Global solutions in higher dimensions to a fourth order parabolic equation modeling epitaxial thin film growth

Michael Winkler*
Fakultät für Mathematik, Universität Duisburg-Essen,
45117 Essen, Germany

Abstract

The initial-value problem for

$$u_t = -\Delta^2 u - \mu\Delta u - \lambda\Delta|\nabla u|^2 + f(x) \qquad (\star)$$

is studied under the conditions $\frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u = 0$ on the boundary of a bounded convex domain $\Omega \subset \mathbb{R}^n$ with smooth boundary. This problem arises in the modeling of the evolution of a thin surface when exposed to molecular beam epitaxy. Correspondingly the physically most relevant spatial setting is obtained when $n = 2$, but previous mathematical results appear to concentrate on the case $n = 1$.
In this work it is proved that when $n \le 3$, $\mu \ge 0, \lambda > 0$ and $f \in L^\infty(\Omega)$ satisfies $\int_\Omega f \ge 0$, for each prescribed initial distribution $u_0 \in L^\infty(\Omega)$ fulfilling $\int_\Omega u_0 \ge 0$, there exists at least one global weak solution $u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ satisfying $\int_\Omega u(\cdot,t) \ge 0$ for a.e. $t > 0$, and moreover it is shown that this solution can be obtained through a Rothe-type approximation scheme.
Furthermore, under an additional smallness condition on μ and $\|f\|_{L^\infty(\Omega)}$ it is shown that there exists a bounded set $S \subset L^1(\Omega)$ which is absorbing for $(\star)$ in the sense that for any such solution we can pick $T > 0$ such that $e^{2\lambda u(\cdot,t)} \in S$ for all $t > T$, provided that Ω is a ball and u_0 and f are radially symmetric with respect to $x = 0$. This partially extends similar absorption results known in the spatially one-dimensional case.
The techniques applied to derive appropriate compactness properties via a priori estimates include straightforward testing procedures which lead to integral inequalities involving, for instance, the functional $\int_\Omega e^{2\lambda u}dx$, but also the use of a maximum principle for second-order elliptic equations.

Key words: fourth-order parabolic equation, absorbing set, molecular beam epitaxy, surface diffusion, Rothe approximation
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*email: michael.winkler@uni-due.de

1 Introduction

In this work we study the fourth order parabolic equation

$$u_t = -\Delta^2 u - \mu\Delta u - \lambda\Delta|\nabla u|^2 + f(x) \tag{1.1}$$

for the unknown function $u = u(x,t)$, where $\mu \geq 0$ and $\lambda > 0$ are given parameters and f is a prescribed function depending on the spatial variable only.

This evolution equation is used as a model for the growth of thin surfaces when exposed to molecular beam epitaxy; in this context, $u(x,t)$ can either represent the absolute thickness of the film, or rather the relative surface height, that is, the deviation of the film height at the point x from the mean film thickness at time t. The model incorporates the linear effects of surface diffusion ([13], [18]) and so-called Schwoebel barriers ([24]), reflected by the biharmonic term $-\Delta^2$ and the second-order backward diffusion term $-\mu\Delta u$ in (1.1), respectively. Apart from that, the nonlinear term $-\lambda\Delta|\nabla u|^2$ accounts for the presence of an energy barrier that particles need to cross before they diffuse ([15]). Finally, in (1.1) it is assumed that the surface is exposed to an external source in the form of a molecular beam that is supposed to be constant in time but possibly inhomogeneous in space. Further details on the model and its physical framework can be found in [24], [17], [23].

Previous work shows that numerical simulations based on (1.1) can be well fitted to experimental data, and that (1.1) adequately describes the phenomena of coarsening and roughening that are characteristic for the growth of corresponding surfaces on intermediate time scales ([23], [25]). The mathematical theory, however, appears to lag somewhat behind in that satisfactory rigorous analytic results on (1.1) up to now seem to be restricted to the spatially one-dimensional setting. In order to point out the technical difficulties related to (1.1), let us briefly comment on some of its basic mathematical features. For definiteness, here and in the sequel we shall complement (1.1) by initial conditions and homogeneous boundary conditions of Neumann type, and accordingly consider the initial-boundary value problem

$$\begin{cases} u_t = -\Delta^2 u - \mu\Delta u - \lambda\Delta|\nabla u|^2 + f(x), \qquad x \in \Omega,\ t > 0, \\ \frac{\partial}{\partial\nu} u = \frac{\partial}{\partial\nu}\Delta u = 0, \qquad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{1.2}$$

in a bounded physical domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where u_0 is a given initial distribution, and where $\frac{\partial}{\partial\nu}$ denotes differentiation with respect to the outward normal ν on $\partial\Omega$.

A first observation is that the nonlinear term $-\lambda\Delta|\nabla u|^2$ is of lower order as related to the biharmonic linear diffusion part, whence by standard methods involving well-established semigroup techniques it is possible to construct local-in-time smooth solutions of (1.2) whenever the initial data u_0 are sufficiently regular. However, in view of the quadratic growth of this nonlinearity it is a priori not clear whether such solutions can be extended to exist for all times, or if finite-time blow-up phenomena may occur. In this respect, a mathematical drawback of (1.1) as compared to related surface growth equations appears to be the lack of a Lyapunov functional involving derivatives of the solution. Indeed, it seems that the derivation of (1.1) does not immediately suggest any dissipated quantity which is corresponds to a genuine physical energy. As contrasted to this, when the nonlinearity $\Psi(u) := -\Delta|\nabla u|^2$ in (1.1) is replaced by $\nabla\cdot(|\nabla u|^2\nabla u)$ ([21], [16]), or – as is the case in the Cahn-Hilliard equation – by $\Delta(u^3)$ ([20], [19]), or also by $\partial^2_{ij}(\frac{\partial_i u \partial_j u}{u})$ like in the Derrida-Lebowitz-Speer-Spohn equation ([11]), ([14]), the availability of corresponding energy-like dissipative quantities provides natural candidates for the derivation of global properties of solutions. More sophisticated techniques, though still relying on dissipation of certain functionals, apply to the Kuramoto-Sivashinsky equation in one dimension, where either $\Psi(u) = -\frac{1}{2}u_x^2$ or $\Psi(u) = -\frac{1}{2}(u^2)_x$ ([19], [9], [22]). The use of energies (and entropies) is also essential to analytical approaches to related fourth-order parabolic problems with nonlinear diffusion such as the thin-film equation $u_t = -\nabla\cdot(u^m\nabla\Delta u)$ ([2], [7]).

Thus first focussing on the question of global solvability, let us recall that in the spatially one-dimensional setting a basic global-in-time information can be derived from the fact that in this case the nonlinearity in (1.2) is orthogonal to the solution itself in the sense of $L^2(\Omega)$. Indeed, if $\Omega = (a,b) \subset \mathbb{R}$ and $f \equiv 0$ for simplicity, integrating by parts we have

$$\int_a^b (u_x^2)_{xx} u = \frac{1}{3} u_x^3 \Big|_a^b = 0 \tag{1.3}$$

for any smooth u satisfying $u_x|_{\partial\Omega} = 0$, and thus formally testing (1.2) by u we obtain that the exponential bound

$$\int_\Omega u^2(x,t)dx \le \Big(\int_\Omega u_0^2(x)dx\Big)\cdot e^{\frac{\mu^2}{2}t} \qquad \text{for all } t \ge 0$$

should hold for any solution of (1.2). In fact, this a priori estimate for solutions in $L^\infty_{loc}([0,\infty); L^2(\Omega))$ can be used as a starting point for a bootstrap procedure to obtain higher order regularity properties and finally construct global weak solutions (see [25] and [3] for a stochastic variant of (1.2)). However, when $n \ge 2$ a similar reasoning is impossible due to the lack of an appropriate analogue of (1.3), and thus the question whether or not (1.2) admits global solutions appears to be open in the higher-dimensional framework.

The first of our main results addresses this problem in space dimensions $n \le 3$. On the one hand it asserts the existence of a global solution to (1.2) in the sense of Definition 2.1 below, and on the other hand makes sure that this solution can be numerically obtained by a Rothe-type approximation scheme.

Theorem 1.1 *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain in $\mathbb{R}^n$, $n \le 3$, and that $\mu \ge 0$ and $\lambda > 0$. Also suppose that $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$ fulfill $\int_\Omega u_0 \ge 0$ and $\int_\Omega f \ge 0$. Then (1.2) possesses at least one global weak solution u that satisfies $\int_\Omega u(x,t)dx \ge 0$ for a.e. $t > 0$.*
This solution can be obtained as the limit of a sequence of functions $\hat{u}^{(\tau)}$ determined by

solutions of the Rothe scheme (2.2), (2.4) below in the sense that $\hat{u}^{(\tau)} \to u$ holds in $L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ and a.e. in $\Omega \times (0,\infty)$ along an appropriate sequence of numbers $\tau = \tau_j \searrow 0$.
Furthermore, this solution has the additional regularity property $e^{\lambda u} \in L^2_{loc}([0,\infty); W^{2,2}(\Omega))$ with $\frac{\partial}{\partial \nu} e^{\lambda u} = 0$ on $\partial\Omega$, and we have $e^{\lambda \hat{u}^{(\tau)}} \rightharpoonup e^{\lambda u}$ in $L^2_{loc}([0,\infty); W^{2,2}(\Omega))$ as $\tau = \tau_j \searrow 0$.

Let us remark here that since (1.2) is invariant under the Galilean transformation $u \mapsto u - a - bt$ for any real a and b, the assumptions $\int_\Omega f \geq 0$ and $\int_\Omega u_0 \geq 0$ do not restrict the generality of the above statement which of course is to be read as referring to u as the relative film height. In fact, if u_0 and f are arbitrary functions from $L^\infty(\Omega)$, then $\tilde{u}_0 := u_0 - a$ and $\tilde{f} := f - b$ with $a := \frac{1}{|\Omega|}\int_\Omega u_0$ and $b := \frac{1}{|\Omega|}\int_\Omega f$ satisfy $\int_\Omega \tilde{u}_0 = \int_\Omega \tilde{f} = 0$, and hence an application of Theorem 1.1 yields a corresponding solution $\tilde{u}$ which can be transformed to a solution u of the original problem via the relation $\tilde{u} = u - a - bt$.

Concerning the qualitative behavior of solutions, the one-dimensional version of (1.2) again possesses a favorable absorption property which underlines the stabilizing effect of the nonlinearity as opposed to the destabilizing second-order backward diffusive term: Namely, in [25] it was shown that when $n = 1$ and $f \equiv 0$, (1.2) possesses a bounded absorbing set in the sense that there exist a bounded set $S \subset L^1(\Omega)$ and a diffeomorphism $\Phi : \mathbb{R} \to (0,\infty)$ such that for any of the weak solutions u constructed there, the function $\Phi(u(\cdot,t))$ belongs to S for all sufficiently large t. This diffeomorphism asymptotically behaves according to $\Phi(u) \simeq e^{\beta u}$ as $u \to -\infty$ and $\Phi(u) \simeq u^p$ as $u \to +\infty$, where $p > 1$ and $\beta \in (0, \frac{8}{3})$ are arbitrary. In particular, this indicates that *the positive part* of each solution will eventually enjoy some boundedness property, whereas the emergence of peaks in *the negative part* is not ruled out. This is in good accordance with numerical simulations ([25]) and provides motivation for the study of blow-up properties of solutions possibly evolving into singularities in their negative part ([6]).

It is the second goal of the present work to establish an analogue of the latter absorption result in higher space dimensions. For technical reasons, our procedure requires that besides the above hypotheses we assume the spatial framework to be radially symmetric. We also need a smallness condition on both μ and the size of f, and as before we simplify the setting by assuming $\int_\Omega u_0 = \int_\Omega f = 0$.

Theorem 1.2 *Suppose that $n \leq 3$ and that $\Omega = B_R(0) \subset \mathbb{R}^n$ for some $R > 0$. Then for any $\lambda > 0$ there exist $c > 0$ and $C > 0$ with the following property: For all $f \in L^\infty(\Omega)$ and $u_0 \in L^\infty(\Omega)$ that are radially symmetric with respect to $x = 0$ and fulfill $\int_\Omega f = \int_\Omega u_0 = 0$, and each $\mu \geq 0$ such that*

$$\|f\|_{L^\infty(\Omega)} + \mu^2 \leq c, \tag{1.4}$$

the problem (1.2) possesses at least one global weak solution u that is radially symmetric and satisfies $\int_\Omega u(x,t)dx = 0$ for a.e. $t > 0$, and moreover there exists $T > 0$, possibly depending on u, such that

$$\int_\Omega e^{2\lambda u(x,t)} dx \leq C \qquad \text{for a.e. } t > T \tag{1.5}$$

and
$$\int_t^{t+1}\int_\Omega e^{2\lambda u}\Big(|D^2u|^2+|\nabla u|^4\Big)\le C \qquad \textit{for all } t>T. \tag{1.6}$$
Furthermore, any sequence $(t_j)_{j\in\mathbb{N}}$ *of numbers* $t_j\to\infty$ *contains a subsequence* $(t_{j_i})_{i\in\mathbb{N}}$ *along which*
$$\int_{t_{j_i}}^{t_{j_i}+1} e^{\lambda u(\cdot,t)}dt \rightharpoonup w \qquad \textit{in } W^{2,2}(\Omega) \tag{1.7}$$
and
$$\int_{t_{j_i}}^{t_{j_i}+1} e^{\lambda u(\cdot,t)}dt \to w \qquad \textit{uniformly in } \bar\Omega \tag{1.8}$$
holds as $i\to\infty$ *with some* $w\in W^{2,2}(\Omega)$ *satisfying* $\frac{\partial}{\partial\nu}w=0$ *on* $\partial\Omega$.

We point out that it still remains an open problem whether or not (1.2) possesses solutions which blow up in finite time, even in the one-dimensional setting. Although it is not clear whether such solutions are meaningful in the applications, for both estimating the precise validity of the model (1.2) and obtaining refined numerical approximation statements it would be useful to either exclude or confirm the possibility of singularity formation. In any event, both numerical evidence in one space dimension ([25]) and formal considerations (see (1.10) and [6]) suggest that if u blows up in $L^\infty(\Omega)$ in finite time then the negative part u_- is a more likely candidate for becoming unbounded than u_+.

We also leave untouched here the question of uniqueness of solutions to (1.2), which is to be expected when solutions are required to lie in suitable classes of sufficiently regular functions (cf. [6] for a detailed discussion in a one-dimensional setting). We believe, however, that our notion of solution (Definition 2.1) is too weak to enforce uniqueness.

Another open problem is whether or not solutions of (1.2) stabilize on large time scales. Our result in Theorem 1.2 provides a rather weak affirmative statement in this direction, but this is probably far from optimal. In particular it would be interesting to know whether or not restrictions on the size of μ and f are really necessary. It seems that even the structure of the set of steady states for (1.2) is unknown. Some results for the one-dimensional case are contained in [6] (cf. also [5] for a stochastic analogue).

Our procedure is organzied in such a way that the proofs of existence and approximation by the Rothe scheme are closely linked to each other. It is an interesting mathematical question whether the mere existence of weak solutions can alternatively be obtained by a different approach based on approximation by suitably regularized *parabolic* problems. For instance, when the quadratic nonlinearity in (1.1) is truncated, it seems that no estimate in the flavor of (1.10) is valid any longer, so that it is not clear whether solutions of the corresponding approximate problems in fact converge to a solution of (1.1). We cannot exclude that in this sense, in higher dimensions the equation (1.1) plays a special role among all its neighboring equations.

Let us finally mention that we expect most of our techniques to apply also to the case when the Neumann boundary conditions are replaced by periodic boundary conditions;

as then the boundary integral in (1.9) formally disappears, the analysis should even be simplified in some places. However, we do not know if corresponding results can achieved for the corresponding Cauchy problem in $\mathbb{R}^n$ under suitable spatial decay conditions.

1.1 Outline of the strategy

Since the proofs of these main results are technically quite involved, before going in medias res let us briefly outline our approach. The proof of Theorem 1.1 is based on semi-discrete versions of the identities

$$\frac{d}{dt}\int_\Omega u(x,t)dx = -\lambda\int_{\partial\Omega}\frac{\partial}{\partial\nu}|\nabla u|^2 + \int_\Omega f \qquad \text{for all } t>0 \tag{1.9}$$

and

$$\frac{1}{2\lambda}\frac{d}{dt}\int_\Omega e^{2\lambda u(x,t)}dx + 2\lambda\int_\Omega e^{2\lambda u}|D^2u|^2 = -\mu\int_\Omega e^{2\lambda u}\Delta u + \int_\Omega e^{2\lambda u} f \qquad \text{for all } t>0 \tag{1.10}$$

which are formally obtained when testing (1.2) against $\varphi\equiv 1$ and $\varphi = e^{2\lambda u}$, respectively (cf. Lemma 2.2 below). Here we shall use the convexity of Ω in deriving from (1.9) that $t\mapsto\int_\Omega u(x,t)dx$ is nondecreasing and hence nonnegative. Since (1.10) states that $\int_\Omega e^{2\lambda u(x,t)}dx$ grows at most exponentially with time, this shows boundedness of $\|u(\cdot,t)\|_{L^1(\Omega)} \equiv 2\int_\Omega u_+(\cdot,t) - \int_\Omega u(\cdot,t)$ in bounded time intervals, and moreover from (1.10) we infer some higher regularity in each of the sets $\{u>c\}$ for $c\in\mathbb{R}$. In order to control ∇u also in regions where possibly u is small, let us test (1.2) by $A^{-1}(u-\overline{u})$, where $\overline{u} := \frac{1}{|\Omega|}\int_\Omega u$ and A denotes the realization of $-\Delta$ in the subspace of functions in $L^2(\Omega)$ having zero average. This formally yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2 + \int_\Omega|\nabla u|^2 = \lambda\int_\Omega u|\nabla u|^2 + \mu\int_\Omega u^2 + \int_\Omega f\cdot A^{-1}(u-\overline{u}) \quad \text{for all } t>0, \tag{1.11}$$

and indeed a corresponding semi-discrete version will be provided by Lemma 2.6. Since here the first term on the right can be bounded *from above* by a term involving the second integral on the left of (1.10), this will lead to an estimate for ∇u in $L^2_{loc}([0,\infty);L^2(\Omega))$ (Lemma 3.1). In a final step, the resulting weak compactness property of ∇u can be turned into strong compactness, where again (1.10) and (1.11) will be utilized (Lemma 3.3).

Finally, Theorem 1.2 will rely on (1.10) and a Morrey's inequality which in the present situation will allow for estimating the right-hand side of (1.10) by a negative multiple of $\int_\Omega e^{2\lambda u}$ modulo an additive constant. Here it will be used that in the radial framework the conditions $\int_\Omega f = 0$ and $\int_\Omega u_0 = 0$ will ensure that $\int_\Omega u(x,t)dx = 0$, and hence $u(\cdot,t)$ must maintain at least one zero for all times.

As a technical detail possibly worth being announced, let us mention that in proving solvability of the elliptic problems occurring in our Rothe-type approximation scheme for (1.2) (see (2.2)), one main step will consist of deriving a one-sided bound by means of an elliptic comparison argument (see Lemma 2.7). Although, of course, the corresponding full fourth-order problem does not allow for a maximum principle, it is essentially an implicitly performed reduction to a second-order system that will enable us to apply this powerful tool.

2 A semi-discrete approximation

Throughout this paper we shall pursue the following concept of weak solutions in which, roughly speaking, as many derivatives as possible are carried over to test functions. Therefore the notion *very weak solution* would be slightly more adequate, but we refrain from distinguishing different solution concepts here.

Definition 2.1 *Let $\mu \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, and assume that $f \in L^1(\Omega)$ and $u_0 \in L^1(\Omega)$. Then by a* global weak solution *of (1.2) we mean a function $u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ that satisfies*

$$\int_0^\infty \int_\Omega u\varphi_t + \int_\Omega u_0\varphi(\cdot,0) = \int_0^\infty \int_\Omega u\Delta^2\varphi + \mu \int_0^\infty \int_\Omega u\Delta\varphi + \lambda \int_0^\infty \int_\Omega |\nabla u|^2 \Delta\varphi - \int_0^\infty \int_\Omega f\varphi \tag{2.1}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ fulfilling $\frac{\partial}{\partial\nu}\varphi = \frac{\partial}{\partial\nu}\Delta\varphi = 0$ on $\partial\Omega$.

In order to construct a solution of (1.2), we shall follow [25] and employ a Rothe-type approximation procedure based on a discretization of (1.2) with respect to the time variable: For $\tau \in (0,1)$, we consider the sequence of elliptic boundary value problems given by

$$\begin{cases} \frac{u_k^{(\tau)} - u_{k-1}^{(\tau)}}{\tau} = -\Delta^2 u_k^{(\tau)} - \mu\Delta u_k^{(\tau)} - \lambda\Delta|\nabla u_k^{(\tau)}|^2 + f, \qquad x \in \Omega \\ \frac{\partial}{\partial\nu} u_k^{(\tau)} = \frac{\partial}{\partial\nu}\Delta u_k^{(\tau)} = 0, \qquad x \in \partial\Omega, \end{cases} \tag{2.2}$$

for $k = 1, 2, ...$, where $(u_0^{(\tau)})_{\tau\in(0,1)}$ is a family of functions approximating u_0 as $\tau \searrow 0$ in the sense that the following requirements are fulfilled, which shall collectively be refererred to as (H1) in the sequel, and which implicitly require that $u_0 \in L^\infty(\Omega)$.

(H1a) For all $\tau \in (0,1)$, the function $u_0^{(\tau)}$ belongs to $W^{2,\infty}(\Omega)$ and satisfies $\int_\Omega u_0^{(\tau)} = \int_\Omega u_0$.

(H1b) We have $u_0^{(\tau)} \to u_0$ a.e. in Ω as $\tau \searrow 0$ and $\sup_{\tau\in(0,1)} \|u_0^{(\tau)}\|_{L^\infty(\Omega)} < \infty$.

(H1c) As $\tau \searrow 0$, we have $\tau\|u_0^{(\tau)}\|^2_{W^{1,2}(\Omega)} \to 0$.

Our plan is to assert solvability of (2.2) and at the same time prepare τ-independent estimates for the corresponding continuous and step-type Rothe functions $u^{(\tau)}$ and $\hat u^{(\tau)}$ defined by

$$u^{(\tau)}(x,t) := \Big(k - \frac{t}{\tau}\Big) \cdot u_{k-1}^{(\tau)}(x) + \Big(\frac{t}{\tau} - (k-1)\Big) \cdot u_k^{(\tau)}(x) \tag{2.3}$$

and

$$\hat u^{(\tau)}(x,t) := u_k^{(\tau)}(x), \tag{2.4}$$

respectively, for $x \in \Omega$ and $t \in [(k-1)\tau, k\tau)$, $k = 1, 2, ...$
For fixed $\tau > 0$ and $k \geq 1$, (2.2) is conveniently rewritten as

$$\begin{cases} \Delta^2 u + \frac{1}{\tau} u = -\mu \Delta u - \lambda \Delta |\nabla u|^2 + \frac{1}{\tau} v + f, \qquad x \in \Omega, \\ \frac{\partial}{\partial \nu} u = \frac{\partial}{\partial \nu} \Delta u = 0, \qquad x \in \partial\Omega, \end{cases} \tag{2.5}$$

where $v := u_{k-1}^{(\tau)}$ is supposed to be given and $u := u_k^{(\tau)}$ is the unknown.
It turns out that an appropriate setting for studying the semilinear problem (2.5) is made up by the following fixed point framework: Let us fix $p > n$ and set

$$X_p := W_N^{3,p}(\Omega) := \Big\{ w \in W^{3,p}(\Omega) \ \Big| \ \frac{\partial}{\partial \nu} w = 0 \text{ on } \partial\Omega \Big\}. \tag{2.6}$$

Then $X_p \hookrightarrow C^2(\bar{\Omega})$, and hence for fixed $v \in L^\infty(\Omega)$ the nonlinear operator N_v defined by

$$N_v u := -\mu \Delta u - \lambda \Delta |\nabla u|^2 + \frac{1}{\tau} v + f, \qquad u \in X_p, \tag{2.7}$$

is continuous from X_p into $L^p(\Omega)$. Next, as is well-known ([10]), the linear operator L defined by

$$Lu := \Delta^2 u + \frac{1}{\tau} u \tag{2.8}$$

acts as a homeomorphism from $W_N^{4,p}(\Omega) := \{w \in W^{4,p}(\Omega) \mid \frac{\partial}{\partial \nu} w = \frac{\partial}{\partial \nu} \Delta w = 0 \text{ on } \partial\Omega\}$ to $L^p(\Omega)$, and in particular its inverse L^{-1} exists and is bounded from $L^p(\Omega)$ into $W_N^{4,p}(\Omega)$. Thereby looking for a solution $u \in W_N^{4,p}(\Omega)$ of (2.5) becomes equivalent to finding a solution $u \in X_p$ of

$$u = L^{-1} N_v u, \tag{2.9}$$

that is, a fixed point of the operator $F := L^{-1} N_v$ when regarded as a mapping from X_p into itself. Evidently, F is continuous, and since $X_p \hookrightarrow C^2(\bar{\Omega})$ implies that N_v maps bounded sets from X_p into bounded sets in $L^p(\Omega)$, it follows that F maps bounded sets from X_p into bounded sets in $W_N^{4,p}(\Omega)$ and hence into compact sets in X_p.
Our goal is to apply the Leray-Schauder fixed point theorem, which in the present context will assert the existence of a solution to (2.9) as soon as we can show that the set

$$\bigcup_{\theta \in (0,1]} Fix(\theta \cdot L^{-1} N_v) \equiv \Big\{ u \in X_p \ \Big| \exists\, \theta \in (0,1] \text{ such that } u = \theta \cdot L^{-1} N_v u \Big\} \tag{2.10}$$

is bounded in X_p. This will be achieved through a series of steps, which are organzied as follows: In the next Section 2.1 we shall provide some a priori estimates on solutions of the equation $u = \theta \cdot L^{-1} N_v u$ which are independent of $\theta \in (0,1]$ *and* $\tau \in (0,1)$. In Section 2.2 we proceed to derive further estimates for such solutions which may depend on τ but not on θ. Combining both types of estimates will enable us to infer the desired boundedness result for the set in (2.10) and thereby establish solvability of (2.2) for $\tau \in (0, \tau_\star)$ and some sufficiently small $\tau_\star > 0$. After that, going back to Section 2.1 and applying its results to the particular choice $\theta = 1$, we will obtain τ-independent estimates and hence favorable compactness properties of the families $(u^{(\tau)})_{\tau \in (0,\tau_\star)}$ and $(\hat{u}^{(\tau)})_{\tau \in (0,\tau_\star)}$ in (2.3) and (2.4).

2.1 Basic a priori estimates

Let us start by recalling from [7] the following identity which can be obtained upon integration by parts, and which plays the role of a higher-dimensional analogue of (1.3).

Lemma 2.1 *Let $u \in C^2(\bar{\Omega})$ be such that $\frac{\partial}{\partial\nu}u = 0$ on $\partial\Omega$. Then for all $\rho \in C^2(\mathbb{R})$ we have*

$$\begin{aligned}\int_\Omega \rho'(u)|\nabla u|^2\Delta u &= -\frac{2}{3}\int_\Omega \rho(u)|\Delta u|^2 + \frac{2}{3}\int_\Omega \rho(u)|D^2u|^2 - \frac{1}{3}\int_\Omega \rho''(u)|\nabla u|^4 \\ &\quad -\frac{1}{3}\int_{\partial\Omega} \rho(u)\cdot\frac{\partial}{\partial\nu}|\nabla u|^2.\end{aligned}$$

We now state a useful identity for solutions of (2.2) that will serve as a starting point for most of our estimates.

Lemma 2.2 *Let $\mu \geq 0, \lambda > 0, \tau > 0, v \in L^\infty(\Omega)$ and $f \in L^1(\Omega)$, and suppose that for some $\theta > 0$, $u \in W^{4,2}(\Omega) \cap C^2(\bar{\Omega})$ is a solution of*

$$\begin{cases} \frac{u-\theta v}{\tau} = -\Delta^2 u - \theta\mu\Delta u - \theta\lambda\Delta|\nabla u|^2 + \theta f(x), & x \in \Omega, \\ \frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u = 0, & x \in \partial\Omega. \end{cases} \tag{2.11}$$

Then the identities

$$\int_\Omega u = \theta\int_\Omega v - \theta\lambda\tau\int_{\partial\Omega}\frac{\partial}{\partial\nu}|\nabla u|^2 + \theta\tau\int_\Omega f \tag{2.12}$$

and

$$\int_\Omega ue^{2\theta\lambda u} + 2\theta\lambda\tau\int_\Omega e^{2\theta\lambda u}|D^2u|^2 = -\theta\mu\tau\int_\Omega e^{2\theta\lambda u}\Delta u + \theta\int_\Omega ve^{2\theta\lambda u} + \theta\tau\int_\Omega fe^{2\theta\lambda u} \tag{2.13}$$

hold.

Proof. In view of the boundary conditions in (2.11), (2.12) immediately results upon an integration of the PDE in (2.11).
To obtain (2.13), we let $\phi(s) := e^{2\theta\lambda s}$ for $s \in \mathbb{R}$ and test (2.11) by $\phi(u)$ to find

$$\begin{aligned}\frac{1}{\tau}\int_\Omega u\phi(u) - \frac{\theta}{\tau}\int_\Omega v\phi(u) &= -\int_\Omega \phi(u)\Delta^2u - \theta\mu\int_\Omega \phi(u)\Delta u - \theta\lambda\int_\Omega \phi(u)\Delta|\nabla u|^2 \\ &\quad +\theta\int_\Omega \phi(u)f.\end{aligned} \tag{2.14}$$

Here, integrating by parts we see that

$$-\int_\Omega \phi(u)\Delta^2u = -\int_\Omega \Delta\phi(u)\Delta u - \int_{\partial\Omega}\phi(u)\frac{\partial}{\partial\nu}\Delta u + \int_{\partial\Omega}\phi'(u)\Delta u\frac{\partial}{\partial\nu}u \tag{2.15}$$

and

$$-\theta\lambda\int_\Omega \phi(u)\Delta|\nabla u|^2 = -\theta\lambda\int_\Omega |\nabla u|^2\Delta\phi(u) - \theta\lambda\int_{\partial\Omega}\phi(u)\frac{\partial}{\partial\nu}|\nabla u|^2 + \theta\lambda\int_{\partial\Omega}\phi'(u)|\nabla u|^2\frac{\partial}{\partial\nu}u. \tag{2.16}$$

Since $\frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u = 0$ on $\partial\Omega$, and since

$$\Delta\phi(u) = \phi'(u)\Delta u + \phi''(u)|\nabla u|^2,$$

from (2.14)-(2.16) we thus obtain

$$\begin{aligned}
\frac{1}{\tau}\int_\Omega u\phi(u) - \frac{\theta}{\tau}\int_\Omega v\phi(u) &= -\int_\Omega \phi'(u)|\Delta u|^2 - \int_\Omega \Big[\phi''(u) + \theta\lambda\phi'(u)\Big]\cdot|\nabla u|^2\Delta u \\
&\quad -\theta\lambda\int_\Omega \phi''(u)|\nabla u|^4 - \theta\mu\int_\Omega \phi(u)\Delta u - \theta\lambda\int_{\partial\Omega}\phi(u)\frac{\partial}{\partial\nu}|\nabla u|^2 \\
&\quad +\theta\int_\Omega \phi(u)f. \qquad (2.17)
\end{aligned}$$

An application of Lemma 2.1 to $\rho(s) := \phi'(s) + \theta\lambda\phi(s)$ shows that

$$\begin{aligned}
-\int_\Omega \Big[\phi''(u) + \theta\lambda\phi'(u)\Big]\cdot|\nabla u|^2\Delta u &= \frac{2}{3}\int_\Omega \Big[\phi'(u) + \theta\lambda\phi(u)\Big]\cdot|\Delta u|^2 \\
&\quad -\frac{2}{3}\int_\Omega \Big[\phi'(u) + \theta\lambda\phi(u)\Big]\cdot|D^2u|^2 \\
&\quad +\frac{1}{3}\int_\Omega \Big[\phi'''(u) + \theta\lambda\phi''(u)\Big]\cdot|\nabla u|^4 \\
&\quad +\frac{1}{3}\int_{\partial\Omega} \Big[\phi'(u) + \theta\lambda\phi(u)\Big]\cdot\frac{\partial}{\partial\nu}|\nabla u|^2,
\end{aligned}$$

which inserted into (2.17) yields

$$\begin{aligned}
\frac{1}{\tau}\int_\Omega u\phi(u) - \frac{\theta}{\tau}\int_\Omega v\phi(u) &= \int_\Omega \Big[-\frac{1}{3}\phi'(u) + \frac{2}{3}\theta\lambda\phi(u)\Big]\cdot|\Delta u|^2 \\
&\quad -\int_\Omega \Big[\frac{2}{3}\phi'(u) + \frac{2}{3}\theta\lambda\phi(u)\Big]\cdot|D^2u|^2 \\
&\quad +\int_\Omega \Big[\frac{1}{3}\phi'''(u) - \frac{2}{3}\theta\lambda\phi''(u)\Big]\cdot|\nabla u|^4 \\
&\quad -\theta\mu\int_\Omega \phi(u)\Delta u \\
&\quad +\int_{\partial\Omega} \Big[\frac{1}{3}\phi'(u) - \frac{2}{3}\theta\lambda\phi(u)\Big]\cdot\frac{\partial}{\partial\nu}|\nabla u|^2 \\
&\quad +\theta\int_\Omega \phi(u)f. \qquad (2.18)
\end{aligned}$$

Since by definition of ϕ we have

$$\frac{1}{3}\phi' \equiv \frac{2}{3}\theta\lambda\phi \qquad \text{and} \qquad \frac{1}{3}\phi''' \equiv \frac{2}{3}\theta\lambda\phi'' \qquad \text{on } \mathbb{R}$$

and

$$\frac{2}{3}\phi'(s) + \frac{2}{3}\theta\lambda\phi(s) = 2\theta\lambda e^{2\theta\lambda s} \qquad \text{for all } s\in\mathbb{R},$$

(2.18) is equivalent to (2.13). ////

In order to cope with the boundary integral in (2.12), we remember the following fact ([7]) which is the reason for our convexity requirement.

Lemma 2.3 *Suppose that Ω is convex, and that $u \in C^2(\bar{\Omega})$ satisfies $\frac{\partial}{\partial\nu}u = 0$ on $\partial\Omega$. Then*

$$\frac{\partial}{\partial\nu}|\nabla u|^2 \le 0 \qquad \text{on } \partial\Omega.$$

The next lemma will ensure that the term on the left-hand side of (2.13) in fact can be used to gain a certain control of the first-oder spatial derivative of u.

Lemma 2.4 *Suppose that $u \in C^2(\bar{\Omega})$ satisfies $\frac{\partial}{\partial\nu}u = 0$ on $\partial\Omega$. Then for all $\alpha > 0$ we have*

$$\int_\Omega e^{\alpha u}|\nabla u|^4 \le \frac{d_n}{\alpha^2}\int_\Omega e^{\alpha u}|D^2u|^2 \tag{2.19}$$

with $d_n := (2+\sqrt{n})^2$.

PROOF. Integrating by parts, we see that

$$\begin{aligned}\int_\Omega e^{\alpha u}|\nabla u|^4 &= \frac{1}{\alpha}\int_\Omega |\nabla u|^2 \nabla e^{\alpha u}\cdot\nabla u \\ &= -\frac{1}{\alpha}\int_\Omega e^{\alpha u}\nabla u\cdot\nabla|\nabla u|^2 - \frac{1}{\alpha}\int_\Omega e^{\alpha u}|\nabla u|^2\Delta u, \end{aligned}\tag{2.20}$$

because $\frac{\partial}{\partial\nu}u = 0$ on $\partial\Omega$. Since $\nabla|\nabla u|^2 = 2D^2u\cdot\nabla u$, we can invoke Hölder's inequality to estimate

$$-\frac{1}{\alpha}\int_\Omega e^{\alpha u}\nabla u\cdot\nabla|\nabla u|^2 \le \frac{2}{\alpha}\Big(\int_\Omega e^{\alpha u}|\nabla u|^4\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega e^{\alpha u}|D^2u|^2\Big)^{\frac{1}{2}}$$

and moreover

$$-\frac{1}{\alpha}\int_\Omega e^{\alpha u}|\nabla u|^2\Delta u \le \frac{1}{\alpha}\Big(\int_\Omega e^{\alpha u}|\nabla u|^4\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega e^{\alpha u}|\Delta u|^2\Big)^{\frac{1}{2}}.$$

Thus, from the elementary pointwise inequality $|\Delta u|^2 \le n|D^2u|^2$ and (2.20) we infer that

$$\int_\Omega e^{\alpha u}|\nabla u|^4 \le \Big(\frac{2}{\alpha}+\frac{\sqrt{n}}{\alpha}\Big)\cdot\Big(\int_\Omega e^{\alpha u}|\nabla u|^4\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega e^{\alpha u}|D^2u|^2\Big)^{\frac{1}{2}},$$

which immediately results in (2.19). ////

Collecting the above three lemmata, we can establish a weighted a priori estimate for solutions of (2.11).

Lemma 2.5 *Suppose that Ω is convex, that $\lambda > 0$, and that $f \in L^\infty(\Omega)$ satisfies $\int_\Omega f \ge 0$. Then there exist positive constants k, K and C such that if for some $\mu \ge 0$, $\tau \in (0,1)$, $\theta \in (0,1]$ and $v \in L^\infty(\Omega)$ with $\int_\Omega v \ge 0$, we are given a solution $u \in W^{4,2}(\Omega) \cap C^2(\bar{\Omega})$ of (2.11), then the estimate*

$$\begin{aligned}\Big(1 - K(\|f\|_{L^\infty(\Omega)} + \mu^2)\theta\tau\Big) \cdot \int_\Omega e^{2\theta\lambda u} \quad &+ \quad k\theta\tau \int_\Omega e^{2\theta\lambda u}|D^2 u|^2 + k\theta^3\tau \int_\Omega e^{2\theta\lambda u}|\nabla u|^4 \\ &\le \quad \int_\Omega e^{2\theta\lambda v} \end{aligned} \tag{2.21}$$

is valid, and if in addition $\tau < \tau_0 := \frac{1}{2K(\|f\|_{L^\infty(\Omega)} + \mu^2)}$ then furthermore

$$\|u\|_{L^1(\Omega)} \le \frac{C}{\theta} \cdot e^{2\theta\lambda\|v\|_{L^\infty(\Omega)}} \tag{2.22}$$

holds. Moreover, for any such solution we have

$$\int_\Omega u \ge 0. \tag{2.23}$$

PROOF. Following [1], we observe that by convexity of $\Phi(s) := e^{2\theta\lambda s}$, $s \in \mathbb{R}$, we have the pointwise inequality

$$\Phi(u) - \Phi(v) \le \Phi'(u) \cdot (u - v) \qquad \text{in } \Omega,$$

which yields

$$\int_\Omega e^{2\theta\lambda u} - \int_\Omega e^{2\theta\lambda v} \le 2\theta\lambda \int_\Omega u e^{2\theta\lambda u} - 2\theta\lambda \int_\Omega v e^{2\theta\lambda u}$$

or, equivalently,

$$\theta \int_\Omega v e^{2\theta\lambda u} \le \theta \int_\Omega u e^{2\theta\lambda u} - \frac{1}{2\lambda} \int_\Omega e^{2\theta\lambda u} + \frac{1}{2\lambda} \int_\Omega e^{2\theta\lambda v}.$$

Therefore (2.13) upon an integration by parts entails

$$\begin{aligned}(1-\theta) \cdot \int_\Omega u e^{2\theta\lambda u} \quad &+ \quad \frac{1}{2\lambda} \int_\Omega e^{2\theta\lambda u} + 2\theta\lambda\tau \int_\Omega e^{2\theta\lambda u}|D^2 u|^2 \\ &\le \quad -\theta\mu\tau \int_\Omega e^{2\theta\lambda u}\Delta u + \frac{1}{2\lambda} \int_\Omega e^{2\theta\lambda v} + \theta\tau \int_\Omega e^{2\theta\lambda u} f \\ &\le \quad 2\theta^2\lambda\mu\tau \int_\Omega e^{2\theta\lambda u}|\nabla u|^2 + \frac{1}{2\lambda} \int_\Omega e^{2\theta\lambda v} + \theta\tau\|f\|_{L^\infty(\Omega)} \cdot \int_\Omega e^{2\theta\lambda u}.\end{aligned}$$

In view of the estimate

$$\theta\lambda\tau \int_\Omega e^{2\theta\lambda u}|D^2 u|^2 \ge \frac{4\theta^3\lambda^3\tau}{d_n} \int_\Omega e^{2\theta\lambda u}|\nabla u|^4$$

asserted by Lemma 2.4, upon a multiplication by 2λ we immediately obtain

$$\begin{aligned}
2\lambda(1-\theta)\int_\Omega ue^{2\theta\lambda u} \quad &+ \quad (1-2\theta\lambda\|f\|_{L^\infty(\Omega)}\tau)\cdot\int_\Omega e^{2\theta\lambda u} \\
&\quad +2\theta\lambda^2\tau\int_\Omega e^{2\theta\lambda u}|D^2u|^2+\frac{8\theta^3\lambda^4\tau}{d_n}\int_\Omega e^{2\theta\lambda u}|\nabla u|^4 \\
&\leq \quad 4\theta^2\lambda^2\mu\tau\int_\Omega e^{2\theta\lambda u}|\nabla u|^2+\int_\Omega e^{2\theta\lambda v}, \qquad (2.24)
\end{aligned}$$

where by Young's inequality

$$4\theta^2\lambda^2\mu\tau\int_\Omega e^{2\theta\lambda u}|\nabla u|^2 \leq \frac{4\theta^3\lambda^4\tau}{d_n}\int_\Omega e^{2\theta\lambda u}|\nabla u|^4+\theta\mu^2\tau d_n\int_\Omega e^{2\theta\lambda u}. \qquad (2.25)$$

Now since Ω is convex and $\frac{\partial}{\partial\nu}u=0$ on $\partial\Omega$, we have $\frac{\partial}{\partial\nu}|\nabla u|^2\leq 0$ on $\partial\Omega$ by Lemma 2.3, and hence (2.12) implies that (2.23) holds, because $\int_\Omega u\geq\theta\int_\Omega v+\theta\tau\int_\Omega f\geq 0$ according to our assumptions on v and f. Therefore, using that $e^{2\theta\lambda u}\leq 1$ if and only if $u\leq 0$, we obtain

$$\begin{aligned}
\int_\Omega ue^{2\theta\lambda u} \quad &= \quad \int_{\{u>0\}} ue^{2\theta\lambda u}+\int_{\{u<0\}} ue^{2\theta\lambda u} \\
&\geq \quad \int_{\{u>0\}} ue^{2\theta\lambda u}+\int_{\{u<0\}} u \\
&= \quad \int_{\{u>0\}} ue^{2\theta\lambda u}+\int_\Omega u-\int_{\{u>0\}} u \\
&\geq \quad \int_{\{u>0\}} ue^{2\theta\lambda u}-\int_{\{u>0\}} u \\
&= \quad \int_{\{u>0\}} u\cdot(e^{2\theta\lambda u}-1) \\
&\geq \quad 0,
\end{aligned}$$

and thus for $\theta\in(0,1]$, (2.21) easily results from (2.24) and (2.25) upon appropriate choices of k and K.
Finally, if $\tau<\tau_0$ then from (2.21) we obtain that

$$\frac{1}{2}\int_\Omega e^{2\theta\lambda u}\leq\int_\Omega e^{2\theta\lambda v}\leq|\Omega|\cdot e^{2\lambda\|v\|_{L^\infty(\Omega)}}.$$

Since $e^{2\theta\lambda u_+}\geq 2\theta\lambda u_+$ and, by (2.23), $\|u\|_{L^1(\Omega)}=\int_{\{u>0\}}u-\int_{\{u<0\}}u\leq 2\int_\Omega u_+$, we see that (2.22) will be valid if we pick $C:=\frac{|\Omega|}{\lambda}$. ////

For what follows, let us recall the well-known fact that the operator $A:=-\Delta$ is self-adjoint in $L^2_\perp(\Omega):=\{w\in L^2(\Omega)\ |\ \int_\Omega w=0\}$, with domain of definition given by $D(A)=W^{2,2}_{N,\perp}(\Omega):=W^{2,2}_N(\Omega)\cap L^2_\perp(\Omega)$, and that its discrete spectrum $\sigma(A)$ lies on the positive real axis $\sigma(A)\subset(0,\infty)$. In particular, A posesses self-adjoint real powers A^α for any $\alpha\in\mathbb{R}$, and A^α is bounded from $L^2_\perp(\Omega)$ into $L^2_\perp(\Omega)$ whenever $\alpha<0$.

Lemma 2.6 *Let $\tau > 0$, and assume that $v \in L^2(\Omega)$ and $f \in L^2(\Omega)$ are such that $\int_\Omega v \geq 0$ and $\int_\Omega f \geq 0$. Suppose that for some $\theta > 0$, $u \in W^{4,2}(\Omega)$ is a solution of (2.11). Then for all $\eta > 0$, the estimate*

$$(1-\theta^2\mu^2\tau-\eta\tau)\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2 \;+\; \tau\int_\Omega |\nabla u|^2 - 2\theta\lambda\tau\int_\Omega u|\nabla u|^2$$
$$\leq\; \theta^2\int_\Omega |A^{-\frac{1}{2}}(v-\overline{v})|^2 + \frac{\theta\tau}{\eta}\int_\Omega |A^{-\frac{1}{2}}(f-\overline{f})|^2 \qquad (2.26)$$

holds, where we have set $\overline{w} := \frac{1}{|\Omega|}\int_\Omega w$ for $w \in L^1(\Omega)$.

PROOF. Since $u-\overline{u} \in L^2_\perp(\Omega)$, we may apply A^{-1} to $u-\overline{u}$ and test (2.11) by $A^{-1}(u-\overline{u})$ to obtain

$$\begin{aligned}
\frac{1}{\tau}\int_\Omega u\cdot A^{-1}(u-\overline{u}) - \frac{\theta}{\tau}\int_\Omega v\cdot A^{-1}(u-\overline{u}) \;&=\; -\int_\Omega \Delta^2 u\cdot A^{-1}(u-\overline{u}) \\
&\quad -\theta\mu\int_\Omega \Delta u\cdot A^{-1}(u-\overline{u}) \\
&\quad -\theta\lambda\int_\Omega \Delta|\nabla u|^2\cdot A^{-1}(u-\overline{u}) \\
&\quad +\theta\int_\Omega f\cdot A^{-1}(u-\overline{u}). \qquad (2.27)
\end{aligned}$$

Here, using that $-\Delta u = -\Delta(u-\overline{u}) = A(u-\overline{u})$ and integrating by parts, we find

$$-\int_\Omega \Delta^2 u\cdot A^{-1}(u-\overline{u}) = \int_\Omega \Delta(u-\overline{u})\cdot(u-\overline{u}) = -\int_\Omega |\nabla(u-\overline{u})|^2 = -\int_\Omega |\nabla u|^2, \quad (2.28)$$

and similarly we have

$$\begin{aligned}
-\theta\mu\int_\Omega \Delta u\cdot A^{-1}(u-\overline{u}) \;&=\; \theta\mu\int_\Omega A(u-\overline{u})\cdot A^{-1}(u-\overline{u}) \\
&=\; \theta\mu\int_\Omega A^{\frac{1}{2}}(u-\overline{u})\cdot A^{-\frac{1}{2}}(u-\overline{u}),
\end{aligned}$$

because $A^{-\frac{1}{2}}$ is self-adjoint. Hence, by Young's inequality and the identity $\|A^{\frac{1}{2}}w\|_{L^2(\Omega)} = \|\nabla w\|_{L^2(\Omega)}$ for $w \in L^2_\perp(\Omega)$,

$$\begin{aligned}
-\theta\mu\int_\Omega \Delta u\cdot A^{-1}(u-\overline{u}) \;&\leq\; \frac{1}{2}\int_\Omega |A^{\frac{1}{2}}(u-\overline{u})|^2 + \frac{\theta^2\mu^2}{2}\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2 \\
&=\; \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{\theta^2\mu^2}{2}\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2. \qquad (2.29)
\end{aligned}$$

Similarly, since $\int_\Omega A^{-1}(u-\overline{u}) = 0$, for any $\eta > 0$ we have

$$\begin{aligned}
\theta\int_\Omega f\cdot A^{-1}(u-\overline{u}) \;&=\; \theta\int_\Omega (f-\overline{f})\cdot A^{-1}(u-\overline{u}) \\
&=\; \theta\int_\Omega A^{-\frac{1}{2}}(f-\overline{f})\cdot A^{-\frac{1}{2}}(u-\overline{u}) \\
&\leq\; \frac{\eta}{2}\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2 + \frac{\theta^2}{2\eta}\int_\Omega |A^{-\frac{1}{2}}(f-\overline{f})|^2. \qquad (2.30)
\end{aligned}$$

Moreover,

$$
\begin{aligned}
-\theta\lambda \int_\Omega \Delta|\nabla u|^2 A^{-1}(u-\overline{u}) &= -\theta\lambda \int_\Omega \Delta\Big(|\nabla u|^2 - \overline{|\nabla u|^2}\Big)\cdot A^{-1}(u-\overline{u}) \\
&= \theta\lambda \int_\Omega \Big(|\nabla u|^2 - \overline{|\nabla u|^2}\Big)\cdot (u-\overline{u}) \\
&= \theta\lambda \int_\Omega u|\nabla u|^2 - \frac{\theta\lambda}{|\Omega|}\Big(\int_\Omega u\Big)\cdot\Big(\int_\Omega |\nabla u|^2\Big) \\
&\quad -\frac{\theta\lambda}{|\Omega|}\Big(\int_\Omega |\nabla u|^2\Big)\cdot\Big(\int_\Omega (u-\overline{u})\Big).
\end{aligned}
$$

Since $\int_\Omega (u-\overline{u}) = 0$, and since $\int_\Omega v \ge 0$ and $\int_\Omega f \ge 0$ imply that $\int_\Omega u \ge 0$ in view of Lemma 2.2 and Lemma 2.3, from this we obtain that

$$
-\theta\lambda \int_\Omega \Delta|\nabla u|^2 \cdot A^{-1}(u-\overline{u}) \le \theta\lambda \int_\Omega u|\nabla u|^2. \tag{2.31}
$$

Finally, on the left-hand side of (2.27) we again use that $A^{-\frac{1}{2}}$ is self-adjoint and that $\int_\Omega A^{-1}(u-\overline{u}) = 0$ in computing

$$
\begin{aligned}
\frac{1}{\tau}\int_\Omega u\cdot A^{-1}(u-\overline{u}) &= \frac{1}{\tau}\int_\Omega (u-\overline{u})\cdot A^{-1}(u-\overline{u}) \\
&= \frac{1}{\tau}\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2
\end{aligned}
\tag{2.32}
$$

and

$$
\begin{aligned}
\frac{\theta}{\tau}\int_\Omega v\cdot A^{-1}(u-\overline{u}) &= \frac{\theta}{\tau}\int_\Omega (v-\overline{v})\cdot A^{-1}(u-\overline{u}) \\
&= \frac{\theta}{\tau}\int_\Omega A^{-\frac{1}{2}}(v-\overline{v})\cdot A^{-\frac{1}{2}}(u-\overline{u}).
\end{aligned}
$$

Therefore another application of Young's inequality shows that

$$
\frac{\theta}{\tau}\int_\Omega v\cdot A^{-1}(u-\overline{u}) \le \frac{1}{2\tau}\int_\Omega |A^{-\frac{1}{2}}(u-\overline{u})|^2 + \frac{\theta^2}{2\tau}\int_\Omega |A^{-\frac{1}{2}}(v-\overline{v})|^2,
$$

which together with (2.28)-(2.32) inserted into (2.27) immediately yields (2.26) upon a multiplictaion by 2τ. ////

2.2 Solvability of the semi-discrete problem

In this part our purpose is to fix τ and assert solvability of (2.5) by proving boundedness of the set in (2.10). For this purpose we provide further estimates for solutions of (2.11) which, unlike those previously obtained, may depend on τ.

The following uniform lower bound will be essential in turning the estimates in Lemma 2.5 into corresponding estimates without weights. It may be worth emphasizing that despite the lack of comparison for the full fourth-order problem (2.11) our argument strongly relies on comparison techniques. The underlying idea how this can be made possible is based upon a splitting of (2.11) into two second-order elliptic problems.

Lemma 2.7 *Let $\mu \ge 0, \lambda > 0$, and suppose that Ω is convex and that $f \in L^\infty(\Omega)$ is such that $\int_\Omega f \ge 0$. Then there exists $\tau_1 > 0$ with the following property: For any $\tau \in (0, \tau_1)$ and each $v \in L^\infty(\Omega)$ satisfying $\int_\Omega v \ge 0$ one can find $c > 0$ such that if $u \in W^{4,2}(\Omega) \cap C^2(\bar\Omega)$ is a solution of (2.11) for some $\theta \in (0, 1]$ then*

$$u(x) \ge -\frac{c}{\theta} \qquad \text{for all } x \in \Omega. \tag{2.33}$$

PROOF. We fix any $\tau_1 \in (0, 1)$ such that $\mu^2\tau_1 \le 1$ and $\tau_1 \le \tau_0$ with τ_0 as in Lemma 2.5, and observe that τ_1 can be chosen independent of v. Then using that v and f belong to $L^\infty(\Omega)$ together with the continuity of $A^{-\frac{1}{2}}$ as an operator in $L^2_\perp(\Omega)$, from Lemma 2.6 we obtain that whenever $\tau < \tau_1$,

$$\int_\Omega |\nabla u|^2 \le 2\theta\lambda \int_\Omega u|\nabla u|^2 + c_1 \le 2\theta\lambda \int_\Omega u_+|\nabla u|^2 + c_1 \tag{2.34}$$

with c_1, as all constants $c_2, c_3, ...$ appearing below, possibly depending on τ and on $\mu, \lambda, \|v\|_{L^\infty(\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$. In order to estimate the rightmost integral in (2.34) from above, we note that as a particular consequence of Lemma 2.5 we have

$$\int_\Omega e^{2\theta\lambda u}|\nabla u|^4 \le \frac{c_2}{\theta^3}$$

for some $c_2 > 0$, because $\tau < \tau_0$. By Hölder's inequality,

$$\int_\Omega u_+|\nabla u|^2 \le \Big(\int_\Omega e^{2\theta\lambda u}|\nabla u|^4\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega u_+^2 e^{-2\theta\lambda u}\Big)^{\frac{1}{2}},$$

so that since $s^2e^{-2\theta\lambda s} \le (\frac{1}{\theta\lambda e})^2$ for all $s \ge 0$, we find that

$$\int_\Omega u_+|\nabla u|^2 \le \frac{c_3}{\theta^{\frac{5}{2}}}$$

and hence, by (2.34), that

$$\int_\Omega |\nabla u|^2 \le \frac{c_4}{\theta^{\frac{3}{2}}} \tag{2.35}$$

are valid, where c_3 and c_4 are appropriately large constants. Our goal is to derive (2.33) from this and the estimate

$$\|u\|_{L^1(\Omega)} \le \frac{c_5}{\theta}, \tag{2.36}$$

the latter guaranteed by (2.22) in Lemma 2.5 for some $c_5 > 0$. To this end, we introduce

$$g(x) := \operatorname{sign}(\Delta u(x)) - \overline{\operatorname{sign}(\Delta u)}, \qquad x \in \Omega,$$

where bars again indicate spatial averages. Then clearly $g \in L^\infty(\Omega)$ with $\|g\|_{L^\infty(\Omega)} \le 2$ and $\int_\Omega g = 0$, and consequently the problem

$$\begin{cases} -\Delta w = g, & x \in \Omega, \\ \frac{\partial}{\partial\nu} w = 0, & x \in \partial\Omega, \\ \int_\Omega w = 0, \end{cases}$$

has a unique solution $w \in W^{2,2}(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\|w\|_{L^\infty(\Omega)} \le c_6 \tag{2.37}$$

for some $c_6 > 0$. Testing (2.11) by $\hat{w} := w + c_6$ yields

$$\begin{aligned} \frac{1}{\tau}\int_\Omega u\hat{w} - \frac{\theta}{\tau}\int_\Omega v\hat{w} &= -\int_\Omega \Delta^2 u \cdot \hat{w} - \theta\mu\int_\Omega \Delta u \cdot \hat{w} - \theta\lambda\int_\Omega \Delta|\nabla u|^2 \cdot \hat{w} + \theta\int_\Omega f\hat{w} \\ &= -\int_\Omega \Delta u \Delta\hat{w} - \theta\mu\int_\Omega u\Delta\hat{w} - \theta\lambda\int_\Omega |\nabla u|^2\Delta\hat{w} \\ &\quad -\theta\lambda\int_{\partial\Omega} \frac{\partial}{\partial\nu}|\nabla u|^2 \cdot \hat{w} + \theta\int_\Omega f\hat{w}, \end{aligned}$$

because $\frac{\partial}{\partial\nu}u = \frac{\partial}{\partial\nu}\Delta u = \frac{\partial}{\partial\nu}\hat{w} = 0$ on $\partial\Omega$. Since $\hat{w}$ was constructed in such a way that $\hat{w} \ge 0$, and since thanks to the convexity of Ω we have $\frac{\partial}{\partial\nu}|\nabla u|^2 \le 0$ on $\partial\Omega$ by Lemma 2.3, the second last term is nonnegative. Using that $-\Delta\hat{w} = -\Delta w = g$, we thus infer that

$$\begin{aligned} \frac{1}{\tau}\int_\Omega u\hat{w} - \frac{\theta}{\tau}\int_\Omega v\hat{w} &\ge \int_\Omega |\Delta u| - \Big(\overline{\operatorname{sign}(\Delta u)}\Big) \cdot \int_\Omega \Delta u \\ &\quad -\theta\mu\int_\Omega ug - \theta\lambda\int_\Omega |\nabla u|^2 g + \theta\int_\Omega f\hat{w}. \end{aligned}$$

Since $\int_\Omega \Delta u = \int_{\partial\Omega}\frac{\partial}{\partial\nu}u = 0$ and $|g| \le 2$, by (2.35) and (2.36) we therefore conclude

$$\begin{aligned} \int_\Omega |\Delta u| &\le \frac{1}{\tau}u(w + c_6) - \frac{\theta}{\tau}\int_\Omega v(w + c_6) \\ &\quad +\theta\mu\int_\Omega ug + \theta\lambda\int_\Omega |\nabla u|^2 g - \theta\int_\Omega f(w + c_6) \\ &\le \frac{2c_6}{\tau}\cdot\frac{c_5}{\theta} + \frac{2\theta c_6|\Omega|}{\tau}\cdot\|v\|_{L^\infty(\Omega)} \\ &\quad +2\theta\mu\cdot\frac{c_5}{\theta} + 2\theta\lambda\cdot\frac{c_4}{\theta^{\frac{3}{2}}} + 2c_6|\Omega|\cdot\|f\|_{L^\infty(\Omega)} \\ &\le \frac{c_7}{\theta} \end{aligned} \tag{2.38}$$

with suitably large c_7. Now going back to (2.11), we see that $z := \Delta u + \theta\mu u + \theta\lambda|\nabla u|^2$ satisfies

$$\begin{cases} -\Delta z = h := \frac{u - \theta v}{\tau} - \theta f, & x \in \Omega, \\ \frac{\partial}{\partial\nu} z = \theta\lambda\frac{\partial}{\partial\nu}|\nabla u|^2 \le 0, & x \in \partial\Omega, \end{cases} \tag{2.39}$$

and as a particular consequence of (2.35), (2.36) and (2.38) we now know that

$$\int_\Omega z_+ \le \int_\Omega |z| \le \frac{c_7}{\theta} + \theta\mu\cdot\frac{c_5}{\theta} + \theta\lambda\cdot\frac{c_4}{\theta^{\frac{3}{2}}} \le \frac{c_8}{\theta}$$

with some $c_8 > 0$. Picking any integer $q > \frac{n}{2}$, from the one-sided pointwise estimate

$$e^{2\theta\lambda u_+} \ge \frac{(2\theta\lambda u_+)^q}{q!}$$

and Lemma 2.5 we easily derive that when $\tau < \tau_0$,

$$\frac{(2\theta\lambda)^q}{q!}\int_\Omega u_+^q \le 2\int_\Omega e^{2\theta\lambda v} \le c_9,$$

which implies that

$$\|h_+\|_{L^q(\Omega)} \le \frac{c_{10}}{\theta}$$

with certain $c_9 > 0$ and $c_{10} > 0$.
An application of Lemma 5.1 from the appendix to (2.39) (with $a := a_0 := 0$) shows that

$$z \le c_{11}(\|z_+\|_{L^1(\Omega)} + \|h_+\|_{L^q(\Omega)}) \le \frac{c_{12}}{\theta}$$

for positive c_{11} and c_{12}, and hence the definition of z yields

$$\begin{aligned} -\Delta(-u) &= \Delta u \le \theta\mu\cdot(-u) - \theta\lambda|\nabla u|^2 + \frac{c_{12}}{\theta} \\ &\le \theta\mu\cdot(-u) + \frac{c_{12}}{\theta}. \end{aligned}$$

Once more invoking Lemma 5.1, now with $a_0 := \mu$ and $a := \theta\mu$, we infer that there exists $c_{13} > 0$ such that

$$-u(x) \le c_{13}\Big(\|u\|_{L^1(\Omega)} + \frac{c_{12}}{\theta}\Big) \qquad \text{for all } x \in \Omega.$$

In view of (2.36), this establishes (2.33). ////

With the above lemma at hand, we are now ready to prove an a priori estimate, depending on τ but not on θ, for solutions of (2.11).

Lemma 2.8 *Suppose that $n \le 3$ and that Ω is convex, and let $\mu \ge 0, \lambda > 0$ and $f \in L^\infty(\Omega)$ be such that $\int_\Omega f \ge 0$. Let $\tau \in (0, \min\{\tau_0, \tau_1\})$ with $\tau_0 > 0$ and $\tau_1 > 0$ as provided by Lemma 2.5 and Lemma 2.7, respectively. Then for each $v \in L^\infty(\Omega)$ satisfying $\int_\Omega v \ge 0$ there exists $c > 0$ such that whenever $u \in W^{4,2}(\Omega)$ is a solution of (2.11) for some $\theta \in (0,1]$, we have*

$$\|u\|_{W^{3,\infty}(\Omega)} \le c. \qquad (2.40)$$

PROOF. Throughout the proof, all appearing constants may depend on $n, \Omega, \mu, \lambda, \tau, v$ and f, but neither on θ nor on the solution u in question. Our argument will follow a bootstrap procedure, deriving from Lemma 2.5 and Lemma 2.7 estimates which will then be used to successively improve themselves.

Step 1. Let us first make sure that there exists $c_1 > 0$ such that

$$\|u\|_{W^{2,2}(\Omega)} \le \frac{c_1}{\theta^{\frac{1}{2}}} \tag{2.41}$$

holds for any such solution.

To see this, we note that since $\tau < \tau_0$, from (2.21) and (2.22) we infer the existence of $c_2 > 0$ such that

$$\theta \int_\Omega e^{2\theta\lambda u} |D^2 u|^2 \le c_2 \qquad \text{and} \qquad \theta^3 \int_\Omega e^{2\theta\lambda u} |\nabla u|^4 \le c_2. \tag{2.42}$$

In particular, this implies the one-sided estimate

$$2\theta\lambda \int_\Omega u|D^2 u|^2 \le \frac{c_2}{\theta}, \tag{2.43}$$

because $e^z \ge z$ for all $z \in \mathbb{R}$. Apart from that, Lemma 2.7 ensures that

$$u \ge -\frac{c_3}{\theta} \qquad \text{in } \Omega \tag{2.44}$$

for some $c_3 > 0$, so that the factors $e^{2\theta\lambda u}$ in (2.42) enjoy a uniform positive bound from below, and hence we also have

$$\int_\Omega |D^2 u|^2 \le \frac{c_4}{\theta} \qquad \text{and} \qquad \int_\Omega |\nabla u|^4 \le \frac{c_4}{\theta^3} \tag{2.45}$$

for some $c_4 > 0$ which, as we once more emphasize, will certainly depend on τ.

Now in order to obtain (2.41), we multiply (2.11) by u and integrate by parts to see that

$$\begin{aligned} \int_\Omega |\Delta u|^2 + \frac{1}{\tau}\int_\Omega u^2 &= -\theta\mu \int_\Omega u\Delta u - \theta\lambda \int_\Omega u\Delta|\nabla u|^2 \\ &\quad + \frac{\theta}{\tau}\int_\Omega uv + \theta \int_\Omega uf. \end{aligned} \tag{2.46}$$

Here, using Young's inequality and (2.45) we easily find that

$$\begin{aligned} -\theta\mu \int_\Omega u\Delta u + \frac{\theta}{\tau}\int_\Omega uv + \theta\int_\Omega uf &\le \frac{1}{2\tau}\int_\Omega u^2 + c_5\theta^2\Big(\int_\Omega |\Delta u|^2 + \int_\Omega v^2 + \int_\Omega f^2\Big) \\ &\le \frac{1}{2\tau}\int_\Omega u^2 + c_6 \end{aligned} \tag{2.47}$$

with certain $c_5 > 0$ and $c_6 > 0$. In the second integral on the right of (2.46), we split

$$\Delta|\nabla u|^2 = 2|D^2 u|^2 + 2\nabla u \cdot \nabla\Delta u \tag{2.48}$$

and integrate by parts in the resulting latter integral. In view of (2.43), (2.45) and the Hölder inequality we then obtain

$$\begin{aligned} I &:= -2\theta\lambda \int_\Omega u\nabla u\cdot\nabla\Delta u \\ &= 2\theta\lambda \int_\Omega |\nabla u|^2\Delta u + 2\theta\lambda\int_\Omega u|\Delta u|^2 \\ &\le 2\theta\lambda\Big(\int_\Omega |\nabla u|^4\Big)^{\frac12}\cdot\Big(\int_\Omega |\Delta u|^2\Big)^{\frac12} + 2\theta\lambda\int_\Omega u|\Delta u|^2 \\ &\le 2\theta\lambda\cdot\Big(\frac{c_4}{\theta^3}\Big)^{\frac12}\cdot\Big(\frac{nc_4}{\theta}\Big)^{\frac12} + \frac{nc_2}{\theta}, \end{aligned}$$

because $|\Delta u|^2 \le n|D^2u|^2$. Therefore using (2.44) we find that

$$\begin{aligned} -\theta\lambda\int_\Omega u\Delta|\nabla u|^2 &= -2\theta\lambda\int_\Omega u|D^2u|^2 + I \\ &\le 2\lambda c_3\cdot\int_\Omega |D^2u|^2 + I \\ &\le \frac{c_7}{\theta} \end{aligned}$$

with suitable $c_7 > 0$, and thus (2.46) and (2.47) show that

$$\int_\Omega |\Delta u|^2 + \frac{1}{2\tau}\int_\Omega u^2 \le c_6 + \frac{c_7}{\theta},$$

whereupon (2.41) follows.

Step 2. We next claim that one can find $c_8 > 0$ such that

$$\|u\|_{W^{3,2}(\Omega)} \le \frac{c_8}{\theta^{\frac12}}. \tag{2.49}$$

Indeed, multiplying (2.11) by Δu and integrating by parts we see that

$$\begin{aligned} \int_\Omega |\nabla\Delta u|^2 + \frac{1}{\tau}\int_\Omega |\nabla u|^2 &= \theta\mu\int_\Omega |\Delta u|^2 + \theta\lambda\int_\Omega \Delta|\nabla u|^2\Delta u \\ &\quad -\frac{\theta}{\tau}\int_\Omega v\Delta u - \theta\int_\Omega f\Delta u, \end{aligned} \tag{2.50}$$

where again by Young's inequality and (2.41) we obtain $c_9 > 0$ such that

$$\begin{aligned} \theta\mu\int_\Omega |\Delta u|^2 - \frac{\theta}{\tau}\int_\Omega v\Delta u - \theta\int_\Omega f\Delta u &\le \Big(\theta\mu + \frac{\theta^2}{2\tau^2} + \frac{\theta^2}{2}\Big)\cdot\int_\Omega |\Delta u|^2 + \frac12\int_\Omega v^2 + \frac12\int_\Omega f^2 \\ &\le c_9. \end{aligned} \tag{2.51}$$

Once more using (2.48) and integrating by parts, we have

$$\begin{aligned}
\theta\lambda\int_\Omega \Delta|\nabla u|^2\Delta u &= 2\theta\lambda\int_\Omega |D^2u|^2\Delta u + 2\theta\lambda\int_\Omega (\nabla u\cdot\nabla\Delta u)\Delta u \\
&= 2\theta\lambda\int_\Omega |D^2u|^2\Delta u - \theta\lambda\int_\Omega (\Delta u)^3 \\
&\le c_{10}\theta\|u\|^3_{W^{2,3}(\Omega)}
\end{aligned}$$

for some $c_{10}>0$. Since $n\le 3$, by the Gagliardo-Nirenberg inequality ([10]) and Young's inequality we can interpolate

$$\begin{aligned}
c_{10}\theta\|u\|^3_{W^{2,3}(\Omega)} &\le c_{11}\theta\Big(\|\nabla\Delta u\|^{\frac{n}{2}}_{L^2(\Omega)}\cdot\|u\|^{\frac{6-n}{2}}_{W^{2,2}(\Omega)} + \|u\|^3_{W^{2,2}(\Omega)}\Big) \\
&\le \frac{1}{2}\|\nabla\Delta u\|^2_{L^2(\Omega)} + c_{12}\Big(\theta^{\frac{4}{4-n}}\cdot\|u\|^{\frac{2(6-n)}{4-n}}_{W^{2,2}(\Omega)} + \theta\|u\|^3_{W^{2,2}(\Omega)}\Big)
\end{aligned}$$

with certain positive constants c_{11} and c_{12}. Recalling the outcome of Step 1, we infer that

$$\begin{aligned}
\theta\lambda\int_\Omega \Delta|\nabla u|^2\Delta u &\le \frac{1}{2}\int_\Omega|\nabla\Delta u|^2 + c_{12}\Big(c_1^{\frac{2(6-n)}{4-n}}\theta^{\frac{n-2}{4-n}} + c_1^3\theta^{-\frac{1}{2}}\Big) \\
&\le \frac{1}{2}\int_\Omega|\nabla\Delta u|^2 + \frac{c_{13}}{\theta^{\frac{1}{2}}}
\end{aligned}$$

holds with some constant c_{13}, whence (2.49) easily results from (2.50), (2.51) and (2.41).

Step 3. We are now in the position to prove that there exists $c_{14}>0$ such that

$$\|u\|_{W^{4,2}(\Omega)} \le c_{14}. \tag{2.52}$$

To this end, we observe that in the rearranged version of (2.11),

$$\Delta^2 u + \frac{1}{\tau}u = -\theta\mu\Delta u - 2\theta\lambda|D^2u|^2 - 2\theta\lambda\nabla u\cdot\nabla\Delta u + \frac{\theta}{\tau}v + \theta f =: g, \tag{2.53}$$

we can find $c_{15}>0$ such that

$$\Big\| -\theta\mu\Delta u + \frac{\theta}{\tau}v + \theta f\Big\|_{L^2(\Omega)} \le c_{15}$$

by (2.45), and such that moreover

$$\Big\| -2\theta\lambda|D^2u|^2\Big\|_{L^2(\Omega)} = 2\theta\lambda\|D^2u\|^2_{L^4(\Omega)} \le c_{15}\theta\|u\|^2_{W^{3,2}(\Omega)} \le c_8^2c_{15}$$

and

$$\|-2\theta\lambda\nabla u\cdot\nabla\Delta u\|_{L^2(\Omega)} \le 2\theta\lambda\|\nabla u\|_{L^\infty(\Omega)}\cdot\|\nabla\Delta u\|_{L^2(\Omega)} \le c_{15}\theta\|u\|^2_{W^{3,2}(\Omega)} \le c_8^2c_{15}$$

hold, where we have used that since $n\le 3$ we have the embeddings $W^{3,2}(\Omega)\hookrightarrow W^{2,6}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$. This provides a θ-independent bound for g in $L^2(\Omega)$, and thus standard elliptic regularity theory ([10]) applied to (2.53) establishes (2.52).

Step 4. In a final step, we can now show that

$$\|u\|_{W^{4,6}(\Omega)} \le c_{16} \tag{2.54}$$

is valid with some $c_{16} > 0$, which will prove the lemma in view of the embedding $W^{4,6}(\Omega) \hookrightarrow W^{3,\infty}(\Omega)$.
To verify (2.54), we proceed as in Step 3: Going back to (2.53), using (2.52) and the fact that $W^{4,2}(\Omega) \hookrightarrow W^{3,6}(\Omega) \hookrightarrow W^{2,\infty}(\Omega)$, we can now estimate

$$\Big\| - 2\theta\lambda|D^2u|^2\Big\|_{L^6(\Omega)} = 2\theta\lambda\|D^2u\|^2_{L^{12}(\Omega)} \le c_{17}\|u\|^2_{W^{4,2}(\Omega)} \le c_{14}^2 c_{17}$$

and

$$\| - 2\theta\lambda\nabla u\cdot\nabla\Delta u\|_{L^6(\Omega)} \le 2\theta\lambda\|\nabla u\|_{L^\infty(\Omega)}\cdot\|\nabla\Delta u\|_{L^6(\Omega)} \le c_{17}\|u\|^2_{W^{4,2}(\Omega)} \le c_{14}^2 c_{17}$$

for any $\theta \in (0,1]$ and some $c_{17} > 0$. This easily leads to a uniform bound for g in $L^6(\Omega)$ and thereby proves (2.54) upon one more application of elliptic estimates to (2.53). ////

As an immediate consequence we obtain solvability of the Rothe scheme (2.2).

Lemma 2.9 *Let $n \le 3$ and Ω be convex, and let $p > n$. Then for any $\mu \ge 0, \lambda > 0$ and $f \in L^\infty(\Omega)$ satisfying $\int_\Omega f \ge 0$ there exists $\tau_\star \in (0,1)$ with the following property: If $\tau \in (0,\tau_\star)$ and $v \in L^\infty(\Omega)$ satisfies $\int_\Omega v \ge 0$, then*

$$\bigcup_{\theta\in(0,1]} Fix(\theta\cdot L^{-1}N_v) \quad \textit{is bounded in } X_p, \tag{2.55}$$

where L, N_v and X_p are defined in (2.8) (2.7) and (2.6), respectively.
Furthermore, under these conditions the problem (2.5) possesses at least one solution $u \in W^{4,p}(\Omega)$ that satisfies $\int_\Omega u \ge 0$.

Proof. Let c denote the constant provided by Lemma 2.8, and let $\tau_\star := \min\{\tau_0,\tau_1\}$ with τ_0 and τ_1 as in Lemma 2.5 and Lemma 2.7, respectively. Suppose that $u \in X_p$ satisfies $u = \theta\cdot L^{-1}Nu$ for some $\theta \in (0,1]$. By the regularizing properties of L^{-1} we then actually have $u \in W^{4,2}(\Omega)$, so that Lemma 2.8 along with the Hölder inequality implies that $\|u\|_{W^{3,p}(\Omega)} \le c|\Omega|^{\frac{1}{p}}$, which proves (2.55).
According to the continuity and compactness properties of the mappings $L^{-1}N_v$, from (2.55) and the Leray-Schauder fixed point theorem ([27]) we immediately infer the existence of a solution $u \in X_p$ of the equation $u = L^{-1}N_v u$. Since $p > n$ implies that $W^{3,p}(\Omega \hookrightarrow W^{2,\infty}(\Omega)$, it follows from the definition of N_v that $N_v u \in L^p(\Omega)$, and hence we actually have $u = L^{-1}N_v u \in W^{4,p}(\Omega)$ by elliptic regularity theory. Finally, the nonnegativity of $\int_\Omega u$ is asserted by Lemma 2.5. ////

Corollary 2.10 *Suppose that $n \le 3$, that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with smooth boundary, and that $p > n$. Moreover, assume that $\mu \ge 0, \lambda > 0$, that $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$ are such that $\int_\Omega u_0 \ge 0$ and $\int_\Omega f \ge 0$, and that we are given a family $(u_0^{(\tau)})_{\tau\in(0,1)}$ which satisfies (H1). Then there exists $\tau_\star \in (0,1)$ such that for each $\tau \in (0,\tau_\star)$, the scheme (2.2) admits at least one solution sequence $(u_k^{(\tau)})_{k\in\{1,2,...\}} \subset W^{4,p}(\Omega)$ for which we have $\int_\Omega u_k^{(\tau)} \ge 0$, $k = 1, 2, ...$.*

Proof. Let $\tau_\star \in (0,1)$ denote the constant provided by Lemma 2.9. Then since $u_0^{(\tau)} \in L^\infty(\Omega)$ satisfies $\int_\Omega u_0^{(\tau)} \ge 0$ by (H1), we may apply Lemma 2.9 to obtain that whenever $\tau \in (0,\tau_\star)$, the problem (2.2) for $k = 1$ has at least one solution $u_1^{(\tau)} \in W^{4,p}(\Omega)$ with $\int_\Omega u_1^{(\tau)} \ge 0$. Due to the latter property, and since of course $u_1^{(\tau)} \in W^{4,p}(\Omega)$ implies $u_1^{(\tau)} \in L^\infty(\Omega)$, the same argument applies to yield solvability of (2.2) also for $k = 2$. Thus, an inductive argument and an application of Lemma 2.9 complete the proof. ////

Remark. Note that since the constant $\tau_\star$ in Lemma 2.9 does not depend on v, it is possible to avoid the situation that different choices of τ, say, $\tau = \tau_k$, $k = 1, 2, ...$, are necessary in each step in (2.2). This evidently rules out the possibility that the obtained Rothe functions defined by (2.3) and (2.4) cease to exist beyond some finite time.

3 Estimates independent of τ

In order to take $\tau \searrow 0$, we proceed to derive estimates for the family of Rothe functions defined by (2.3) and (2.4).

Lemma 3.1 *Let $n \le 3$, Ω be convex, $p > n$, $\mu \ge 0, \lambda > 0$, and suppose that $u_0 \in L^\infty(\Omega)$ and $f \in L^\infty(\Omega)$ are such that $\int_\Omega u_0 \ge 0$ and $\int_\Omega f \ge 0$. Given a familiy $(u_0^{(\tau)})_{\tau\in(0,1)}$ satisfying (H1), let $\tau_\star \in (0,1)$ and, for $\tau \in (0,\tau_\star)$, $(u_k^{(\tau)})_{k\in\{1,2,...\}} \subset W^{4,p}(\Omega)$ be as provided by Corollary 2.10. Then for all $T > 0$ there exists $C > 0$ such that the corresponding Rothe functions defined by (2.3) and (2.4) satisfy*

$$\int_\Omega e^{2\lambda\hat{u}^{(\tau)}(x,t)}dx \le C \qquad \text{for all } t \in [0,T], \tag{3.1}$$

$$\int_0^T \int_\Omega e^{2\lambda\hat{u}^{(\tau)}}|D^2\hat{u}^{(\tau)}|^2 \le C, \qquad \text{and} \tag{3.2}$$

$$\int_0^T \int_\Omega e^{2\lambda\hat{u}^{(\tau)}}|\nabla\hat{u}^{(\tau)}|^4 \le C \tag{3.3}$$

as well as

$$\int_0^T \int_\Omega |\nabla\hat{u}^{(\tau)}|^2 \le C, \tag{3.4}$$

$$\int_0^T \int_\Omega |\hat{u}^{(\tau)}| \cdot |\nabla \hat{u}^{(\tau)}|^2 \le C \qquad \textit{and} \tag{3.5}$$

$$\int_0^T \int_\Omega |\nabla u^{(\tau)}|^2 \le C \tag{3.6}$$

whenever $\tau \in (0, \tau_\star)$.

PROOF. According to Lemma 2.5 applied to $\theta = 1$, there exist $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \int_\Omega e^{2\lambda u_k^{(\tau)}} - \int_\Omega e^{2\lambda u_{k-1}^{(\tau)}} + c_1\tau \int_\Omega e^{2\lambda u_k^{(\tau)}} |D^2 u_k^{(\tau)}|^2 \quad &+ \quad c_1\tau \int_\Omega e^{2\lambda u_k^{(\tau)}} |\nabla u_k^{(\tau)}|^4 \\ &\le \quad c_2\tau \int_\Omega e^{2\lambda u_k^{(\tau)}} \end{aligned} \tag{3.7}$$

holds for all $k \ge 1$. Here, ignoring nonnegative terms we first obtain that $a_k := \int_\Omega e^{2\lambda u_k^{(\tau)}}$ satisfies

$$a_k - a_{k-1} \le c_2 \tau a_k \qquad \text{for all } k \ge 1. \tag{3.8}$$

By an elementary discrete Gronwall-type inequality, if $c_2\tau \le \frac{1}{2}$ this implies

$$a_k \le a_0 \cdot (1 - c_2\tau)^{-k} \qquad \text{for all } k \ge 1$$

and hence

$$a_k \le a_0 \cdot e^{2c_2 k\tau} \qquad \text{for all } k \ge 1,$$

because $(1-z)^{-k} \le e^{2kz}$ for all $z \in [0, \frac{1}{2}]$. If we fix $\tau \in (0,1)$ and take $N_\tau \in \mathbb{N}$ such that $T \le N_\tau \cdot \tau < T + 1$, this yields

$$\int_\Omega e^{2\lambda u_k^{(\tau)}} \le c_3 := \Big(\sup_{\tau \in (0,1)} \int_\Omega e^{2\lambda u_0^{(\tau)}} \Big) \cdot e^{2c_2(T+1)} \qquad \text{for all } k \in \{0, ..., N_\tau\} \tag{3.9}$$

and thereby proves (3.1) in view of (H1b). Now going back to (3.7), upon summing up over $k \in \{1, ..., N_\tau\}$, using (3.9) we find

$$\begin{aligned} \int_\Omega e^{2\lambda u_{N_\tau}^{(\tau)}} + c_1\tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega e^{2\lambda u_k^{(\tau)}} |D^2 u_k^{(\tau)}|^2 \quad &+ \quad c_1\tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega e^{2\lambda u_k^{(\tau)}} |\nabla u_k^{(\tau)}|^4 \\ &\le \quad \int_\Omega e^{2\lambda u_0^{(\tau)}} + c_2\tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega e^{2\lambda u_k^{(\tau)}} \\ &\le \quad c_3 \cdot (1 + c_2 \cdot (T+1)), \end{aligned} \tag{3.10}$$

which implies (3.2) and (3.3).

Proceeding similarly, from an application of Lemma 2.6 to $\theta = 1$ and arbitrary $\eta > 0$ we obtain that for some $c_4 > 0$ we have

$$\begin{aligned}\int_\Omega \left|A^{-\frac12}\left(u_k^{(\tau)} - \overline{u_k^{(\tau)}}\right)\right|^2 &- \int_\Omega \left|A^{-\frac12}\left(u_{k-1}^{(\tau)} - \overline{u_{k-1}^{(\tau)}}\right)\right|^2 + \tau\int_\Omega |\nabla u_k^{(\tau)}|^2 \\ &\le c_4\tau\left(1 + \int_\Omega \left|A^{-\frac12}\left(u_k^{(\tau)} - \overline{u_k^{(\tau)}}\right)\right|^2\right) \\ &\quad +2\lambda\tau\int_\Omega u_k^{(\tau)}|\nabla u_k^{(\tau)}|^2. \qquad (3.11)\end{aligned}$$

Here, the last integral can be estimated by using the inequality $e^z \ge z$ for $z \in \mathbb{R}$, leading to

$$\begin{aligned}I_k := 2\lambda\tau\int_\Omega u_k^{(\tau)}|\nabla u_k^{(\tau)}|^2 &\le \tau\int_\Omega e^{2\lambda u_k^{(\tau)}}|\nabla u_k^{(\tau)}|^2 \\ &\le \frac{\tau}{2}\int_\Omega e^{2\lambda u_k^{(\tau)}}|\nabla u_k^{(\tau)}|^4 + \frac{\tau}{2}\int_\Omega e^{2\lambda u_k^{(\tau)}} \qquad (3.12)\end{aligned}$$

in view of Young's inequality. Therefore, (3.12) shows that defining $b_k := \int_\Omega |A^{-\frac12}(u_k^{(\tau)} - \overline{u_k^{(\tau)}})|^2$, from (3.11) we thus obtain

$$b_k - b_{k-1} \le c_4\tau b_k + B_k \qquad \text{for all } k \ge 1,$$

where $B_k := c_4\tau + I_k$ and hence, thanks to (3.9), (3.10) and (3.12),

$$B := \sum_{k=1}^{N_\tau} B_k \le c_5 \qquad (3.13)$$

with some $c_5 > 0$ independent of τ. Again using a discrete Gronwall inequality, which can easily be derived by induction, from this we infer that

$$b_k \le (1 - c_4\tau)^{-k} \cdot b_0 + \sum_{l=1}^{k}(1 - c_4\tau)^{l-k-1}B_l \qquad \text{for all } k \ge 1,$$

which if $c_4\tau \le \frac12$ implies

$$b_k \le e^{2c_4k\tau} \cdot b_0 + \sum_{l=1}^{k} e^{2c_4(k+1-l)\tau}B_l \qquad \text{for all } k \ge 1.$$

This proves that

$$\begin{aligned}\int_\Omega \left|A^{-\frac12}\left(u_k^{(\tau)} - \overline{u_k^{(\tau)}}\right)\right|^2 \le c_6 &:= \left(\sup_{\tau\in(0,1)}\int_\Omega \left|A^{-\frac12}\left(u_0^{(\tau)} - \overline{u_0^{(\tau)}}\right)\right|^2\right)\cdot e^{2c_4(T+1)} \\ &\quad +Be^{2c_4(T+1)} \qquad \text{for all } k \in \{0, ..., N_\tau\}, \qquad (3.14)\end{aligned}$$

and thus from (3.11) we infer

$$
\begin{aligned}
\int_\Omega \Big| A^{-\frac{1}{2}} \Big(u_{N_\tau}^{(\tau)} - \overline{u_{N_\tau}^{(\tau)}} \Big) \Big|^2 &+ \tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega |\nabla u_k^{(\tau)}|^2 - 2\lambda\tau \cdot \sum_{k=1}^{N_\tau} u_k^{(\tau)} |\nabla u_k^{(\tau)}|^2 \\
&\le \int_\Omega \Big| A^{-\frac{1}{2}} \Big(u_0^{(\tau)} - \overline{u_0^{(\tau)}} \Big) \Big|^2 + c_4\tau \cdot \sum_{k=1}^{N_\tau} \Big| A^{-\frac{1}{2}} \Big(u_k^{(\tau)} - \overline{u_k^{(\tau)}} \Big) \Big|^2 \\
&\quad + c_4\tau \cdot N_\tau \\
&\le c_6 \cdot \Big(1 + 2c_4(T+1) \Big).
\end{aligned}
\tag{3.15}
$$

In view of (3.12) and (3.13), this establishes (3.4) and (3.5). Moreover, since

$$
\begin{aligned}
\sum_{k=1}^{N_\tau} \int_{(k-1)\tau}^{k\tau} |\nabla u^{(\tau)}|^2 &= \sum_{k=1}^{N_\tau} \int_{(k-1)\tau}^{k\tau} \int_\Omega \Big| \Big(k - \frac{t}{\tau} \Big) \cdot \nabla u_{k-1}^{(\tau)} + \Big(\frac{t}{\tau} - (k-1) \Big) \cdot \nabla u_k^{(\tau)} \Big|^2 \\
&\le 2\tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega |\nabla u_{k-1}^{(\tau)}|^2 + 2\tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega |\nabla u_k^{(\tau)}|^2 \\
&\le 4\tau \cdot \sum_{k=1}^{N_\tau} \int_\Omega |\nabla u_k^{(\tau)}|^2 + 2\tau \int_\Omega |\nabla u_0^{(\tau)}|^2,
\end{aligned}
$$

from (3.15) and (H1c) we finally also obtain (3.6). ////

By standard compactness arguments, the above boundedness properties allow for the extraction of suitably converging subsequences.

Lemma 3.2 *Under the assumptions of Lemma 3.1, there exist $(\tau_j)_{j\in\mathbb{N}} \subset (0,1)$ and a function $u \in L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ such that $\tau_j \searrow 0$ as $j \to \infty$ and*

$$
u^{(\tau)} \to u \quad \textit{and} \quad \hat{u}^{(\tau)} \to u \qquad \textit{in } L^2_{loc}([0,\infty); L^2(\Omega)) \textit{ and a.e. in } \Omega \times (0,\infty) \tag{3.16}
$$

and

$$
\hat{u}^{(\tau)} \rightharpoonup u \qquad \textit{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \tag{3.17}
$$

as $\tau = \tau_j \searrow 0$.

Proof. Let $T > 0$. By (3.1), (3.4), (3.6) and the fact that $\int_\Omega u_k^{(\tau)} \ge 0$ for all $k \ge 0$,

$$
(\hat{u}^{(\tau)})_{\tau\in(0,\tau_\star)} \quad \text{and} \quad (u^{(\tau)})_{\tau\in(0,\tau_\star)} \qquad \text{are bounded in } L^2((0,T); W^{1,2}(\Omega)). \tag{3.18}
$$

We claim that moreover

$$
(\partial_t u^{(\tau)})_{\tau\in(0,\tau_\star)} \quad \text{is bounded in } L^1\Big((0,T); (W_0^{m,2}(\Omega))^\star\Big), \tag{3.19}
$$

holds, where $m \in \mathbb{N}$ is large enough such that the Hilbert space $W_0^{m,2}(\Omega)$ is continuously embedded into $W^{2,\infty}(\Omega)$. In fact, rewriting (2.2) in the form

$$
\partial_t u^{(\tau)} = -\Delta^2 \hat{u}^{(\tau)} - \mu\Delta\hat{u}^{(\tau)} - \lambda\Delta|\nabla\hat{u}^{(\tau)}|^2 + f, \tag{3.20}
$$

and testing this by an arbitrary $\psi \in C_0^\infty(\Omega)$ we obtain

$$\int_\Omega \partial_t u^{(\tau)}(\cdot,t)\cdot\psi = -\int_\Omega \hat{u}^{(\tau)}\Delta^2\psi - \mu\int_\Omega \hat{u}^{(\tau)}\Delta\psi - \lambda\int_\Omega |\nabla\hat{u}^{(\tau)}|^2\Delta\psi + \int_\Omega f\psi$$

for $t \in (0,T)\setminus\{k\tau \mid k\in\mathbb{N}\}$, because $\frac{\partial}{\partial\nu}\hat{u}^{(\tau)} = \frac{\partial}{\partial\nu}\Delta\hat{u}^{(\tau)} = 0$. Since

$$\begin{aligned}
&\Big| -\int_\Omega \hat{u}^{(\tau)}(\cdot,t)\Delta^2\psi\Big| \le \|\hat{u}^{(\tau)}(\cdot,t)\|_{L^2(\Omega)}\cdot\|\Delta^2\psi\|_{L^2(\Omega)},\\
&\Big| -\mu\int_\Omega \hat{u}^{(\tau)}(\cdot,t)\Delta\psi\Big| \le \mu\|\hat{u}^{(\tau)}(\cdot,t)\|_{L^2(\Omega)}\cdot\|\Delta\psi\|_{L^2(\Omega)},\\
&\Big| -\lambda\int_\Omega |\nabla\hat{u}^{(\tau)}|^2\Delta\psi\Big| \le \lambda\|\nabla\hat{u}^{(\tau)}(\cdot,t)\|_{L^2(\Omega)}^2\cdot\|\Delta\psi\|_{L^\infty(\Omega)} \qquad \text{and}\\
&\Big|\int_\Omega f\psi\Big| \le \|f\|_{L^2(\Omega)}\cdot\|\psi\|_{L^2(\Omega)},
\end{aligned}$$

and since $\|\Delta\psi\|_{L^\infty(\Omega)} \le c_1\|\psi\|_{W^{m,2}(\Omega)}$ for some $c_1>0$ according to our assumption on m, we infer that there exists $c_2>0$ such that

$$\Big|\partial_t u^{(\tau)}(\cdot,t)\cdot\psi\Big| \le c_2(1+\|\hat{u}^{(\tau)}(\cdot,t)\|_{W^{1,2}(\Omega)}^2)\cdot\|\psi\|_{W^{m,2}(\Omega)}$$

and hence

$$\Big\|\partial_t u^{(\tau)}(\cdot,t)\Big\|_{(W_0^{m,2}(\Omega))^\star} \le c_2(1+\|\hat{u}^{(\tau)}(\cdot,t)\|_{W^{1,2}(\Omega)}^2) \qquad \text{for a.e. } t>0.$$

In view of (3.18), this immediately yields (3.19). As a consequence, the Aubins-Lions lemma ([26]) states that $(u^{(\tau)})_{\tau\in(0,\tau_\star)}$ is relatively compact with respect to the strong topology in $L^2((0,T);L^2(\Omega))$. Thus, along a suitable sequence $\tau_j \searrow 0$ we have $u^{(\tau)} \to u$ in $L^2_{loc}([0,\infty);L^2(\Omega))$. Now (3.16) and (3.17) easily follow from this, the technical Lemma 5.2 below and (3.18). ////

In order to be prepared for an adequate limit process $\tau=\tau_j\searrow 0$ in the nonlinear term in (1.2), we shall require the following statement on *strong* compactness of $(\nabla\hat{u}^{(\tau)})_{\tau\in(0,\tau_\star)}$ in $L^2_{loc}([0,\infty);L^2_{loc}(\Omega))$.

Lemma 3.3 *Under the assumptions from Lemma 3.1,*

$$\nabla\hat{u}^{(\tau)} \to \nabla u \qquad \text{in } L^2_{loc}([0,\infty);L^2(\Omega)) \qquad \text{as } \tau=\tau_j\searrow 0 \tag{3.21}$$

holds, where $(\tau_j)_{j\in\mathbb{N}} \subset (0,\tau_\star)$ is the sequence provided by Lemma 3.2.
Moreover, we have $e^{\lambda u} \in L^2_{loc}([0,\infty);W_N^{2,2}(\Omega))$ and

$$e^{\lambda\hat{u}^{(\tau)}} \rightharpoonup e^{\lambda u} \qquad \text{in } L^2_{loc}([0,\infty);W^{2,2}(\Omega)) \qquad \text{as } \tau=\tau_j\searrow 0. \tag{3.22}$$

PROOF. Given $T > 0$, by (3.2), (3.3) and the identity

$$(e^{\lambda v})_{x_i x_j} = \lambda e^{\lambda v} v_{x_i x_j} + \lambda^2 e^{\lambda v} v_{x_i} v_{x_j}, \qquad v \in W^{2,2}(\Omega) \cap L^\infty(\Omega), \tag{3.23}$$

there exists $c_1 > 0$ such that for all sufficiently small $\tau > 0$ we have

$$\int_0^T \int_\Omega \left| D^2 e^{\lambda \hat{u}^{(\tau)}} \right|^2 \le c_1.$$

In conjunction with (3.1), this entails that

$$\| e^{\lambda \hat{u}^{(\tau)}} \|_{L^2((0,T);W^{2,2}(\Omega))} \le c_2 \tag{3.24}$$

for some $c_2 > 0$ and all τ small enough.
Now let $c_3 > 0$ be such that, according to Lemma 3.1,

$$\int_0^T \int_\Omega |\hat{u}^{(\tau)}| \cdot |\nabla \hat{u}^{(\tau)}|^2 \le c_3 \tag{3.25}$$

for small τ. Then, given $\varepsilon > 0$, we fix $M > 0$ large such that $\frac{c_3}{M} < \frac{\varepsilon}{4}$, and next pick a cut-off function $\zeta \in C_0^\infty(\mathbb{R})$ satisfying $\zeta(s) = 1$ for $|s| \le M$ and $\zeta(s) = 0$ for $|s| \ge M + 1$. Decomposing

$$\begin{aligned} \int_0^T \int_\Omega |\nabla \hat{u}^{(\tau)}|^2 &= \int_0^T \int_\Omega \zeta(\hat{u}^{(\tau)}) \cdot |\nabla \hat{u}^{(\tau)}|^2 + \int_0^T \int_\Omega (1 - \zeta(\hat{u}^{(\tau)})) \cdot |\nabla \hat{u}^{(\tau)}|^2 \\ &=: I_1(\tau) + I_2(\tau), \end{aligned} \tag{3.26}$$

from (3.25) we infer that

$$\begin{aligned} I_2(\tau) &\le \int_0^T \int_\Omega \chi_{\{|\hat{u}^{(\tau)}| \ge M\}} \cdot |\nabla \hat{u}^{(\tau)}|^2 \\ &\le \frac{1}{M} \cdot \int_0^T \int_\Omega |\hat{u}^{(\tau)}| \cdot |\nabla \hat{u}^{(\tau)}|^2 \\ &\le \frac{c_3}{M} \\ &\le \frac{\varepsilon}{4} \end{aligned} \tag{3.27}$$

for small τ. Recalling that

$$\hat{u}^{(\tau)} \to u \qquad \text{a.e. in } \Omega \times (0,T) \text{ as } \tau = \tau_j \searrow 0 \tag{3.28}$$

due to Lemma 3.2, from the dominated convergence theorem we obtain that $1 - \zeta(\hat{u}^{(\tau)}) \to 1 - \zeta(u)$ in $L^2(\Omega \times (0,T))$, which together with (3.17) ensures that $\sqrt{1 - \zeta(\hat{u}^{(\tau)})} \nabla \hat{u}^{(\tau)} \rightharpoonup \sqrt{1 - \zeta(u)} \nabla u$ in $L^2(\Omega \times (0,T))$ as $\tau = \tau_j \searrow 0$. Thus, by lower semicontinuity of the L^2 norm with respect to weak convergence, (3.27) entails that also

$$I_2(0) := \int_0^T \int_\Omega (1 - \zeta(u)) |\nabla u|^2 \le \frac{\varepsilon}{4}. \tag{3.29}$$

As to $I_1(\tau)$, writing

$$H(s) := \frac{1}{\lambda} \int_{-\infty}^{s} e^{-\lambda\sigma} \zeta(\sigma) d\sigma, \qquad s \in \mathbb{R}, \tag{3.30}$$

we have

$$\begin{aligned} I_1(\tau) &= \int_0^T \int_\Omega \Big(\frac{1}{\lambda} e^{-\lambda \hat{u}^{(\tau)}} \zeta(\hat{u}^{(\tau)}) \nabla \hat{u}^{(\tau)} \Big) \cdot \Big(\lambda e^{\lambda \hat{u}^{(\tau)}} \nabla \hat{u}^{(\tau)} \Big) \\ &= \int_0^T \int_\Omega \nabla H(\hat{u}^{(\tau)}) \cdot \nabla e^{\lambda \hat{u}^{(\tau)}} \\ &= - \int_0^T \int_\Omega H(\hat{u}^{(\tau)}) \Delta e^{\lambda \hat{u}^{(\tau)}} \end{aligned} \tag{3.31}$$

upon an integration by parts, because $\frac{\partial}{\partial\nu} e^{\lambda \hat{u}^{(\tau)}} = 0$ on $\partial\Omega$. As a consequence of (3.24), $(e^{\lambda \hat{u}^{(\tau)}})_{\tau \in (\tau_j)_{j\in\mathbb{N}}}$ is weakly precompact in $L^2((0,T); W_N^{2,2}(\Omega))$, and by (3.28) we obtain that whenever $(\tau_{j_i})_{i\in\mathbb{N}} \subset (\tau_j)_{j\in\mathbb{N}}$ is such that $e^{\lambda u^{(\tau)}} \rightharpoonup z$ in $L^2((0,T); W_N^{2,2}(\Omega))$ as $\tau = \tau_{j_i} \searrow 0$, we must have $z = e^{\lambda u}$ by Egorov's theorem. This implies that

$$e^{\lambda u} \in L^2((0,T); W_N^{2,2}(\Omega)) \qquad \text{and} \qquad e^{\lambda \hat{u}^{(\tau)}} \rightharpoonup e^{\lambda u} \quad \text{in } L^2((0,T); W_N^{2,2}(\Omega)) \tag{3.32}$$

along the whole sequence $\tau = \tau_j \searrow 0$.

Next, from the definition (3.30) of H and the construction of ζ we know that $0 \le H \le c_4 := \frac{1}{\lambda} \cdot 2(M+1) \cdot e^{\lambda(M+1)}$ on $\mathbb{R}$, which combined with the fact that $H(\hat{u}^{(\tau)}) \to H(u)$ a.e. in $\Omega \times (0,T)$ as $\tau = \tau_j \searrow 0$ by (3.28) guarantees that $\int_0^T \int_\Omega H^2(\hat{u}^{(\tau)}) \to \int_0^T \int_\Omega H^2(u)$ and in particular

$$H(\hat{u}^{(\tau)}) \to H(u) \qquad \text{in } L^2(\Omega \times (0,T)) \tag{3.33}$$

as $\tau = \tau_j \searrow 0$. It follows from (3.32) and (3.33) that in (3.31) we may safely go to the limit to find that

$$I_1(\tau) \to - \int_0^T \int_\Omega H(u) \Delta e^{\lambda u} \qquad \text{as } \tau = \tau_j \searrow 0. \tag{3.34}$$

However, since $e^{\lambda u} \in L^2((0,T); W_N^{2,2}(\Omega))$, we have $\frac{\partial}{\partial\nu} e^{\lambda u} = 0$ on $\partial\Omega$ and therefore we may integrate by parts on the right-hand side of (3.34) to infer that in fact

$$\begin{aligned} I_1(\tau) &\to \int_0^T \int_\Omega \nabla H(u) \cdot \nabla e^{\lambda u} \\ &= \lambda \int_0^T \int_\Omega H'(u) e^{\lambda u} |\nabla u|^2 \\ &= \int_0^T \int_\Omega \zeta(u) |\nabla u|^2 =: I_1(0) \qquad \text{as } \tau = \tau_j \searrow 0 \end{aligned} \tag{3.35}$$

in view of (3.30). All in all, from (3.26), (3.27), (3.29) and (3.35) we obtain that

$$\begin{aligned} \Big| \int_0^T \int_\Omega |\nabla \hat{u}^{(\tau)}|^2 - \int_0^T \int_\Omega |\nabla u|^2 \Big| &= \Big| I_1(\tau) + I_2(\tau) - I_1(0) - I_2(0) \Big| \\ &\le \frac{\varepsilon}{2} + |I_1(\tau) - I_1(0)| \\ &< \varepsilon \end{aligned}$$

whenever $\tau \in (\tau_j)_{j\in\mathbb{N}}$ is sufficiently small, and therefore conclude that $\nabla \hat{u}^{(\tau)} \to \nabla u$ in $L^2(\Omega \times (0,T))$, which proves the lemma. ////

We are now ready for the proof of our main statement on existence and approximation of global weak solutions to (1.2).

Proof (of Theorem 1.1). We let $(\tau_j)_{j\in\mathbb{N}} \subset (0,1)$ be as provided by Lemma 3.2. Then a straightforward testing procedure applied to (2.2) yields

$$\begin{aligned}\int_0^\infty \int_\Omega u^{(\tau)}\varphi_t + \int_\Omega u_0^{(\tau)}\varphi(\cdot,0) \quad &= \quad \int_0^\infty \int_\Omega \hat{u}^{(\tau)}\Delta^2\varphi + \mu \int_0^\infty \int_\Omega \hat{u}^{(\tau)}\Delta\varphi \\ &\quad +\lambda \int_0^\infty \int_\Omega |\nabla \hat{u}^{(\tau)}|^2 \Delta\varphi - \int_0^\infty \int_\Omega f\varphi\end{aligned}$$

for all $\varphi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$ fulfilling $\frac{\partial}{\partial\nu}\varphi = \frac{\partial}{\partial\nu}\Delta\varphi = 0$ on $\partial\Omega$. Here, in view of Lemma 3.2, Lemma 3.3 and (H1b) we may let $\tau = \tau_j \searrow 0$ separately in each integral to conclude that u indeed satisfies (2.1). Finally, from Corollary 2.10) we obtain $\int_\Omega u(\cdot,t) \ge 0$ for a.e. $t > 0$, whereas the statements involving $e^{\lambda u}$ are immediate consequences of (3.32). ////

4 Large time behavior of radial solutions

Without further preparations we can immediately proceed to the proof of our result on the existence of absorbing sets. In order to shorten notation, we shall throughout abbreviate the phrase 'radially symmetric with respect to $x = 0$' by simply saying 'radial'.

Proof (of Theorem 1.2). Our goal is to refine the first part of the proof of Lemma 3.1 and thereby derive an improved version of (3.8) under the present assumptions.
To this end, we first observe that since u_0 and f are radial, we may assume that the same holds for $u_0^{(\tau)}$, $\tau \in (0,1)$. Then restricting all the above considerations to the respective subclasses of radial functions, we clearly obtain that (2.2) admits at least one radial solution sequence $(u_k^{(\tau)})_{k\in\{1,2,\dots\}}$ whenever $\tau \in (0,1)$ is sufficiently small. Moreover, the radial symmetriy along with the boundary condition $\frac{\partial}{\partial\nu}u_k^{(\tau)} = 0$ on $\partial\Omega$ implies that even $\nabla u_k^{(\tau)} = 0$ on $\partial\Omega$ and hence $\frac{\partial}{\partial\nu}|\nabla u_k^{(\tau)}|^2$ vanishes on $\partial\Omega$. Thus, Lemma 2.2 ensures that $\int_\Omega u_k^{(\tau)} = 0$ for all $k \ge 1$, so that in particular

$$\text{for all } \tau \in (0,1) \text{ and } k \ge 1 \text{ there exists } x_k^{(\tau)} \in \bar{\Omega} \text{ such that } u_k^{(\tau)}\Big(x_k^{(\tau)}\Big) \le 0. \tag{4.1}$$

Now by Lemma 2.5, there exist $c_1 > 0$ and $c_2 > 0$ such that for small τ we have

$$\begin{aligned}\int_\Omega e^{2\lambda u_k^{(\tau)}} - \int_\Omega e^{2\lambda u_{k-1}^{(\tau)}} \quad &+ \quad c_1\tau \int_\Omega e^{2\lambda u_k^{(\tau)}} |D^2 u_k^{(\tau)}|^2 + c_1\tau \int_\Omega e^{2\lambda u_k^{(\tau)}} |\nabla u_k^{(\tau)}|^4 \\ &\le \quad c_2(\|f\|_{L^\infty(\Omega)} + \mu^2)\tau \cdot \int_\Omega e^{2\lambda u_k^{(\tau)}} \qquad \text{for all } k \ge 1.\end{aligned} \tag{4.2}$$

Here we use that as a consequence of the fact that $n \le 3$, Morrey's inequality implies that

$$|v(x) - v(y)| \le c_3 \Big(\int_\Omega |\nabla v|^4 \Big)^{\frac{1}{4}} \qquad \text{for all } x \in \bar\Omega \text{ and } y \in \bar\Omega,$$

is valid with some $c_3 > 0$ for each $v \in W^{1,4}(\Omega)$. We apply this to $v := e^{\frac{\lambda}{2} u_k^{(\tau)}}$ and $y := x_k^{(\tau)}$ to achieve the estimate

$$e^{\frac{\lambda}{2} u_k^{(\tau)}(x)} \le 1 + c_3 \cdot \frac{\lambda}{2} \cdot \Big(\int_\Omega e^{2\lambda u_k^{(\tau)}} |\nabla u_k^{(\tau)}|^4 \Big)^{\frac{1}{4}}$$

and therefore

$$\int_\Omega e^{2\lambda u_k^{(\tau)}} \le c_4 \Big(1 + \int_\Omega e^{2\lambda u_k^{(\tau)}} |\nabla u_k^{(\tau)}|^4 \Big)$$

with some $c_4 > 0$. Accordingly, (4.2) shows that there exist $c_5 > 0$ and $c_6 > 0$ such that if we require

$$c_2 (\|f\|_{L^\infty(\Omega)} + \mu^2) \cdot c_4 \le \frac{c_1}{4}, \tag{4.3}$$

then

$$\begin{aligned} \int_\Omega e^{2\lambda u_k^{(\tau)}} - \int_\Omega e^{2\lambda u_{k-1}^{(\tau)}} \quad &+ \quad c_1 \tau \int_\Omega e^{2\lambda u_k^{(\tau)}} |D^2 u_k^{(\tau)}|^2 + \frac{c_1}{2} \tau \int_\Omega e^{2\lambda u_k^{(\tau)}} |\nabla u_k^{(\tau)}|^4 \\ &\le \quad -c_5 \tau \int_\Omega e^{2\lambda u_k^{(\tau)}} + c_6 \tau \qquad \text{for all } k \ge 1. \end{aligned} \tag{4.4}$$

In particular, for $a_k := \int_\Omega e^{2\lambda u_k^{(\tau)}}$, $k \in \{0, 1, 2, ...\}$, instead of (3.8) we now have

$$a_k - a_{k-1} \le -c_5 \tau a_k + c_6 \tau \qquad \text{for all } k \ge 1,$$

whence by a straightforward induction

$$\begin{aligned} a_k \quad &\le \quad \frac{a_0}{(1+c_5\tau)^k} + c_6 \tau \cdot \sum_{j=1}^k \frac{1}{(1+c_5\tau)^j} \\ &= \quad \frac{a_0}{(1+c_5\tau)^k} + \frac{c_6}{c_5} \Big(1 - \frac{1}{(1+c_5\tau)^k} \Big) \qquad \text{for all } k \ge 1. \end{aligned}$$

Since $1 + z \ge e^{\frac{z}{2}}$ whenever $0 \le z \le 2 \ln 2$, this shows that if $\tau \le \frac{2 \ln 2}{c_5}$ then

$$\int_\Omega e^{2\lambda u_k^{(\tau)}} \le \Big(\sup_{\tau \in (0,1)} \int_\Omega e^{2\lambda u_0^{(\tau)}} \Big) \cdot e^{-\frac{c_5}{2} k\tau} + \frac{c_6}{c_5} \qquad \text{for all } k \ge 1. \tag{4.5}$$

With $(\tau_j)_{j \in \mathbb{N}}$ as provided by Theorem 1.1, we let $\tau = \tau_j \searrow 0$ in (4.5) and thereby easily verify (1.5) for any $C \ge 2\frac{c_6}{c_5}$ and sufficiently large T.
In order to obtain (1.6), we let $t > T$ be given and fix $k_2 > k_1 > 1$ such that

$$T \le k_1 \tau \le t < (k_1 + 1)\tau < (k_2 - 1)\tau < t + 1 \le k_2 \tau,$$

which is possible whenever $\tau > 0$ is sufficiently small. Proceeding similarly to the proof of Lemma 3.1, we sum up in (4.4) to find

$$\begin{aligned}\int_\Omega e^{2\lambda u_{k_2}^{(\tau)}} &+ c_1\tau \cdot \sum_{j=k_1}^{k_2} \int_\Omega e^{2\lambda u_j^{(\tau)}} |D^2 u_j^{(\tau)}|^2 + \frac{c_1}{2}\tau \cdot \sum_{j=k_1}^{k_2} \int_\Omega e^{2\lambda u_j^{(\tau)}} |\nabla u_j^{(\tau)}|^4 \\ &\le \int_\Omega e^{2\lambda u_{k_1}^{(\tau)}} + c_6\tau \cdot (k_2 - k_1).\end{aligned}$$

In view of (4.5) and the fact that $(k_2 - k_1)\tau < 1 + 2\tau$, upon taking $\tau \searrow 0$ along a suitable sequence this easily leads to (1.6) if we adequately enlarge C.

Now the convergence statements (1.7) and (1.8) are straightforward consequences of (1.5) and (1.6): Given a sequence of numbers $t_j \to \infty$, for $j \in \mathbb{N}$ we let

$$w_j(x) := \int_{t_j}^{t_j+1} e^{\lambda u(x,t)} dt, \qquad x \in \Omega.$$

Then by the Hölder inequality we have

$$\int_\Omega w_j^2 \le \int_\Omega \Big(\int_{t_j}^{t_j+1} e^{2\lambda u(x,t)} dt \Big) dx \le C \qquad \text{for all } j \in \mathbb{N}$$

due to (1.5). Moreover, using the identity (3.23) above and (1.6) we find

$$\int_\Omega |D^2 w_j|^2 \le c_7 \int_{t_j}^{t_j+1} \int_\Omega e^{2\lambda u} \Big(|D^2 u|^2 + |\nabla u|^4 \Big) \le c_7 C \qquad \text{for all } j \in \mathbb{N}.$$

This shows that $(w_j)_{j\in\mathbb{N}}$ is bounded in $W^{2,2}(\Omega)$ and hence we have (1.7) along a suitable subsequence with some $w \in W_N^{2,2}(\Omega)$. Since $W^{2,2}(\Omega)$ is compactly embedded into $C^0(\bar\Omega)$ again because of the fact that $n \le 3$, this at the same time entails (1.8). ////

5 Appendix

5.1 One-sided uniform estimates implied by second-order elliptic inequalities

The following lemma basically is a variant of [12, Theorem 8.15]. Since we could not find a precise reference but essentially need the statement in its form presented here, we include a short proof for the reader's convenience.

Lemma 5.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $a_0 \ge 0$ and $q > \max\{1, \frac{n}{2}\}$. Then there exists a constant $c > 0$ such that if $g \in L^q(\Omega)$ and $w \in C^2(\bar\Omega)$ satisfy*

$$\begin{cases} -\Delta w \le aw + g, & x \in \Omega, \\ \frac{\partial}{\partial\nu} w \le 0, & x \in \partial\Omega \end{cases} \tag{5.1}$$

for some $a \in [0, a_0]$, *then*

$$w(x) \le c \cdot \Big(\|w_+\|_{L^1(\Omega)} + \|g_+\|_{L^q(\Omega)} \Big). \tag{5.2}$$

PROOF. Since $q > 1$, the operator $-\Delta + 1$ acts as a homeomorphism from $W^{2,q}_N(\Omega) = \{z \in W^{2,q}(\Omega) \mid \frac{\partial}{\partial\nu} z = 0 \text{ on } \partial\Omega\}$ onto $L^q(\Omega)$ and hence there exists $c_1 > 0$ such that

$$\|(-\Delta + 1)^{-1} g\|_{W^{2,q}(\Omega)} \le c_1 \|g\|_{L^q(\Omega)} \qquad \text{for all } g \in L^q(\Omega). \tag{5.3}$$

Moreover, the additional assumption $q > \frac{n}{2}$ guarantees that $W^{2,q}(\Omega) \hookrightarrow L^\infty(\Omega)$, so that

$$\|z\|_{L^\infty(\Omega)} \le c_2 \|z\|_{W^{2,q}(\Omega)} \qquad \text{for all } z \in W^{2,q}(\Omega) \tag{5.4}$$

is valid with some $c_2 > 0$. Now (5.1) implies that

$$-\Delta w + w \le (a+1) w + g \le (a_0 + 1) w_+ + g_+ \qquad \text{in } \Omega. \tag{5.5}$$

Thus, if we let $z \in W^{2,q}(\Omega)$ denote the solution of

$$\begin{cases} -\Delta z + z = (a_0 + 1) w_+ + g_+, & x \in \Omega, \\ \frac{\partial}{\partial\nu} z = 0, & x \in \partial\Omega, \end{cases} \tag{5.6}$$

then

$$\|z\|_{W^{2,q}(\Omega)} \le c_1 \Big((a_0 + 1) \|w_+\|_{L^q(\Omega)} + \|g_+\|_{L^q(\Omega)} \Big)$$

by (5.3). Since $-\Delta + 1$ under Neumann boundary conditions allows for a comparison principle, (5.5) and (5.6) entail that $w \le z$ in Ω, whence we conclude that

$$\begin{aligned} \|w_+\|_{L^\infty(\Omega)} &\le \|z\|_{L^\infty(\Omega)} \le c_2 \|z\|_{W^{2,q}(\Omega)} \\ &\le c_1 c_2 \Big((a_0 + 1) \|w_+\|_{L^q(\Omega)} + \|g_+\|_{L^q(\Omega)} \Big). \end{aligned}$$

Interpolating $\|w_+\|_{L^q(\Omega)} \le \eta \|w_+\|_{L^\infty(\Omega)} + c_\eta \|w_+\|_{L^1(\Omega)}$ with suitable $\eta > 0$ and $c_\eta > 0$ independent of w, we easily arrive at (5.2). ////

5.2 Estimating the distance between continuous and step-type Rothe functions

In this section we shall establish a smallness statement on the difference between the Rothe functions $u^{(\tau)}$ and $\hat{u}^{(\tau)}$ from above. Extending this to a slightly more general setting, we suppose that $u^{(\tau)}_k, k = 0, 1, 2, ...,$ are given elements from a Banach space Y, and correspondingly we set

$$u^{(\tau)}(t) := \Big(k - \frac{t}{\tau} \Big) \cdot u^{(\tau)}_{k-1} + \Big(\frac{t}{\tau} - (k-1) \Big) \cdot u^{(\tau)}_k \tag{5.7}$$

and

$$\hat{u}^{(\tau)}(t) := u_k^{(\tau)} \tag{5.8}$$

for $t \in [(k-1)\tau, k\tau)$ and $k \in \{1, 2, ...\}$. Then the following lemma says that relative compactness of the $u^{(\tau)}$ in $L^q((0,T);Y)$ for some $q \in [1,\infty)$ (where $\tau \searrow 0$ along some sequence) is essentially sufficient to guarantee that $\hat{u}^{(\tau)}$ and $u^{(\tau)}$ will not differ too much in the limit $\tau = \tau_j \searrow 0$.

Lemma 5.2 *Let Y be a Banach space and $(\tau_j)_{j\in\mathbb{N}}$ be such that $\tau_j \searrow 0$ as $j \to \infty$. Suppose that for each $\tau \in (\tau_j)_{j\in\mathbb{N}}$ we are given $u_k^{(\tau)} \in Y,\ k \in \{0, 1, 2, ...\}$, and let $u^{(\tau)} : [0,\infty) \to Y$ and $\hat{u}^{(\tau)} : [0,\infty) \to Y$ be defined by (5.7) and (5.8). Then if $(u^{(\tau)})_{\tau\in(\tau_j)_{j\in\mathbb{N}}}$ is relatively compact in $L^q((0,T);Y)$ for some $q \in [1,\infty)$ and $T > 0$, and if*

$$\tau \|u_0^{(\tau)}\|_Y^q \to 0 \qquad \textit{as } \tau = \tau_j \searrow 0, \tag{5.9}$$

we have

$$\|u^{(\tau)} - \hat{u}^{(\tau)}\|_{L^q((0,T');Y)} \to 0 \qquad \textit{as } \tau = \tau_j \searrow 0 \tag{5.10}$$

for all $T' \in (0,T)$.

Proof. Given $T' < T$, for all sufficiently small $\tau > 0$ we can pick $N_\tau \in \mathbb{N}$ such that $T' < N_\tau \tau \le T$. Thus, for $\tau = \tau_j$ with suitably large $j \in \mathbb{N}$, using (2.3) and (2.4) we have

$$\begin{aligned}
\|u^{(\tau)} - \hat{u}^{(\tau)}\|^q_{L^q((0,T');Y)} &= \int_0^{T'} \|u^{(\tau)}(t) - \hat{u}^{(\tau)}(t)\|_Y^q dt \\
&\le \int_0^{N_\tau\cdot\tau} \|u^{(\tau)}(t) - \hat{u}^{(\tau)}(t)\|_Y^q dt \\
&= \sum_{k=1}^{N_\tau} \int_{(k-1)\tau}^{k\tau} \Big\| \frac{k\tau - t}{\tau} (u_k^{(\tau)} - u_{k-1}^{(\tau)}) \Big\|_Y^q dt \\
&= \sum_{k=1}^{N_\tau} \bigg(\int_{(k-1)\tau}^{k\tau} \Big(\frac{k\tau - t}{\tau}\Big)^q dt \bigg) \cdot \|u_k^{(\tau)} - u_{k-1}^{(\tau)}\|_Y^q \\
&= \frac{\tau}{q+1} \cdot \sum_{k=1}^{N_\tau} \|u_k^{(\tau)} - u_{k-1}^{(\tau)}\|_Y^q.
\end{aligned} \tag{5.11}$$

In order to show that the right-hand side in (5.11) tends to zero as $\tau = \tau_j \searrow 0$, let us fix $c_1 > 0$ large enough such that $(1-z)^q \ge \frac{1}{2} - c_1 z^q$ for all $z \in [0,1]$. It can then easily be checked that $\|x+y\|_Y^q \ge \frac{1}{2}\|x\|_Y^q - c_1\|y\|_Y^q$ for all $x \in Y$ and $y \in Y$. Taking now $\delta \in (0,1)$ so small that

$$c_2 := \frac{\delta(1-\delta)^q}{2} - c_1 \delta^{q+1} > 0,$$

we can estimate

$$\int_0^T \|u^{(\tau)}(t+\tau) - u^{(\tau)}(t)\|_Y^q dt$$

$$
\begin{aligned}
&\geq \int_0^{N_\tau\cdot\tau} \|u^{(\tau)}(t+\tau)-u^{(\tau)}(t)\|_Y^q dt \\
&= \sum_{k=1}^{N_\tau}\int_{(k-1)\tau}^{k\tau} \Big\|\frac{t-(k-1)\tau}{\tau}(u_{k+1}^{(\tau)}-u_k^{(\tau)})+\frac{k\tau-t}{\tau}(u_k^{(\tau)}-u_{k-1}^{(\tau)})\Big\|_Y^q \\
&\geq \sum_{k=1}^{N_\tau}\int_{(k-\delta)\tau}^{k\tau} \Big\|\frac{t-(k-1)\tau}{\tau}(u_{k+1}^{(\tau)}-u_k^{(\tau)})+\frac{k\tau-t}{\tau}(u_k^{(\tau)}-u_{k-1}^{(\tau)})\Big\|_Y^q \\
&\geq \sum_{k=1}^{N_\tau}\int_{(k-\delta)\tau}^{k\tau} \Big\{\frac{1}{2}\Big(\frac{t-(k-1)\tau}{\tau}\Big)^q\cdot\|u_{k+1}^{(\tau)}-u_k^{(\tau)}\|_Y^q \\
&\quad -c_1\cdot\Big(\frac{k\tau-t}{\tau}\Big)^q\cdot\|u_k^{(\tau)}-u_{k-1}^{(\tau)}\|_Y^q\Big\}dt \\
&\geq \sum_{k=1}^{N_\tau}\Big\{\delta\tau\cdot\frac{1}{2}\cdot\Big(\frac{(k-\delta)\tau-(k-1)\tau}{\tau}\Big)^q\cdot\|u_{k+1}^{(\tau)}-u_k^{(\tau)}\|_Y^q \\
&\quad -\delta\tau\cdot c_1\cdot\Big(\frac{k\tau-(k-\delta)\tau}{\tau}\Big)^q\cdot\|u_k^{(\tau)}-u_{k-1}^{(\tau)}\|_Y^q\Big\} \\
&= \sum_{k=2}^{N_\tau}\Big\{\frac{\delta(1-\delta)^q\tau}{2}-c_1\delta^{q+1}\tau\Big\}\cdot\|u_k^{(\tau)}-u_{k-1}^{(\tau)}\|_Y^q \\
&\quad +\frac{\delta(1-\delta)^q\tau}{2}\cdot\|u_{N_\tau+1}^{(\tau)}-u_{N_\tau}^{(\tau)}\|_Y^q-c_1\delta^{q+1}\tau\cdot\|u_1^{(\tau)}-u_0^{(\tau)}\|_Y^q \\
&\geq c_2\tau\cdot\sum_{k=1}^{N_\tau}\|u_k^{(\tau)}-u_{k-1}^{(\tau)}\|_Y^q-(c_1\delta^{q+1}+c_2)\tau\cdot\|u_1^{(\tau)}-u_0^{(\tau)}\|_Y^q. \qquad (5.12)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\int_0^\tau \|u^{(\tau)}(t)\|_Y^q dt &= \int_0^\tau \Big\|\frac{t}{\tau}(u_1^{(\tau)}-u_0^{(\tau)})+u_0^{(\tau)}\Big\|_Y^q dt \\
&\geq \int_0^\tau \Big\{\frac{1}{2}\Big(\frac{t}{\tau}\Big)^q\cdot\|u_1^{(\tau)}-u_0^{(\tau)}\|_Y^q-c_1\|u_0^{(\tau)}\|_Y^q\Big\}dt \\
&= \frac{\tau}{2(q+1)}\|u_1^{(\tau)}-u_0^{(\tau)}\|_Y^q-c_1\tau\|u_0^{(\tau)}\|_Y^q. \qquad (5.13)
\end{aligned}
$$

Combining (5.12) with (5.13), we obtain

$$
\begin{aligned}
c_2\tau\cdot\sum_{k=1}^{N_\tau}\|u_k^{(\tau)}-u_{k-1}^{(\tau)}\|_Y^q &\leq \int_0^T \|u^{(\tau)}(t+\tau)-u^{(\tau)}(t)\|_Y^q \\
&\quad +c_3\cdot\Big(\int_0^\tau \|u^{(\tau)}(t)\|_Y^q dt+\tau\|u_0^{(\tau)}\|_Y^q\Big) \qquad (5.14)
\end{aligned}
$$

for some $c_3 > 0$ independent of τ. Since $\tau\|u_0^{(\tau)}\|_Y^q \to 0$ as $\tau = \tau_j \searrow 0$ by (5.9), and since $(u^{(\tau)})_{\tau\in(\tau_j)_{j\in\mathbb{N}}}$ is relatively compact in $L^q((0,T);Y)$, it follows from the Kolmogorov compactness criterion that the right-hand side of (5.14) tends to zero as $\tau = \tau_j \searrow 0$. In conjunction with (5.11) this proves the lemma. ////

6 Conclusion

We have considered the higher-dimensional version of a model for surface evolution in presence of molecular beam epitaxy. As compared to the spatially one-dimensional analogue that has been studied in the existing literature, this model appears to be more realistic, but even the basic problem of well-posedness brings about a number of mathematical obstacles which can not be overcome by methods used in previous work. We have thus pursued a novel approach that essentially relied on a combination of (1.9) and (1.10) with the apparently new identity (1.11). A careful analysis has shown that these ingredients, originally gained in a purely formal manner, indeed are valid in a certain weakened sense, and that they can be used to establish the existence of a global generalized solution of (1.2) through a numerically usable approximation scheme under mild conditions on the initial data.

As an example of how the above tools can be applied to address further questions, we have shown a stabilization result for solutions of (1.2) under convenient additional assumptions, essentially requiring radial symmetry.

Of course, this is to be understood only as a first step towards a complete understanding of the global dynamical properties of (1.2), but we believe that the approach presented here might offer some new possibilities of how (1.2) can be analyzed rigorously. It is likely to be expected, for instance, that appropriately localized variants of our estimates can be used to describe local properties of solutions to a satisfactory extent.

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References

[1] H.-W. Alt and s. Luckhaus, Quasilinear Elliptic-Parabolic Differential Equations, Math. Z. **183** (1983), 311–341.

[2] F. Bernis, A. Friedman, Higher order nonlinear degenerate parabolic equations, J. Differ. Equations **83**, no. 1 (1990), 179–206.

[3] D. Blömker and C. Gugg, On the existence of solutions for amorphous molecular beam epitaxy, Nonlin. Anal.: Real World Appl. **3** (2002), 61–73.

[4] D. Blömker, D., C. Gugg, C., and M. Raible, Thin-film-growth models: Roughness and correlation functions, Eur. J. Appl. Math. **13**, 4 (2002), 385–402.

[5] D. Blömker and M. Hairer, Stationary solutions for a model of amorphous thin-film growth, Stochastic Anal. Appl. **22**, 4 (2004), 903–922.

[6] D. Blömker and M. Romito, Regularity and blow-up in a surface growth model, Dyn. Partial Differ. Equ. **6**, 2 (2009), 227–252.

[7] R. Dal Passo, H. Garcke, and G. Grün, On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions, SIAM J. Math. Anal. **29**, 2 (1998), 321–342.

[8] S.F. Edwards and D.R. Wilkinson, The surface statistics of a granular aggregate, Proceedings of the Royal Society of London A **381** (1982), 17–31.

[9] C. Foias, B. Nicolaenko, G.R. Sell, and R. Temam, Inertial manifolds for the Kuramoto-Sivashinsky equation and an estimate of their lowest dimension, J. Math. Pures Appl. **67**, no. 3 (1988), 197–226.

[10] A. Friedman, Partial Differential Equations, Holt, Rinehart & Winston, New York, 1969.

[11] U. Gianazza, G. Savaré, and G. Toscani, The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation, Arch. Ration. Mech. Anal. **194**, no. 1 (2009), 133–220.

[12] D. Gilbarg and N.S. Trudinger, Elliptic partial differential equations of second order, Springer-Verlag, Berlin, 1998.

[13] C. Herring, Surface tension as a motivation for sintering, in: W.E. Kingston (ed.) *The Physics of Powder Metallurgy.* New York. McGraw-Hill, 1951.

[14] A. Jüngel and D. Matthes, The Derrida-Lebowitz-Speer-Spohn equation: existence, nonuniqueness, and decay rates of the solutions, SIAM J. Math. Anal. **39**, no. 6 (2008), 1996–2015.

[15] M. Kardar, G. Parisi, G., and Y.-C. Zhang, Dynamic Scaling of Growing Interfaces, Phys. Rev. Lett. **56** (1986), 889–892.

[16] B.B. King, O. Stein, and M. Winkler, A fourth order parabolic equation modeling epitaxial thin film growth, J. Math. Anal. Appl. **286** (2003), 459–490.

[17] S.G. Mayr, M. Moske, and K. Samwer, Early stages in amorphous $Zr_{65}Al_{7.5}Cu_{27.5}$ film growth on HOPG, Europhys. Lett. **44** (1998), 465–470.

[18] W.W. Mullins, Theory of thermal grooving, Journal of Applied Physics **28** /1957), 333.

[19] B. Nicolaenko, B. Scheurer, and R. Temam, Some global dynamical properties of the Kuramoto-Sivashinsky equations: Nonlinear stability and attractors, Physica **16D** /1985), 155–183.

[20] A. Novick-Cohen, On Cahn-Hilliard type equations, Nonlinear Anal., Theory Methods Appl. **15**, 9 (1990), 797–814.

[21] M. Ortiz, E.A. Repetto, and H. Si, A continuum model of kinetic roughening and coarsening in thin films, J. Mech. Phys. Solids **47** (1999), 697–730.

[22] F. Otto, Optimal bounds on the Kuramoto-Sivashinsky equation, J. Functional Analysis **257** /2009), 2188–2245.

[23] B. Reinker, S.G. Mayr, S.J. Linz, M. Moske, P. Hänggi, and K. Samwer, Amorphous thin film growth: Theory compared with experiment, Europhys. Lett. **50** (2000), 61–67.

[24] N. Siegert and M. Plischke, Solid-on-solid models of molecular-beam epitaxy, Phys. Rev. **E50** (1994), 917–931.

[25] O. Stein and M. Winkler, Amorphous molecular beam epitaxy: global solutions and absorbing sets, Eur. J. Appl. Math. **16** (2005), 767–798.

[26] R. Temam, Navier-Stokes equations. Theory and numerical analysis, Studies in Mathematics and its Applications. Vol. 2. North-Holland, Amsterdam, 1977.

[27] E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems, Springer-Verlag, New York, 1986.