

# Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction

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## Abstract

We study radially symmetric solutions of a class of chemotaxis systems generalizing the prototype

$$(\star) \quad \begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \lambda u - \mu u^\kappa, & x \in \Omega, t > 0, \\ 0 = \Delta v - m(t) + u, & x \in \Omega, t > 0, \end{cases}$$

in a ball  $\Omega \subset \mathbb{R}^n$ , with parameters  $\chi > 0$ ,  $\lambda \geq 0$ ,  $\mu \geq 0$  and  $\kappa > 1$ , and  $m(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx$ . It is shown that when  $n \geq 5$  and

$$\kappa < \frac{3}{2} + \frac{1}{2n-2},$$

then there exist initial data such that the smooth local-in-time solution of  $(\star)$  blows up in finite time. This indicates that even superlinear growth restrictions may be insufficient to rule out a chemotactic collapse, as is known to occur in the corresponding system without any proliferation.

**Key words:** chemotaxis, logistic proliferation, blow-up, singularity formation

**AMS Classification:** 35K55, 35B40, 35K65, 92C17

## Introduction

We consider the time evolution of a cell population under the influence of chemotaxis, diffusion, and cell kinetics of logistic type. In view of numerous applications ([8]), we assume that the chemotactic movement is directed towards increasing concentrations of the chemical signal substance, and that this substance is produced by the cells themselves. A celebrated model for such a process was proposed by Keller and Segel ([13]), and a simplification thereof, based on the assumption that chemicals diffuse much faster than cells, leads to the parabolic-elliptic problem

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ 0 = \Delta v - m(t) + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, t > 0, \quad \int_{\Omega} v(x, t) dx = 0, \quad t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \quad (0.1)$$

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for the cell density  $u = u(x, t)$  and the concentration  $v = v(x, t)$  of the chemical. Here  $u_0$  is a given nonnegative function and  $f(u)$  models proliferation and death of cells, a prototype being

$$f(u) = \lambda u - \mu u^\kappa, \quad u \geq 0, \quad (0.2)$$

with  $\lambda \geq 0, \mu \geq 0$  and  $\kappa > 1$ . The function  $m(t)$  denotes the time-dependent spatial mean of  $u(\cdot, t)$ , that is, if the physical domain  $\Omega$  is bounded then we set

$$m(t) := \frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx, \quad t > 0. \quad (0.3)$$

In the special case  $f \equiv 0$  with trivial cell kinetics, (0.1) was introduced in [12], and it was shown there that in the two-dimensional setting this problem exhibits a mass threshold phenomenon in respect of singularity formation: Indeed, if  $\Omega \subset \mathbb{R}^2$  is a ball and  $f \equiv 0$ , then all radially symmetric initial data with small total mass  $\int_{\Omega} u_0$  exist globally in time and remain bounded, whereas if  $\int_{\Omega} u_0$  is large then  $(u, v)$  may blow up in finite time in the sense that  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$  becomes unbounded within finite time (cf. also [1]). Similar conclusions are true in the nonradial setting, when  $\Omega = \mathbb{R}^2$ , or when the second PDE in (0.1) is replaced with the parabolic equation  $v_t = \Delta v - v + u$  ([15], [1], [16], [10], [7], cf. also [21]).

In the higher-dimensional case when  $n \geq 3$ , certain alternative smallness assumptions, involving stronger norms of the initial data, still guarantee boundedness of solutions, but small total mass of cells appears to be insufficient to rule out blow-up if  $f \equiv 0$  ([23], [27]).

On the other hand, logistic-type growth restrictions in the style of (0.2) have been detected to prevent any chemotactic collapse in some systems closely related to (0.1): In [19] it was shown that if  $f(u) = \mu u(1 - u)$  with some  $\mu > 0$ , then in planar bounded domains  $\Omega \subset \mathbb{R}^2$ , all solutions of the corresponding initial-boundary value problem for

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \quad (0.4)$$

are global and bounded. If  $n \geq 3$  and  $\Omega \subset \mathbb{R}^n$  is bounded and convex, the same boundedness result is available under the additional assumption that  $\mu$  be sufficiently large ([26]). Similar conclusions in both the two-dimensional and the higher-dimensional framework were derived in [24] for the parabolic-elliptic system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + f(u), & x \in \Omega, t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega, t > 0, \end{cases} \quad (0.5)$$

with  $f$  generalizing the choice  $f(u) = \mu u(1 - u)$ .

Less restrictive growth restrictions of type (0.2) with  $\kappa$  possibly being smaller than two were considered in [25], where some global "very weak" solutions of (0.5) were constructed for rather arbitrary initial data under the assumption that  $\kappa > 2 - \frac{1}{n}$ ,  $n \geq 2$ . In the recent paper [18], such proliferation terms were shown to assert boundedness of solutions in a variant of (0.5) where the production term  $u$  in the second PDE is replaced by sublinear signal kinetics term  $u^\alpha$  with certain  $\alpha < 1$ .

Going beyond these boundedness statements, a number of results is available which show that the interplay of chemotactic cross-diffusion and cell kinetics of type (0.2) may lead to quite a colorful dynamics. For instance, the dynamical system associated with (0.4) with  $f(u) = \mu u(1 - u)$  was proved to possess an exponential attractor when  $n = 2$  ([19]). Moreover, recent numerical evidence indicates that even in the

spatially one-dimensional case this problem seems to allow for chaotic behavior ([9]). A rigorous statement on the global dynamical properties of (0.4) was obtained in [5] and [6] for the related special case  $f(u) = \mu u^2(1 - u)$  of a Fisher-type source with cubic absorption; there, namely, the authors established a two-sided estimate  $c_1\chi \leq \dim \mathcal{A} \leq c_2\chi^2$  (for some  $c_2 > c_1 > 0$  when  $\chi \geq 1$ ) for the fractal dimension of the corresponding global attractor.

In summary, the available analytical results on chemotaxis systems with logistic sources as in (0.2) concentrate on providing conditions, mostly on  $f$ , that are sufficient to assert boundedness of solutions, possibly along with more detailed dynamical properties; to the best of our knowledge, up to now no exploding solution has been proved to exist under such circumstances. It is the goal of the present paper to show that such a chemotactic collapse need not be ruled out by any logistic-type growth restriction of type (0.2). Indeed, let us assume that the cell kinetics term  $f$  in (0.1) satisfies the following set of hypotheses.

$$(H1) \quad f \in C^0([0, \infty)) \cap C^1((1, \infty)).$$

$$(H2) \quad f(u) \geq -\mu u^\kappa \text{ for all } u \geq 0 \text{ and some } \mu \geq 0 \text{ and } \kappa > 1.$$

$$(H3) \quad f(u) \leq A \cdot (1 + u) \text{ for all } u \geq 0 \text{ with some } A \geq 0.$$

Our main result then reads as follows.

**Theorem 0.1** *Assume that  $n \geq 5$  and that  $\Omega \subset \mathbb{R}^n$  is a ball, and suppose that  $f$  satisfies (H1)-(H3) with some  $\mu \geq 0$ ,  $A \geq 0$  and  $\kappa > 1$  such that*

$$\kappa < \frac{3}{2} + \frac{1}{2n-2}.$$

*Then for all  $m_0 > 0$  and each  $T_0 > 0$  there exist radially symmetric positive initial data  $u_0 \in C^\infty(\bar{\Omega})$  such that  $\frac{1}{|\Omega|} \int_\Omega u_0 = m_0$ , and such that (0.1), (0.3) possesses a unique classical solution  $(u, v)$  in  $\Omega \times (0, T)$  for some  $T \in (0, T_0)$  which fulfills*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty \quad \text{as } t \nearrow T.$$

**Remark.** i) In particular, the above hypotheses (H1)-(H3) cover the case of a Gompertz-type cell kinetics given by

$$f(u) = -\mu u \ln \frac{u}{A}, \quad u \geq 0,$$

for each  $\mu > 0$  and  $A > 0$ . Accordingly, it is asserted that in any of the resulting chemotaxis-Gompertz models blow-up occurs despite the superlinear absorptive character of the kinetic term.

ii) The exponent  $\kappa_n := \frac{3}{2} + \frac{1}{2n-2}$  below which the statement of Theorem 0.1 applies decreases to  $\frac{3}{2}$  as  $n \rightarrow \infty$  and satisfies  $\kappa_n < 2$  for all  $n \geq 5$ , in accordance with the mentioned boundedness results available for  $\kappa = 2$  when  $\mu$  is small ([24]).

iii) Let us also mention that the choice  $f \equiv 0$  is as well consistent with (H1)-(H3), and that correspondingly as a by-product of Theorem 0.1 we thereby obtain a rigorous blow-up result for the Keller-Segel model without cell proliferation in space dimensions  $n \geq 5$ .

The question whether or not such explosion phenomena may occur on the one hand appears to be of

evident biological relevance. Indeed, they apparently represent the most drastic version of the effect that a population exceeds its "carrying capacity", which for (0.1), (0.2) would be the finite number  $(\frac{\lambda}{\mu})^{\frac{1}{\kappa-1}}$ . Whereas the parabolic comparison principle essentially rules out such a behavior in the case when  $\chi = 0$  in (0.1), it is known that chemotactic cross-diffusion of the form in (0.1) may lead to at least bounded exceedance of the carrying capacity in the large time limit ([14], [24]).

On the other hand, the mentioned question also gives rise to the mathematical challenge of proving blow-up in a diffusive system with superlinear absorption in the PDE for the component which is supposed to become unbounded. In fact, it seems that only few techniques have turned out to be efficient tools for proving blow-up in Keller-Segel-type systems. These either require the availability of a nontrivial free energy ([10], [2], [27]), the access of the  $n$ -th moment functional  $t \mapsto \int_{\Omega} |x|^n u(x, t) dx$  to an ODI analysis ([15], [1]), or the circumstance that (radial) solutions can be transformed into solutions of a single scalar parabolic equation allowing for a comparison principle ([12], [3], [4]). More sophisticated techniques involving matched asymptotics procedures and degree arguments to detect non-generic blow-up behavior seem to rely on the very specific structure of the non-forced Keller-Segel model ([7], [22]). However, none of these requirements appear to be met in presence of logistic sources, not even in the simple model (0.1).

Accordingly our analysis will be based on a different approach which may be viewed as an ODI analysis of some  $L^p$  seminorms, with singular weights and  $p$  smaller than one, of the mass distribution function

$$w(s, t) := \int_{B_{\frac{1}{s^n}}(0)} u(x, t) dx, \quad s \in [0, R], \quad t \geq 0,$$

of radial solutions in the spatial domain  $\Omega = B_R(0) \subset \mathbb{R}^n$ . In fact,  $w$  satisfies a degenerate scalar parabolic equation that contains, unlike in the well-studied case  $f \equiv 0$ , a nonlocal nonlinearity involving the spatial derivative  $w_s$  (cf. (1.4) below).

## 1 Preliminaries

### 1.1 Local existence

The question of local solvability of (0.1) for sufficiently smooth initial data can be addressed by adapting methods that are well-established in the context of Keller-Segel-type systems. We therefore may confine ourselves with a statement of the main result in this direction, and refer the interested reader e.g. to [3], [28] and [11] for detailed reasonings in closely related situations.

**Lemma 1.1** *Suppose that  $f \in C^0([0, \infty)) \cap C^1((0, \infty))$  is such that  $f(0) \geq 0$ . Let  $n \geq 1$ , and assume that  $\Omega = B_R(0) \subset \mathbb{R}^n$  for some  $R > 0$ , and that  $u_0 \in C^1(\bar{\Omega})$  is positive and radially symmetric with respect to  $x = 0$ . Then there exist  $T_{max} \in (0, \infty]$  and a unique pair*

$$(u, v) \in (C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})))^2$$

*which solves (0.1), (0.3) in the classical sense in  $\Omega \times (0, T_{max})$ . Moreover, we have  $u > 0$  in  $\Omega \times (0, T_{max})$ , and both  $u(\cdot, t)$  and  $v(\cdot, t)$  are radially symmetric with respect to  $x = 0$  for all  $t \geq 0$ . Finally,*

$$\text{if } T_{max} < \infty \text{ then } \|u(\cdot, t)\|_{L^\infty(\Omega)} \rightarrow \infty \quad \text{as } t \nearrow T_{max}. \quad (1.1)$$

Unlike in the case when  $f \equiv 0$ , no complete information of the evolution of mass is available for (0.1) when  $f$  is not affine. As a substitute which will be sufficient for our purpose, we can easily derive an exponential upper bound for the growth of  $m(t)$ .

**Lemma 1.2** *Suppose that (H1) and (H3) hold with some  $A \geq 0$ , and that  $f(0) \geq 0$ . Then the function  $m$  in (0.3) satisfies*

$$m(t) \leq (m_0 + 1) \cdot e^{At} \quad \text{for all } t \in (0, T_{max}), \quad (1.2)$$

where  $m_0 := \frac{1}{|\Omega|} \int_{\Omega} u_0$ .

PROOF. Integrating the first equation in (0.1) in space yields

$$\frac{d}{dt} \int_{\Omega} u(\cdot, t) = \int_{\Omega} f(u) \leq A|\Omega| + A \int_{\Omega} u \quad \text{for all } t \in (0, T_{max})$$

because of (H3). Upon a time integration, this shows that  $\int_{\Omega} u(x, t) dx \leq (\int_{\Omega} u_0 + |\Omega|) \cdot e^{At} - |\Omega|$  for  $t \in (0, T_{max})$ . Dividing by  $|\Omega|$  and omitting the last nonpositive term, we immediately arrive at (1.2). ////

## 1.2 Transformation to a nonlocal scalar parabolic equation

Let us assume that  $\Omega = B_R(0)$  with some  $R > 0$  and that  $u_0 \in C^1(\bar{\Omega})$  is radially symmetric with respect to  $x = 0$ , and let  $(u, v)$  denote the corresponding radial solution in  $\Omega \times (0, T_{max})$  asserted by Lemma 1.1. Without danger of confusion, we may write  $u = u(r, t)$  and  $v = v(r, t)$  with  $r = |x| \in [0, R]$ .

Following [12], we introduce

$$w(s, t) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t) d\rho, \quad s = r^n \in [0, R^n], \quad t \in [0, T_{max}), \quad (1.3)$$

and then compute

$$w_s(s, t) = \frac{1}{n} \cdot u(s^{\frac{1}{n}}, t), \quad w_{ss}(s, t) = \frac{1}{n^2} \cdot s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t).$$

Since from the second equation in (0.1) we see that

$$r^{n-1} v_r(r, t) = - \int_0^r \rho^{n-1} u(\rho, t) d\rho + \frac{m(t)r^n}{n},$$

we thus see, again using (0.1), that

$$\begin{aligned} w_t(s, t) &= \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_t(\rho, t) d\rho \\ &= \int_0^{s^{\frac{1}{n}}} (\rho^{n-1} u_r)_r(\rho, t) d\rho - \int_0^{s^{\frac{1}{n}}} (\rho^{n-1} u v_r)_r(\rho, t) d\rho + \int_0^{s^{\frac{1}{n}}} \rho^{n-1} f(u(\rho, t)) d\rho \\ &= s^{1-\frac{1}{n}} u_r(s^{\frac{1}{n}}, t) - u(s^{\frac{1}{n}}, t) \cdot \left( - \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u(\rho, t) d\rho + \frac{m(t)s}{n} \right) + \int_0^{s^{\frac{1}{n}}} \rho^{n-1} f(u(\rho, t)) d\rho. \end{aligned}$$

Upon evident substitutions we thereby obtain that  $w$  satisfies the scalar parabolic equation

$$w_t = n^2 s^{2-\frac{2}{n}} w_{ss} + n\chi w w_s - \chi m(t) s w_s + \frac{1}{n} \cdot \int_0^s f(n \cdot w_s(\sigma, t)) d\sigma, \quad s \in (0, R^n), t \in (0, T_{max}), \quad (1.4)$$

with nonlocal nonlinearity. In light of (H2) and (0.3), we thus see that  $w$  solves

$$\begin{cases} w_t \geq n^2 s^{2-\frac{2}{n}} w_{ss} + n\chi w w_s - \chi m(t) s w_s - \mu n^{\kappa-1} \cdot \int_0^s w_s^\kappa(\sigma, t) d\sigma, & s \in (0, R^n), t \in (0, T_{max}), \\ w(0, t) = 0, \quad w(R^n, t) = \frac{m(t) R^n}{n}, & t \in (0, T_{max}), \\ w(s, 0) = w_0(s), & s \in (0, R^n), \end{cases} \quad (1.5)$$

where we have set

$$w_0(s) := \int_0^{s^{\frac{1}{n}}} \rho^{n-1} u_0(\rho) d\rho \quad \text{for } s \in [0, R^n]. \quad (1.6)$$

We observe that by nonnegativity of  $u$ , we have  $w_s \geq 0$  in  $(0, R^n) \times (0, T_{max})$ . In particular, in view of the boundary condition at  $s = R^n$  we have the rough estimate

$$w(s, t) \leq \frac{m(t) R^n}{n} \quad \text{for all } s \in [0, R^n] \text{ and } t \in [0, T_{max}]. \quad (1.7)$$

## 2 Blow-up

### 2.1 An integral inequality for $t \mapsto \int_0^{R^n} s^{-\alpha} w^p(s, t) ds$

Our analysis will focus on the time evolution of the functional  $\int_0^{R^n} s^{-\alpha} w^p$  for suitable  $\alpha > 1$  and  $p < 1$  which will be fixed in Lemma 2.3 below. A first observation concerning this will be made in the following lemma which still requires much milder assumptions than needed for the proof of our main results.

**Lemma 2.1** *Suppose that (H1)-(H3) hold with some  $\kappa > 1$ ,  $\mu \geq 0$  and  $A \geq 0$ . Assume that  $\Omega = B_R(0) \subset \mathbb{R}^n$  with some  $R > 0$  and  $n \geq 2$ . Let  $u_0 \in C^1(\bar{\Omega})$  be radial, and let  $(u, v)$  denote the solution of (0.1) in  $\Omega \times (0, T_{max})$ . Then for all  $\alpha > 0$  and  $p \in (0, 1)$ , the function  $w$  defined by (1.3) satisfies*

$$\begin{aligned} & \frac{1}{p} \int_0^{R^n} s^{-\alpha} w^p(s, t) ds + 2n(n-1) \int_0^t \int_0^{R^n} s^{1-\frac{2}{n}-\alpha} w^{p-1} w_s ds d\tau \\ & + \chi \int_0^t m(\tau) \cdot \int_0^{R^n} s^{1-\alpha} w^{p-1} w_s ds d\tau \\ & + \mu n^{\kappa-1} \int_0^t \int_0^{R^n} s^{-\alpha} w^{p-1}(s, \tau) \cdot \left( \int_0^s w_s^\kappa(\sigma, \tau) d\sigma \right) ds d\tau \\ & \geq \frac{1}{p} \int_0^{R^n} s^{-\alpha} w_0^p(s) ds \\ & + n^2(1-p) \int_0^t \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} w^{p-2} w_s^2 ds d\tau \end{aligned}$$

$$\begin{aligned}
& + \frac{n\chi}{2} \int_0^t \int_0^{R^n} s^{-\alpha} w^p w_s ds d\tau \\
& + \frac{n\chi\alpha}{2(p+1)} \int_0^t \int_0^{R^n} s^{-\alpha-1} w^{p+1} ds d\tau
\end{aligned} \tag{2.1}$$

for all  $t \in (0, T_{max})$ , where  $w_0$  is as in (1.6).

PROOF. We multiply (1.5) by  $(s + \varepsilon)^{-\alpha} w^{p-1}(s, \tau)$  and integrate over  $s \in (0, R^n)$  to obtain

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \int_0^{R^n} (s + \varepsilon)^{-\alpha} w^p(s, t) ds & \geq n^2 \int_0^{R^n} s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} w^{p-1} w_{ss} + n\chi \int_0^{R^n} (s + \varepsilon)^{-\alpha} w^p w_s \\
& - \chi m(t) \int_0^{R^n} s (s + \varepsilon)^{-\alpha} w^{p-1} w_s \\
& - \mu n^{\kappa-1} \cdot \int_0^{R^n} (s + \varepsilon)^{-\alpha} w^{p-1}(s, t) \left( \int_0^s w_s^\kappa(\sigma, t) d\sigma \right) ds \\
& =: I_1 + I_2 + I_3 + I_4 \quad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{2.2}$$

An integration by parts yields

$$\begin{aligned}
I_1 & = n^2(1-p) \cdot \int_0^{R^n} s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} w^{p-2} w_s^2 - n^2 \cdot \int_0^{R^n} \frac{d}{ds} \left\{ s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} \right\} \cdot w^{p-1} w_s ds \\
& + n^2 s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} w^{p-1} w_s \Big|_0^{R^n}.
\end{aligned} \tag{2.3}$$

In order to show that

$$n^2 s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} w^{p-1} w_s \Big|_0^{R^n} \geq 0, \tag{2.4}$$

we observe that due to the strong maximum principle we have  $u > 0$  in  $\bar{\Omega} \times (0, T_{max})$ , and hence also  $w_s(s, t) = \frac{1}{n} u(s^{\frac{1}{n}}, t)$  is positive for  $s \in [0, R^n]$  and  $t \in (0, T_{max})$ . In particular, this implies that for all  $t \in (0, T_{max})$  we can find  $c_1(t) > 0$  such that  $w(s, t) \geq c_1(t) \cdot s$  for all  $s \in [0, R^n]$ . Since moreover  $w_s(\cdot, t)$  is bounded in  $L^\infty((0, R^n))$  for any fixed  $t \in (0, T_{max})$ , this ensures that indeed

$$s^{2-\frac{2}{n}} \cdot w^{p-1}(s, t) \cdot w_s(s, t) \leq \frac{\|w_s(\cdot, t)\|_{L^\infty((0, R^n))}}{c_1^{1-p}(t)} \cdot s^{p+1-\frac{2}{n}} \rightarrow 0 \quad \text{as } s \rightarrow 0$$

because of the fact that  $p + 1 - \frac{2}{n} > 1 - \frac{2}{n} \geq 0$ . By positivity of  $w_s$ , this proves (2.4). Therefore, since

$$\frac{d}{ds} \left\{ s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} \right\} = \left( 2 - \frac{2}{n} \right) s^{1-\frac{2}{n}} (s + \varepsilon)^{-\alpha} - \alpha s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha-1} \leq \left( 2 - \frac{2}{n} \right) s^{1-\frac{2}{n}} (s + \varepsilon)^{-\alpha}$$

for all  $s > 0$ , we infer from (2.3) that

$$I_1 \geq n^2(1-p) \cdot \int_0^{R^n} s^{2-\frac{2}{n}} (s + \varepsilon)^{-\alpha} w^{p-2} w_s^2 - 2n(n-1) \cdot \int_0^{R^n} s^{1-\frac{2}{n}} (s + \varepsilon)^{-\alpha} w^{p-1} w_s \tag{2.5}$$

for all  $t \in (0, T_{max})$ . As to  $I_2$ , we split  $I_2 = \frac{I_2}{2} + \frac{I_2}{2}$  and again integrate by parts to find

$$\begin{aligned} \frac{I_2}{2} &= \frac{n\chi}{2(p+1)} \cdot \int_0^{R^n} (s+\varepsilon)^{-\alpha} (w^{p+1})_s \\ &\geq \frac{n\chi\alpha}{2(p+1)} \cdot \int_0^{R^n} (s+\varepsilon)^{-\alpha-1} w^{p+1}, \end{aligned} \quad (2.6)$$

because

$$\frac{n\chi}{2(p+1)} (s+\varepsilon)^{-\alpha} w^{p+1} \Big|_0^{R^n} \geq 0$$

due to the fact that  $w(0, t) = 0$  for all  $t \in (0, T_{max})$ . Combining (2.2)-(2.6) and integrating over  $(0, t)$  yields upon an obvious rearrangement

$$\begin{aligned} &\frac{1}{p} \int_0^{R^n} (s+\varepsilon)^{-\alpha} w^p(s, t) ds + 2n(n-1) \int_0^t \int_0^{R^n} s^{1-\frac{2}{n}} (s+\varepsilon)^{-\alpha} w^{p-1} w_s ds d\tau \\ &+ \chi \int_0^t m(\tau) \cdot \int_0^{R^n} (s+\varepsilon)^{1-\alpha} w^{p-1} w_s ds d\tau \\ &+ \mu n^{\kappa-1} \int_0^t \int_0^{R^n} (s+\varepsilon)^{-\alpha} w^{p-1}(s, \tau) ds d\tau \cdot \int_0^s w_s^\kappa(\sigma, \tau) d\sigma ds d\tau \\ &\geq \frac{1}{p} \int_0^{R^n} (s+\varepsilon)^{-\alpha} w_0^p(s) ds \\ &+ n^2(1-p) \int_0^t \int_0^{R^n} s^{2-\frac{2}{n}} (s+\varepsilon)^{-\alpha} w^{p-2} w_s^2 ds d\tau \\ &+ \frac{n\chi}{2} \int_0^t \int_0^{R^n} (s+\varepsilon)^{-\alpha} w^p w_s ds d\tau \\ &+ \frac{n\chi\alpha}{2(p+1)} \int_0^t \int_0^{R^n} (s+\varepsilon)^{-\alpha-1} w^{p+1} ds d\tau \end{aligned}$$

for  $t \in (0, T_{max})$ . Here in each term we apply the monotone convergence theorem in taking  $\varepsilon \searrow 0$  to arrive at (2.1). ////

The next lemma will enable us to estimate the crucial term in (2.1) which stems from the nonlocal nonlinearity in (1.4). It is based on an application of Fubini's theorem, followed by a straightforward three-step interpolation argument.

**Lemma 2.2** *Let  $n \geq 3$ ,  $R > 0$ ,  $\kappa \in (1, \frac{3}{2} + \frac{1}{2n-2})$ ,  $\alpha > 1$  and  $p \in (0, 1)$ . Moreover, let  $\theta \in (\kappa - 1, \frac{\kappa}{2})$  be such that*

$$\theta < \frac{n}{n-2} \cdot (2 - \kappa). \quad (2.7)$$

*Then for all  $\varepsilon > 0$  there exists  $C_\varepsilon > 0$  such that for any nondecreasing  $\varphi \in C^1([0, R^n])$  fulfilling  $\varphi > 0$  in  $(0, R^n)$  we have*

$$\int_0^{R^n} s^{-\alpha} \varphi^{p-1}(s) \cdot \left( \int_0^s \varphi_s^\kappa(\sigma) d\sigma \right) ds \leq \varepsilon \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} \varphi^{p-2} \varphi_s^2 + \varepsilon \int_0^{R^n} s^{-\alpha} \varphi^p \varphi_s + \varepsilon \int_0^{R^n} s^{-1-\alpha} \varphi^{p+1}$$

$$+C_\varepsilon \int_0^{R^n} \varphi^{p+\frac{\frac{2}{n}\theta-(2-\theta-\kappa)\alpha}{2-\theta+\frac{2}{n}\theta-\kappa}}.$$

**Remark.** The above restriction  $\kappa < \frac{3}{2} + \frac{1}{2n-2}$  precisely asserts that in fact it will be possible to pick  $\theta > \kappa - 1$  such that (2.7) holds (cf. also the proof of Lemma 2.3 below).

PROOF. By Fubini's theorem we have

$$I := \int_0^{R^n} s^{-\alpha} \varphi^{p-1}(s) \cdot \left( \int_0^s \varphi_s^\kappa(\sigma) d\sigma \right) ds = \int_0^{R^n} \left( \int_\sigma^{R^n} s^{-\alpha} \varphi^{p-1}(s) ds \right) \cdot \varphi_s^\kappa(\sigma) d\sigma.$$

Since  $\varphi^{p-1}(s)$  decreases with  $s$  due to the fact that  $p < 1$ , and since  $\alpha > 1$ , we can estimate

$$\begin{aligned} I &\leq \int_0^{R^n} \left( \int_\sigma^{R^n} s^{-\alpha} ds \right) \cdot \varphi^{p-1}(\sigma) \varphi_s^\kappa(\sigma) d\sigma \\ &\leq \frac{1}{\alpha-1} \int_0^{R^n} \sigma^{1-\alpha} \varphi^{p-1}(\sigma) \varphi_s^\kappa(\sigma) d\sigma \quad := \tilde{I}. \end{aligned}$$

We now let  $\varepsilon > 0$  be given and apply Young's inequality with exponents  $\frac{1}{\theta}$  and  $\frac{1}{1-\theta}$  to obtain  $c_1 > 0$  such that

$$\begin{aligned} \tilde{I} &= \frac{1}{\alpha-1} \int_0^{R^n} \left( s^{2-\frac{2}{n}-\alpha} \varphi^{p-2} \varphi_s^2 \right)^\theta \cdot \left( s^{1-\alpha-(2-\frac{2}{n}-\alpha)\theta} \varphi^{p-1-(p-2)\theta} \varphi_s^{\kappa-2\theta} \right) \\ &\leq \varepsilon \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} \varphi^{p-2} \varphi_s^2 + c_1 \int_0^{R^n} s^{-\alpha-\frac{2\theta-\frac{2}{n}\theta-1}{1-\theta}} \varphi^{p+\frac{2\theta-1}{1-\theta}} \varphi_s^{\frac{\kappa-2\theta}{1-\theta}}. \end{aligned}$$

Here since  $\theta \in (\kappa - 1, \frac{\kappa}{2})$  we have  $0 < \frac{\kappa-2\theta}{1-\theta} < 1$ , so that another application of Young's inequality, with exponents  $\frac{1-\theta}{\kappa-2\theta}$  and  $\frac{1-\theta}{1+\theta-\kappa}$  yields  $c_2 > 0$  such that

$$c_1 \int_0^{R^n} s^{-\alpha-\frac{2\theta-\frac{2}{n}\theta-1}{1-\theta}} \varphi^{p+\frac{2\theta-1}{1-\theta}} \varphi_s^{\frac{\kappa-2\theta}{1-\theta}} \leq \varepsilon \int_0^{R^n} s^{-\alpha} \varphi^p \varphi_s + c_2 \int_0^{R^n} s^{-\alpha-\frac{2\theta-\frac{2}{n}\theta-1}{1+\theta-\kappa}} \varphi^{p+\frac{2\theta-1}{1+\theta-\kappa}}.$$

Finally, according to our assumption (2.7) we know that  $\beta := \frac{2\theta-\frac{2}{n}\theta-1}{1+\theta-\kappa}$  satisfies  $\beta < 1$ , whence employing Young's inequality with exponents  $\frac{\alpha+1}{\alpha+\beta}$  and  $\frac{\alpha+1}{1-\beta}$  provides  $c_3 > 0$  fulfilling

$$c_2 \int_0^{R^n} s^{-\alpha-\frac{2\theta-\frac{2}{n}\theta-1}{1+\theta-\kappa}} \varphi^{p+\frac{2\theta-1}{1+\theta-\kappa}} \leq \varepsilon \int_0^{R^n} s^{-\alpha-1} \varphi^{p+1} + c_3 \int_0^{R^n} \varphi^{p+\frac{\frac{2}{n}\theta-(2-\theta-\kappa)\alpha}{2-\theta+\frac{2}{n}\theta-\kappa}}.$$

This completes the proof. ////

With the above statement at hand, we can proceed to derive from Lemma 2.1 a favorable integral inequality for  $t \mapsto \int_0^{R^n} s^{-\alpha} w^p(s, t) ds$  for suitable  $\alpha > 1$  and  $p < 1$ , provided that  $n \geq 5$  and  $\kappa < \frac{3}{2} + \frac{1}{2n-2}$ .

**Lemma 2.3** *Let  $n \geq 5, R > 0, m_0 > 0$  and suppose that (H1)-(H3) are valid with some  $\mu \geq 0, A \geq 0$  and  $\kappa \in (1, \frac{3}{2} + \frac{1}{2n-2})$ . Then there exist  $\alpha > 1, p \in (0, 1), \delta > 0, \Lambda > 0$  and  $C > 0$  such that whenever*

$u_0 = u_0(r)$  is nonnegative in  $\Omega = B_R(0) \subset \mathbb{R}^n$  such that  $\frac{1}{|\Omega|} \int_{\Omega} u_0 = m_0$ , then for the corresponding solution  $(u, v)$  of (0.1) in  $\Omega \times (0, T_{max})$  and  $w$  as defined by (1.3) we have

$$\int_0^{R^n} s^{-\alpha} w^p(s, t) ds \geq \int_0^{R^n} s^{-\alpha} w_0^p(s) ds + \delta \int_0^t \left( \int_0^{R^n} s^{-\alpha} w^p(s, \tau) ds \right)^{\frac{p+1}{p}} d\tau - C e^{\Lambda t} \quad \text{for all } t \in (0, T_{max}), \quad (2.8)$$

where  $w_0$  is as given by (1.6).

**Remark.** It is important here to observe that according to the above formulation, the constants  $\delta, \Lambda$  and  $C$  depend on  $u_0$  only through its total mass  $m_0$ .

PROOF. Since  $\kappa < \frac{3}{2} + \frac{1}{2n-2}$ , we have  $\kappa - 1 < \frac{n}{n-2}(2 - \kappa)$ , so that it is possible to pick  $\theta \in (\kappa - 1, \frac{\kappa}{2})$  such that

$$\theta < \frac{n}{n-2} \cdot (2 - \kappa). \quad (2.9)$$

Then  $2 - \theta + \frac{2}{n}\theta - \kappa > 0$ , and hence

$$\gamma(p, \alpha) := p + \frac{\frac{2}{n}\theta - (2 - \theta - \kappa)\alpha}{2 - \theta + \frac{2}{n}\theta - \kappa}, \quad p \in [0, 1], \quad \alpha \geq 1, \quad (2.10)$$

satisfies

$$\gamma(1, 1) = \frac{2 - \theta + \frac{2}{n}\theta - \kappa + \frac{2}{n}\theta - 2 + \theta + \kappa}{2 - \theta + \frac{2}{n}\theta - \kappa} = \frac{4}{n} \cdot \frac{\theta}{2 - \theta + \frac{2}{n}\theta - \kappa} > 0,$$

so that by continuity we can find  $p_0 \in (0, 1)$  and  $\alpha_0 > 1$  such that

$$\gamma(p, \alpha) \geq 0 \quad \text{for all } p \in [p_0, 1) \text{ and } \alpha \in (1, \alpha_0]. \quad (2.11)$$

Now the fact that  $n \geq 5$  allows us to fix  $\alpha \in (1, \alpha_0]$  such that

$$\alpha < 2 - \frac{4}{n},$$

which ensures that  $\alpha - 1 + \frac{2}{n} < \frac{n-2}{n}$ . Hence, we can finally choose  $p \in [p_0, 1)$  fulfilling

$$p > \frac{n}{n-2} \cdot \left( \alpha - 1 + \frac{2}{n} \right). \quad (2.12)$$

We now suppose that  $u_0 = u_0(r)$  is nonnegative with  $\frac{1}{|\Omega|} \int_{\Omega} u_0 = m_0$ , and let  $w$  and  $w_0$  be defined by (1.3) and (1.6), respectively. Then Lemma 2.1 says that

$$\begin{aligned} \frac{1}{p} \int_0^{R^n} s^{-\alpha} w^p(s, t) ds &\geq \frac{1}{p} \int_0^{R^n} s^{-\alpha} w_0^p(s) ds \\ &+ c_1 \int_0^t \int_0^{R^n} s^{2 - \frac{2}{n} - \alpha} w^{p-2} w_s^2 \\ &+ c_1 \int_0^t \int_0^{R^n} s^{-\alpha} w^p w_s \end{aligned}$$

$$\begin{aligned}
& +c_1 \int_0^t \int_0^{R^n} s^{-1-\alpha} w^{p+1} \\
& -2n(n-1) \int_0^t \int_0^{R^n} s^{1-\frac{2}{n}-\alpha} w^{p-1} w_s \\
& -\chi \int_0^t m(\tau) \cdot \int_0^{R^n} s^{1-\alpha} w^{p-1} w_s \\
& -\mu n^{\kappa-1} \int_0^t \int_0^{R^n} s^{-\alpha} w^{p-1}(s, \tau) \cdot \left( \int_0^s w_s^\kappa(\sigma, \tau) d\sigma \right) ds d\tau \\
=: & J_1 + J_2 + J_3 + J_4 - J_5 - J_6 - J_7 \quad \text{for all } t \in (0, T_{max}) \quad (2.13)
\end{aligned}$$

holds with  $c_1 := \min\{n^2(1-p), \frac{n\chi}{2}, \frac{n\chi\alpha}{2(p+1)}\}$ . Here, Young's inequality first provides  $c_2 > 0$  such that

$$J_5 \leq \frac{c_1}{3} \int_0^t \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} w^{p-2} w_s^2 + c_2 \int_0^t \int_0^{R^n} s^{-\frac{2}{n}-\alpha} w^p,$$

and then yields  $c_3 > 0$  such that

$$c_2 \int_0^t \int_0^{R^n} s^{-\frac{2}{n}-\alpha} w^p \leq \frac{c_1}{4} \int_0^t \int_0^{R^n} s^{-1-\alpha} w^{p+1} + c_3 \int_0^t \int_0^{R^n} s^{-\frac{2}{n}-\alpha+\frac{n-2}{n}\cdot p}.$$

According to (2.12), we know that  $-\frac{2}{n}-\alpha+\frac{n-2}{n}\cdot p > -1$ , so that we conclude that

$$J_5 \leq \frac{1}{3}J_2 + \frac{1}{4}J_4 + c_4 t \quad \text{for all } t \in (0, T_{max}) \quad (2.14)$$

holds with some  $c_4 > 0$ .

Next, in order to estimate  $J_6$  we recall (1.2) which states that

$$m(t) \leq (m_0 + 1) e^{At} \quad \text{for all } t \in (0, T_{max})$$

with  $A$  as in (H3). Therefore, using Young's inequality as before we find  $c_5 > 0$  and  $c_6 > 0$  fulfilling

$$\begin{aligned}
J_6 & \leq (m_0 + 1)\chi \int_0^t \int_0^{R^n} e^{A\tau} s^{1-\alpha} w^{p-1} w_s \\
& \leq \frac{c_1}{3} \int_0^t \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} w^{p-2} w_s^2 + c_5 \int_0^t \int_0^{R^n} e^{2A\tau} \cdot s^{\frac{2}{n}-\alpha} w^p \\
& \leq \frac{c_1}{3} \int_0^t \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} w^{p-2} w_s^2 + \frac{c_1}{4} \int_0^t \int_0^{R^n} s^{-1-\alpha} w^{p+1} \\
& \quad + c_6 \int_0^t \int_0^{R^n} e^{2(p+1)A\tau} \cdot s^{\frac{2}{n}-\alpha+\frac{n+2}{n}\cdot p}.
\end{aligned}$$

Since clearly (2.12) implies that  $\frac{2}{n}-\alpha+\frac{n+2}{n}\cdot p > -1$ , for some  $c_7 > 0$  we have

$$J_6 \leq \frac{1}{3}J_2 + \frac{1}{4}J_4 + c_7 e^{2(p+1)At} \quad \text{for all } t \in (0, T_{max}). \quad (2.15)$$

Finally, an application of Lemma 2.2 with sufficiently small  $\varepsilon > 0$  (for instance, with  $\varepsilon = \frac{c_1}{4}$ ) yields  $c_8 > 0$  such that

$$J_7 \leq \frac{c_1}{3} \int_0^t \int_0^{R^n} s^{2-\frac{2}{n}-\alpha} w^{p-2} w_s^2 + c_1 \int_0^t \int_0^{R^n} s^{-\alpha} w^p w_s + \frac{c_1}{4} \int_0^t \int_0^{R^n} s^{-1-\alpha} w^{p+1} + c_8 \int_0^t \int_0^{R^n} w^{\gamma(p,\alpha)}(s,\tau) ds d\tau$$

with  $\gamma(p,\alpha)$  as given by (2.10). Since  $\gamma(p,\alpha) \geq 0$  thanks to (2.11) and our choice of  $p$  and  $\alpha$ , the pointwise estimate

$$w(s,\tau) \leq \frac{m(\tau)R^n}{n} \leq \frac{(m_0+1)R^n}{n} e^{A\tau} \quad \text{for all } s \in (0, R^n) \text{ and } \tau \in (0, T_{max})$$

asserted by (1.7) and (1.2) shows that

$$J_7 \leq \frac{1}{3} J_2 + J_3 + \frac{1}{4} J_4 + c_9 e^{\gamma(p,\alpha) \cdot At} \quad \text{for all } t \in (0, T_{max}) \quad (2.16)$$

holds with some  $c_9 > 0$ . Collecting (2.13)-(2.16), we see that

$$\int_0^{R^n} s^{-\alpha} w^p(s,t) ds \geq \int_0^{R^n} s^{-\alpha} w_0^p(s) ds + \frac{pc_1}{4} \int_0^t \int_0^{R^n} s^{-1-\alpha} w^{p+1} - C e^{\Lambda t} \quad \text{for all } t \in (0, T_{max}) \quad (2.17)$$

is valid with  $\Lambda := \max\{2(p+1), \gamma(p,\alpha)\} \cdot A$  and some  $C > 0$ . Here, by the Hölder inequality we have

$$\begin{aligned} \int_0^{R^n} s^{-\alpha} w^p &= \int_0^{R^n} s^{-\alpha + \frac{p(\alpha+1)}{p+1}} \cdot \left( s^{-1-\alpha} w^{p+1} \right)^{\frac{p}{p+1}} \\ &\leq \left( \int_0^{R^n} s^{-\alpha+p} ds \right)^{\frac{1}{p+1}} \cdot \left( \int_0^{R^n} s^{-1-\alpha} w^{p+1} \right)^{\frac{p}{p+1}}, \end{aligned}$$

so that since (2.12) entails that  $-\alpha + p > -1$ , we obtain

$$\frac{pc_1}{4} \int_0^t \int_0^{R^n} s^{-1-\alpha} w^{p+1} \geq \frac{pc_1}{4} \cdot \left( \frac{p+1-\alpha}{R^{n(p+1-\alpha)}} \right)^{\frac{1}{p}} \cdot \int_0^t \left( \int_0^{R^n} s^{-\alpha} w^p \right)^{\frac{p+1}{p}} \quad \text{for all } t \in (0, T_{max}).$$

Therefore (2.17) implies (2.8) upon an evident choice of  $\delta$ . ////

## 2.2 Proof of the main results

In order to be able to apply a convenient comparison argument to the integral inequality provided by Lemma 2.3, as a last preparation let us state an elementary lemma of Gronwall type.

**Lemma 2.4** *Let  $a > 0, \delta > 0$  and  $\beta > 0$ , and suppose that for some  $T > 0$ ,  $y \in C^0([0, T])$  is a nonnegative function satisfying*

$$y(t) \geq a + \delta \cdot \int_0^t y^{1+\beta}(\tau) d\tau \quad \text{for all } t \in (0, T). \quad (2.18)$$

Then

$$T \leq \frac{1}{\beta\delta a^\beta}. \quad (2.19)$$

PROOF. We first claim that whenever  $\varepsilon \in (0, a)$ , for the solution  $z_\varepsilon$ , defined up to its maximal existence time  $T_\varepsilon$ , of

$$\begin{cases} z'_\varepsilon(t) = \delta z_\varepsilon^{1+\beta}(t), & t \in (0, T_\varepsilon), \\ z_\varepsilon(0) = a - \varepsilon, \end{cases} \quad (2.20)$$

we have  $T_\varepsilon \geq T$  and  $z_\varepsilon < y$  in  $(0, T)$ . Indeed, if this was false then there would exist  $t_0 \in (0, \min\{T, T_\varepsilon\})$  such that  $z_\varepsilon < y$  on  $(0, t_0)$ , but  $z_\varepsilon(t_0) = y(t_0)$ . Since  $z_\varepsilon(t) = a - \varepsilon + \delta \int_0^t z_\varepsilon^{1+\beta}(\tau) d\tau$  for all  $t \in (0, T_\varepsilon)$ , however, this would imply that

$$a - \varepsilon + \delta \int_0^{t_0} z_\varepsilon^{1+\beta}(\tau) d\tau = y(t_0) \geq a + \delta \int_0^{t_0} y^{1+\beta}(\tau) d\tau > a + \delta \int_0^{t_0} z_\varepsilon^{1+\beta}(\tau) d\tau,$$

which is absurd.

Now integrating (2.20), we explicitly find that

$$T_\varepsilon = \frac{1}{\beta\delta(a - \varepsilon)^\beta} \quad \text{for all } \varepsilon \in (0, a),$$

so that (2.19) results from the inequality  $T_\varepsilon \geq T$  on taking  $\varepsilon \searrow 0$ . ////

We can now pass to the proof of our main results.

PROOF (of Theorem 0.1). We fix  $n \geq 5$  and may assume that  $\Omega = B_R(0) \subset \mathbb{R}^n$  with some  $R > 0$ . Then given  $\kappa \in (1, \frac{3}{2} + \frac{1}{2n-2})$  and  $m_0 > 0$ , we let  $\alpha > 1, p \in (0, 1), \delta > 0, \Lambda > 0$  and  $C$  be as provided by Lemma 2.3. Now for fixed  $T > 0$  we pick  $a > 0$  large such that

$$a > \left(\frac{p}{\delta T}\right)^p. \quad (2.21)$$

Next,

$$\psi_\varepsilon(s) := \frac{m_0}{n} \cdot \frac{R^n + \varepsilon}{s + \varepsilon} \cdot s, \quad s \in [0, R^n], \quad \varepsilon > 0,$$

is nonnegative and satisfies

$$\psi_\varepsilon(s) \nearrow \frac{m_0 R^n}{n} \quad \text{for all } s \in (0, R^n] \quad \text{as } \varepsilon \searrow 0,$$

so that the monotone convergence theorem asserts that

$$\int_0^{R^n} s^{-\alpha} \psi_\varepsilon^p(s) ds \rightarrow \infty \quad \text{as } \varepsilon \searrow 0.$$

Thus, we can find some sufficiently small  $\varepsilon > 0$  such that

$$\int_0^{R^n} s^{-\alpha} \psi_\varepsilon^p(s) ds \geq a + C e^{\Lambda T}. \quad (2.22)$$

With this value of  $\varepsilon$  fixed henceforth, we let

$$w_0(s) := \psi_\varepsilon(s), \quad s \in [0, R^n], \quad (2.23)$$

and then immediately see that  $w_0$  belongs to  $C^\infty([0, R^n])$  and satisfies  $w_0(0) = 0, w_0(R^n) = \frac{m_0 R^n}{n}$  and  $w_{0s}(s) > 0$  for all  $s \in [0, R^n]$ . Accordingly, the function  $u_0$  defined by  $u_0(x) := n \cdot w_{0s}(|x|^n)$  for  $x \in \bar{\Omega}$  is radially symmetric, smooth and positive in  $\bar{\Omega}$  with  $\frac{1}{|\bar{\Omega}|} \int_{\Omega} u_0 = m_0$ .

We claim that the maximal existence time  $T_{max}$  of the corresponding solution  $(u, v)$  of (0.1) satisfies  $T_{max} < T$ . To see this, we let  $w$  be given by (1.3) and apply Lemma 2.3 and use (2.23) and (2.22) to obtain the inequality

$$\begin{aligned} \int_{\Omega} s^{-\alpha} w^p(s, t) ds &\geq \int_0^{R^n} s^{-\alpha} w_0^p(s) ds + \delta \int_0^t \left( \int_0^{R^n} s^{-\alpha} w^p(s, \tau) ds \right)^{\frac{p+1}{p}} d\tau - C e^{\Lambda t} \\ &\geq a + \delta \int_0^t \left( \int_0^{R^n} s^{-\alpha} w^p(s, \tau) ds \right)^{\frac{p+1}{p}} d\tau \quad \text{for all } t \in (0, T_{max}). \end{aligned} \quad (2.24)$$

This means that  $y(t) := \int_0^{R^n} s^{-\alpha} w^p(s, t) ds$ ,  $t \in (0, T_{max})$ , satisfies (2.18) with  $\beta := \frac{1}{p}$ , so that Lemma 2.4 states that

$$T_{max} \leq \frac{1}{\frac{1}{p} \cdot \delta \cdot a^{\frac{1}{p}}}.$$

In conjunction with our largeness assumption (2.21) on  $a$ , this entails that indeed  $T_{max} < T$  and thereby completes the proof. ////

### 3 Discussion

We have performed a rigorous blow-up analysis for the system (0.1) which acts as a model for the interplay between self-diffusion, chemotactic cross-diffusion, and cell proliferation and death. We concentrated on the spatially radial framework, where (0.1) is equivalent to the scalar degenerate nonlocal PDE (1.4) for the unknown  $w$  defined in (1.3). We have thereby identified a class of nonlinearities  $f$  which exhibit superlinear absorption at large densities, but which are unable to prevent chemotactic collapse in (0.1).

Our approach was based on an analysis of the functional  $y(t) := \int_0^{R^n} s^{-\alpha} w^p(s, t) ds$  with appropriate  $\alpha > 1$  and, which seems to be novel in this context, with (positive)  $p$  less than one. This allowed us not only to make sure that the dissipative properties in (0.1) can be overbalanced by the cross-diffusive part, but also to show that *some* of the diffusive terms contributing to the evolution of  $y(t)$  itself essentially dominates the absorptive nonlocal nonlinearity (cf. (2.1), Lemma 2.2 and Lemma 2.3). In particular, the technical condition (2.12), which is closely related to our overall restrictions on  $n$  and  $\kappa$ , was crucial for the estimates for the integrals  $J_5$  and  $J_6$  in (2.13). This finally led to the observation that the evolution of  $y(t)$  will eventually be determined by the chemotactic term, which according to (2.24) gives rise to an integral inequality of the form (2.18) for  $y(t)$  with a nonlinearity growing with  $y$  in a superlinear way (that actually is close to the naturally expected quadratic behavior according to the fact that  $p$  lies close to 1).

Of course, our result is to be understood as a first step towards a more general theory of Keller-Segel systems with sources of logistic type. Although it is restricted to space dimensions of no physical relevance, it confirms that the aggregative tendency of the cross-diffusive term in (0.1) need not be compensated by any superlinear death term. We conjecture that actually the critical growth exponent in (H2) should be  $\kappa = 2$  in the sense that for any  $\kappa < 2$  blow-up may occur. We moreover believe that also in the relevant space dimension  $n = 3$  a critical superlinear absorption behavior of  $f(u)$  exists that distinguishes between occurrence and impossibility of blow-up.

## References

- [1] BILER, P.: *Local and global solvability of some parabolic systems modelling chemotaxis*. Adv. Math. Sci. Appl. **8**, 715-743 (1998)
- [2] CIEŚLAK, T., LAURENÇOT, PH.: *Finite time blow-up for a one-dimensional quasilinear parabolic-parabolic chemotaxis system*. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **27** (1), 437-446 (2010)
- [3] CIEŚLAK, T., WINKLER, M.: *Finite-time blow-up in a quasilinear system of chemotaxis*. Nonlinearity **21**, 1057-1076 (2008)
- [4] DJIE, K., WINKLER, M.: *Boundedness and finite-time collapse in a chemotaxis system with volume-filling effect*. Nonlinear Analysis, Theory Meth. Appl. **72** (2), 1044-1064 (2010)
- [5] EFENDIEV, M., NAKAGUCHI, E.: *On a new dimension estimate of the global attractor for chemotaxis-growth systems*. Osaka J. Math. **45**, 273-281 (2008)
- [6] EFENDIEV, M., NAKAGUCHI, E., WENDLAND, W.L.: *Uniform estimate of dimension of the global attractor for a semi-discretized chemotaxis-growth system*. Discrete Cont. Dyn. Syst. S, 334-343 (2007)
- [7] HERRERO, M. A., VELÁZQUEZ, J. J. L.: *A blow-up mechanism for a chemotaxis model*. Ann. Scuola Normale Superiore **24**, 663-683 (1997)
- [8] HILLEN, TH., PAINTER, K.: *A users' guide to PDE models for chemotaxis*. J. Math. Biol. **58** (1-2), 183-217 (2009)
- [9] HILLEN, TH., PAINTER, K.: *Spatio-Temporal Chaos in a Chemotaxis Model*. Physica D **240** (4-5), 363-375 (2011)
- [10] HORSTMANN, D., WANG, G.: *Blow-up in a chemotaxis model without symmetry assumptions*. Eur. J. Appl. Math. **12**, 159-177 (2001)
- [11] HORSTMANN, D., WINKLER, M.: *Boundedness vs. blow-up in a chemotaxis system*. J. Differential Equations **215** (1), 52-107 (2005)
- [12] JÄGER, W., LUCKHAUS, S.: *On Explosions of Solutions to a System of Partial Differential Equations Modelling Chemotaxis*. Trans. Amer. Math. Soc. **329** (2), 819-824 (1992)
- [13] KELLER, E. F., SEGEL, L. A.: *Initiation of slime mold aggregation viewed as an instability*. J. Theor. Biol. **26**, 399-415 (1970)

- [14] MARCINIAK-CZOCHRA, A., PTASHNYK, M.: *Boundedness in a haptotaxis model.* Math. Mod. Meth. Appl. Sci. **20** (3), 449-476 (2010)
- [15] NAGAI, T.: *Blowup of nonradial solutions to parabolicelliptic systems modeling chemotaxis in two-dimensional domains.* J. Inequal. Appl. **6**, 37-55 (2001)
- [16] NAGAI, T., SENBA, T., YOSHIDA, K.: *Application of the Moser-Trudinger inequality to a parabolic system of chemotaxis.* Funkc. Ekvacioj, Ser. Int. **40**, 411-433 (1997)
- [17] NAGAI, T., SENBA, T., YOSHIDA, K.: *Global existence of solutions to the parabolic systems of chemotaxis.* RIMS Kokyuroku **1009**, 22-28 (1997)
- [18] NAKAGUCHI, E., OSAKI, K.: *Global Existence of Solutions to a Parabolic-Parabolic System for Chemotaxis with Weak Degradation.* Nonlinear Anal. **74**, 286-297 (2011)
- [19] OSAKI, K., TSUJIKAWA, T, YAGI, A., MIMURA, M.: *Exponential attractor for a chemotaxis-growth system of equations.* Nonlinear Anal. **51**, 119-144 (2002)
- [20] OSAKI, K., YAGI, A.: *Finite dimensional attractors for one-dimensional Keller-Segel equations,* Funkcialaj Ekvacioj **44**, 441 - 469 (2001)
- [21] PERTHAME, B.: *Transport Equations in Biology.* Birkhäuser-Verlag, Basel, 2007
- [22] SENBA, T.: *Type II blowup of solutions to a simplified Keller-Segel system in two dimensional domains.* Nonlinear Anal. TMA **66** (8), 1817-1839 (2007)
- [23] SUGIYAMA, Y.: *Blow-up criterion via scaling invariant quantities with effect on coefficient growth in Keller-Segel system.* Diff. Int. Eq. **23** (7-8), 619-634 (2010)
- [24] TELLO, J.I., WINKLER, M.: *A chemotaxis system with logistic source.* Communications in Partial Differential Equations **32** (6), 849-877 (2007)
- [25] WINKLER, M.: *Chemotaxis with logistic source: Very weak global solutions and boundedness properties.* Journal of Math. Analysis and Applications **348**, 708-729 (2008)
- [26] WINKLER, M.: *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source.* Communications in Partial Differential Equations **35**, 1516-1537 (2010)
- [27] WINKLER, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889-2905 (2010)
- [28] WRZOSEK, D.: *Long-time behaviour of solutions to a chemotaxis model with volume-filling effect.* Proceedings of the Royal Society of Edinburgh A **136** (2), 431-444 (2006)