Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source

Michael Winkler
Fakultät für Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany

Abstract

We consider nonnegative solutions of the Neumann boundary value problem for the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

in a smooth bounded convex domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$, where $\tau > 0, \chi \in \mathbb{R}$ and f is a smooth function generalizing the logistic source

$$f(s) = \kappa s - \mu s^2, \quad s \ge 0, \qquad \text{with } \kappa > 0,\ \mu > 0.$$

It is shown that if μ is sufficiently large then for all sufficiently smooth initial data the problem possesses a unique global-in-time classical solution that is bounded in $\Omega \times (0, \infty)$. Known results, asserting boundedness under the additional restriction $n \le 2$, are thereby extended to arbitrary space dimensions.

Key words: chemotaxis, logistic source, global existence, boundedness
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Introduction

We consider nonnegative solutions of the Neumann initial-boundary value problem for two coupled parabolic equations,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \big(u \nabla v\big) + f(u), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{0.1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, where $\tau > 0$, $\chi \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies $f(0) \ge 0$ as well as

$$f(s) \le a - \mu s^2 \qquad \text{for all } s \ge 0 \tag{0.2}$$

with some $a \geq 0$ and $\mu > 0$.
The PDE system in (0.1) is used in mathematical biology to model the mechanism of chemotaxis, that is, the movement of cells in response to the presence of a chemical signal substance which is inhomogeneously distributed in space. Here, u represents the cell density and v the concentration of the chemical ([KS]). When $\chi > 0$, cells exhibit a tendency to move towards higher signal concentrations (chemoattraction), while conversely the choice $\chi < 0$ leads to a model for chemorepulsion, where cells prefer to move away from the chemical in question. A concise derivation of the PDE framework in (0.1) is presented in [HP], where the reader is provided with a wide survey on various chemotaxis models and their respective biological background, as well as with a rich selection of references.
As compared to the so-called minimal model obtained when $f \equiv 0$, (0.1) comprises a possible proliferation of cells, a growth restriction of logistic type being included by the assumption (0.2); accordingly, one might expect that (0.2) prevents an unlimited increase of the cell density. This conjecture is supported by numerical experiments (see e.g. [PH] or [S]), which in addition indicate that (0.1), though apparently simple as a two-component parabolic system, possesses quite a large variety of dynamical properties, especially in respect of the spontaneous emergence of patterns. Further evidence, both numerically and analytically, on the self-organizing features of (0.1) can be found in [MT], where shock-type movements of interfaces are detected when $f(u) = u(1-u)(u-a)$ ($0 < a < \frac{1}{2}$) and when cell kinetics take place much faster than cell movement (see also [HHS]).
Unfortunately, the literature seems to provide only partial results concerning the question whether or not solutions of (0.1) indeed remain bounded, or if under appropriate circumstances blow-up may occur. It is known, at least, that all solutions of (0.1) are bounded when $n = 1$ (which can be seen as in [OY1]) or $n = 2$ ([OTYM], see also [OY2]). In these cases, the problem even could be made accessible to strong tools of dynamical systems theory so as to shed light on its global dynamics, and in the one- and two-dimensional framework there are results on finite dimensional attractors available not only for (0.1) but also for a number of variants with more general cross-diffusion terms in the PDE for the cell density ([OY1], [OTYM], [AOTYM]).

However, results addressing the case $n \geq 3$ so far only cover simplified versions of (0.1). One of the most common among such simplifications concerns the choice $\tau = 0$ that reflects and takes to a limit the physically reasonable model assumption that chemicals diffuse much faster than cells move ([JL]). In the accordingly obtained initial-boundary value problem for the parabolic-elliptic analogue of (0.1),

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \big(u \nabla v\big) + f(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{0.3}$$

solutions are global and bounded whenever $\mu > 0$ satisfies $\mu > \frac{n-2}{n}\chi$, while for any $\mu > 0$ one can at least construct globally existing weak solutions. For $\chi > 0$ this is precisely stated in [TW], but the proofs there can easily be carried over to the case $\chi \leq 0$ (cf. also [Wi1] for less restrictive assumptions on f).
In another simplified variant of (0.1) the second PDE contains no production term, and the outcome of [K] states that then a global classical solution exists if $\|v_0\|_{L^\infty(\Omega)}$ is small enough.
Finally, a further possibility of reducing technical difficulties in the proof of boundedness of solutions consists of incorporating so-called vloume-filling effects in the diffusive and the cross-diffusive part of the model, thus taking into account that cell movement is inhibited when cells are densely packed ([PH]). The resulting variant of (0.1) is known to have global bounded solutions, and the

associated dynamical system could be proved to possess a global attractor ([Wr]).

It is the purpose of the present paper to clarify the issue of boundedness for solutions of (0.1) without any restriction on the space dimension. Our main result is the following.

Theorem 0.1 *Suppose $\Omega \subset \mathbb{R}^n$, $n \ge 1$, is a convex bounded domain with smooth boundary, $\tau > 0$ and $\chi \in \mathbb{R}$. Then for all $a > 0$ there exists $\mu_0 > 0$ with the following property: If $f \in W^{1,\infty}_{loc}(\mathbb{R})$ satisfies $f(0) \ge 0$ and (0.2) holds with some $\mu \ge \mu_0$, then for any nonnegative u_0 and v_0 fulfilling the inclusions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\theta}(\Omega)$ for some $\theta > n$, (0.1) possesses a uniquely determined global solution (u, v) for which both u and v are bounded in $\Omega \times (0,\infty)$.*

Let us once again point out that when $n \le 2$ there is no novelty in this, but in the case $n \ge 3$ we can exclude unbounded solutions whenever μ in (0.2) is large enough, which is consistent with the results for (0.3) mentioned above. We have to leave open the question whether this is optimal in the sense that also in higher space dimensions boundedness can be asserted for arbitrarily small $\mu > 0$. That this should be true at least for negative χ is suggested by the fact that in that case it is known that in the borderline case $f \equiv 0$ at least global weak solutions always exist ([CLM-R]).

Along with the method by which it is obtained, Theorem 0.1 may serve as a starting point for further investigation addressing dynamical features of (0.1), in particular the existence of global attractors. Let us therefore briefly outline the main idea of our proof which can be obtained by a rather simple observation. Formally differentiating the second equation in (0.1) we obtain

$$\frac{\tau}{2}\frac{d}{dt}|\nabla v|^2 = \nabla v \cdot \nabla \Delta v - |\nabla v|^2 + \nabla u \cdot \nabla v.$$

Focussing on the most interesting case $\chi > 0$, we can express the last term using the first equation in (0.1) and rewrite $\nabla v \cdot \nabla \Delta v = \Delta\Big(\frac{1}{2}|\nabla v|^2\Big) - |D^2 v|^2$ to arrive at the identity

$$\frac{d}{dt}\Big(\frac{\tau}{2}|\nabla v|^2 + \frac{1}{\chi}u\Big) = \Delta\Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{\chi}u\Big) - |D^2 v|^2 - |\nabla v|^2 - u\Delta v + \frac{1}{\chi}f(u).$$

Since $|\Delta v|^2 \le n|D^2 v|^2$, by Young's inequality we obtain $-u\Delta v \le |D^2 v|^2 + \frac{n}{4}u^2$ and hence, in view of (0.2),

$$\frac{d}{dt}\Big(\frac{\tau}{2}|\nabla v|^2 + \frac{1}{\chi}u\Big) \le \Delta\Big(\frac{1}{2}|\nabla v|^2 + \frac{1}{\chi}u\Big) - |\nabla v|^2 - \Big(\frac{\mu}{\chi} - \frac{n}{4}\Big)u^2 + \frac{a}{\chi}.$$

Here we see that in the special case $\tau = 1$, essentially meaning that both cells and signal diffuse at the same speed, under the assumption $\mu > \frac{n\chi}{4}$ the function $z := \frac{\tau}{2}|\nabla v|^2 + \frac{1}{\chi}u$ satisfies a scalar parabolic inequality of the form

$$z_t \le \Delta z - bz + c \tag{0.4}$$

with some positive b and c. Now, for instance, the maximum principle becomes applicable to ensure that z is bounded – at least when Ω is convex, which entails that $\frac{\partial|\nabla v|^2}{\partial\nu} \le 0$ on $\partial\Omega$. This immediately yields boundedness of both u and ∇v.
When $\tau \ne 1$ (or $\chi \le 0$), this simple procedure fails. However, we note that the maximum principle was not the only way to derive boundedness from (0.4). One possible alternative would have been a Moser-type iteration, based on multiplying (0.4) by z^{m-1}, integrating, estimating suitably

and carefully tracking the appearing constants when $m \to \infty$. Here it turns out that the first step, namely integrating powers of z, after certain modifications is somewhat stable with respect to changing τ. More precisely, the core of our approach will consist of deriving a bound for the quantity

$$\sum_{k=0}^{m} b_k \cdot \int_\Omega u^k |\nabla v|^{2m-2k} \tag{0.5}$$

with arbitrarily large $m \in \mathbb{N}$ and appropriately constructed positive $b_0, ..., b_m$. Upon applying suitable inequalities to this term from above and below, one can easily check that estimating the sum in (0.5) is equivalent to estimating the norm of z in $L^m(\Omega)$, so that in this sense our procedure indeed is a variant of the method of integrating powers of z. Once this crucial step has been accomplished (cf. Lemma 2.7), we will know that the term $u\nabla v$ in the first equation in (0.1) is uniformly bounded in any space $L^p(\Omega)$, $1 < p < \infty$, for all times, so that it will be possible to apply standard regularity arguments to conclude that u and hence v must be bounded.

1 Preliminaries

Let us first briefly recall standard arguments (see [HW], for instance) to assert local well-posedness of (0.1) in a sense sufficient for our purpose. Moreover, we make sure that a solution can cease to exist in finite time only when it blows up in a certain norm.

Lemma 1.1 *Suppose that $\tau > 0$, $\chi \in \mathbb{R}$, and that $f \in W^{1,\infty}_{loc}(\mathbb{R})$ satisfies $f(0) \ge 0$. Moreover, assume that $u_0 \in C^0(\bar\Omega)$ and $v_0 \in W^{1,\theta}(\Omega)$ are nonnegative, where $\theta > n$. Then there exist a maximal $T_{max} \in (0,\infty]$ and a uniquely determined pair (u,v) of nonnegative functions*

$$u \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})),$$
$$v \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,\theta}(\Omega))$$

that solve (0.1) in the classical sense in $\Omega \times (0,T_{max})$. Moreover,

$$\textit{if } T_{max} < \infty \textit{ then } \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\theta}(\Omega)} \to \infty \qquad \textit{as } t \nearrow T_{max}. \tag{1.1}$$

Proof. i) **Existence.** We claim that for all $R > 0$ there exists $T = T(R) > 0$ such that if in addition to the above hypotheses we have $\|u_0\|_{L^\infty(\Omega)} \le R$ and $\|v_0\|_{W^{1,\theta}(\Omega)} \le R$, then (0.1) is classically solvable in $\Omega \times (0,T)$. In view of a standard extension argument, this will imply the existence of a maximal existence time T_{max} satisfying (1.1). To this end, according to well-known estimates for the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ (see [Wi2, Lemma 1.3], for instance, for a formulation adequate for our purpose) we first pick $K > 0$ such that $\|e^{t\Delta} z\|_{W^{1,\theta}(\Omega)} \le K\|z\|_{W^{1,\theta}(\Omega)}$ for all $z \in W^{1,\theta}(\Omega)$. For small $T \in (0,1)$ to be specified below, we introduce the Banach space

$$X := C^0([0,T]; C^0(\bar\Omega)) \times C^0([0,T]; W^{1,\theta}(\Omega))$$

along with its closed subset

$$S := \Big\{ (u,v) \in X \ \Big|\ \|u\|_{L^\infty((0,T);L^\infty(\Omega))} \le R+1 \text{ and } \|v\|_{L^\infty((0,T);W^{1,\theta}(\Omega))} \le KR+1 \Big\}.$$

For $(u,v) \in S$ and $t \in [0,T]$, we let

$$\Phi(u,v)(t) := \begin{pmatrix} \Phi_1(u,v)(t) \\ \Phi_2(u,v)(t) \end{pmatrix} := \begin{pmatrix} e^{t\Delta}u_0 - \chi \int_0^t e^{(t-s)\Delta}\nabla \cdot (u(s)\nabla v(s))ds + \int_0^t e^{(t-s)\Delta} f(u(s))ds \\ e^{\frac{t}{\tau}(\Delta-1)}v_0 + \frac{1}{\tau}\int_0^t e^{(t-s)(\Delta-1)}u(s)ds \end{pmatrix}.$$

Then

$$\begin{aligned} \|\Phi_1(u,v)(t)\|_{L^\infty(\Omega)} &\leq \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + |\chi| \int_0^t \|e^{(t-s)\Delta}\nabla \cdot (u(s)\nabla v(s))\|_{L^\infty(\Omega)} ds \\ &\quad + \int_0^t \|e^{(t-s)\Delta} f(u(s))\|_{L^\infty(\Omega)} ds, \end{aligned} \tag{1.2}$$

where by the maximum principle

$$\|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} \leq R \tag{1.3}$$

and

$$\int_0^t \|e^{t\Delta} f(u(s))\|_{L^\infty(\Omega)} ds \leq \int_0^t \|f(u(s))\|_{L^\infty(\Omega)} ds \leq \|f\|_{L^\infty((-R-1,R+1))} \cdot T \tag{1.4}$$

for all $t \in (0,T)$. Moreover, picking any $p > \frac{n\theta}{\theta - n}$ and then $\alpha \in (\frac{n}{p}, \frac{1}{2} - \frac{n}{2}(\frac{1}{\theta} - \frac{1}{p}))$, we have $p\alpha > n$ and therefore the fractional power A^α of the sectorial operator $A := -\Delta + 1$ with Neumann data in $L^p(\Omega)$ satisfies $\|z\|_{L^\infty(\Omega)} \leq C\|A^\alpha z\|_{L^p(\Omega)}$ as well as $\|A^\alpha e^{\sigma\Delta} z\|_{L^p(\Omega)} \leq C\sigma^{-\alpha}\|z\|_{L^p(\Omega)}$ for all $z \in C_0^\infty(\Omega)$ (cf. [H]), where here and below C denotes a generic positive constant. As a consequence,

$$\begin{aligned} |\chi| \int_0^t \|e^{(t-s)\Delta}\nabla \cdot (u(s)\nabla v(s))\|_{L^\infty(\Omega)} ds &\leq C \int_0^t \|A^\alpha e^{(t-s)\Delta}\nabla \cdot (u(s)\nabla v(s))\|_{L^p(\Omega)} ds \\ &\leq C \int_0^t (t-s)^{-\alpha} \|e^{\frac{t-s}{2}\Delta}\nabla \cdot (u(s)\nabla v(s))\|_{L^p(\Omega)} ds \\ &\leq C \int_0^t (t-s)^{-\alpha} \cdot (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{\theta} - \frac{1}{p})} \|u(s)\nabla v(s)\|_{L^\theta(\Omega)} ds \\ &\leq CT^{\frac{1}{2} - \alpha - \frac{n}{2}(\frac{1}{\theta} - \frac{1}{p})} \cdot (R+1) \cdot (KR+1) \end{aligned} \tag{1.5}$$

for all $t \in (0,T)$, where we have used that $T < 1$, that $\alpha < \frac{1}{2} - \frac{n}{2}(\frac{1}{\theta} - \frac{1}{p})$, and that $\|e^{\sigma\Delta}\nabla \cdot z\|_{L^p(\Omega)} \leq C\sigma^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{\theta} - \frac{1}{p})}\|z\|_{L^\theta(\Omega)}$ for $\sigma < 1$ and all ($\mathbb{R}^n$-valued) $z \in C_0^\infty(\Omega)$ (cf. [Wi2, Lemma 1.3]). Similarly,

$$\begin{aligned} \|\Phi_2(u,v)(t)\|_{W^{1,\theta}(\Omega)} &\leq e^{-\frac{t}{\tau}}\|e^{\frac{t}{\tau}\Delta}v_0\|_{W^{1,\theta}(\Omega)} + C\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^\theta(\Omega)} ds \\ &\leq K\|v_0\|_{W^{1,\theta}(\Omega)} + C\int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^\infty(\Omega)} ds \\ &\leq KR + CT^{\frac{1}{2}} \cdot (R+1) \qquad \text{for all } t \in (0,T), \end{aligned} \tag{1.6}$$

so that it follows from (1.2)–(1.6) that if we fix $T_0 \in (0,1)$ small enough and T satisfies $T \in (0, T_0)$ then Φ maps S into itself.

Furthermore, using the same ideas, for $(u,v) \in S$ and $(\bar{u},\bar{v}) \in S$ we estimate

$$\begin{aligned}
\|\Phi_1(u,v)(t) - \Phi_1(\bar{u},\bar{v})(t)\|_{L^\infty(\Omega)} &\le C\int_0^t \|A^\alpha e^{(t-s)\Delta}\nabla\cdot[u(s)\nabla v(s) - \bar{u}(s)\nabla\bar{v}(s)]\|_{L^p(\Omega)}ds \\
&\quad + \int_0^t \|e^{(t-s)\Delta}(f(u(s)) - f(\bar{u}(s)))\|_{L^\infty(\Omega)}ds \\
&\le C\int_0^t (t-s)^{-\alpha-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{\theta})}\|u(s)\nabla v(s) - \bar{u}(s)\nabla\bar{v}(s)\|_{L^\theta(\Omega)}ds \\
&\quad + \int_0^t \|f(u(s)) - f(\bar{u}(s))\|_{L^\infty(\Omega)}ds \\
&\le C\int_0^t (t-s)^{-\alpha-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{\theta})}\Big(\|u(s)\|_{L^\infty(\Omega)}\|\nabla v(s) - \nabla\bar{v}(s)\|_{L^\theta(\Omega)} \\
&\qquad\qquad + \|u(s) - \bar{u}(s)\|_{L^\infty(\Omega)}\|\nabla\bar{v}(s)\|_{L^\theta(\Omega)}\Big)ds \\
&\quad + \|f'\|_{L^\infty((-R-1,R+1))}\int_0^t \|u(s) - \bar{u}(s)\|_{L^\infty(\Omega)}ds \\
&\le CT^{\frac{1}{2}-\alpha-\frac{n}{2}(\frac{1}{p}-\frac{1}{\theta})}\Big((R+1) + (KR+1)\Big)\cdot\|(u,v) - (\bar{u},\bar{v})\|_X \\
&\quad + \|f'\|_{L^\infty((-R-1,R+1))}\cdot T\cdot\|(u,v) - (\bar{u},\bar{v})\|_X
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi_2(u,v)(t) - \Phi_2(\bar{u},\bar{v})(t)\|_{W^{1,\theta}(\Omega)} &\le C\int_0^t (t-s)^{-\frac{1}{2}}\|u(s) - \bar{u}(s)\|_{L^\theta(\Omega)}ds \\
&\le CT^{\frac{1}{2}}\cdot\|(u,v) - (\bar{u},\bar{v})\|_X
\end{aligned}$$

for all $t \in (0,T)$, which shows that if $T \in (0,T_0)$ is chosen sufficiently small then Φ acts as a contraction on S. Accordingly, the Banach fixed point theorem asserts the existence of some $(u,v) \in S$ such that $\Phi(u,v) = (u,v)$. Once again using standard arguments involving semigroup estimates, it can easily be checked that in fact (u,v) lies in the asserted regularity class and is a classical solution of (0.1) in $\Omega\times(0,T)$. Since $f(0) \ge 0$, the maximum principle moreover ensures that both u and v are nonnegative.

ii) **Uniqueness.** Proceeding as in [GZ], given $T > 0$ and two solutions (u,v) and $(\bar{u},\bar{v})$ in $\Omega\times(0,T)$, we fix $T_0 \in (0,T)$, let $w := u - \bar{u}$ and $z := v - \bar{v}$ and obtain by applying straightforward testing procedures to (0.1) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 = \chi\int_\Omega w\nabla v\cdot\nabla w + \chi\int_\Omega \bar{u}\nabla w\cdot\nabla z + \int_\Omega (f(u) - f(\bar{u}))w \tag{1.7}$$

and

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega |\nabla z|^2 + \int_\Omega |\Delta z|^2 + \int_\Omega |\nabla z|^2 = -\int_\Omega w\Delta z \tag{1.8}$$

for $t \in (0, T_0)$. By Hölder's, Young's and the Gagliardo-Nirenberg inequalities,

$$\begin{aligned}\chi \int_\Omega w \nabla v \cdot \nabla w &\leq |\chi| \Big(\int_\Omega |\nabla w|^2 \Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\nabla v|^\theta \Big)^{\frac{1}{\theta}} \cdot \Big(\int_\Omega w^{\frac{2\theta}{\theta-2}} \Big)^{\frac{\theta-2}{2\theta}} \\ &\leq C \Big(\int_\Omega |\nabla w|^2 \Big)^{\frac{1}{2}+\frac{n}{2\theta}} \cdot \Big(\int_\Omega |\nabla v|^\theta \Big)^{\frac{1}{\theta}} \cdot \Big(\int_\Omega w^2 \Big)^{\frac{\theta-n}{2\theta}} \\ &\leq \frac{1}{2} \int_\Omega |\nabla w|^2 + C \int_\Omega w^2,\end{aligned}$$

where we have used that $\int_\Omega w = 0$ by simple integration of (0.1), that $\|\nabla v(t)\|_{L^\theta(\Omega)} \leq C$ for $t \in (0, T_0)$, and that $\theta > n \geq 2$. Furthermore,

$$\chi \int_\Omega \bar{u} \nabla w \cdot \nabla z \leq \frac{1}{2} \int_\Omega |\nabla w|^2 + C \int_\Omega |\nabla z|^2$$

and

$$\int_\Omega (f(u) - f(\bar{u}))w \leq C \int_\Omega w^2$$

in view of the boundedness of u and $\bar{u}$ in $\Omega \times (0, T_0)$ and the local Lipschitz continuity of f. Since finally

$$-\int_\Omega w \Delta z \leq \int_\Omega |\Delta z|^2 + \frac{1}{4} \int_\Omega w^2,$$

we conclude upon adding (1.7) and (1.8) that

$$\frac{d}{dt} \Big(\int_\Omega w^2 + \tau \int_\Omega |\nabla z|^2 \Big) \leq C \cdot \Big(\int_\Omega w^2 + \tau \int_\Omega |\nabla z|^2 \Big) \qquad \text{for all } t \in (0, T_0),$$

which implies $w \equiv 0$ and $z \equiv 0$ in $\Omega \times (0, T_0)$ and hence $(u, v) \equiv (\bar{u}, \bar{v})$ in $\Omega \times (0, T)$, because $T_0 \in (0, T)$ was arbitrary. ////

2 A priori estimates

In this section we shall develop our main ingredient for the proof of Theorem 0.1. As a first step towards this, let us derive the following, yet rather weak, a priori bounds for solutions of (0.1).

Lemma 2.1 *Let f satisfy (0.2). Then there exists $A > 0$ such that the solution (u, v) of (0.1) satisfies*

$$\int_\Omega u(x,t)dx \leq A \qquad \textit{and} \qquad \int_\Omega |\nabla v(x,t)|^2 dx \leq A \qquad \textit{for all } t \in (0, T_{max}). \tag{2.1}$$

PROOF. Integrating the first equation in (0.1) using (0.2) gives

$$\frac{d}{dt}\int_\Omega u = \int_\Omega f(u) \le a|\Omega| - \mu \int_\Omega u^2 \qquad \text{for all } t \in (0, T_{max}). \tag{2.2}$$

Since $u^2 \ge 2u - 1$, this yields

$$\frac{d}{dt}\int_\Omega u \le (\mu + a)|\Omega| - 2\mu \int_\Omega u \qquad \text{for all } t \in (0, T_{max}). \tag{2.3}$$

Next, multiplying the second equation in (0.1) by $-\Delta v$ and integrating, we see that

$$\frac{\tau}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 = -\int_\Omega u\Delta v \le \frac{1}{4}\int_\Omega u^2 + \int_\Omega |\Delta v|^2$$

for all $t \in (0, T_{max})$ by Young's inequality. Hence, in view of (2.2) and (2.3) we obtain

$$\begin{aligned} \frac{d}{dt}\Big(\frac{\tau}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2\mu}\int_\Omega u\Big) + \int_\Omega |\nabla v|^2 &\le \frac{1}{4}\int_\Omega u^2 + \frac{1}{4\mu}\Big(a|\Omega| - \mu\int_\Omega u^2\Big) + \frac{1}{4\mu}\frac{d}{dt}\int_\Omega u \\ &= \frac{a|\Omega|}{4\mu} + \frac{1}{4\mu}\Big((\mu + a)|\Omega| - 2\mu\int_\Omega u\Big) \end{aligned}$$

for all $t \in (0, T_{max})$. Thus, $y(t) := \frac{\tau}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2\mu}\int_\Omega u$ satisfies the ODI $y' \le -cy + C$ in $(0, T_{max})$ with $c := \min\{\frac{2}{\tau}, \mu\} > 0$ and $C := \frac{(\mu+2a)|\Omega|}{4\mu}$. Now an integration shows that $y(t) \le \max\{y(0), \frac{C}{c}\}$ for all $t \in (0, T_{max})$ and thereby completes the proof. ////

Still having in mind the sum in (0.5), we prepare a proof of its boundedness by following estimate for its summands. The following lemma is the only place in this paper in which we immediately require that Ω be convex.

Lemma 2.2 *Suppose that Ω is convex, $\tau > 0$ and $\chi \in \mathbb{R}$, and that f satisfies (0.2). Then for all $m \in \mathbb{N}$ and $k \in \{0, ..., m\}$, the solution (u, v) of (0.1) satisfies the inequality*

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^k|\nabla v|^{2m-2k} &+k(k-1)\int_\Omega u^{k-2}|\nabla u|^2|\nabla v|^{2m-2k} \\ &+\frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}|D^2 v|^2 \\ &+\frac{(m-k)(m-k-1)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-4}\Big|\nabla|\nabla v|^2\Big|^2 \\ &+k\mu\int_\Omega u^{k+1}|\nabla v|^{2m-2k} \\ \le\ & \frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}\nabla u\cdot\nabla v \\ &+k(k-1)\chi\int_\Omega u^{k-1}|\nabla v|^{2m-2k}\nabla u\cdot\nabla v \\ &+k(m-k)\chi\int_\Omega u^k|\nabla v|^{2m-2k-2}\nabla v\cdot\nabla|\nabla v|^2 \end{aligned}$$

$$-k(m-k)\Big(1+\frac{1}{\tau}\Big)\int_\Omega u^{k-1}|\nabla v|^{2m-2k-2}\nabla u\cdot\nabla|\nabla v|^2$$
$$+ak\int_\Omega u^{k-1}|\nabla v|^{2m-2k} \qquad \textit{for all } t\in(0,T_{max}). \tag{2.4}$$

PROOF. By direct differentiation we obtain, using the fact that (u,v) is a smooth solution of (0.1) in $\bar\Omega\times(0,T_{max})$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^k|\nabla v|^{2m-2k} &= k\int_\Omega u^{k-1}|\nabla v|^{2m-2k}\cdot\Big[\Delta u-\chi\nabla\cdot(u\nabla v)+f(u)\Big]\\ &\quad +\frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}\nabla v\cdot\nabla(\Delta v-v+u)\\ &=: I_1+I_2.\end{aligned} \tag{2.5}$$

Here, I_1 gives a nonzero contribution only in the case $k\ge 1$, in which we integrate by parts to obtain

$$\begin{aligned}k\int_\Omega u^{k-1}|\nabla v|^{2m-2k}\Delta u &= -k(k-1)\int_\Omega u^{k-2}|\nabla u|^2|\nabla v|^{2m-2k}\\ &\quad -k(m-k)\int_\Omega u^{k-1}|\nabla v|^{2m-2k-2}\nabla u\cdot\nabla|\nabla v|^2\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}k\chi\int_\Omega u^{k-1}|\nabla v|^{2m-2k}\nabla\cdot(u\nabla v) &= k(k-1)\chi\int_\Omega u^{k-1}|\nabla v|^{2m-2k}\nabla u\cdot\nabla v\\ &\quad +k(m-k)\chi\int_\Omega u^k|\nabla v|^{2m-2k-2}\nabla v\cdot\nabla|\nabla v|^2,\end{aligned} \tag{2.7}$$

where we have used that both $\frac{\partial u}{\partial\nu}$ and $\frac{\partial v}{\partial\nu}$ vanish on $\partial\Omega$. Moreover, by (0.2) we have

$$k\int_\Omega u^{k-1}|\nabla v|^{2m-2k}f(u) \le ka\int_\Omega u^{k-1}|\nabla v|^{2m-2k}-k\mu\int_\Omega u^{k+1}|\nabla v|^{2m-2k}. \tag{2.8}$$

In treating I_2, we make use of the pointwise identity

$$\nabla v\cdot\nabla\Delta v=\frac12\Delta(|\nabla v|^2)-|D^2v|^2$$

to obtain

$$\begin{aligned}I_2 &= \frac{m-k}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}\Delta|\nabla v|^2-\frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}|D^2v|^2\\ &\quad -\frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k}+\frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}\nabla u\cdot\nabla v\\ &=: I_{21}+I_{22}+I_{23}+I_{24},\end{aligned} \tag{2.9}$$

Clearly,

$$I_{23}\le 0, \tag{2.10}$$

whereas I_{22} and I_{24} exactly coincide with the third term on the left and the first term on the right of (2.4). As to I_{21}, another integration by parts reveals that

$$\begin{aligned} I_{21} \quad = \quad & -\frac{k(m-k)}{\tau}\int_\Omega u^{k-1}|\nabla v|^{2m-2k-2}\nabla u\cdot\nabla|\nabla v|^2 \\ & -\frac{(m-k)(m-k-1)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-4}\Big|\nabla|\nabla v|^2\Big|^2 \\ & +\frac{m-k}{\tau}\int_{\partial\Omega} u^k|\nabla v|^{2m-2k-2}\frac{\partial|\nabla v|^2}{\partial\nu}. \end{aligned} \tag{2.11}$$

Since the convexity of $\partial\Omega$ along with the relation $\frac{\partial v}{\partial\nu}=0$ on $\partial\Omega$ implies that $\frac{\partial|\nabla v|^2}{\partial\nu}\le 0$ on $\partial\Omega$ (cf. [DalPGG]), collecting (2.5)–(2.11) yields (2.4) after obvious rearrangements. ////

Our plan is to estimate the right-hand side of (2.4) appropriately. This will be done separately for the cases $k=0, k=1, k\in\{2,...,m-1\}$ and $k=m$ in the next four lemmata. To begin with, let us start with the case when u does not appear on the left of (2.4).

Lemma 2.3 *Suppose that Ω is convex, $\tau>0$ and $\chi\in\mathbb{R}$, and that f satisfies (0.2). Then for all $m\in\mathbb{N}$ with $m\ge 2$ there exist positive constants c_0 and C_0 such that*

$$\frac{d}{dt}\int_\Omega|\nabla v|^{2m}+c_0\int_\Omega|\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2\le C_0\int_\Omega u^2|\nabla v|^{2m-2} \qquad \textit{for all } t\in(0,T_{max}). \tag{2.12}$$

PROOF. When applied to $k=0$, (2.4) takes the form

$$\begin{aligned} \frac{d}{dt}\int_\Omega|\nabla v|^{2m} \quad + \quad & \frac{2m}{\tau}\int_\Omega|\nabla v|^{2m-2}|D^2v|^2+\frac{m(m-1)}{\tau}\int_\Omega|\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 \\ \le \quad & \frac{2m}{\tau}\int_\Omega|\nabla v|^{2m-2}\nabla u\cdot\nabla v. \end{aligned} \tag{2.13}$$

Here an integration by parts yields

$$\frac{2m}{\tau}\int_\Omega|\nabla v|^{2m-2}\nabla u\cdot\nabla v=-\frac{2m(m-1)}{\tau}\int_\Omega u|\nabla v|^{2m-4}\nabla v\cdot\nabla|\nabla v|^2-\frac{2m}{\tau}\int_\Omega u|\nabla v|^{2m-2}\Delta v, \tag{2.14}$$

because $\frac{\partial u}{\partial\nu}=0$ on $\partial\Omega$. By Young's inequality,

$$\begin{aligned} -\frac{2m(m-1)}{\tau}\int_\Omega u|\nabla v|^{2m-4}\nabla v\cdot\nabla|\nabla v|^2 \quad \le \quad & \frac{m(m-1)}{2\tau}\int_\Omega|\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 \\ & +\frac{2m(m-1)}{\tau}\int_\Omega u^2|\nabla v|^{2m-2} \end{aligned} \tag{2.15}$$

and

$$-\frac{2m}{\tau}\int_\Omega u|\nabla v|^{2m-2}\Delta v\le\frac{2m}{n\tau}\int_\Omega|\nabla v|^{2m-2}|\Delta v|^2+\frac{mn}{2\tau}\int_\Omega u^2|\nabla v|^{2m-2}. \tag{2.16}$$

Recalling that the Cauchy-Schwarz inequality entails the pointwise estimate

$$|\Delta v|^2 = \Big|\sum_{i=1}^n \frac{\partial^2 v}{\partial x_i^2}\Big|^2 \le n \cdot \sum_{i=1}^n \Big(\frac{\partial^2 v}{\partial x_i^2}\Big)^2 \le n|D^2 v|^2,$$

we may combine (2.13)–(2.16) to arrive at

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2m} + \frac{m(m-1)}{2\tau}\int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 \le \Big(\frac{2m(m-1)}{\tau} + \frac{mn}{2\tau}\Big)\cdot \int_\Omega u^2|\nabla v|^{2m-2},$$

which coincides with (2.12) upon evident choices of c_0 and C_0. ////

In order to absorb the term on the right of (2.12), our sum in (0.5) will contain a suitable multiple of the term in (2.4) obtained for $k=1$. This is made possible by the next lemma.

Lemma 2.4 *Suppose that Ω is convex, $\tau > 0$ and $\chi \in \mathbb{R}$, and that f satisfies (0.2). Then for all $m \in \mathbb{N}$ with $m \ge 2$ there exists $C_1 > 0$ such that*

$$\begin{aligned}\frac{d}{dt}\int_\Omega u|\nabla v|^{2m-2} + \Big(\mu - C_1\Big)\cdot\int_\Omega u^2|\nabla v|^{2m-2} \quad &\le \quad \int_\Omega |\nabla u|^2|\nabla v|^{2m-4} + C_1\int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 \\ &\qquad + C_1\int_\Omega |\nabla v|^{2m-2} \quad \textit{for all } t \in (0,T_{max}). \end{aligned} \tag{2.17}$$

Proof. By (2.4) with $k=1$,

$$\begin{aligned}\frac{d}{dt}\int_\Omega u|\nabla v|^{2m-2} \quad &+ \frac{2(m-1)}{\tau}\int_\Omega u|\nabla v|^{2m-4}|D^2v|^2 + \frac{(m-1)(m-2)}{\tau}\int_\Omega u|\nabla v|^{2m-6}\Big|\nabla|\nabla v|^2\Big|^2 \\ &+\mu\int_\Omega u^2|\nabla v|^{2m-2} \\ \le \quad &\frac{2(m-1)}{\tau}\int_\Omega u|\nabla v|^{2m-4}\nabla u\cdot\nabla v + (m-1)\chi\int_\Omega u|\nabla v|^{2m-4}\nabla v\cdot\nabla|\nabla v|^2 \\ &-(m-1)\Big(1+\frac{1}{\tau}\Big)\int_\Omega |\nabla v|^{2m-4}\nabla u\cdot\nabla|\nabla v|^2 + a\int_\Omega |\nabla v|^{2m-2}. \end{aligned} \tag{2.18}$$

Here, the first three terms on the right can be estimated by Young's inequality according to

$$\frac{2(m-1)}{\tau}\int_\Omega u|\nabla v|^{2m-4}\nabla u\cdot\nabla v \le \frac12\int_\Omega |\nabla u|^2|\nabla v|^{2m-4} + \frac{2(m-1)^2}{\tau^2}\int_\Omega u^2|\nabla v|^{2m-2}$$

and

$$(m-1)\chi\int_\Omega u|\nabla v|^{2m-4}\nabla v\cdot\nabla|\nabla v|^2 \le \int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 + \frac{(m-1)^2\chi^2}{4}\int_\Omega u^2|\nabla v|^{2m-2}$$

as well as

$$\begin{aligned}-(m-1)\Big(1+\frac{1}{\tau}\Big)\int_\Omega |\nabla v|^{2m-4}\nabla u\cdot\nabla|\nabla v|^2 \quad &\le \quad \frac12\int_\Omega |\nabla u|^2|\nabla v|^{2m-4} \\ &\qquad + \frac{(m-1)^2(1+\frac{1}{\tau})^2}{2}\int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2. \end{aligned}$$

Dropping the second and the third term on the left of (2.18), we thus easily find that (2.17) holds if we let $C_1 := \max\{\frac{(m-1)^2\chi^2}{4} + \frac{2(m-1)^2}{\tau^2}, \frac{(m-1)^2(1+\frac{1}{\tau})^2}{2} + 1, a\}$. ////

Again, the right-hand side in (2.17) contains ill-signed terms that have to be coped with. Treating the third will be postponed until the proof of Corollary 2.8, while the second already appeared with a favorable sign in (2.12). The first will be estimated using the following lemma.

Lemma 2.5 *Suppose that Ω is convex, $\tau > 0$ and $\chi \in \mathbb{R}$, and that f satisfies (0.2). Then for all $m \in \mathbb{N}$ with $m \geq 3$ and each $k \in \{2, ..., m-1\}$ one can find $c_k > 0$ and $C_k > 0$ such that the solution (u, v) of (0.1) fulfils the estimate*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^k|\nabla v|^{2m-2k} \quad & +c_k\int_\Omega u^{k-2}|\nabla u|^2|\nabla v|^{2m-2k} + \big(k\mu - C_k\big)\cdot\int_\Omega u^{k+1}|\nabla v|^{2m-2k} \\
& \leq \int_\Omega u^{k-1}|\nabla u|^2|\nabla v|^{2m-2k-2} + C_k\int_\Omega u^2|\nabla v|^{2m-2} \\
& \quad +C_k\int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 \\
& \quad +C_k\int_\Omega |\nabla v|^{2m-2} + C_k \quad \textit{for all } t\in(0,T_{max}). \qquad (2.19)
\end{aligned}$$

PROOF. For $k \in \{2, ..., m-1\}$, estimating the terms on the right of (2.4) turns out to be slightly more involved than it was in Lemma 2.3 and Lemma 2.4. Unlike the situation in those lemmata, however, the second term on the left of (2.4) fortunately now appears with the positive factor $k(k-1)$ and thus may be used to absorb some of the terms from the right-hand side.
First, by Young's inequality, the last term in (2.4) satisfies

$$\begin{aligned}
I_5 \quad &:= \quad ak\int_\Omega u^{k-1}|\nabla v|^{2m-2k} \\
&\leq \quad ak\cdot\frac{k-1}{k+1}\int_\Omega u^{k+1}|\nabla v|^{2m-2k} + ak\cdot\frac{2}{k+1}\int_\Omega |\nabla v|^{2m-2k} \\
&\leq \quad ak\cdot\frac{k-1}{k+1}\int_\Omega u^{k+1}|\nabla v|^{2m-2k} \\
&\quad\quad +ak\cdot\frac{2}{k+1}\cdot\frac{m-k}{m-1}\int_\Omega |\nabla v|^{2m-2} + ak\cdot\frac{2}{k+1}\cdot\frac{k-1}{k+1}|\Omega|, \qquad (2.20)
\end{aligned}$$

whereas also by straightforward interpolation,

$$\begin{aligned}
I_1 := \frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}\nabla u\cdot\nabla v \quad &\leq \quad \frac{1}{2}\int_\Omega u^{k-1}|\nabla u|^2|\nabla v|^{2m-2k-2} \\
&\quad +\frac{2(m-k)^2}{\tau^2}\int_\Omega u^{k+1}|\nabla v|^{2m-2k}. \qquad (2.21)
\end{aligned}$$

Similarly,

$$\begin{aligned}
I_2 := k(k-1)\chi\int_\Omega u^{k-1}|\nabla v|^{2m-2k}\nabla u\cdot\nabla v \quad &\leq \quad \frac{k(k-1)}{2}\int_\Omega u^{k-2}|\nabla u|^2|\nabla v|^{2m-2k} \\
&\quad +\frac{k(k-1)\chi^2}{2}\int_\Omega u^k|\nabla v|^{2m-2k-2} \qquad (2.22)
\end{aligned}$$

and

$$I_3 := k(m-k)\chi \int_\Omega u^k |\nabla v|^{2m-2k-2} \nabla v \cdot \nabla |\nabla v|^2 \le \frac{m-k}{4\tau} \int_\Omega u^k |\nabla v|^{2m-2k-4} \Big|\nabla |\nabla v|^2\Big|^2 + k^2 (m-k)\chi^2 \tau \int_\Omega u^k |\nabla v|^{2m-2k+2}. \qquad (2.23)$$

In order to estimate the last terms in (2.22) and (2.23), we note that once again by Young's inequality,

$$\begin{aligned}
\int_\Omega u^k |\nabla v|^{2m-2k+2} &= \int_\Omega \Big(u^2 |\nabla v|^{2m-2}\Big)^{\frac{1}{k-1}} \cdot \Big(u^{k+1} |\nabla v|^{2m-2k}\Big)^{\frac{k-2}{k-1}} \\
&\le \frac{1}{k-1} \int_\Omega u^2 |\nabla v|^{2m-2} + \frac{k-2}{k-1} \int_\Omega u^{k+1} |\nabla v|^{2m-2k},
\end{aligned}$$

so that in view of (2.20)–(2.23), (2.4) yields

$$\begin{aligned}
\frac{d}{dt} \int_\Omega u^k |\nabla v|^{2m-2k} &+ \frac{k(k-1)}{2} \int_\Omega u^{k-2} |\nabla u|^2 |\nabla v|^{2m-2k} + \frac{2(m-k)}{\tau} \int_\Omega u^k |\nabla v|^{2m-2k-2} |D^2 v|^2 \\
&+ (k\mu - C) \int_\Omega u^{k+1} |\nabla v|^{2m-2k} \\
&\le \frac{1}{2} \int_\Omega u^{k-1} |\nabla u|^2 |\nabla v|^{2m-2k-2} + \frac{m-k}{4\tau} \int_\Omega u^k |\nabla v|^{2m-2k-4} \Big|\nabla |\nabla v|^2\Big|^2 \\
&+ C\Big(\int_\Omega u^2 |\nabla v|^{2m-2} + \int_\Omega |\nabla v|^{2m-2} + 1\Big) + I_4 \qquad (2.24)
\end{aligned}$$

with some $C > 0$ and

$$I_4 := -k(m-k)\Big(1 + \frac{1}{\tau}\Big) \int_\Omega u^{k-1} |\nabla v|^{2m-2k-2} \nabla u \cdot \nabla |\nabla v|^2.$$

Observe that in (2.24) we have dropped the nonnegative fourth term on the left of (2.4), which might have been useful in coping with the second term on the right of (2.24). However, since we want to present a uniform treatment for all k up to the value $k = m-1$, we shall rather rely on the remaining term involving $|D^2 v|^2$ here (cf.(2.25) below).
As to I_4, upon another two-step interpolation we obtain

$$I_4 \le \frac{1}{2} \int_\Omega u^{k-1} |\nabla u|^2 |\nabla v|^{2m-2k-2} + \frac{k^2 (m-k)^2 (1+\frac{1}{\tau})^2}{2} \int_\Omega u^{k-1} |\nabla v|^{2m-2k-2} \Big|\nabla |\nabla v|^2\Big|^2$$

and then estimate

$$\begin{aligned}
\int_\Omega u^{k-1} |\nabla v|^{2m-2k-2} \Big|\nabla |\nabla v|^2\Big|^2 &= \int_\Omega \Big(\varepsilon u^k |\nabla v|^{2m-2k-4}\Big)^{\frac{k-1}{k}} \cdot \Big(\varepsilon^{-(k-1)} |\nabla v|^{2m-4}\Big)^{\frac{1}{k}} \cdot \Big|\nabla |\nabla v|^2\Big|^2 \\
&\le \frac{(k-1)\varepsilon}{k} \int_\Omega u^k |\nabla v|^{2m-2k-4} \Big|\nabla |\nabla v|^2\Big|^2 \\
&+ \frac{1}{k\varepsilon^{k-1}} \int_\Omega |\nabla v|^{2m-4} \Big|\nabla |\nabla v|^2\Big|^2
\end{aligned}$$

for arbitrary $\varepsilon > 0$. Choosing ε appropriately small, we can hence achieve that

$$\begin{aligned} I_1 &\le \frac{1}{2}\int_\Omega u^{k-1}|\nabla u|^2|\nabla v|^{2m-2k-2} + \frac{m-k}{4\tau}\int_\Omega u^k|\nabla v|^{2m-2k-4}\Big|\nabla|\nabla v|^2\Big|^2 \\ &\quad +\hat{C}\int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2 \end{aligned}$$

holds with some $\hat{C} > 0$. Since

$$\Big|\nabla|\nabla v|^2\Big|^2 = \big|2D^2v\cdot\nabla v\big|^2 \le 4|D^2v|^2\cdot|\nabla v|^2 \tag{2.25}$$

by the Cauchy-Schwarz inequality, (2.24) thereupon turns into the relation

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^k|\nabla v|^{2m-2k} &+\frac{k(k-1)}{2}\int_\Omega u^{k-2}|\nabla u|^2|\nabla v|^{2m-2k} + \frac{2(m-k)}{\tau}\int_\Omega u^k|\nabla v|^{2m-2k-2}|D^2v|^2 \\ &+(k\mu - C)\int_\Omega u^{k+1}|\nabla v|^{2m-2k} \\ &\le \int_\Omega u^{k-1}|\nabla u|^2|\nabla v|^{2m-2k-2} + 2\cdot\frac{m-k}{4\tau}\cdot 4\int_\Omega u^k|\nabla v|^{2m-2k-2}|D^2v|^2 \\ &\quad +C\Big(\int_\Omega u^2|\nabla v|^{2m-2} + \int_\Omega |\nabla v|^{2m-2}+1\Big) + \hat{C}\int_\Omega |\nabla v|^{2m-4}\Big|\nabla|\nabla v|^2\Big|^2, \end{aligned}$$

which immediately gives (2.19) if we let $c_k := \frac{k(k-1)}{2}$ and $C_k := \max\{C, \hat{C}\}$. ////

Finally, in order to digest the first term appearing on the right of (2.19) when $k = m-1$, we shall employ (2.4) for $k = m$ with the following result.

Lemma 2.6 *Let Ω be convex, $\tau > 0$, $\chi \in \mathbb{R}$, and assume (0.2). Then for all $m \in \mathbb{N}$ with $m \ge 2$ the inequality*

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^m &+c_m\int_\Omega u^{m-2}|\nabla u|^2 + (m\mu - C_m)\int_\Omega u^{m+1} \\ &\le C_m\int_\Omega u^2|\nabla v|^{2m-2} + C_m \quad \textit{for all } t\in(0,T_{max}) \end{aligned} \tag{2.26}$$

holds for suitable positive constants c_m and C_m.

Proof. We let $k = m$ in (2.4) to obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^m + m(m-1)\int_\Omega u^{m-2}|\nabla u|^2 + m\mu\int_\Omega u^{m+1} &\le m(m-1)\chi\int_\Omega u^{m-1}\nabla u\cdot\nabla v \\ &\quad +am\int_\Omega u^{m-1}, \end{aligned} \tag{2.27}$$

where Young's inequality gives

$$am\int_\Omega u^{m-1} \le am\cdot\frac{m-1}{m+1}\int_\Omega u^{m+1} + am\cdot\frac{2}{m+1}|\Omega| \tag{2.28}$$

and

$$m(m-1)\chi \int_\Omega u^{m-1} \nabla u \cdot \nabla v \le \frac{m(m-1)}{2} \int_\Omega u^{m-2}|\nabla u|^2 + \frac{m(m-1)\chi^2}{2} \int_\Omega u^m |\nabla v|^2. \tag{2.29}$$

Interpolating once more by the same token, we have

$$\begin{aligned} \int_\Omega u^m |\nabla v|^2 &= \int_\Omega u^{\frac{(m+1)(m-2)}{m-1}} \cdot \Big(u^2 |\nabla v|^{2m-2}\Big)^{\frac{1}{m-1}} \\ &\le \frac{m-2}{m-1} \int_\Omega u^{m+1} + \frac{1}{m-1} \int_\Omega u^2 |\nabla v|^{2m-2}. \end{aligned}$$

Combining this with (2.29), (2.28) and (2.27) directly results in (2.26). ////

We now sum up all the inequalities obtained so far, each one provided with an appropriate weight.

Lemma 2.7 *Let Ω be convex, $\tau > 0$, $\chi \in \mathbb{R}$ and $m \in \mathbb{N}$ be such that $m \ge 3$. Then there exists $\mu_0 = \mu_0(\tau, \chi, m) > 0$ with the property that whenever $\mu \ge \mu_0$, one can find positive constants c, C and $b_0, ..., b_m$ such that the solution (u, v) of (0.1) satisfies*

$$\begin{aligned} \frac{d}{dt}\Big\{ \sum_{k=0}^m b_k \cdot \int_\Omega u^k |\nabla v|^{2m-2k} \Big\} &+ c \int_\Omega |\nabla v|^{2m-4} \Big|\nabla |\nabla v|^2\Big|^2 \\ &+ c \int_\Omega u^{m-2} |\nabla u|^2 \\ &\le C\Big(1 + \int_\Omega |\nabla v|^{2m-2}\Big) \qquad \textit{for all } t \in (0, T_{max}). \end{aligned} \tag{2.30}$$

PROOF. We let $c_0, c_2, ..., c_m$ and $C_0, C_1, ..., C_m$ denote the constants provided by Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6. It is then possible to pick a number $M > 1$ large enough such that

$$M c_k > 2 \qquad \text{for all } k \in \{2, ..., m\} \tag{2.31}$$

and after that $\varepsilon > 0$ small enough fulfilling

$$\sum_{k=1}^{m-1} \varepsilon M^k C_k < \frac{c_0}{2} \tag{2.32}$$

as well as

$$\sum_{k=2}^{m} \varepsilon M^k C_k < 1. \tag{2.33}$$

We next fix $\mu_0 > 0$ large enough satisfying

$$\varepsilon M \cdot (\mu_0 - C_1) > C_0 + 1 \tag{2.34}$$

and

$$k\mu_0 > C_k \qquad \text{for all } k \in \{2, ..., m\}. \tag{2.35}$$

We now define

$$b_0 := 1 \qquad \text{and} \qquad b_k := \varepsilon M^k, \quad k \in \{1, ..., m\}, \tag{2.36}$$

and claim that then for any choice of $\mu \geq \mu_0$, (2.30) holds for sufficiently small $c > 0$ and suitably large $C > 0$.

Indeed, in the sum on the left of (2.30) let us apply Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.6 separately to the terms corresponding to $k = 0, k = 1, k \in \{2, ..., m-1\}$ and $k = m$, respectively. The consequence thus obtained then reads

$$\begin{aligned}
J := \left\{ \frac{d}{dt} \sum_{k=0}^{m} b_k \cdot \int_\Omega u^k |\nabla v|^{2m-2k} \right\} &= \frac{d}{dt} \int_\Omega |\nabla v|^{2m} + b_1 \frac{d}{dt} \int_\Omega u |\nabla v|^{2m-2} \\
&\quad + \sum_{k=2}^{m-1} b_k \cdot \frac{d}{dt} \int_\Omega u^k |\nabla v|^{2m-2k} + b_m \frac{d}{dt} \int_\Omega u^m \\
&\leq -c_0 \int_\Omega |\nabla v|^{2m-4} \Big| \nabla |\nabla v|^2 \Big|^2 + C_0 \int_\Omega u^2 |\nabla v|^{2m-2} \\
&\quad + b_1 \cdot \bigg\{ -(\mu - C_1) \int_\Omega u^2 |\nabla v|^{2m-2} + \int_\Omega |\nabla u|^2 |\nabla v|^{2m-4} \\
&\quad + C_1 \int_\Omega |\nabla v|^{2m-4} \Big| \nabla |\nabla v|^2 \Big|^2 + C_1 \int_\Omega |\nabla v|^{2m-2} \bigg\} \\
&\quad + \sum_{k=2}^{m-1} b_k \cdot \bigg\{ -c_k \int_\Omega u^{k-2} |\nabla u|^2 |\nabla v|^{2m-2k} \\
&\quad - (k\mu - C_k) \int_\Omega u^{k+1} |\nabla v|^{2m-2k} \\
&\quad + \int_\Omega u^{k-1} |\nabla u|^2 |\nabla v|^{2m-2k-2} + C_k \int_\Omega u^2 |\nabla v|^{2m-2} \\
&\quad + C_k \int_\Omega |\nabla v|^{2m-4} \Big| \nabla |\nabla v|^2 \Big|^2 + C_k \int_\Omega |\nabla v|^{2m-2} + C_k \bigg\} \\
&\quad + b_m \cdot \bigg\{ -c_m \int_\Omega u^{m-2} |\nabla u|^2 - (m\mu - C_m) \int_\Omega u^{m+1} \\
&\quad + C_m \int_\Omega u^2 |\nabla v|^{2m-2} + C_m \bigg\}, \qquad t \in (0, T_{max}).
\end{aligned}$$

In view of (2.35), we have $k\mu - C_k \geq 0$ for $k \in \{2, ..., m\}$, whence after rearranging we obtain

$$\begin{aligned}
J &\leq -\bigg\{ c_0 - b_1 C_1 - \sum_{k=2}^{m-1} b_k C_k \bigg\} \cdot \int_\Omega |\nabla v|^{2m-4} \Big| \nabla |\nabla v|^2 \Big|^2 \\
&\quad - \bigg\{ b_1 (\mu - C_1) - C_0 - \sum_{k=2}^{m-1} b_k C_k - b_m C_m \bigg\} \cdot \int_\Omega u^2 |\nabla v|^{2m-2} \\
&\quad + b_1 \int_\Omega |\nabla u|^2 |\nabla v|^{2m-4}
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{k=2}^{m-1} b_k c_k \int_\Omega u^{k-2}|\nabla u|^2 |\nabla v|^{2m-2k} \\
&+\sum_{k=2}^{m-1} b_k \int_\Omega u^{k-1}|\nabla u|^2 |\nabla v|^{2m-2k-2} \\
&-b_m c_m \int_\Omega u^{m-2}|\nabla u|^2 \\
&+\Big\{ b_1 C_1 + \sum_{k=2}^{m-1} b_k C_k \Big\} \cdot \int_\Omega |\nabla v|^{2m-2} \\
&+\sum_{k=2}^{m-1} b_k C_k + b_m C_m. \qquad (2.37)
\end{aligned}
$$

Here, (2.32) and the definition (2.36) of $b_1, ..., b_m$ assert that

$$
-\Big\{ c_0 - b_1 C_1 - \sum_{k=2}^{m-1} b_k C_k \Big\} \cdot \int_\Omega |\nabla v|^{2m-4} \Big|\nabla|\nabla v|^2\Big|^2 \le -\frac{c_0}{2} \int_\Omega |\nabla v|^{2m-4} \Big|\nabla|\nabla v|^2\Big|^2, \qquad (2.38)
$$

whereas (2.33) in conjunction with (2.34) guarantees that

$$
b_1(\mu - C_1) - C_0 - \sum_{k=2}^{m-1} b_k C_k - b_m C_m > b_1(\mu - C_1) - C_0 - 1 > 0, \qquad (2.39)
$$

implying nonpositivity of the second term on the right of (2.37). The sum of the third, fourth, fifth and sixth terms on the right of (2.37) can readily be rewritten according to

$$
\begin{aligned}
J_{3-6} \quad := \quad & b_1 \int_\Omega |\nabla u|^2 |\nabla v|^{2m-4} - \sum_{k=2}^{m-1} b_k c_k \int_\Omega u^{k-2} |\nabla u|^2 |\nabla v|^{2m-2k} \\
& + \sum_{k=2}^{m-1} b_k \int_\Omega u^{k-1} |\nabla u|^2 |\nabla v|^{2m-2k-2} - b_m c_m \int_\Omega u^{m-2} |\nabla u|^2 \\
= \quad & -\sum_{k=2}^{m} (b_k c_k - b_{k-1}) \cdot \int_\Omega u^{k-2} |\nabla u|^2 |\nabla v|^{2m-2k}. \qquad (2.40)
\end{aligned}
$$

Since (2.36) and the restriction (2.31) on $M > 1$ entail that

$$
b_k c_k - b_{k-1} = \varepsilon M^{k-1}(M c_k - 1) > \varepsilon M^{k-1} > \varepsilon M \qquad \text{for all } k \in \{2, ..., m\},
$$

we thus have

$$
J_{3-6} \le -\varepsilon M \cdot \sum_{k=2}^{m} \int_\Omega u^{k-2} |\nabla u|^2 |\nabla v|^{2m-2k}
$$

and hence in particular

$$
J_{3-6} \le -\varepsilon M \int_\Omega u^{m-2} |\nabla u|^2. \qquad (2.41)
$$

Collecting (2.37)–(2.41) and letting $c := \min\{\frac{c_0}{2}, \varepsilon M\}$ and $C := \sum_{k=1}^m b_k C_k$, we immediately end up with (2.30). ////

As a comparatively easy consequence we obtain time-independent bounds for u in $L^m(\Omega)$ and for ∇v in $L^{2m}(\Omega)$ for arbitrarily large m.

Corollary 2.8 *Suppose that Ω is convex, $\tau > 0$, $\chi \in \mathbb{R}$ and $3 \le m \in \mathbb{N}$, and assume that $\mu \ge \mu_0(\tau, \chi, m)$ with $\mu_0(\tau, \chi, m) > 0$ as given by Lemma 2.7. Then there exists $K > 0$ such that the solution (u, v) of (0.1) satisfies*

$$\int_\Omega u^m(x,t)dx \le K \qquad \textit{and} \qquad \int_\Omega |\nabla v(x,t)|^{2m} dx \le K \qquad \textit{for all } t \in (0, T_{max}). \tag{2.42}$$

PROOF. We first use Young's inequality with exponents $\frac{m}{k}$ and $\frac{m}{m-k}$ to estimate

$$\int_\Omega u^k |\nabla v|^{2m-2k} \le K_1 \Big(\int_\Omega u^m + \int_\Omega |\nabla v|^{2m} \Big), \qquad t \in (0, T_{max}),$$

for each $k \in \{1, ..., m-1\}$, where K_1 is a positive constant independent of k and t. Therefore, with $b_0, b_1, ..., b_m$ as in Lemma 2.7, the function

$$y(t) := \sum_{k=0}^m b_k \cdot \int_\Omega u^k |\nabla v|^{2m-2k}, \qquad t \in [0, T_{max}),$$

satisfies

$$y(t) \le K_2 \Big(\int_\Omega u^m + \int_\Omega |\nabla v|^{2m} \Big), \qquad t \in (0, T_{max}), \tag{2.43}$$

with some K_2 depending on m only. We next invoke the Poincaré inequality in the form

$$\int_\Omega z^2 \le K_p \Big\{ \int_\Omega |\nabla z|^2 + \Big(\int_\Omega |z|^{\frac{2}{m}} \Big)^m \Big\} \qquad \text{for all } z \in W^{1,2}(\Omega), \tag{2.44}$$

valid with some $K_p > 0$, and apply this first to $z := u^{\frac{m}{2}}$. Letting A denote the constant provided by Lemma 2.1, we thereby obtain

$$\begin{aligned} \int_\Omega u^m &\le K_p \Big\{ \int_\Omega |\nabla u^{\frac{m}{2}}|^2 + \Big(\int_\Omega u \Big)^m \Big\} \\ &\le \frac{m^2 K_p}{4} \int_\Omega u^{m-2} |\nabla u|^2 + A^m K_p, \qquad t \in (0, T_{max}). \end{aligned} \tag{2.45}$$

Similarly, the choice $z := |\nabla v|^m$ in (2.44) yields

$$\begin{aligned} \int_\Omega |\nabla v|^{2m} &\le K_p \Big\{ \int_\Omega \Big| \nabla (|\nabla v|^2)^{\frac{m}{2}} \Big|^2 + \Big(\int_\Omega |\nabla v|^2 \Big)^m \Big\} \\ &\le \frac{m^2 K_p}{4} \int_\Omega |\nabla v|^{2m-4} \Big| \nabla |\nabla v|^2 \Big|^2 + A^m K_p, \qquad t \in (0, T_{max}). \end{aligned} \tag{2.46}$$

By (2.43), (2.45) and (2.46), Lemma 2.7 states that with $c > 0$ and $C > 0$ taken from (2.30),

$$\begin{aligned} y'(t) &\le -c \cdot \Big\{ \int_\Omega |\nabla v|^{2m-4} \Big|\nabla|\nabla v|^2\Big|^2 + \int_\Omega u^{m-2}|\nabla u|^2 \Big\} + C\Big(1 + \int_\Omega |\nabla v|^{2m-2}\Big) \\ &\le -\frac{4c}{m^2K_p}\Big(\int_\Omega |\nabla v|^{2m} + \int_\Omega u^m\Big) + \frac{8cA^m}{m^2} + C\Big(1 + \int_\Omega |\nabla v|^{2m-2}\Big) \end{aligned}$$

holds for all $t \in (0, T_{max})$. Since by Young's inequality we have

$$C\int_\Omega |\nabla v|^{2m-2} \le \frac{2c}{m^2K_p}\int_\Omega |\nabla v|^{2m} + K_3, \qquad t \in (0, T_{max}),$$

with some $K_3 > 0$, this shows that

$$y'(t) \le -K_4 y(t) + K_5 \qquad \text{for all } t \in (0, T_{max}),$$

where $K_4 = \frac{2c}{m^2K_pK_2}$ and $K_5 = \frac{8cA^m}{m^2} + C + K_3$ are positive. Integrating the latter ODI, we see that

$$y(t) \le \max\Big\{y(0), \frac{K_5}{K_4}\Big\}$$

and thereby conclude the proof. ////

3 Proof of the main result

We can now easily prove our main result.

Proof (of Theorem 0.1). In view of Lemma 1.1, we only need to make sure that $T_{max} = \infty$ and that (u, v) is bounded. To this end, we pick $q > n$ and then infer from Corollary 2.8 the existence of $c_1 > 0$ such that

$$\|u(\cdot,t)\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max}). \tag{3.1}$$

Since in view of (0.2),

$$u_t \le \Delta u - \chi\nabla\cdot(u\nabla v) + a \qquad \text{in } \Omega \times (0, T_{max}),$$

from the order preserving property of the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ we obtain

$$u(\cdot,t) \le e^{(t-t_0)\Delta}u(\cdot,t_0) - \chi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))ds + \int_{t_0}^t e^{(t-s)\Delta}a ds \tag{3.2}$$

for all $t \in (0, T_{max})$ and $t_0 \in [0, t)$. Here, the third term on the right equals $a(t - t_0)$, while the first can be estimated according to

$$\|e^{(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty(\Omega)} \le c_2 A(t-t_0)^{-\frac{n}{2}}$$

with some $c_2 > 0$, A being the constant from Lemma 2.1. As to the second term, we fix any $p \in (q, \infty)$ and apply (3.1) and standard $L^p - L^q$ estimates for $(e^{t\Delta})_{t\geq 0}$ (cf. also [Wi2, Lemma 1.3]) to obtain for suitably large c_3 and c_4

$$\begin{aligned}
\Big\| e^{(t-s)\Delta} \nabla \cdot (u(\cdot,s)\nabla v(\cdot,s)) \Big\|_{L^\infty(\Omega)} &\leq c_3 (t-s)^{-\frac{n}{2p}} \Big\| e^{\frac{t-s}{2}\Delta} \nabla \cdot (u(\cdot,s)\nabla v(\cdot,s)) \Big\|_{L^p(\Omega)} \\
&\leq c_4 (t-s)^{-\frac{n}{2p}} \cdot (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})} \|u(\cdot,s)\nabla v(\cdot,s)\|_{L^q(\Omega)} \\
&\leq c_4 c_1 (t-s)^{-\frac{1}{2}-\frac{n}{2q}} \qquad \text{for all } s \in (t_0,t) \text{ with } s \geq t-1.
\end{aligned}$$

Since $\frac{1}{2} + \frac{n}{2q} < 1$, we conclude from (3.2) that if $t_0 \in [0,t]$ is such that $t_0 \geq t-1$ then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq c_5 \Big\{ (t-t_0)^{-\frac{n}{2}} + (t-t_0)^{\frac{1}{2}-\frac{n}{2q}} + (t-t_0) \Big\}$$

for some $c_5 > 0$. Thus, picking $t_0 := \max\{0, t-1\}$ here shows that with some $c_6 > 0$ we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq c_6 \Big(1 + t^{-\frac{n}{2}} \Big) \qquad \text{for all } t \in (0, T_{max}).$$

Since u is bounded in $\Omega \times (0, \frac{T_{max}}{2})$ by (1.1), this shows that u is bounded in $\Omega \times (0, T_{max})$. In view of standard parabolic theory applied to the second equation in (0.1), we also obtain that v is bounded in $L^\infty((0,T_{max}); W^{1,\theta}(\Omega))$. Along with (1.1), this proves that $T_{max} = \infty$ and that in fact, (u,v) is bounded in $\Omega \times (0,\infty)$. ////

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