Global solutions in a fully parabolic chemotaxis system with singular sensitivity

Michael Winkler
Fakultät für Mathematik, Universität Duisburg-Essen, 45117 Essen, Germany

Abstract

The Neumann boundary value problem for the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

is considered in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, with initial data $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geq 0$ and $v_0 > 0$ in $\bar{\Omega}$.
It is shown that if $0 < \chi < \sqrt{\frac{2}{n}}$ then for any such data there exists a global-in-time classical solution, generalizing a previous result which asserts the same for $n = 2$ only. Furthermore, it is seen that the range of admissible χ can be enlarged upon relaxing the solution concept. More precisely, global existence of weak solutions is established whenever $0 < \chi < \sqrt{\frac{n+2}{3n-4}}$.
Key words: chemotaxis, global existence, weak solutions, singularity formation
AMS Classification: 92C17, 35K55, 35B35, 35B40

Introduction

We consider the Neumann initial-boundary value problem for two coupled parabolic equations,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v\right), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{0.1}$$

with parameter $\chi > 0$, in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth outward normal vector field ν on $\partial\Omega$. The initial functions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are assumed to satisfy $u_0 \geq 0$ and $v_0 > 0$ in $\bar{\Omega}$.

This problem is used in the theoretical description of chemotaxis, the biological phenomenon of oriented, or partially oriented, movement of cells in response to a chemical signal that is produced by themselves. Being a special case of a more general PDE system originally introduced by Keller and Segel ([KS]), (0.1) especially focuses on the modeling of those chemotaxis processes where movement towards higher signal concentrations is inhibited at points where these concentrations are high; in [SL], [OS] and [HilP], the reader can find both a derivation of the crucial first equation in (0.1) and a number of examples where incorporating the above form of signal-dependence of chemotactic sensitivity is used as an essential ingredient in the respective model. For various relatives of (0.1) with the second equation replaced by an ODE, quite a rich understanding has been gained both at a numerical ([OS], [LS]) and at an analytical level ([LS], [SL]); however, the knowledge appears to be much less complete as soon as diffusion is accounted for in the equation for the signal.

Mathematically, the system (0.1) of two diffusion equations may be viewed as a borderline case among basically well-understood models. To demonstrate this, let us replace the sensitivity function $\chi(v)$, chosen as $\chi(v) = \frac{\chi}{v}$ in (0.1), by $\chi_{\varepsilon,\delta}(v) = \frac{\chi}{(v+\varepsilon)^{1+\delta}}$ for $\varepsilon \geq 0$ and $\delta \geq 0$. Then known results state that if $n = 2$, $\varepsilon > 0$, $\delta \geq 0$ and $\chi \leq (1+\delta)\varepsilon^{\delta}$, then all reasonable solutions of (0.1) are global in time ([B, Theorem 3 and Remark 4]). Moreover, in the case when both ε and δ are positive, the result in [W1] goes beyond this and states that for arbitrary $n \geq 1$ and $\chi > 0$, all solutions will be global and even bounded. Hence, either of the sets of conditions $n \geq 1, \varepsilon > 0, \delta > 0, \chi > 0$ and $n = 2, \varepsilon > 0, \delta = 0, \chi \leq 1$ is sufficient to prevent a chemotactic collapse in the sense of finite-time blow-up such as it may occur in the so-called *minimal* Keller-Segel model formally obtained when $\delta = -1$ ([HV]).
However, in the limit case $\varepsilon = 0$ and $\delta = 0$ to be considered here, it seems that the picture is more involved: Whereas all solutions of (0.1) are global in time when either $n = 1$ ([OY]), or when $n = 2$ and $\chi \leq 1$ ([B], [NSY2]), the results in [NS] give rise to the conjecture that unbounded solutions might exist for large values of χ: In that work, namely, the authors consider the parabolic-elliptic analogue of (0.1) obtained upon replacing the second PDE with the elliptic equation $0 = \Delta v - v + u$, and it is shown that if $n \geq 3$ and $\chi > \frac{2n}{n-2}$ then finite-time blow-up does occur for some solutions. In addition, it is proved there that in the same simplified problem, all radially symmetric solutions are global in time if either $n \geq 3$ and $\chi < \frac{2}{n-2}$, or $n = 2$ and $\chi > 0$ is arbitrary ([NS]). In the two-dimensional setting, the latter may be regarded as an indication for absence of blow-up also in the full parabolic-parabolic problem (0.1) for any $\chi > 0$.

It is the purpose of the present work to investigate the question of global existence of solutions to (0.1), which has been posted as an open problem in [HilP]. The mathematical challenging issue behind this is to estimate the effectiveness of the growth inhibition that is to be expected when passing from the minimal Keller-Segel model to (0.1) by replacing $\chi(v) \equiv const.$ with $\chi(v) = \frac{\chi}{v}$. Our first result in this direction states that

- if $\chi < \sqrt{\frac{2}{n}}$ then (0.1) has a global classical solution (Theorem 2.5).

In respect of classical solvability, this essentially extends the mentioned two-dimensional result to arbitrary space dimensions. In particular, as to the biologically relevant case $n = 3$, this means that global smooth solutions exist for small χ below an explicit threshold.

Next, in order to gain further insight into the global dynamics of (0.1) for a wider range of χ, we introduce a generalized solution concept. This will allow us to also include solutions that might become unbounded in space at some time, but continue to exist as a solution in a meaningful sense afterwards. More precisely, we consider a notion of weak solutions that essentially is the natural one obtained upon carrying over as many derivatives as possible to sufficiently smooth test functions (see Definition 3.1). As to solvability within this framework, we shall derive that

- if $\chi < \sqrt{\frac{n+2}{3n-4}}$ then there exists a global weak solution of (0.1) (Theorem 3.5).

As we shall see below, this solution has additional regularity properties; for instance, there exists $\alpha = \alpha(n,\chi) > 1$ such that $u \in L^\alpha_{loc}([0,\infty); L^\alpha(\Omega))$. However, we believe that our explicit lower estimates for such α are not optimal. Therefore we prefer not to present them here, and rather refer the reader to Corollary 3.2 and the proofs of Lemma 3.3 and Lemma 3.4, where supplementary information about integrability of our weak solutions and their gradients can be found.

We have to leave open here whether blow-up with respect to the norm in $L^\infty(\Omega)$ may or may not occur for large χ, including the question of criticality of the particular value $\chi = \sqrt{\frac{2}{n}}$ arising here. If such blow-up solutions indeed exist, our results indicate that the decay of $\chi(v)$ for large v should influence the explosion mechanism to some extent, and it would of course be interesting to see quantitative details of these effects. Moreover, our approach does not rule out the possibility that solutions, though existing globally and being smooth throughout, may become unbounded as $t \to \infty$. We believe, but cannot prove, that such a blow-up in infinite time does not occur, at least not for $\chi < \sqrt{\frac{2}{n}}$.

The plan of the paper is as follows: In Section 1 we will provide some preparatory material, including a useful integral identity (Lemma 1.3) that will be fundamental for our subsequent analysis. We then consider the case of small χ in Section 2 and demonstrate global existence of classical solutions. Our approach here differs from well-established regularity-providing procedures in the case of signal-independent sensitivity functions as can be found in [GZ], [NSY1] or [HW], for instance: Namely, our initial step will rely on a certain *weighted* integral estimate (Corollary 2.2) which is turned into a bound for $u(\cdot,t)$ in $L^\infty(\Omega)$ for $0 < t < T < \infty$ by a recursive argument. Next, Section 3 is devoted to the study of weak solutions, where the basic difficulty consists of proving a bound for $\frac{u}{v}\nabla v$ in $L^\beta(\Omega\times(0,T))$ for some $\beta > 1$ (Lemma 3.3), which itself is prepared by an estimate for u in $L^\alpha(\Omega\times(0,T))$ for some $\alpha > 2 - \frac{1}{n}$ (Lemma 3.4).

Let us finally mention that all of our methods can be extended so as to cover the case when the first equation in (0.1) is replaced with the more general equation $u_t = \Delta u - \chi\nabla\cdot(\frac{u}{v+\beta}\nabla v)$ for any $\beta \ge 0$, as proposed in [OS] and [HilP], but for simplicity in notation we refrain from giving the technical details here.

1 Preliminaries

We intend to construct a solution of (0.1) as the limit of a sequence of solutions to the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi\nabla\cdot\left(\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon\right), & x\in\Omega,\ t>0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x\in\Omega,\ t>0, \\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\varepsilon \in (0,1)$ and

$$\rho_\varepsilon(s) := s + \varepsilon(1+s^2), \qquad vs \ge 0. \tag{1.2}$$

Then $\chi(s) := \frac{\chi}{\rho_\varepsilon(s)}$ is smooth and nonnegative in $[0,\infty)$ and satisfies $\limsup_{s\to\infty} s^2\chi(s) < \infty$. Therefore we may invoke Theorem 2.2 in [W1] to obtain global existence of classical solutions to (1.1):

Lemma 1.1 *For all $\varepsilon \in (0,1)$, (1.1) possesses a global bounded solution $(u_\varepsilon, v_\varepsilon)$ satisfying $u_\varepsilon \ge 0$ and $v_\varepsilon \ge 0$ in $\Omega \times (0,\infty)$.*

For later reference, let us note that for each $\varepsilon \in (0,1)$, u_ε and v_ε enjoy the properties

$$\int_\Omega u_\varepsilon(x,t)dx = \int_\Omega u_0 \qquad \text{for all } t>0 \tag{1.3}$$

and

$$\int_\Omega v_\varepsilon(x,t)dx = \int_\Omega u_0 + \Big(\int_\Omega v_0 - \int_\Omega u_0\Big)\cdot e^{-t} \qquad \text{for all } t>0 \tag{1.4}$$

hold; in fact, these mass identities immediately result from integrating the first and the second PDE in (1.1).
Moreover, since we assume v_0 to be strictly positive, we can strengthen the above nonnegativity statement as follows.

Lemma 1.2 *For any $\varepsilon \in (0,1)$, we have*

$$v_\varepsilon(x,t) \ge \Big(\inf_{y\in\Omega} v_0(y)\Big)\cdot e^{-t} \qquad \textit{for all } t>0 \textit{ and } x \in \Omega. \tag{1.5}$$

PROOF. By nonnegativity of u_ε, the right-hand side of (1.5) is a spatially homogeneous subsolution of the second PDE in (1.1). Therefore the claim results from the comparison principle. ////

The following lemma will serve as a source for several ε-independent integral estimates in the sequel. It will be applied with certain parameters $p>1$ to provide global smooth solutions in Section 2 (cf. Lemma 2.1), and for $p \in (0,1)$ in establishing global weak solvability in Section 3 (Lemma 3.1 and Corollary 3.2).

Lemma 1.3 *Let $p \in \mathbb{R}$ and $q \in \mathbb{R}$. Then for all $\varepsilon \in (0,1)$, the identity*

$$\begin{aligned} \frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q \ + \ q\int_\Omega u_\varepsilon^p v_\varepsilon^q - q\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1} &= -p(p-1)\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 \\ &\quad + \int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}\cdot\Big[-q(q-1) + pq\chi\cdot\frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big]\cdot|\nabla v_\varepsilon|^2 \\ &\quad + \int_\Omega u_\varepsilon^{p-1}v_\varepsilon^{q-1}\cdot\Big[-2pq + p(p-1)\chi\cdot\frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big]\nabla u_\varepsilon\cdot\nabla v_\varepsilon \end{aligned} \tag{1.6}$$

holds for all $t>0$.

PROOF. By direct computation using (1.1), we find

$$
\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q &= p\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q u_{\varepsilon t} + q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1} v_{\varepsilon t} \\
&= p\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q \Delta u_\varepsilon - p\chi \int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q \nabla\cdot\Big(\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon\Big) \\
&\quad +q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1}\Delta v_\varepsilon - q\int_\Omega u_\varepsilon^p v_\varepsilon^q + q\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1}. \qquad (1.7)
\end{aligned}
$$

Integrating by parts, wee see that

$$
p\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q \Delta u_\varepsilon = -p(p-1)\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q |\nabla u_\varepsilon|^2 - pq\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon \qquad (1.8)
$$

and

$$
-p\chi\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q \nabla\cdot\Big(\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon\Big) = p(p-1)\chi\int_\Omega u_\varepsilon^{p-1}\frac{v_\varepsilon^q}{\rho_\varepsilon(v_\varepsilon)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon + pq\chi\int_\Omega u_\varepsilon^p \frac{v_\varepsilon^{q-1}}{\rho_\varepsilon(v_\varepsilon)}|\nabla v_\varepsilon|^2 \qquad (1.9)
$$

as well as

$$
q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1}\Delta v_\varepsilon = -pq\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - q(q-1)\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2, \qquad (1.10)
$$

because $\frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0$ on $\partial\Omega$. Inserting (1.8)-(1.10) into (1.7), after obvious rearrangements we end up with (1.6). ////

The following regularity properties of v_ε in dependence on boundedness features of u_ε can be derived by a straightforward application of well-known smoothing estimates for the heat semigroup under homogeneous Neumann boundary conditions (cf. [W2, Lemma 1.3] for sharp versions thereof, for instance).

Lemma 1.4 *Let $T > 0$ and $1 \le \theta, \mu \le \infty$.*
i) If $\frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$ then there exists $C > 0$ such that

$$
\|v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1 + \sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1). \qquad (1.11)
$$

ii) If $\frac{1}{2} + \frac{n}{2}(\frac{1}{\theta} - \frac{1}{\mu}) < 1$ then

$$
\|\nabla v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le C\Big(1 + \sup_{s\in(0,t)}\|u_\varepsilon(\cdot,s)\|_{L^\theta(\Omega)}\Big) \qquad \textit{for all } t\in(0,T) \textit{ and } \varepsilon\in(0,1) \qquad (1.12)
$$

is valid with some $C > 0$.

PROOF. i) We represent v_ε according to

$$
v_\varepsilon(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)ds, \qquad t > 0, \qquad (1.13)
$$

where $(e^{t\Delta})_{t\ge 0}$ denotes the Neumann heat semigroup. By standard smoothing estimates, we see that if $\mu \ge \theta$ then

$$\|v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le c_1 \bigg(\|v_0\|_{L^\infty(\Omega)} + \int_0^t (t-s)^{-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})} \|u_\varepsilon(\cdot,s)\|_{L^\theta(\Omega)} ds \bigg)$$

for some $c_1 > 0$. This proves (1.11) in the case $\mu \ge \theta$, whereas for $\mu < \theta$ the assertion results from this and Hölder's inequality.

ii) Applying ∇ to both sides in (1.13) and invoking corresponding smoothing properties involving gradients ([W2, Lemma 1.3]), we similarly find that

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le c_2 \bigg(\|\nabla v_0\|_{L^\infty(\Omega)} + \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{\theta}-\frac{1}{\mu})} \|u_\varepsilon(\cdot,s)\|_{L^\theta(\Omega)} ds \bigg)$$

with a certain $c_2 > 0$ and conclude as before. ////

2 Global smooth solutions for $\chi < \sqrt{\frac{2}{n}}$

The technical advantage of small values of χ is that these will allow us to pick some $p > 1$ in Lemma 1.3 in such a way that the right-hand side in (1.6) becomes nonpositive. Relabeling $q = -r$, we can thereupon derive the following.

Lemma 2.1 *Assume that $\chi < 1$. Then for all $p \in (1, \frac{1}{\chi^2})$, each $r \in (r_-(p), r_+(p))$ and any $T > 0$ one can find $C > 0$ such that*

$$\int_\Omega u_\varepsilon^p(x,t) v_\varepsilon^{-r}(x,t) dx \le C \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1) \tag{2.1}$$

and

$$\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-r-1} \le C \qquad \text{for all } \varepsilon \in (0,1), \tag{2.2}$$

where

$$r_\pm(p) := \frac{p-1}{2} \Big(1 \pm \sqrt{1 - p\chi^2} \Big).$$

Proof. Choosing $q := -r$ in (1.6), we obtain

$$\begin{aligned} I := \frac{d}{dt} \int_\Omega u_\varepsilon^p v_\varepsilon^{-r} \; - \; r \int_\Omega u_\varepsilon^p v_\varepsilon^{-r} + r \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-r-1} &= -p(p-1) \int_\Omega u_\varepsilon^{p-2} v_\varepsilon^{-r} |\nabla u_\varepsilon|^2 \\ &\quad - \int_\Omega u_\varepsilon^p v_\varepsilon^{-r-2} \Big[r(r+1) + pr\chi \cdot \frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)} \Big] \cdot |\nabla v_\varepsilon|^2 \\ &\quad + \int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{-r-1} \Big[2pr + p(p-1)\chi \frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)} \Big] \nabla u_\varepsilon \cdot \nabla v_\varepsilon \end{aligned} \tag{2.3}$$

for $t > 0$. Here, Young's inequality says that we can estimate the last term according to

$$\Big| \int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{-r-1} \Big[2pr + p(p-1)\chi \frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)} \Big] \nabla u_\varepsilon \cdot \nabla v_\varepsilon \Big| \le p(p-1) \int_\Omega u_\varepsilon^{p-2} v_\varepsilon^{-r} |\nabla u_\varepsilon|^2 + \frac{1}{4p(p-1)} \int_\Omega u_\varepsilon^p v_\varepsilon^{-r-2} \Big[2pr + p(p-1)\chi \cdot \frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)} \Big]^2 \cdot |\nabla v_\varepsilon|^2.$$

Therefore (2.3) yields

$$I \le - \int_\Omega u_\varepsilon^p v_\varepsilon^{-r-2} h_\varepsilon(v_\varepsilon) |\nabla v_\varepsilon|^2, \tag{2.4}$$

where we have set

$$h_\varepsilon(s) := r(r+1) + pr\chi \cdot \frac{s}{\rho_\varepsilon(s)} - \frac{[2pr + p(p-1)\chi \cdot \frac{s}{\rho_\varepsilon(s)}]^2}{4p(p-1)} \qquad \text{for } s \ge 0.$$

By direct computation, for $s \ge 0$ and $\varepsilon \in (0,1)$ we find

$$\begin{aligned} 4(p-1)h_\varepsilon(s) &= 4(p-1)r(r+1) + 4p(p-1)r\chi \cdot \frac{s}{\rho_\varepsilon(s)} - 4pr^2 \\ &\quad - p(p-1)^2\chi^2 \frac{s^2}{\rho_\varepsilon^2(s)} - 4p(p-1)r\chi \cdot \frac{s}{\rho_\varepsilon(s)} \\ &= -4r^2 + 4(p-1)r - p(p-1)^2\chi^2 \cdot \frac{s^2}{\rho_\varepsilon^2(s)}. \end{aligned}$$

Since $\rho_\varepsilon(s) \ge s$ for all $s \ge 0$, we thus obtain

$$\begin{aligned} 4(p-1)h_\varepsilon(s) &\ge -4r^2 + 4(p-1)r - p(p-1)^2\chi^2 \\ &= 4\big(r_+(p) - r\big)\big(r - r_-(p)\big) \\ &> 0 \end{aligned}$$

for all $s \ge 0$ and $\varepsilon \in (0,1)$. In view of (2.4), this shows that $I \le 0$, so that (2.3) implies

$$\frac{d}{dt} \int_\Omega u_\varepsilon^p v_\varepsilon^{-r} + r \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-r-1} \le r \int_\Omega u_\varepsilon^p v_\varepsilon^{-r} \tag{2.5}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Observing that r is positive, upon integrating this differential inequality we first infer that

$$\int_\Omega u_\varepsilon^p v_\varepsilon^{-r} \le \Big(\int_\Omega u_0^p v_0^{-r} \Big) \cdot e^{rT} \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1)$$

and then conclude, again using (2.5), that both (2.1) and (2.2) hold for a suitable $C > 0$. ////

Let us state an immediate consequence for the available p and r in the case when $\chi < \sqrt{\frac{2}{n}}$.

Corollary 2.2 *Assume that $\chi < \sqrt{\frac{2}{n}}$. Then there exist $p > \frac{n}{2}$ and $r \in (0, \frac{n}{2})$ with the property that for all $T > 0$ one can pick $C > 0$ such that*

$$\int_\Omega u_\varepsilon^p(x,t) v_\varepsilon^{-r}(x,t)dx \le C \qquad \textit{for all } t \in (0,T) \textit{ and } \varepsilon \in (0,1).$$

Using that boundedness properties of u_ε imply (even better) boundedness properties of v_ε by Lemma 1.4, in the latter estimate we actually can get rid of the factor involving v_ε:

Corollary 2.3 *Suppose that $\chi < \sqrt{\frac{2}{n}}$. Then there exists $p > \frac{n}{2}$ such that for all $T > 0$ one can find $C > 0$ satisfying*

$$\int_\Omega u_\varepsilon^p(x,t)dx \le C \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{2.6}$$

Proof. According to Corollary 2.2, we pick $p_0 > \frac{n}{2}$ and $r \in (0, \frac{n}{2})$ such that for all $T > 0$,

$$\int_\Omega u_\varepsilon^{p_0}(x,t)v_\varepsilon^{-r}(x,t)dx \le c_1 \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1) \tag{2.7}$$

holds with some $c_1 = c_1(T) > 0$. Since $r < \frac{n}{2}$ and $p_0 > \frac{n}{2}$, it is possible to fix $p \in (\frac{n}{2}, p_0)$ such that $p < \frac{n(p_0-r)}{n-2r}$. Applying Hölder's inequality, we find that

$$\int_\Omega u_\varepsilon^p \le \Big(\int_\Omega u_\varepsilon^{p_0} v_\varepsilon^{-r}\Big)^{\frac{p}{p_0}} \cdot \Big(\int_\Omega v_\varepsilon^{\frac{pr}{p_0-p}}\Big)^{\frac{p_0-p}{p_0}},$$

so that (2.7) implies

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_1^{\frac{1}{p_0}} \|v_\varepsilon(\cdot,t)\|_{L^{\frac{pr}{p_0-p}}(\Omega)}^{\frac{r}{p_0}} \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1). \tag{2.8}$$

Now since $\frac{n}{2}(\frac{1}{p} - \frac{p_0-p}{pr}) < 1$ in view of our restriction $p < \frac{n(p_0-r)}{n-2r}$, it follows from Lemma 1.4 that

$$\sup_{t\in(0,T)} \|v_\varepsilon(\cdot,t)\|_{L^{\frac{pr}{p_0-p}}(\Omega)} \le c_2\Big(1 + \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\Big)$$

and accordingly, by (2.8),

$$\sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_3\bigg(1 + \Big(\sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\Big)^{\frac{r}{p_0}}\bigg)$$

with appropriate constants c_2 and c_3. Upon the observation that $\frac{r}{p_0} < 1$ due to $p_0 > \frac{n}{2} > r$, we can conclude the proof. ////

We proceed to turn this into an estimate in $L^\infty(\Omega \times (0,T))$ by a recursive argument. Note that the following lemma applies to any $\chi > 0$ – in fact, it would hold for negative χ as well.

Lemma 2.4 *Let $\chi > 0$, and suppose that there exist $p_0 > \frac{n}{2}$, $T > 0$ and $c_1 > 0$ such that*

$$\int_\Omega u_\varepsilon^{p_0}(x,t)dx \le c_1 \qquad \text{for all } t \in (0,T) \text{ and all } \varepsilon \in (0,1). \tag{2.9}$$

Then there exists $c_2 > 0$ with the property

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t \in (0,T) \text{ and all } \varepsilon \in (0,1). \tag{2.10}$$

PROOF. We first claim that

$$\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_3 \qquad \text{for all } t\in(0,T) \text{ and all } \varepsilon\in(0,1) \tag{2.11}$$

for some $p>1$ and $c_3>0$ implies that

$$\begin{cases} \text{for all } \bar p < \frac{np}{2(n-p)} & \text{if } p\le n,\\ \text{for } \bar p=\infty & \text{if } p>n,\end{cases}$$

we can find $c_4>0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^{\bar p}(\Omega)} \le c_4 \qquad \text{for all } t\in(0,T) \text{ and all } \varepsilon\in(0,1). \tag{2.12}$$

To show this, we first consider the case $p\le n$. Then, if $\frac12+\frac n2(\frac1p-\frac1\mu)<1$, that is, if $\mu<\frac{np}{n-p}$, in view of Lemma 1.4 and (2.11), we can find $c_5(\mu)>0$ fulfilling

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)} \le c_5(\mu) \qquad \text{for all } t\in(0,T) \text{ and all } \varepsilon\in(0,1). \tag{2.13}$$

Combining (2.11) with (2.13) and invoking Lemma 1.2, from Hölder's inequality we gain some $c_6>0$ such that

$$\begin{aligned}\Big\|\frac{u_\varepsilon(\cdot,t)}{\rho_\varepsilon(v_\varepsilon(\cdot,t))}\nabla v_\varepsilon(\cdot,t)\Big\|_{L^\theta(\Omega)} &\le c_6\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}\|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{p\theta}{p-\theta}}(\Omega)}\\ &\le c_6c_3c_5(\tfrac{p\theta}{p-\theta}) \qquad \text{for all } t\in(0,T) \text{ and all } \varepsilon\in(0,1)\end{aligned} \tag{2.14}$$

whenever θ is such that $\frac{p\theta}{p-\theta}<\frac{np}{n-p}$, that is, when $\theta<\frac{np}{2n-p}$. We now use the representation formula

$$u_\varepsilon(\cdot,t)=e^{t\Delta}u_0-\chi\int_0^t e^{(t-s)\Delta}\nabla\cdot\Big(\frac{u_\varepsilon(\cdot,s)}{\rho_\varepsilon(v_\varepsilon(\cdot,s))}\nabla v_\varepsilon(\cdot,s)\Big)ds, \qquad t>0, \tag{2.15}$$

and the smoothing estimate ([W2, Lemma 1.3])

$$\|e^{\tau\Delta}\nabla\cdot w\|_{L^{\bar p}(\Omega)} \le c_7\tau^{-\frac12-\frac n2(\frac1\theta-\frac1{\bar p})}\|w\|_{L^\theta(\Omega)}, \tag{2.16}$$

valid for all $w\in C^1(\bar\Omega)$, $\tau\in(0,T)$ and $\bar p\ge\theta>1$ with some $c_7>0$. We thus find that if $\theta<\frac{np}{2n-p}$ then

$$\|u_\varepsilon(\cdot,t)\|_{L^{\bar p}(\Omega)} \le c_8\Big(\|u_0\|_{L^\infty(\Omega)}+\int_0^t(t-s)^{-\frac12-\frac n2(\frac1\theta-\frac1{\bar p})}ds\Big)$$

and, consequently,

$$\|u_\varepsilon(\cdot,t)\|_{L^{\bar p}(\Omega)} \le c_{10} \qquad \text{for all } t\in(0,T) \text{ and all } \varepsilon\in(0,1)$$

with some c_8 and c_9, provided that $\frac12+\frac n2(\frac1\theta-\frac1{\bar p})<1$ or, equivalently, $\bar p<\frac{n\theta}{n-\theta}$. Choosing θ close to $\frac{np}{2n-p}$, we thereby see that (2.12) will hold for any $\bar p<\frac{np}{2(n-p)}$ if $p\le n$.

In the case $p>n$ the procedure is quite similar: Then (2.13) holds for $\mu=\infty$ and, consequently, (2.14) is true for $\theta=p>n$. Therefore, if we pick $m>n$ large and then $\delta>0$ small such that

$0 < \delta - \frac{n}{2m} < \frac{1}{2} - \frac{n}{2p}$ then applying the fractional power $(-\Delta+1)^\delta$ to the integral in (2.15) shows that

$$\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\leq \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} + c_{10}\int_0^t \Big\|(-\Delta+1)^\delta e^{(t-s)\Delta}\nabla\cdot\Big(\frac{u_\varepsilon(\cdot,s)}{\rho_\varepsilon(v_\varepsilon(\cdot,s))}\nabla v_\varepsilon(\cdot,s)\Big)\Big\|_{L^m(\Omega)}ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + c_{11}\int_0^t (t-s)^{-\delta}\Big\|e^{\frac{1}{2}(t-s)\Delta}\nabla\cdot\Big(\frac{u_\varepsilon(\cdot,s)}{\rho_\varepsilon(v_\varepsilon(\cdot,s))}\nabla v_\varepsilon(\cdot,s)\Big)\Big\|_{L^m(\Omega)}ds \\
&\leq \|u_0\|_{L^\infty(\Omega)} + c_{12}\int_0^t (t-s)^{-\delta-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{m})}ds \qquad (2.17)
\end{aligned}$$

with constants c_{10}, c_{11} and c_{12}, where we have used the maximum principle and the well-known fact that the domain of definition of $(-\Delta+1)^\delta$ in $L^m(\Omega)$ is embedded into $L^\infty(\Omega)$ when $\delta - \frac{n}{2m} > 0$. Since $\delta + \frac{1}{2} + \frac{n}{2}(\frac{1}{p} - \frac{1}{m}) < 1$ by our choice of δ, we conclude that (2.12) is valid with $\bar{p} = \infty$ if $p > n$, as claimed.

Now the rest of the proof is a straightforward iteration of the implication (2.11) $\Rightarrow$ (2.12). Namely, if we fix $\eta > 0$ small such that $p_0 > \frac{n}{2} + \eta$ and define

$$p_{k+1} := \left\{ \begin{array}{ll} \frac{np_k}{2(n-p_k+\eta)} & \text{if } p_k \leq n, \\ \infty & \text{if } p_k > n \end{array} \right\} \qquad \text{for } k = 0, 1, 2, ...,$$

then we easily see than $p_{k+1} \geq p_k$ for all $k \geq 0$, and that $p_k = \infty$ holds for all sufficiently large k. Moreover, successive applications of the above result to $p = p_k$ and $\bar{p} := p_{k+1}$ shows that $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,T); L^{p_k}(\Omega))$ for all k and hence, in particular, in $L^\infty((0,T); L^\infty(\Omega))$. ////

Now standard parabolic theory allows us to turn the above L^∞ estimate into global solvability in the classical pointwise sense.

Theorem 2.5 *Suppose that $\chi < \sqrt{\frac{2}{n}}$. Then for all $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ satisfying $u_0 \geq 0$ and $v_0 > 0$ in $\bar{\Omega}$, (0.1) has a global classical solution.*

Proof. Since $(u_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty_{loc}(\bar{\Omega}\times[0,\infty))$, standard parabolic Schauder estimates ([LSU]) imply that both $(u_\varepsilon)_{\varepsilon\in(0,1)}$ and $(v_\varepsilon)_{\varepsilon\in(0,1)}$ are bounded in $C^{2+\theta,1+\frac{\theta}{2}}_{loc}(\bar{\Omega}\times(0,\infty))$ for some $\theta > 0$, whence along a suitable sequence of numbers $\varepsilon = \varepsilon_k \searrow 0$ we have $u_\varepsilon \to u$ and $v_\varepsilon \to v$ in $C^{2,1}_{loc}(\bar{\Omega}\times(0,\infty))$ for some limit pair (u,v) that solves the PDE system and the boundary conditions in (0.1). Also by parabolic regularity theory (or by going back to (2.13) for $\mu = \infty$), we obtain that $(\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty_{loc}(\bar{\Omega}\times[0,\infty))$, whence arguing as in (2.17), for each $\eta > 0$ we can find $\tau > 0$ such that

$$\|u_\varepsilon(\cdot,t) - e^{t\Delta}u_0\|_{L^\infty(\Omega)} < \eta \qquad \text{and} \qquad \|v_\varepsilon(\cdot,t) - e^{(t-1)\Delta}v_0\|_{L^\infty(\Omega)} < \eta$$

for all $t \in (0,\tau)$. We therefore easily conclude that for any $T > 0$, both u_ε and v_ε converge to their respective limits uniformly in $\Omega\times(0,T)$ as $\varepsilon = \varepsilon_k \searrow 0$, and that (u,v) also satisfies the initial conditions in (0.1). ////

3 Weak solutions for $\chi < \sqrt{\frac{n+2}{3n-4}}$

By straightforward testing procedures, we obtain a natural concept of weak solutions.

Definition 3.1 *Assume that $\chi > 0$, $u_0 \in L^1(\Omega)$, $v_0 \in L^1(\Omega)$ and $T > 0$. By a* weak solution *of (0.1) in $\Omega \times (0,T)$ we mean a pair (u,v) of nonnegative functions*

$$u \in L^1_{loc}(\bar\Omega \times [0,T)), \qquad v \in L^1_{loc}(\bar\Omega \times [0,T))$$

with the additional property

$$\frac{u}{v}\nabla v \in L^1_{loc}(\bar\Omega \times [0,T)), \tag{3.1}$$

that satisfy

$$-\int_0^T\int_\Omega u\varphi_t - \int_0^T\int_\Omega u\Delta\varphi - \chi\int_0^T\int_\Omega \frac{u}{v}\nabla v\cdot\nabla\varphi = \int_\Omega u_0\varphi(\cdot,0)$$

as well as

$$-\int_0^T\int_\Omega v\varphi_t - \int_0^T\int_\Omega v\Delta\varphi + \int_0^T\int_\Omega v\varphi - \int_0^T\int_\Omega u\varphi = \int_\Omega v_0\varphi(\cdot,0)$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,T)$.
If u and v are defined in $\Omega \times (0,\infty)$ then (u,v) will be said to be a global weak solution *of (0.1) if (u,v) is a weak solution of (0.1) in $\Omega \times (0,T)$ for all $T > 0$.*

The main obstacles in proving global existence of weak solutions will be connected to the above regularity requirements, in particular to (3.1). In order to achieve appropriate estimates also in the case $\chi > 1$, we need to develop a strategy other than that leading to Lemma 2.1. Our approach will still rely on Lemma 1.3, but we shall now choose p to be *smaller* than one and prove that the right-hand side of (1.6) will be *positive* for appropriate q.

Lemma 3.1 *Given $\chi > 0$ and $p \in (0,1)$ satisfying $p < \frac{1}{\chi^2}$, let $q_+(p)$ and $q_-(p)$ be defined by*

$$q_\pm(p) := \frac{1-p}{2}\Big(1 \pm \sqrt{1-p\chi^2}\Big).$$

Then for all $q \in (q_-(p), q_+(p))$ there exists $C > 0$ such that for any $\varepsilon \in (0,1)$ we have

$$\begin{aligned}\int_0^t\int_\Omega u_\varepsilon^{p-2}v_\varepsilon^q|\nabla u_\varepsilon|^2 \; &+ \int_0^t\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 + \int_0^t\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1} \\ &\le \; C\int_\Omega u_\varepsilon^p(\cdot,t)v_\varepsilon^q(\cdot,t) \qquad \text{for all } t > 0.\end{aligned} \tag{3.2}$$

Proof. As in the proof of Lemma 2.1, the basic idea is to use Lemma 1.3, but now with reversed sign of the factor in front of those integrals in (1.6) that involve gradients: Indeed, from (1.6) we obtain

$$
\begin{aligned}
I \quad &:= \quad \frac{d}{dt}\int_\Omega u_\varepsilon^p v_\varepsilon^q + q\int_\Omega u_\varepsilon^p v_\varepsilon^q - q\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} \\
&= \quad p(1-p)\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q |\nabla u_\varepsilon|^2 \\
&\qquad + \int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}\Big[q(1-q) + pq\chi\frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big]\cdot|\nabla v_\varepsilon|^2 \\
&\qquad - \int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1}\Big[2pq + p(1-p)\chi\frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big]\nabla u_\varepsilon\cdot\nabla v_\varepsilon \qquad (3.3)
\end{aligned}
$$

for $t > 0$, and our goal is to estimate the right-hand side from below, up to lower-order terms, by positive multiples of the integrals involving squares of gradients. To this end, we note that Young's inequality ensures that for all $\eta \in (0,1)$ and $\mu > 0$,

$$
\begin{aligned}
\left| - \int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1}\Big[2pq + p(1-p)\chi\frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big]\nabla u_\varepsilon\cdot\nabla v_\varepsilon\right| \quad &\le \quad \eta p(1-p)\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q|\nabla u_\varepsilon|^2 \\
&\qquad + \frac{1}{4\eta p(1-p)}\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}\Big[2pq + p(1-p)\chi\frac{v_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big]^2|\nabla v_\varepsilon|^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
I \quad &\ge \quad (1-\eta-\mu)p(1-p)\int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q|\nabla u_\varepsilon|^2 \\
&\qquad + \int_\Omega u_\varepsilon^p v_\varepsilon^{q-2} h_{\varepsilon,\eta,\mu}(v_\varepsilon)|\nabla v_\varepsilon|^2, \qquad (3.4)
\end{aligned}
$$

where

$$
h_{\varepsilon,\eta,\mu}(s) = q(1-q) + pq\chi\frac{s}{\rho_\varepsilon(s)} - \frac{[2pq + p(1-p)\chi\frac{s}{\rho_\varepsilon(s)}]^2}{4\eta p(1-p)} - \mu.
$$

Therefore, (3.2) will result upon an integration as soon as we have shown that for all $q \in (q_-(p), q_+(p))$ we can pick $\eta \in (0,1)$, $\mu > 0$ and $c > 0$ such that

$$
h_{\varepsilon,\eta,\mu}(s) \ge c \qquad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1). \qquad (3.5)
$$

For this purpose, we compute

$$
\begin{aligned}
4\eta(1-p)\big(h_{\varepsilon,\eta,\mu}(s) + \mu\big) \quad &= \quad 4\eta(1-p)q(1-q) + 4\eta(1-p)\cdot pq\chi\cdot\frac{s}{\rho_\varepsilon(s)} \\
&\qquad -4pq^2 - p(1-p)^2\chi^2\cdot\frac{s^2}{\rho_\varepsilon^2(s)} - 4p(1-p)q\chi\cdot\frac{s}{\rho_\varepsilon(s)} \\
&= \quad 4\eta(1-p)q(1-q) - 4pq^2 \\
&\qquad -4(1-\eta)p(1-p)q\chi\cdot\frac{s}{\rho_\varepsilon(s)} - p(1-p)^2\chi^2\cdot\frac{s^2}{\rho_\varepsilon^2(s)}.
\end{aligned}
$$

We can now use that $\rho_\varepsilon(s) \ge s$ for all $s \ge 0$ to estimate

$$4\eta(1-p)\big(h_{\varepsilon,\eta,\mu}(s)+\mu\big) \ \ge \ 4\eta(1-p)q(1-q) - 4pq^2 \\ -4(1-\eta)p(1-p)q\chi - p(1-p)^2\chi^2 =: J(\eta) \qquad (3.6)$$

for all $s \ge 0, \varepsilon \in (0,1), \eta \in (0,1)$ and $\mu > 0$. Here, as $\eta \nearrow 1$ we have

$$\begin{aligned} J(\eta) &\to 4(1-p)q(1-q) - 4pq^2 - p(1-p)^2\chi^2 \\ &= -4q^2 + 4(1-p)q - p(1-p)^2\chi^2 \\ &= 4\big(q_+(p) - q\big)\big(q - q_-(p)\big) \\ &> 0, \end{aligned}$$

whence it follows from (3.6) that (3.5) holds if we first pick $\eta \in (0,1)$ sufficiently close to 1 and then $\mu > 0$ appropriately small. ////

We shall need the following consequence.

Corollary 3.2 *Let $\chi > 0$ and $p \in (0,1)$ be such that $p < \frac{1}{\chi^2}$, and define $q_\pm(p)$ as in Lemma 3.1. Then for all $q \in (q_-(p), q_+(p))$ and each $T > 0$ there exists $C_1 > 0$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^{p-2} v_\varepsilon^q |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^{q-2} |\nabla v_\varepsilon|^2 \le C_1 \qquad (3.7)$$

for all $\varepsilon \in (0,1)$. In particular, for all such p and each $T > 0$ one can find $C_2 > 0$ with the property

$$\int_0^T \int_\Omega u_\varepsilon^{p-2} |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^{-2} |\nabla v_\varepsilon|^2 \le C_2 \qquad (3.8)$$

for all $\varepsilon \in (0,1)$.

Proof. We apply Lemma 3.1 and use Hölder's inequality and the fact that $p+q < p+q_+(p) < 1$ in estimating

$$\int_\Omega u_\varepsilon^p(\cdot,t) v_\varepsilon^q(\cdot,t) \le \Big(\int_\Omega u_\varepsilon(\cdot,t)\Big)^p \Big(\int_\Omega v_\varepsilon(\cdot,t)\Big)^q |\Omega|^{\frac{1}{1-p-q}}$$

for all $t > 0$ and $\varepsilon \in (0,1)$. Thereupon, (3.2) implies (3.7), which in turn entails (3.8) in view of Lemma 1.2. ////

Now if χ lies below another threshold (which is larger than that in the previous section), using the above along with results on maximal Sobolev regularity in scalar parabolic equations ([HieP]), we can find a space-time integral estimate for some power of u_ε that will be large enough to yield the desired regularity of $\frac{u}{v}\nabla v$.

Lemma 3.3 *Suppose that $\chi < \sqrt{\frac{n+2}{3n-4}}$. Then there exists $\alpha > 2 - \frac{1}{n}$ with the property that for all $T > 0$ one can find $C > 0$ such that*

$$\int_0^T \int_\Omega u_\varepsilon^\alpha \le C \qquad \textit{for all } \varepsilon \in (0,1). \qquad (3.9)$$

Proof. Let us first consider the case $n = 2$, in which the condition $\chi < \sqrt{\frac{n+2}{3n-4}} = \sqrt{2}$ allows us to pick some $p < 1$ such that $p < \frac{1}{\chi^2}$ and $p > \frac{1}{2}$. The latter requirement ensures that we can now fix a number $\alpha < p+1$ satisfying $\alpha > 2 - \frac{1}{n} > 1$. Then in view of the mass conservation property (1.3), Corollary 3.2 yields uniform boundedness of $u_\varepsilon^{\frac{p}{2}}$ in $L^2((0,T);W^{1,2}(\Omega))$ and hence in $L^2((0,T);L^\kappa(\Omega))$ for all $\kappa \in (0,\infty)$ by the Sobolev embedding theorem. This means that for such κ,

$$\int_0^T \|u_\varepsilon(\cdot,t)\|^p_{L^{\frac{p\kappa}{2}}(\Omega)} dt \le c_1 \qquad \text{for all } \varepsilon \in (0,1) \tag{3.10}$$

holds with some $c_1 > 0$. Specifying $\kappa := \frac{2}{p+1-\alpha}$ here, we have $\kappa \in (0,\infty)$ and $\frac{p\kappa}{2} > \alpha$ because of $\alpha < p+1$ and $p < 1 < \alpha$, and thus we may interpolate between (1.3) and (3.10) to obtain

$$\int_0^T \|u_\varepsilon(\cdot,t)\|^\alpha_{L^\alpha(\Omega)} dt \le \int_0^T \|u_\varepsilon(\cdot,t)\|^p_{L^{\frac{p\kappa}{2}}(\Omega)} \|u_\varepsilon(\cdot,t)\|^{\alpha-p}_{L^1(\Omega)} \le c_1 \Big(\int_\Omega u_0 \Big)^{\alpha-p}$$

for all $\varepsilon \in (0,1)$, which proves (3.9) in the two-dimensional setting.

If $n \ge 4$, the hypothesis $\chi < \sqrt{\frac{n+2}{3n-4}}$ enables us to choose some $p > \frac{3n-4}{n+2} \ge 1$ such that $p < \frac{1}{\chi^2}$. Since $p > \frac{3n-4}{n+2}$, it is then possible to pick some $\alpha > 2 - \frac{1}{n}$ fulfilling $\alpha < \frac{(n+2)(p+1)}{2n}$, where we evidently can also achieve that $\alpha > \frac{p+1}{2}$ and $\alpha < \frac{n}{2}$, the latter because of the fact that $n \ge 4$. We now apply Lemma 2.1 with $r := \frac{p-1}{2}$ to obtain

$$\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-\frac{p+1}{2}} \le c_2 \qquad \text{for all } \varepsilon \in (0,1) \tag{3.11}$$

with some $c_2 > 0$. By parabolic Sobolev regularity theory ([HieP, Theorem 3.1]) applied to the second equation in (1.1), we furthermore find $c_3 > 0$ such that

$$\|v_\varepsilon - e^{t(\Delta-1)}v_0\|_{L^\alpha((0,T);W^{2,\alpha}(\Omega))} \le c_3 \|u_\varepsilon\|_{L^\alpha(\Omega\times(0,T))}.$$

Hence, invoking the embedding $W^{2,\alpha}(\Omega) \hookrightarrow L^{\frac{n\alpha}{n-2\alpha}}(\Omega)$, valid since $\alpha < \frac{n}{2}$, and the fact that $e^{t(\Delta-1)}v_0$ is bounded since $v_0 \in L^\infty(\Omega)$,

$$\|v_\varepsilon\|_{L^\alpha((0,T);L^{\frac{n\alpha}{n-2\alpha}}(\Omega))} \le c_4 \Big(1 + \|u_\varepsilon\|_{L^\alpha(\Omega\times(0,T))} \Big) \tag{3.12}$$

for all $\varepsilon \in (0,1)$ with suitable $c_4 > 0$. Also, Lemma 1.4 implies that for any $\mu \in (1, \frac{n}{n-2})$ there exists $c_5 > 0$ such that

$$\|v_\varepsilon\|_{L^\infty((0,T);L^\mu(\Omega))} \le c_5 \qquad \text{for all } \varepsilon \in (0,1).$$

In conjunction with (3.11), for all such μ this gives

$$\begin{aligned} \int_0^T \int_\Omega u_\varepsilon^\alpha &\le \Big(\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{-\frac{p+1}{2}} \Big)^{\frac{\alpha}{p+1}} \cdot \Big(\int_0^T \|v_\varepsilon(\cdot,t)\|^{\frac{(p+1)\alpha}{2(p+1-\alpha)}}_{L^{\frac{(p+1)\alpha}{2(p+1-\alpha)}}(\Omega)} \Big)^{\frac{p+1-\alpha}{p+1}} \\ &\le c_2^{\frac{\alpha}{p+1}} \Big(\int_0^T \|v_\varepsilon(\cdot,t)\|^{\frac{(p+1)\alpha}{2(p+1-\alpha)}\cdot a}_{L^{\frac{n\alpha}{n-2\alpha}}(\Omega)} \cdot c_5^{\frac{(p+1)\alpha}{2(p+1-\alpha)}\cdot(1-a)} \Big)^{\frac{p+1-\alpha}{p+1}} \end{aligned} \tag{3.13}$$

by Hölder's inequality, where

$$a = \frac{\frac{1}{\mu} - \frac{2(p+1-\alpha)}{(p+1)\alpha}}{\frac{1}{\mu} - \frac{n-2\alpha}{n\alpha}}. \tag{3.14}$$

Choosing $\mu := \frac{n(2\alpha-p-1)}{4(p+1-\alpha)}$ here, we have $\mu > 0$ since $\alpha > \frac{p+1}{2}$, and moreover $\mu < \frac{n}{n-2}$ because of the fact that $\alpha < \frac{(n+2)(p+1)}{2n}$ implies

$$\frac{n-2}{n}\mu < \frac{(n-2)\Big(\frac{(n+2)(p+1)}{n} - p - 1\Big)}{4\Big(p+1-\frac{(n+2)(p+1)}{2n}\Big)} = \frac{(n-2)\cdot\frac{2}{n}}{4(1-\frac{n+2}{2n})} = 1.$$

Moreover, by (3.14) we then have $\frac{(p+1)\alpha}{2(p+1-\alpha)} \cdot a = \alpha$ and therefore (3.13) combined with (3.12) shows that

$$\int_0^T \int_\Omega u_\varepsilon^\alpha \le c_6\bigg(1 + \Big(\int_0^T \int_\Omega u_\varepsilon^\alpha\Big)^{\frac{p+1-\alpha}{p+1}}\bigg) \qquad \text{for all } \varepsilon \in (0,1)$$

with some $c_6 > 0$. This proves (3.9) if the spatial dimension is at least four.

In the remaining case $n = 3$ we apply a mixture of both above procedures. Let us first pick $\alpha > 2 - \frac{1}{n} = \frac{5}{3}$ close enough to $\frac{5}{3}$ such that $\alpha < 2$ and

$$\alpha < \frac{10(1+\frac{1}{\chi^2})}{11+\frac{1}{\chi^2}}, \tag{3.15}$$

which is possible since now our assumption on χ precisely means that $\frac{1}{\chi^2} > 1$. Next, we can fix a number $p > 1$ satisfying $p < \frac{1}{\chi^2}$,

$$p > \frac{11\alpha - 10}{10 - \alpha} \tag{3.16}$$

and

$$p < \frac{7\alpha - 6}{6 - \alpha}, \tag{3.17}$$

since (3.15) ensures that $\frac{11\alpha-10}{10-\alpha} < \frac{1}{\chi^2}$, and since $\frac{7\alpha-6}{6-\alpha} > \frac{11\alpha-10}{10-\alpha}$ thanks to the positivity of α. We finally choose some small $\delta \in (0, \frac{2}{3})$ fulfilling

$$\delta < \frac{5}{3} - \frac{(p+1)\alpha}{6(p+1-\alpha)} \tag{3.18}$$

and

$$\delta < \frac{1}{3} + \frac{2(p+1-\alpha)}{(p+1)\alpha}, \tag{3.19}$$

where we note that $\alpha < p + 1$ because $\alpha < 2$ and $p > 1$, and that the expression on the right of (3.18) is positive due to our restriction (3.16). We now apply Corollary 3.2 to obtain $c_7 > 0$ and $c_8 > 0$ such that

$$\int_0^T \|u_\varepsilon^{\frac{1-\delta}{2}}(\cdot,t)\|^2_{W^{1,2}(\Omega)} dt \le c_7$$

and, by an embedding estimate,

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^{3(1-\delta)}(\Omega)}^{1-\delta} dt = \int_0^T \|u_\varepsilon^{\frac{1-\delta}{2}}(\cdot,t)\|_{L^6(\Omega)}^2 dt \le c_8.$$

Using the mass conservation property (1.3) and the Hölder inequality, from this we infer that

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\beta(\Omega)}^\gamma dt \le c_9 \tag{3.20}$$

holds for

$$\beta := \frac{1}{\frac{2(p+1-\alpha)}{(p+1)\alpha} + \frac{2}{3}} \qquad \text{and} \qquad \gamma := \frac{\frac{2}{3} - \delta}{1 - \frac{1}{\beta}} = \frac{\frac{2}{3} - \delta}{\frac{1}{3} - \frac{2(p+1-\alpha)}{(p+1)\alpha}} \tag{3.21}$$

and some $c_9 > 0$. Here it can easily be checked that $\beta > 1$ by (3.17) and $\gamma > 1$ according to (3.19). By parabolic regularity theory, (3.20) entails a bound for v_ε in $L^\gamma((0,T);W^{2,\beta}(\Omega))$ and thus also in $L^\gamma((0,T);L^{\frac{3\beta}{3-2\beta}}(\Omega))$, because we have $\beta < \frac{3}{2}$ by (3.21) and hence $W^{2,\beta}(\Omega) \hookrightarrow L^{\frac{3\beta}{3-2\beta}}(\Omega)$. Computing

$$\frac{3\beta}{3-2\beta} = \frac{1}{\frac{1}{\beta} - \frac{2}{3}} = \frac{(p+1)\alpha}{2(p+1-\alpha)}$$

and using (3.18) in estimating

$$\frac{2(p+1-\alpha)}{(p+1)\alpha} \cdot \gamma > \frac{2(p+1-\alpha)}{(p+1)\alpha} \cdot \frac{\frac{2}{3} - \left(\frac{5}{3} - \frac{(p+1)\alpha}{6(p+1-\alpha)}\right)}{\frac{1}{3} - \frac{2(p+1-\alpha)}{(p+1)\alpha}} = 1,$$

we thereby obtain

$$\int_0^T \|v_\varepsilon(\cdot,t)\|_{L^{\frac{(p+1)\alpha}{2(p+1-\alpha)}}(\Omega)}^{\frac{(p+1)\alpha}{2(p+1-\alpha)}} dt \le c_{10} \tag{3.22}$$

with some $c_{10} > 0$. We now only need to observe that the first line in (3.13) is still valid to derive (3.9) from (3.22). ////

We can next establish sufficient regularity of the cross-diffusion term in (1.1).

Lemma 3.4 *If $\chi < \sqrt{\frac{n+2}{3n-4}}$ then there exists $\beta > 1$ such that for all $T > 0$,*

$$\int_0^T \int_\Omega \left| \frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)} \nabla v_\varepsilon \right|^\beta \le C \qquad \textit{for all } \varepsilon \in (0,1) \tag{3.23}$$

holds for some $C > 0$ independent of ε.

Proof. According to Lemma 3.3, we can pick $\alpha \in (2 - \frac{1}{n}, 2)$ such that

$$(u_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{is bounded in} \quad L^\alpha_{loc}(\bar\Omega \times [0,\infty)). \tag{3.24}$$

We then can find $\theta \in (1, \alpha)$ satisfying $\theta > \frac{n\alpha}{n\alpha-(n-\alpha)}$, which is possible since $\alpha > 2 - \frac{1}{n} > \frac{2n}{n+1}$. By a continuity argument, we can now fix $\beta_0 \in (1, \alpha)$ such that

$$\theta > \frac{n\alpha\beta_0}{n\alpha - \beta_0(n-\alpha)} \qquad \text{for all } \beta \in [1, \beta_0], \tag{3.25}$$

where we can also achieve

$$\beta_0 < \frac{\alpha - 1}{1 - \frac{1}{\theta}}, \tag{3.26}$$

because $\alpha > 2 - \frac{1}{n}$ and $\theta < \alpha < 2 \le n$ imply that $\frac{\alpha-1}{1-\frac{1}{\theta}} > 1$. Next, since (3.25) and $\beta_0 < \alpha$ guarantee that

$$\begin{aligned} \theta &> \frac{n\beta_0}{n - \beta_0(\frac{n}{\alpha} - 1)} \\ &= \frac{\beta_0}{\alpha - \frac{n-1}{n}\beta_0 + (\alpha-1)(\frac{\beta_0}{\alpha} - 1)} \\ &> \frac{\beta_0}{\alpha - \frac{n-1}{n}\beta_0}, \end{aligned}$$

we see upon rearranging that $\frac{n-1}{n} > \frac{\frac{\beta_0}{\theta} - \frac{\alpha}{n}}{\alpha - \beta_0}$. Hence, for all $\mu \in (0, \frac{n}{n-1}]$ we have $\frac{1}{\mu} > \frac{\frac{\beta_0}{\theta} - \frac{\alpha}{n}}{\alpha - \beta_0}$ or, equivalently,

$$\alpha\Big(\frac{1}{\mu} + \frac{1}{n}\Big) > \beta_0\Big(\frac{1}{\mu} + \frac{1}{\theta}\Big).$$

Therefore, for all $\beta \in [1, \beta_0]$ and $\mu \in (0, \frac{n}{n-1}]$ the expression

$$I(\beta, \mu) := \frac{\alpha - 1}{\beta(1 - \frac{1}{\theta})} - \frac{\alpha(\frac{1}{\mu} + \frac{1}{n}) - 1}{\alpha(\frac{1}{\mu} + \frac{1}{n}) - \beta(\frac{1}{\mu} + \frac{1}{\theta})}$$

is well-defined. Since

$$I\Big(1, \frac{n}{n-1}\Big) = \frac{\alpha-1}{1 - \frac{1}{\theta}} - \frac{\alpha - 1}{\alpha - \frac{n-1}{n} - \frac{1}{\theta}} = \frac{(\alpha-1)(\alpha - 2 + \frac{1}{n})}{(1 - \frac{1}{\theta})(\alpha - \frac{n-1}{n} - \frac{1}{\theta})} > 0$$

in view of (3.25) and the fact that $\alpha > 2 - \frac{1}{n}$, again concluding by continuity we can fix some $\beta \in (1, \beta_0)$ and $\mu \in (0, \frac{n}{n-1})$ such that $I(\beta, \mu) > 0$. Then $s := \frac{\alpha-1}{\beta(1-\frac{1}{\theta})}$ satisfies $s > 1$ by (3.26), and by positivity of $I(\beta, \mu)$, $s' := \frac{1}{1-\frac{1}{s}}$ fulfils

$$\frac{1}{s'} = 1 - \frac{1}{s} > 1 - \frac{\alpha(\frac{1}{\mu} + \frac{1}{n}) - \beta(\frac{1}{\mu} + \frac{1}{\theta})}{\alpha(\frac{1}{\mu} + \frac{1}{n}) - 1} = \frac{\beta(\frac{1}{\mu} + \frac{1}{\theta}) - 1}{\alpha(\frac{1}{\mu} + \frac{1}{n}) - 1}. \tag{3.27}$$

We now let $T > 0$ be given and apply Hölder's inequality to estimate, using that $\rho_\varepsilon(v_\varepsilon) \ge v_\varepsilon$ and Lemma 1.2,

$$\int_0^T \int_\Omega \Big| \frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)} \nabla v_\varepsilon \Big|^\beta \le c_1 \int_0^T \int_\Omega |u_\varepsilon \nabla v_\varepsilon|^\beta$$

$$
\begin{aligned}
&\le c_1 \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\theta(\Omega)}^\beta \|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{\beta\theta}{\theta-\beta}}(\Omega)}^\beta dt \\
&\le c_1 \Big(\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\theta(\Omega)}^{\beta s} dt \Big)^{\frac{1}{s}} \cdot \Big(\int_0^T \|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{\beta\theta}{\theta-\beta}}(\Omega)}^{\beta s'} dt \Big)^{\frac{1}{s'}} \qquad (3.28)
\end{aligned}
$$

with some $c_1 > 0$ independent of $\varepsilon \in (0,1)$. Here, recalling $1 < \theta < \alpha$ we can interpolate

$$
\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\theta(\Omega)}^{\beta s} dt \le \int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\alpha(\Omega)}^{\beta s a} \|u_\varepsilon(\cdot,t)\|_{L^1(\Omega)}^{\beta s(1-a)} dt
$$

with $a = \frac{1-\frac{1}{\theta}}{1-\frac{1}{\alpha}}$. Since

$$
\beta s a = \beta \cdot \frac{\alpha - 1}{\beta(1-\frac{1}{\theta})} \cdot \frac{1-\frac{1}{\theta}}{1-\frac{1}{\alpha}} = \alpha,
$$

we infer from (3.24) and the mass identity (1.3) that for some $c_2 > 0$,

$$
\int_0^T \|u_\varepsilon(\cdot,t)\|_{L^\theta(\Omega)}^{\beta s} dt \le c_2 \qquad \text{for all } \varepsilon \in (0,1). \tag{3.29}
$$

Next, from Lemma 1.4 and (1.3) we know that

$$
(\nabla v_\varepsilon)_{\varepsilon \in (0,1)} \quad \text{is bounded in} \quad L^\infty_{loc}([0,\infty); L^\mu(\Omega)), \tag{3.30}
$$

because $\mu < \frac{n}{n-1}$. Also, in view of (3.24) and [HieP, Theorem 3.1], $(v_\varepsilon - e^{t(\Delta-1)}v_0)_{\varepsilon\in(0,1)}$ is bounded in $L^\alpha((0,T); W^{2,\alpha}(\Omega))$ and hence in $L^\alpha((0,T); W^{1,\frac{n\alpha}{n-\alpha}}(\Omega))$. Since by semigroup estimates ([W2, Lemma 1.3]) we have $\|\nabla e^{t(\Delta-1)}v_0\|_{L^\infty(\Omega)} \le c_3 t^{-\frac{1}{2}} \|v_0\|_{L^\infty(\Omega)}$ for some $c_3 > 0$, recalling our restriction $\alpha < 2$ we conclude that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\alpha((0,T); W^{1,\frac{n\alpha}{n-\alpha}}(\Omega))$. We can thus once more employ Hölder's inequality to interpolate between the latter estimate and (3.30). As a result, writing

$$
b := \frac{\frac{1}{\mu} - \frac{\theta-\beta}{\beta\theta}}{\frac{1}{\mu} - \frac{n-\alpha}{n\alpha}},
$$

for some $c_3 > 0$ we have

$$
\begin{aligned}
\int_0^T \|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{\beta\theta}{\theta-\beta}}(\Omega)}^{\beta s'} dt &\le \int_0^T \|\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac{n\alpha}{n-\alpha}}(\Omega)}^{\beta s' b} \|\nabla v_\varepsilon(\cdot,t)\|_{L^\mu(\Omega)}^{\beta s'(1-b)} dt \\
&\le c_4 \qquad \text{for all } \varepsilon \in (0,1)
\end{aligned}
$$

with an appropriate $c_4 > 0$, because according to (3.27),

$$
\beta s' b < \beta \cdot \frac{\alpha(\frac{1}{\mu} + \frac{1}{n}) - 1}{\beta(\frac{1}{\mu} + \frac{1}{\theta}) - 1} \cdot \frac{\frac{1}{\mu} - \frac{\theta-\beta}{\beta\theta}}{\frac{1}{\mu} - \frac{n-\alpha}{n\alpha}} = \alpha.
$$

Together with (3.29) and (3.28), this establishes (3.23). ////

On the basis of the two preceding lemmata, performing standard compactness arguments and extraction procedures we can now find a sequence of solutions to (1.1) that converges to a weak solution of (0.1) in an appropriate sense.

Theorem 3.5 *Assume that $n \ge 2$ and $\chi < \sqrt{\frac{n+2}{3n-4}}$. Then for all $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ with $u_0 \ge 0$ and $v_0 > 0$ in $\bar{\Omega}$, there exists a global weak solution (u, v) of (0.1).*

PROOF. According to Lemma 3.3 and Lemma 3.4, for any $T > 0$,

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is weakly precompact in} \quad L^\alpha(\Omega \times (0,T)) \tag{3.31}$$

and

$$\Big(\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\Big)_{\varepsilon\in(0,1)} \quad \text{is weakly precompact in} \quad L^\beta(\Omega \times (0,T)) \tag{3.32}$$

for some $\beta > 1$. Moreover, from parabolic regularity theory ([HieP, Theorem 3.1]) and the inclusion $v_0 \in W^{1,\infty}(\Omega)$ we gain (cf. also the proof of Lemma 3.4)

$$\|v_\varepsilon\|_{L^\alpha((0,T);W^{2,\alpha}(\Omega))} + \|v_{\varepsilon t}\|_{L^\alpha(\Omega\times(0,T))} \le c_1 \qquad \text{for all } \varepsilon \in (0,1)$$

with some $c_1 > 0$, so that the Aubin-Lions lemma ([T]) ensures that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is strongly precompact in} \quad L^\alpha((0,T); W^{1,\alpha}(\Omega)). \tag{3.33}$$

To derive some type of strong compactness property for $(u_\varepsilon)_{\varepsilon\in(0,1)}$, we apply Corollary 3.2 to obtain that for some $p \in (0,1)$ and $c_2 > 0$,

$$\int_0^T \int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^{-2}|\nabla v_\varepsilon|^2 \le c_2 \qquad \text{for all } \varepsilon \in (0,1). \tag{3.34}$$

We now fix $m \in \mathbb{N}$ large such that $W^{m,2}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ and let $\zeta \in W^{m,2}(\Omega)$. Then multiplying the first equation in (1.1) by $(u_\varepsilon+1)^{\frac{p}{2}-1}\zeta$ and integrating over Ω we see that

$$\begin{aligned} \frac{2}{p}\int_\Omega \partial_t (u_\varepsilon+1)^{\frac{p}{2}}\zeta \;&=\; \Big(1-\frac{p}{2}\Big)\int_\Omega (u_\varepsilon+1)^{\frac{p}{2}-2}|\nabla u_\varepsilon|^2\zeta - \int_\Omega (u_\varepsilon+1)^{\frac{p}{2}-1}\nabla u_\varepsilon\cdot\nabla\zeta \\ &\quad + \Big(1-\frac{p}{2}\Big)\chi\int_\Omega (u_\varepsilon+1)^{\frac{p}{2}-2}\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon\zeta - \chi\int_\Omega (u_\varepsilon+1)^{\frac{p}{2}-1}\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon\cdot\nabla\zeta \end{aligned}$$

and hence, using Hölder's and Young's inequalities,

$$\begin{aligned} \frac{2}{p}\Big|\int_\Omega \partial_t (u_\varepsilon+1)^{\frac{p}{2}}\zeta\Big| \;&\le\; c_3\|\zeta\|_{W^{1,\infty}(\Omega)}\cdot\bigg(\int_\Omega (u_\varepsilon+1)^{\frac{p}{2}-2}|\nabla u_\varepsilon|^2 + \int_\Omega (u_\varepsilon+1)^{\frac{p}{2}} \\ &\qquad + \int_\Omega (u_\varepsilon+1)^{\frac{p}{2}-2}\frac{u_\varepsilon^2}{\rho_\varepsilon(v_\varepsilon)}|\nabla v_\varepsilon|^2\bigg) \end{aligned}$$

with some $c_3 > 0$. Since $\rho_\varepsilon(v_\varepsilon) \ge v_\varepsilon$, $(u_\varepsilon+1)^{\frac{p}{2}} \le (u_\varepsilon+1)^p \le u_\varepsilon + 1$ and $(u_\varepsilon+1)^{p-2}u_\varepsilon^2 \le u_\varepsilon^p$, we therefore obtain

$$\frac{2}{p}\big\|\partial_t(u_\varepsilon+1)^{\frac{p}{2}}\big\|_{L^1\left((0,T);\left(W^{m,2}(\Omega)\right)^\star\right)} \;=\; \frac{2}{p}\int_0^T \sup_{\|\zeta\|_{W^{m,2}(\Omega)}\le 1}\Big|\int_\Omega \partial_t(u_\varepsilon+1)^{\frac{p}{2}}(\cdot,t)\zeta\Big|dt$$

$$\le c_4\Big(\int_0^T\int_\Omega (u_\varepsilon+1)^{p-2}|\nabla u_\varepsilon|^2+\int_0^T\int_\Omega (u_\varepsilon+1) + \int_0^T\int_\Omega u_\varepsilon^p v_\varepsilon^{-2}|\nabla v_\varepsilon|^2\Big)$$

for an appropriate $c_4>0$. Due to (3.34) and (1.3), this entails that $\big(\partial_t (u_\varepsilon+1)^{\frac{p}{2}}\big)_{\varepsilon\in(0,1)}$ is bounded in $L^1\big((0,T);(W^{2,m}(\Omega))^\star\big)$, whereas also by (3.34) and (1.3), $\big((u_\varepsilon+1)^{\frac{p}{2}}\big)_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{1,2}(\Omega))$. By a variant of the Aubin-Lions lemma ([T]), this guarantees that

$$\big((u_\varepsilon+1)^{\frac{p}{2}}\big)_{\varepsilon\in(0,1)} \quad \text{is strongly precompact in} \quad L^1(\Omega\times(0,T)). \tag{3.35}$$

Now according to (3.35) and (3.33), we can extract a sequence of numbers $\varepsilon=\varepsilon_k\searrow 0$ along which

$$u_\varepsilon\to u,\quad v_\varepsilon\to v \quad \text{and} \quad \nabla v_\varepsilon\to\nabla v \qquad \text{a.e. in } \Omega\times(0,\infty), \tag{3.36}$$
$$u_\varepsilon\rightharpoonup u \qquad \text{in } L^\alpha(\Omega\times(0,T)) \qquad \text{for all } T>0, \tag{3.37}$$
$$v_\varepsilon\to v \qquad \text{in } L^\alpha((0,T);W^{1,\alpha}(\Omega)) \qquad \text{for all } T>0 \tag{3.38}$$

and

$$\frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon \rightharpoonup w \qquad \text{in } L^\beta(\Omega\times(0,T)) \qquad \text{for all } T>0 \tag{3.39}$$

hold for some limit functions $u\ge 0, v\ge 0$ and w defined in $\Omega\times(0,\infty)$. Here, (3.36) and Egorov's theorem guarantee that

$$w=\frac{u}{v}\nabla v. \tag{3.40}$$

Now given $T>0$ and $\varphi\in C_0^\infty(\bar\Omega\times[0,T))$ with $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,T)$, from (1.1) we obtain upon obvious testing procedures the identities

$$-\int_0^T\int_\Omega u_\varepsilon\varphi_t-\int_0^T\int_\Omega u_\varepsilon\Delta\varphi-\chi\int_0^T\int_\Omega \frac{u_\varepsilon}{\rho_\varepsilon(v_\varepsilon)}\nabla v_\varepsilon\cdot\nabla\varphi=\int_\Omega u_0\varphi(\cdot,0) \tag{3.41}$$

and

$$-\int_0^T\int_\Omega v_\varepsilon\varphi_t-\int_0^T\int_\Omega v_\varepsilon\Delta\varphi+\int_0^T\int_\Omega v_\varepsilon\varphi-\int_0^T\int_\Omega u_\varepsilon\varphi=\int_\Omega v_0\varphi(\cdot,0) \tag{3.42}$$

for all $\varepsilon\in(0,1)$. Thanks to (3.37)-(3.40), we can take $\varepsilon=\varepsilon_k\searrow 0$ here to find that each of the terms in (3.41) and (3.42) converges to its expected limit, whereby it follows that (u,v) is a weak solution of (0.1) in $\Omega\times(0,T)$ in the sense of Definition 3.1 for arbitrary $T>0$. ////

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