{n}} z_+^2(s,\tau) \le 2\left(C_4 + \frac{C_4^2 C_3}{C_2 n^2}\right) \int_0^\tau \int_0^{R^n} s^{-2+\frac{2}{n}} \cdot (z_+)^2$$

which trivially holds in the other case $C_3 C_4 = 0$. Now, we apply Grönwall's lemma which leads to

$$\int_0^{R^n} s^{-2+\frac{2}{n}} z_+^2(s,\tau) \le 0 \text{ for all } \tau \in [0,T] \ .$$

Because z_+ is continuous and nonnegative, it follows that $z_+ = 0$ and therefore $\underline{W} \le \overline{W}$ on $[0,R^n] \times [0,T]$. ////

In the sequel we additionally assume

$$0 < \phi(s) \le c_1 s^{-p} \ , \ \psi(s) \ge c_2 s(1+s)^{q-1} \ , \ \psi(0) = 0 \tag{3.3}$$

for suitable

$$c_1, c_2 > 0 \ , \ p \ge 0 \ , \ q > 0 \ , \text{ and } \ p + q > \frac{2}{n} \ . \tag{3.4}$$

In order to reduce the PDE in question to an ordinary differential equation we take a selfsimilar ansatz

$$\widetilde{W}(s,t) := (T-t)^\alpha \cdot w((T-t)^{-\beta} s) \ , \ s \in [0,R^n] \ , \ t \in [0,T) \ . \tag{3.5}$$

Let us assume for a moment that

$$w \ge 0 \ , \ w' > 0 \ , \ w'' \le 0 \ .$$

Then this implies with $\xi := (T-t)^{-\beta} s$

$$\begin{aligned}
\mathcal{P}\widetilde{W} &= -\alpha (T-t)^{\alpha-1}[w(\xi)] + \beta (T-t)^{\alpha-1}\xi[w'(\xi)] \\
&\quad -n^2 \xi^{2-\frac{2}{n}} (T-t)^{(2-\frac{2}{n})\beta + \alpha - 2\beta}[w''(\xi)]\phi((T-t)^{\alpha-\beta}w'(\xi)) \\
&\quad -[(T-t)^\alpha w(\xi) - M(T-t)^\beta \xi]\psi((T-t)^{\alpha-\beta}w'(\xi)) \\
&\le -c_1 n^2 \xi^{2-\frac{2}{n}} (T-t)^{(2-\frac{2}{n})\beta - p(\alpha-\beta) + \alpha - 2\beta}[w''(\xi)][w'(\xi)]^{-p} \\
&\quad +\beta (T-t)^{\alpha-1}\xi[w'(\xi)] - \alpha (T-t)^{\alpha-1}[w(\xi)] \\
&\quad -[(T-t)^\alpha w(\xi) - M(T-t)^\beta \xi]\psi((T-t)^{\alpha-\beta}w'(\xi)) \ .
\end{aligned}$$

Because of $q > 0$ we can choose α and β such that $(2-\frac{2}{n})\beta - p(\alpha-\beta) + \alpha - 2\beta = \alpha - 1 = \alpha + q(\alpha-\beta)$, which is equivalent to

$$\alpha = \frac{n(\frac{p}{q}+1)}{2} - \frac{1}{q} \ , \ \beta = \frac{n(\frac{p}{q}+1)}{2} \ . \tag{3.6}$$

We remark at this point that α and β are positive because of $p + q > \frac{2}{n}$. Then we obtain

$$\begin{aligned}
(T-t)^{1-\alpha}\mathcal{P}\widetilde{W} &\le -c_1 n^2 \xi^{2-\frac{2}{n}}[w''(\xi)][w'(\xi)]^{-p} + \beta\xi[w'(\xi)] - \alpha[w(\xi)] \\
&\quad -c_2[w(\xi)][w'(\xi)][(T-t)^{\frac{1}{q}} + w'(\xi)]^{q-1} \cdot \Psi((T-t)^{-\frac{1}{q}}w'(\xi)) \\
&\quad +c_2 M\xi(T-t)^{\frac{1}{q}}[w'(\xi)][(T-t)^{\frac{1}{q}} + w'(\xi)]^{q-1}\Psi((T-t)^{-\frac{1}{q}}w'(\xi)) \\
&=: \mathcal{E}w
\end{aligned} \tag{3.7}$$

with

$$\Psi(s) := \frac{\psi(s)}{c_2 s(1+s)^{q-1}} \ge 1 \ , \ s > 0 \ . \tag{3.8}$$

Lemma 3.2 (Linear functions as subsolutions) *Assume that (3.3) holds with parameters satisfying (3.4), and let m be a positive number fulfilling*

$$m \ge \begin{cases} \left(\frac{2^{2-q}}{c_2 q}\right)^{\frac{1}{q}} \ , & \text{if } 0 < q < 1 \ , \\ \left(\frac{2}{c_2 q}\right)^{\frac{1}{q}} \ , & \text{if } q \ge 1 \ , \end{cases}$$

and let

$$w_1(\xi) := m\xi\ ,\ \xi \in [0,\infty)\ .$$

Then we have $w_1' > 0$, $w_1'' = 0$, *and* $\mathcal{E}w_1 \le 0$ *on* $[0,\infty)$ *for all*

$$T \le \begin{cases} \min\{(\frac{m}{2M})^q, m^q\}\ , & \text{if } 0 < q < 1\ ,\\ (\frac{m}{2M})^q\ , & \text{if } q \ge 1\ ,\end{cases}$$

with $\mathcal{E}$ *as determined by (3.7) and (3.8).*

Proof. Obviously, $w_1' > 0$ and $w_1'' = 0$. Moreover,

$$\mathcal{E}w_1 = m\xi\left[\frac{1}{q} - c_2\Psi\left((T-t)^{-\frac{1}{q}}m\right)\left[(T-t)^{\frac{1}{q}}+m\right]^{q-1}\left(m - M(T-t)^{\frac{1}{q}}\right)\right]\ .$$

Therefore $\mathcal{E}w_1 \le 0$ is equivalent to

$$\frac{1}{qc_2\Psi\left((T-t)^{-\frac{1}{q}}m\right)} \le \left[(T-t)^{\frac{1}{q}}+m\right]^{q-1}\left(m - M(T-t)^{\frac{1}{q}}\right)\ .$$

Since $\Psi \ge 1$, it is sufficient to ensure that

$$\frac{1}{qc_2} \le \left[(T-t)^{\frac{1}{q}}+m\right]^{q-1}\left(m - M(T-t)^{\frac{1}{q}}\right)\ .$$

This holds in the case $0 < q < 1$ because of the estimate

$$\left[(T-t)^{\frac{1}{q}}+m\right]^{q-1}\left(m - M(T-t)^{\frac{1}{q}}\right) \ge (2m)^{q-1}\frac{m}{2}\ ,$$

whereas in the case $q \ge 1$ we find

$$\left[(T-t)^{\frac{1}{q}}+m\right]^{q-1}\left(m - M(T-t)^{\frac{1}{q}}\right) \ge \frac{m^q}{2}\ .$$

This implies the assertion in both cases. ////

Because this value plays an important role in the following, we set

$$\widetilde{\eta} := \begin{cases} \infty & \text{if } np-2 \le 0\ ,\\ \frac{n}{np-2} & \text{if } np-2 > 0\ .\end{cases} \tag{3.9}$$

Lemma 3.3 (Further subsolutions for sufficiently large ξ) *Suppose that (3.3) holds with parameters (3.4), and let* $\eta \in (0,\widetilde{\eta})$ *with* $\widetilde{\eta}$ *from (3.9),* $a \ge 0$, *and* $\xi_0 > a$ *be arbitrary. Then for all*

$$\varepsilon \in \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta+\eta}, \min\left\{\frac{\alpha}{\beta}, \frac{\alpha}{\beta} - 1 + \frac{2}{np}\right\}\right)$$

with α, β *from (3.6) there exists* $A_0 = A_0(a, c_1, c_2, n, p, q, \varepsilon, \eta, \xi_0) > 0$ *such that*

$$w_2(\xi) := A(\xi - a)^{\frac{\alpha}{\beta}-\varepsilon}\ ,\ \xi \in [a,\infty)$$

has the properties $w_2 > 0$, $w_2' > 0$, $w_2'' < 0$, *and* $\mathcal{E}w_2 < 0$ *on* $[\xi_0,\infty)$ *for all* $A > A_0$ *and for all*

$$T \le \min\left\{\left[A\left(\frac{\alpha}{\beta}-\varepsilon\right)(\xi_0-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\right]^q, \left(\frac{\eta}{c_3M[2A(\frac{\alpha}{\beta}-\varepsilon)(\xi_0-a)^{\frac{\alpha}{\beta}-\varepsilon-1}]^{\max\{0,q-1\}}}\right)^{\max\{1,q\}}\right\}$$

with $c_3 := \sup\left\{\frac{\psi(s)}{s(1+s)^{q-1}}\ \middle|\ s \in (0, (\frac{\alpha}{\beta}-\varepsilon)M)\right\} \ge c_2$ *wherein* $\mathcal{E}$ *again is taken from (3.7) and (3.8).*

PROOF. First one notices that $\frac{\alpha}{\beta}-\varepsilon\in(0,1)$, so that we obtain $w_2>0$, $w_2'>0$, and $w_2''<0$. Now, $\frac{\alpha}{\beta}-\frac{\alpha}{(\beta+\eta)}<\frac{\alpha}{\beta}-1+\frac{2}{np}$ is equivalent to

$$\eta(np-2)<n$$

which holds because of $\eta\in(0,\widetilde{\eta})$. Therefore the interval which defines ε is nonempty. Observe that

$$\begin{aligned}
\mathcal{E}w_2 \;=\;& c_1n^2\xi^{2-\frac{2}{n}}(1+\varepsilon-\tfrac{\alpha}{\beta})(\tfrac{\alpha}{\beta}-\varepsilon)^{1-p}A^{1-p}(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-2+p(1+\varepsilon-\frac{\alpha}{\beta})}\\
&+\beta\xi A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\\
&-\alpha A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon}\\
&-c_2[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\Psi((T-t)^{-\frac{1}{q}}w'(\xi))[w(\xi)-M\xi(T-t)^{\frac{1}{q}}]\ .
\end{aligned}$$

We distinguish between two possible cases.

- Case 1: $w(\xi)\geq M\xi(T-t)^{\frac{1}{q}}$:

  Then

$$\begin{aligned}
&-c_2[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\Psi((T-t)^{-\frac{1}{q}}w'(\xi))[w(\xi)-M\xi(T-t)^{\frac{1}{q}}]\\
\leq\;&-c_2[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}[w(\xi)-M\xi(T-t)^{\frac{1}{q}}]\ .
\end{aligned}$$

- Case 2: $w(\xi)<M\xi(T-t)^{\frac{1}{q}}$:

  Then $0<(T-t)^{-\frac{1}{q}}w'(\xi)=(T-t)^{-\frac{1}{q}}\frac{(\frac{\alpha}{\beta}-\varepsilon)}{\xi}w(\xi)<(\frac{\alpha}{\beta}-\varepsilon)M$. Furthermore, $c_3:=\sup\left\{\frac{\psi(s)}{s(1+s)^{q-1}}\mid s\in(0,(\frac{\alpha}{\beta}-\varepsilon)M)\right\}$ is finite because of $\lim\limits_{s\searrow 0}\frac{\psi(s)}{s}=\psi'(0)<\infty$, hence $\Psi(s)\leq\frac{c_3}{c_2}$ for all $s\in(0,(\frac{\alpha}{\beta}-\varepsilon)M)$. This implies

$$\begin{aligned}
&-c_2[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\Psi((T-t)^{-\frac{1}{q}}w'(\xi))[w(\xi)-M\xi(T-t)^{\frac{1}{q}}]\\
\leq\;&-c_2[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}w(\xi)\\
&\quad+c_3[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}M\xi(T-t)^{\frac{1}{q}}\ .
\end{aligned}$$

Because of $c_3\geq c_2$ we can summarize both cases as

$$\begin{aligned}
&-c_2[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\Psi((T-t)^{-\frac{1}{q}}w'(\xi))[w(\xi)-M\xi(T-t)^{\frac{1}{q}}]\\
\leq\;&-c_2A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon}[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\\
&\quad+c_3A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}[(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}M\xi(T-t)^{\frac{1}{q}}\ .
\end{aligned}$$

This implies

$$\begin{aligned}
\mathcal{E}w_2 \;\leq\;& c_1n^2\xi^{2-\frac{2}{n}}(1+\varepsilon-\tfrac{\alpha}{\beta})(\tfrac{\alpha}{\beta}-\varepsilon)^{1-p}A^{1-p}(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-2+p(1+\varepsilon-\frac{\alpha}{\beta})}\\
&+\beta\xi A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\\
&-\alpha A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon}\\
&-c_2A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon}[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\\
&+c_3M\xi A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}(T-t)^{\frac{1}{q}}[(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\ .
\end{aligned}$$

Next, we again consider two different cases:

- Case i): $0<q<1$:

  By the assumption $T\leq\min\left\{[w'(\xi_0)]^q,\frac{\eta}{c_3M}\right\}$ it follows that for all $\xi\geq\xi_0$

$$\begin{aligned}
&-c_2A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon}[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\\
&\quad+c_3M\xi A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}(T-t)^{\frac{1}{q}}[(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1}\\
\leq\;&-c_2[2w'(\xi_0)]^{q-1}(\tfrac{\alpha}{\beta}-\varepsilon)A^2(\xi-a)^{2(\frac{\alpha}{\beta}-\varepsilon)-1}\\
&\quad+c_3M(T-t)^{\frac{1}{q}+\frac{q-1}{q}}(\tfrac{\alpha}{\beta}-\varepsilon)A\xi(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\ .\\
\leq\;&-c_2[2w'(\xi_0)]^{q-1}(\tfrac{\alpha}{\beta}-\varepsilon)A^2(\xi-a)^{2(\frac{\alpha}{\beta}-\varepsilon)-1}\\
&\quad+\eta(\tfrac{\alpha}{\beta}-\varepsilon)A\xi(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\ .
\end{aligned}$$

- Case ii): $q \geq 1$:

  By the assumption $T \leq \min\left\{[w'(\xi_0)]^q, \left(\frac{\eta}{c_3 M[2w'(\xi_0)]^{q-1}}\right)^q\right\}$ we obtain for all $\xi \geq \xi_0$

$$\begin{aligned}
& -c_2 A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon}[w'(\xi)][(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1} \\
& \quad +c_3 M\xi A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}(T-t)^{\frac{1}{q}}[(T-t)^{\frac{1}{q}}+w'(\xi)]^{q-1} \\
\leq\ & -c_2(\tfrac{\alpha}{\beta}-\varepsilon)^q A^{q+1}(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon+q(\frac{\alpha}{\beta}-\varepsilon-1)} \\
& \quad +c_3 M[2w'(\xi_0)]^{q-1}T^{\frac{1}{q}}(\tfrac{\alpha}{\beta}-\varepsilon)A\xi(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\ . \\
\leq\ & -c_2(\tfrac{\alpha}{\beta}-\varepsilon)^q A^{q+1}(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon+q(\frac{\alpha}{\beta}-\varepsilon-1)} \\
& \quad +\eta(\tfrac{\alpha}{\beta}-\varepsilon)A\xi(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\ .
\end{aligned}$$

Collecting both cases, we arrive at

$$\begin{aligned}
\mathcal{E}w_2 \quad \leq\ & c_1 n^2 \xi^{2-\frac{2}{n}}(1+\varepsilon-\tfrac{\alpha}{\beta})(\tfrac{\alpha}{\beta}-\varepsilon)^{1-p}A^{1-p}(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-2+p(1+\varepsilon-\frac{\alpha}{\beta})} \\
& +(\beta+\eta)\xi A(\tfrac{\alpha}{\beta}-\varepsilon)(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon-1} \\
& -\alpha A(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon} \\
& -I_4 \\
=:\ & I_1+I_2-I_3-I_4
\end{aligned}$$

with

$$I_4 := \begin{cases} c_2[2w'(\xi_0)]^{q-1}(\frac{\alpha}{\beta}-\varepsilon)A^2(\xi-a)^{2(\frac{\alpha}{\beta}-\varepsilon)-1}\ , & \text{if } 0<q<1\ , \\ c_2(\frac{\alpha}{\beta}-\varepsilon)^q A^{q+1}(\xi-a)^{\frac{\alpha}{\beta}-\varepsilon+q(\frac{\alpha}{\beta}-\varepsilon-1)}\ , & \text{if } q \geq 1\ . \end{cases}$$

Since $\varepsilon > \frac{\alpha}{\beta} - \frac{\alpha}{(\beta+\eta)}$, there exists $\mu = \mu(n,p,q,\varepsilon,\eta) \in (0,1)$ with

$$\alpha(1-\mu) > \left(\frac{\alpha}{\beta}-\varepsilon\right)(\beta+\eta)\ .$$

Then the expression

$$X = X(a,n,p,q,\varepsilon,\eta) := a + \frac{a}{\frac{\alpha(1-\mu)}{(\frac{\alpha}{\beta}-\varepsilon)(\beta+\eta)}-1}$$

is well-defined. In particular, for all $\xi \geq X$ we have

$$\frac{\xi}{\xi-a} = 1+\frac{a}{\xi-a} \leq 1+\frac{a}{X-a} = \frac{\alpha(1-\mu)}{(\frac{\alpha}{\beta}-\varepsilon)(\beta+\eta)}\ .$$

Now, one has the following equivalences (assuming $A>0$):

$$I_1 < \mu I_3 \Leftrightarrow A^p > \frac{c_1 n^2(\frac{\alpha}{\beta}-\varepsilon)^{1-p}(1+\varepsilon-\frac{\alpha}{\beta})\left(\frac{\xi}{\xi-a}\right)^{2-\frac{2}{n}}(\xi-a)^{-\frac{2}{n}+p(1+\varepsilon-\frac{\alpha}{\beta})}}{\mu\alpha}$$

and

$$I_2 \leq (1-\mu)I_3 \Leftrightarrow \frac{\xi}{\xi-a} \leq \frac{\alpha(1-\mu)}{(\frac{\alpha}{\beta}-\varepsilon)(\beta+\eta)}\ .$$

Due to $-\frac{2}{n}+p(1+\varepsilon-\frac{\alpha}{\beta})<0$ and $\frac{\xi}{\xi-a} = 1+\frac{a}{\xi-a} \leq 1+a$ for all $\xi \geq a+1$ we have

$$I_1 < \mu I_3 \text{ for all } \xi \geq a+1 \text{ and } A > A_1 := \left(\frac{c_1 n^2(\frac{\alpha}{\beta}-\varepsilon)^{1-p}(1+\varepsilon-\frac{\alpha}{\beta})(1+a)^{2-\frac{2}{n}}}{\mu\alpha}\right)^{\frac{1}{p}}$$

and by definition of X

$$I_2 \leq (1-\mu)I_3 \text{ for all } \xi \geq X\ .$$

In order to achieve estimates for I_4 we split into two cases again.

- Case I): $0 < q < 1$:

  Assuming $A > 0$ we obviously have the equivalences

$$I_1 < \frac{1}{2} I_4 \Leftrightarrow A^{p+q} > \frac{2^{2-q} c_1 n^2 (1+\varepsilon-\frac{\alpha}{\beta})}{c_2(\frac{\alpha}{\beta}-\varepsilon)^{p+q-1}(\xi_0-a)^{(1-q)(1+\varepsilon-\frac{\alpha}{\beta})}} \xi^{2-\frac{2}{n}}(\xi-a)^{-2+(p+1)(1+\varepsilon-\frac{\alpha}{\beta})}$$

  and

$$I_2 \le \frac{1}{2} I_4 \Leftrightarrow A^q \ge \frac{2^{2-q}(\beta+\eta)(\frac{\alpha}{\beta}-\varepsilon)^{1-q}}{c_2(\xi_0-a)^{(1-q)(1+\varepsilon-\frac{\alpha}{\beta})}} \xi(\xi-a)^{\varepsilon-\frac{\alpha}{\beta}} \ .$$

  In the case $\xi_0 < a+1$, in particular we obtain $I_1 < \frac{1}{2} I_4$ for all $\xi \in [\xi_0, a+1]$ if

$$A > A_2 := \left(\frac{2^{2-q} c_1 n^2 (1+\varepsilon-\frac{\alpha}{\beta})}{c_2(\frac{\alpha}{\beta}-\varepsilon)^{p+q-1}(\xi_0-a)^{(1-q)(1+\varepsilon-\frac{\alpha}{\beta})}} \max_{\xi \in [\xi_0, a+1]} \left\{ \xi^{2-\frac{2}{n}}(\xi-a)^{-2+(p+1)(1+\varepsilon-\frac{\alpha}{\beta})} \right\} \right)^{\frac{1}{p+q}} \ .$$

  If $\xi_0 < X$, however, we set

$$b = b(a, n, p, q, \varepsilon, \eta, \xi_0) := \max_{\xi \in [\xi_0, X]} \left\{ \xi(\xi-a)^{\varepsilon-\frac{\alpha}{\beta}} \right\} \ .$$

  Then one finally concludes

$$I_2 \le \frac{1}{2} I_4 \text{ for all } \xi \in [\xi_0, X] \text{ and } A \ge A_3 := \left(\frac{2^{2-q}(\beta+\eta)(\frac{\alpha}{\beta}-\varepsilon)^{1-q} b}{c_2(\xi_0-a)^{(1-q)(1+\varepsilon-\frac{\alpha}{\beta})}} \right)^{\frac{1}{q}} \ .$$

- Case II): $q \ge 1$:

  Analogously we consider the equivalences (again assuming $A > 0$)

$$I_1 < \frac{1}{2} I_4 \Leftrightarrow A^{p+q} > \frac{2 c_1 n^2 (1+\varepsilon-\frac{\alpha}{\beta})}{c_2(\frac{\alpha}{\beta}-\varepsilon)^{p+q-1}} \xi^{2-\frac{2}{n}}(\xi-a)^{-2+(p+q)(1+\varepsilon-\frac{\alpha}{\beta})}$$

  and

$$I_2 \le \frac{1}{2} I_4 \Leftrightarrow A^q \ge \frac{2(\beta+\eta)}{c_2(\frac{\alpha}{\beta}-\varepsilon)^{q-1}} \frac{\xi}{(\xi-a)^{1-q(1+\varepsilon-\frac{\alpha}{\beta})}} \ .$$

  In particular, in the case $\xi_0 < a+1$ we have $I_1 < \frac{1}{2} I_4$ for all $\xi \in [\xi_0, a+1]$, provided that

$$A > A_2 := \left(\frac{2 c_1 n^2 (1+\varepsilon-\frac{\alpha}{\beta})}{c_2(\frac{\alpha}{\beta}-\varepsilon)^{p+q-1}} \max_{\xi \in [\xi_0, a+1]} \left\{ \xi^{2-\frac{2}{n}}(\xi-a)^{-2+(p+q)(1+\varepsilon-\frac{\alpha}{\beta})} \right\} \right)^{\frac{1}{p+q}} \ .$$

  If $\xi_0 < X$, we set

$$b = b(a, n, p, q, \varepsilon, \eta, \xi_0) := \max_{\xi \in [\xi_0, X]} \left\{ \frac{\xi}{(\xi-a)^{1-q(1+\varepsilon-\frac{\alpha}{\beta})}} \right\} \ .$$

  Finally, we end up with

$$I_2 \le \frac{1}{2} I_4 \text{ for all } \xi \in [\xi_0, X] \text{ and } A \ge A_3 := \left(\frac{2(\beta+\eta) b}{c_2(\frac{\alpha}{\beta}-\varepsilon)^{q-1}} \right)^{\frac{1}{q}} \ .$$

Therefore the lemma has been proven upon the choice $A_0 := \max\{A_1, A_2, A_3\}$. ////

Corollary 3.4 (Combined subsolution) *Assume (3.3) and (3.4). Then for all* $\xi_1 > 0$, $\eta \in (0, \widetilde{\eta})$ *with* $\widetilde{\eta}$ *taken from (3.9) and*

$$\varepsilon \in \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta + \eta}, \min \left\{ \frac{\alpha}{\beta}, \frac{\alpha}{\beta} - 1 + \frac{2}{np} \right\} \right)$$

with α, β *from (3.6) there exist* $a = a(n,p,q,\varepsilon,\xi_1) \in (0,\xi_1)$, $A = A(c_1,c_2,n,p,q,\varepsilon,\eta,\xi_1)$, *and* $m = m(c_1,c_2,n,p,q,\varepsilon,\eta,\xi_1) > 0$ *such that*

$$w(\xi) := \begin{cases} m\xi \ , & \xi \in [0,\xi_1) \ , \\ A(\xi - a)^{\frac{\alpha}{\beta} - \varepsilon} \ , & \xi \in [\xi_1, \infty) \ , \end{cases}$$

lies in $W^{2,\infty}([0,\infty))$, *and that*

$$\underline{W}(s,t) := (T-t)^{\alpha} \cdot w((T-t)^{-\beta} s) \ , \ s \in [0, R^n] \ , \ t \in [0,T)$$

has the properties $\underline{W} \geq 0$, $\underline{W}_s > 0$, $\underline{W}_{ss} \leq 0$, $\underline{W}(0,t) = 0$ *for all* $t \in [0,T]$ *and* $\mathcal{P}\underline{W} \leq 0$ *almost everywhere for all*

$$T \leq \begin{cases} \min\{(\frac{m}{2M})^q, m^q, \left[A\left(\frac{\alpha}{\beta} - \varepsilon\right)(\xi_1 - a)^{\frac{\alpha}{\beta} - \varepsilon - 1}\right]^q, \frac{\eta}{c_3 M}\} \ , & 0 < q < 1 \ , \\ \min\{(\frac{m}{2M})^q, \left[A\left(\frac{\alpha}{\beta} - \varepsilon\right)(\xi_1 - a)^{\frac{\alpha}{\beta} - \varepsilon - 1}\right]^q, \left(\frac{\eta}{c_3 M [2A(\frac{\alpha}{\beta} - \varepsilon)(\xi_1 - a)^{\frac{\alpha}{\beta} - \varepsilon - 1}]^{q-1}}\right)^q\} \ , & q \geq 1 \ , \end{cases}$$

with $c_3 := \sup\left\{ \frac{\psi(s)}{s(1+s)^{q-1}} \mid s \in \left(0, (\frac{\alpha}{\beta} - \varepsilon)M\right) \right\}$. *Herein* $\mathcal{P}$ *is taken from (3.1) and (3.2).*

Proof. One easily checks that the graphs of

$$w_1(\xi) := m\xi \text{ and } w_2(\xi) := A(\xi - a)^{\frac{\alpha}{\beta} - \varepsilon}$$

touch each other at ξ_1 if

$$a = \left(1 + \varepsilon - \frac{\alpha}{\beta}\right)\xi_1 \text{ and } m = \left(\frac{\alpha}{\beta} - \varepsilon\right)^{\frac{\alpha}{\beta} - \varepsilon} \xi_1^{-(1+\varepsilon-\frac{\alpha}{\beta})} \cdot A \ .$$

In this case, evidently, $w \in W^{2,\infty}([0,\infty))$. Therefore we define $a := \left(1 + \varepsilon - \frac{\alpha}{\beta}\right)\xi_1$ and $\xi_0 := \xi_1$. Then $a \in (0,\xi_1)$. Finally we pick some $A > A_0(a,c_1,c_2,n,p,q,\varepsilon,\eta,\xi_0)$ such that

$$m := \left(\frac{\alpha}{\beta} - \varepsilon\right)^{\frac{\alpha}{\beta} - \varepsilon} \xi_1^{-(1+\varepsilon-\frac{\alpha}{\beta})} \cdot A \geq \begin{cases} \left(\frac{2^{2-q}}{c_2 q}\right)^{\frac{1}{q}} \ , & \text{if } 0 < q < 1 \ , \\ \left(\frac{2}{c_2 q}\right)^{\frac{1}{q}} \ , & \text{if } q \geq 1 \ , \end{cases}$$

so that $\mathcal{E}w_1 \leq 0$ on $[0,\infty)$ by Lemma 3.2. Moreover, Lemma 3.3 implies $\mathcal{E}w_2 \leq 0$ on $[\xi_1,\infty)$. From $w \geq 0$, $w' > 0$, and $w'' \leq 0$ it immediately follows that $\underline{W} \geq 0$, $\underline{W}_s > 0$, and $\underline{W}_{ss} \leq 0$. In particular we obtain $\mathcal{P}\underline{W} \leq 0$ for all T as given above. Obviously, $\underline{W}(0,t) = 0$. ////

Theorem 3.5 (Finite-time blow-up) *Assume (3.3) and (3.4), and suppose that* u_0 *is radially nonincreasing. Then there exists* $R_0 = R_0(c_1,c_2,M,n,p,q,R) \in (0,R]$ *such that* $T_{\max} < \infty$ *for all those* u_0 *which additionally fulfil* $\operatorname{supp}(u_0) \subset \overline{B}(R_0,0)$. *To be more precise: For all* $\varepsilon \in \left(0, \min\left\{1 - \frac{2}{n(p+q)}, \frac{2}{np} - \frac{2}{n(p+q)}\right\}\right)$ *and* $M, R > 0$ *there exists a constant* $C = C(c_1,c_2,n,p,q,\varepsilon) \in (0,1]$ *such that for all*

$$T \leq C \min\left\{1, \frac{1}{M^q}, (c_3 M)^{-\max\{1,q\}}, R^{\frac{2q}{p+q}}, M^{\frac{2q}{n(p+q)\varepsilon}} R^{\frac{2q}{(p+q)\varepsilon}\left(\frac{2}{n(p+q)} + \varepsilon\right)}\right\}$$

with $c_3 := \sup\left\{\frac{\psi(s)}{s(1+s)^{q-1}} \mid s \in \left(0, \left(1 - \frac{2}{n(p+q)} - \varepsilon\right) M\right)\right\}$ *and*

$$R_0 \le \min\{R, CM^{\frac{1}{n}} T^{\frac{1}{nq}} R\} \ ,$$

all solutions for radially nonincreasing initial data u_0 *with* $\operatorname{supp}(u_0) \subset \overline{B}(R_0, 0)$ *and* $M = \frac{1}{|B_R|}\int_{B_R} u_0 \mathrm{d}x$ *undergo a finite-time blow-up at* $T_{\max} \le T$.

Proof. Let $\varepsilon \in \left(0, \min\left\{\frac{\alpha}{\beta}, \frac{\alpha}{\beta} - 1 + \frac{2}{np}\right\}\right)$ be arbitrary. Then there exists $\eta \in (0, \widetilde{\eta})$ with $\varepsilon \in \left(\frac{\alpha}{\beta} - \frac{\alpha}{\beta+\eta}, \min\left\{\frac{\alpha}{\beta}, \frac{\alpha}{\beta} - 1 + \frac{2}{np}\right\}\right)$. Then we apply Corollary 3.4 with $\xi_1 := 1$. Let further be in the case $0 < q < 1$

$$T \le \min\left\{\left(\frac{m}{2M}\right)^q, m^q, \left[A\left(\frac{\alpha}{\beta} - \varepsilon\right)(1-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\right]^q, \frac{\eta}{c_3 M}, R^{\frac{n}{\beta}}, \left(\frac{M}{A} R^{n(1+\varepsilon-\frac{\alpha}{\beta})}\right)^{\frac{1}{\beta\varepsilon}}\right\}$$

and in the case $q \ge 1$

$$\begin{aligned} T \quad \le \quad \min\Bigg\{ & \left(\tfrac{m}{2M}\right)^q, \left[A\left(\tfrac{\alpha}{\beta} - \varepsilon\right)(1-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\right]^q, \\ & \left(\frac{\eta}{c_3 M\left[2A(\frac{\alpha}{\beta}-\varepsilon)(1-a)^{\frac{\alpha}{\beta}-\varepsilon-1}\right]^{q-1}}\right)^q, R^{\frac{n}{\beta}}, \left(\tfrac{M}{A} R^{n(1+\varepsilon-\frac{\alpha}{\beta})}\right)^{\frac{1}{\beta\varepsilon}} \Bigg\} \ . \end{aligned}$$

Apart from this, let

$$R_0 \le \min\left\{R, \left(\frac{MT^{\frac{1}{q}}}{m}\right)^{\frac{1}{n}} R\right\}$$

be arbitrary. Then for all $t \in [0, T)$ we have $(T-t)^{-\beta} R^n \ge T^{-\beta} R^n \ge 1 = \xi_1$ and therefore

$$\begin{aligned} \underline{W}(R^n, t) &= (T-t)^\alpha w((T-t)^{-\beta} R^n) \\ &= (T-t)^\alpha A[(T-t)^{-\beta} R^n]^{\frac{\alpha}{\beta}-\varepsilon} \\ &\le T^{\beta\varepsilon} A R^{n(\frac{\alpha}{\beta}-\varepsilon)} \\ &\le MR^n \ . \end{aligned}$$

Hence,

$$\underline{W}(s, 0) \le \underline{W}(R^n, 0) \le MR^n = W(s, 0) \text{ for all } s \in [R_0^n, R^n]$$

and

$$\underline{W}(s, 0) = T^\alpha w(T^{-\beta} s) \le mT^\alpha T^{-\beta} s = msT^{-\frac{1}{q}}$$

in view of $w(0) = 0$, $w'(0) = m$, and $w'' \le 0$. Next, we show $msT^{-\frac{1}{q}} \le W(s, 0)$ for all $s \in [0, R_0^n)$. In order to verify this, we set

$$f(s) := W(s, 0) = n \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_0(\rho) \mathrm{d}\rho \ .$$

This implies

$$f'(s) = s^{\frac{1}{n}-1} s^{1-\frac{1}{n}} u_0\left(s^{\frac{1}{n}}\right) = u_0\left(s^{\frac{1}{n}}\right)$$

which is nonincreasing in s. Hence we get for all $s \in [0, R_0^n)$

$$W(s, 0) = f(s) \ge \frac{f(R_0^n) - f(0)}{R_0^n - 0} s = \frac{MR^n s}{R_0^n} \ .$$

By the choice of R_0 it follows that $W(s, 0) \ge \underline{W}(s, 0)$ for all $s \in [0, R_0^n)$. Thus, assuming $T_{\max} > T$, we obtain from the weak comparison principle (Theorem 3.1) that for all $\tau \in (0, T)$,

$$\underline{W}(s, t) \le W(s, t) \text{ for all } s \in [0, R^n] \text{ and } t \in [0, \tau] \ .$$

Because $\tau \in (0,T)$ was arbitrary, this inequality therefore holds for all $t \in [0,T)$. Recalling $W(s,t) = n\int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\mathrm{d}\rho$, we infer

$$\frac{W(h,t)}{h} = \frac{n}{h}\int_0^{h^{\frac{1}{n}}} u(\rho,t)\mathrm{d}\rho = \frac{1}{h}\int_0^h u(r^n,t)\mathrm{d}r \to u(0,t) \text{ for } h \searrow 0 \ .$$

Hence, together with

$$\underline{W}_s(s,t) = (T-t)^{\alpha-\beta}w'((T-t)^{-\beta}s)$$

one finally obtains

$$u(0,t) = \lim_{h\searrow 0}\frac{W(h,t)}{h} \geq \lim_{h\searrow 0}\frac{\underline{W}(h,t)}{h} = \underline{W}_s(0,t) = (T-t)^{\alpha-\beta}m = (T-t)^{-\frac{1}{q}}m$$

for all $t \in [0,T)$, which contradicts the assumption $T < T_{\max}$. ////

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