Eventual smoothness and stabilization of large-data solutions in a three-dimensional chemotaxis system with consumption of chemoattractant

Youshan Tao*
Department of Applied Mathematics, Dong Hua University,
Shanghai 200051, P.R. China

Michael Winkler#
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

This paper deals with positive solutions of

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega, \ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in bounded convex domains $\Omega \subset \mathbb{R}^3$ with smooth boundary.
It is shown that for arbitrarily large initial data, this problem admits at least one global weak solution for which there exists $T > 0$ such that (u, v) is bounded and smooth in $\Omega \times (T, \infty)$. Moreover, it is asserted that such solutions approach spatially constant equilibria in the large time limit.

Key words: chemotaxis, global existence, a priori estimates, eventual regularity
AMS Classification: 35A01, 35B40, 35B65, 35K57, 35Q92, 92C17

*taoys@dhu.edu.cn
#michael.winkler@math.uni-paderborn.de

1 Introduction

Processes of directed movement of cells in response to a chemical signal, also referred to as chemotaxis, play an important role in the interaction of cells with their environment, and accordingly there appears to be a growing interest in their theoretical understanding ([15]). Among the possibly most striking implications of such a behavior is the spontaneous formation of aggregates like in *Dictyostelium discoideum*, for instance, and considerable efforts have been made to describe such mechanisms of self-organization mathematically ([11]). Since the origin of the fundamental model introduced by Keller and Segel ([9]), a rich literature on various versions thereof has revealed that its constitutive ingredient of *cross-diffusion* is indeed able to enforce the spontaneous emergence of structures even in the most extreme conceivable mathematical form of blow-up of solutions – provided that the process of cross-diffusive migration is accompanied by a *production* of the signal substance by the cells themselves ([10], [12]).

In the present work we deal with a typical chemotaxis process where the signal is *degraded*, rather than produced, by the cells. More precisely, we consider a population of bacteria which consume oxygen, and study the model

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), \qquad x\in\Omega,\ t>0, \\ v_t = \Delta v - uv, \qquad x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, \qquad x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \qquad x\in\Omega, \end{cases} \tag{1.1}$$

for its spatio-temporal evolution, where $u=u(x,t)$ denotes the cell density and $v=v(x,t)$ represents the oxygen concentration. The cross-diffusive term in the first equation reflects the assumption that individual cells at least partially adapt their motion so as to prefer to migrate toward increasing oxygen concentrations. The second equation in (1.1) accounts for the hypothesis that oxygen is degraded upon contact with bacteria at a fixed rate, and that there is no additional production of oxygen (see [16] for motivation and discussion of a closely related model).

The problem is posed in a bounded domain $\Omega\subset\mathbb{R}^N$ with smooth boundary, where our main focus will be on the physically most relevant case $N=3$, and throughout we shall assume that the initial data u_0 and v_0 satisfy

$$\begin{cases} u_0\in C^0(\bar\Omega), \quad u_0>0 \quad \text{in } \bar\Omega, \\ v_0\in W^{1,\infty}(\Omega), \quad v_0>0 \quad \text{in } \bar\Omega. \end{cases} \tag{1.2}$$

Our main interest is in the question whether this interaction of chemotactic cross-diffusion and signal consumption may support a singular behavior of solutions, as it is the case when the signal is produced by cells.

As a first step toward an answer, it has been shown as part of the results in [21, Theorem 1.1 ii)] that under the above assumptions when $N=2$ or $N=3$, (1.1) possesses a globally defined weak solution (see Definition 2.1 below for a precise formulation of the underlying solution concept). Independently, in [17] it has been proved that if in addition $\|v_0\|_{L^\infty(\Omega)}$ is sufficiently small, then (1.1) even admits a global classical solution which is bounded and smooth for $t>0$.

A natural question connected to the latter two results is whether or not global weak solutions of (1.1) emanating from *large* initial data are bounded and smooth, and if singularities, possibly arising after some finite time, persist or disappear again. Our main result in this direction states that the above weak solutions at least *eventually* become bounded and smooth, and that they approach the unique relevant constant steady state in the large time limit. To be more precise:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded convex domain with smooth boundary, and assume that u_0 and v_0 satisfy (1.2). Then in the sense of Definition 2.1 below, (1.1) possesses a global weak solution. Moreover, there exists $T > 0$ such that this solution is bounded and belongs to $C^{2,1}(\bar{\Omega} \times [T, \infty))$, and we have*

$$u(x,t) \to \overline{u}_0 \quad \text{and} \quad v(x,t) \to 0 \qquad \text{as } t \to \infty, \tag{1.3}$$

uniformly with respect to $x \in \Omega$, where

$$\overline{u}_0 := \frac{1}{|\Omega|} \int_\Omega u_0. \tag{1.4}$$

In addressing the large time behavior of solutions only, Theorem 1.1 does not exclude the possibility of blow-up of a solution in finite time, but it shows that a supposedly occurring explosion of any of our solutions is a temporally restricted phenomenon only, which is followed by a smooth stabilization toward a flat equilibrium. It is an interesting open task to either rule out or prove the existence of such extensible and eventually smooth blow-up solutions.

As a consequence of our analysis when applied to the spatially two-dimensional version of (1.1), we shall easily obtain the following by-product concerning the case $N = 2$. It has been shown in [21, Theorem 1.1 i)] that in any bounded convex domain $\Omega \subset \mathbb{R}^2$ and under the assumption (1.2), (1.1) possesses a unique global classical solution. Here we can further assert that this two-dimensional solution is globally bounded and satisfies (1.3).

Proposition 1.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and assume that u_0 and v_0 satisfy (1.2). Then the unique global classical solution of (1.1) is bounded. Moreover, this global bounded solution enjoys the convergence properties in (1.3).*

Before going into details, let us mention that (1.1) can be regarded as the 'fluid-free' version of the coupled chemotaxis-fluid model

$$\begin{cases} u_t + V \cdot \nabla u = \Delta u - \nabla \cdot (u\chi(v)\nabla v), \qquad x \in \Omega,\ t > 0, \\ v_t + V \cdot \nabla v = \Delta v - uf(v), \qquad x \in \Omega,\ t > 0, \\ V_t + V \cdot \nabla V + \nabla P - \eta \Delta V + u \nabla \phi = 0, \qquad x \in \Omega,\ t > 0, \\ \nabla \cdot V = 0, \qquad x \in \Omega,\ t > 0, \end{cases} \tag{1.5}$$

which was initially proposed by Goldstein et al. ([19]) to describe the motion of oxygen-driven swimming bacteria in an incompressible fluid. Here, u and v are defined as before, and V represents the velocity field of the fluid subject to an incompressible Navier-Stokes equation with pressure P and viscosity η and a gravitational force $\nabla\phi$. The function $\chi(v)$ measures the chemotactic sensitivity, $f(v)$ is the consumption rate of the oxygen by the bacteria, and ϕ is a given potential function. In (1.5), both bacteria and oxygen are transported with the fluid. Evidently, our model (1.1) can be obtained upon the choices $V \equiv 0, \chi \equiv 1$ and $f(v) = v$ in (1.5).
As for (1.5), recent contributions assert global classical solutions near constant steady states when $\Omega = \mathbb{R}^3$ ([5]), global weak solutions with arbitrarily large data in $\Omega = \mathbb{R}^2$ ([13]), or global classical

solutions in bounded convex $\Omega \subset \mathbb{R}^2$ ([21]). As far as we know, the global existence or blow-up of solutions to (1.5) with arbitrarily large initial data remains an open and challenging topic in the three-dimensional case.

The paper is organized as follows. In Section 2, we give a definition of a global weak solution of (1.1), recall the approximation procedure (2.2) used in ([21]) to construct such solutions, and present some preliminary observations. The starting point toward the proof of Theorem 1.1 is a natural energy inequality (cf. [21] and [5]), from which we can infer some fundamental estimates for the solutions $(u_\varepsilon, v_\varepsilon)$ of the regularized problem (2.2) in Section 3. We then provide a weak stabilization result for u in Section 4 before establishing uniform decay of v in Section 5. With this information on asymptotic smallness of v, we can further assert eventual boundedness and regularity of u in Section 6. This assertion strongly depends on a uniform $L^p(\Omega)$ bound for $u_\varepsilon(\cdot,t)$, and the proof for the latter involves a delicate choice of a suitable weight function $\varphi(v_\varepsilon)$. In Section 7, we obtain the desired stabilization result for u and thereby will be able to complete the proof of Theorem 1.1. Finally, in Section 8 we present a short proof of global boundedness in the case $N = 2$, which is based on the two-dimensional version of the above energy inequality. Unlike in the case $N = 3$, we can first establish the stabilization property for u and thereby will be able to give a simple proof of the claimed stabilization property for v in the case $N = 2$.

2 Preliminaries

The following concept of weak solutions appears to be natural in the present setting.

Definition 2.1 *By a* global weak solution *of (1.1) we mean a pair (u, v) of functions*

$$u \in L^1_{loc}([0,\infty); L^1(\Omega)), \qquad v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)),$$

such that

$$uv \ \text{and}\ u\nabla v \quad \text{belong to}\ L^1_{loc}([0,\infty); L^1(\Omega)),$$

and such that the identities

$$\begin{aligned} -\int_0^\infty\int_\Omega u\zeta_t - \int_\Omega u_0\zeta(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\zeta + \int_0^\infty\int_\Omega u\nabla v\cdot\nabla\zeta \qquad \text{and} \\ -\int_0^\infty\int_\Omega v\zeta_t - \int_\Omega v_0\zeta(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla v\cdot\nabla\zeta - \int_0^\infty\int_\Omega uv\zeta \end{aligned} \tag{2.1}$$

hold for all $\zeta \in C_0^\infty(\bar\Omega\times[0,\infty))$.

Throughout the rest of the paper, unless otherwise specified we assume that Ω is a bounded convex domain in $\mathbb{R}^3$ with smooth boundary.

As seen in [21], a global weak solution in the above sense can be obtained as the limit of a sequence of solutions $(u_\varepsilon, v_\varepsilon)$, $\varepsilon = \varepsilon_j \in (0,1)$, of the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (u_\varepsilon F'_\varepsilon(u_\varepsilon)\nabla v_\varepsilon), \qquad x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - F_\varepsilon(u_\varepsilon)v_\varepsilon, \qquad x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), \qquad x \in \Omega, \end{cases} \tag{2.2}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Here,

$$F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1+\varepsilon s), \qquad s \ge 0, \tag{2.3}$$

for $\varepsilon \in (0,1)$.

More precisely, the following statement on global existence of solutions to (2.2) and their limit properties is proved in [21, Lemma 5.4, Lemma 2.1, Theorem 1.1 ii)].

Lemma 2.1 *Let (1.2) hold, and let F_ε be defined by (2.3). Then for each $\varepsilon \in (0,1)$, the problem (2.2) possesses a global classical solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon > 0$ and $v_\varepsilon > 0$ in $\Omega \times [0,\infty)$. Moreover, there exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ of numbers $\varepsilon_j \searrow 0$ such that*

$$u_\varepsilon \to u \quad \textit{and} \quad v_\varepsilon \to v \qquad \textit{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \textit{ and a.e. in } \Omega \times (0,\infty) \qquad \textit{as } \varepsilon = \varepsilon_j \searrow 0 \tag{2.4}$$

for some couple (u,v) of nonnegative functions which form a global weak solution of (1.1) in the sense of Definition 2.1.

The following mass conservation property is easily checked but important.

Lemma 2.2 *For all $\varepsilon \in (0,1)$, the solution of (2.2) has the property*

$$\frac{1}{|\Omega|}\int_\Omega u_\varepsilon(\cdot,t) = \overline{u}_0 \qquad \textit{for all } t \ge 0 \tag{2.5}$$

with $\overline{u}_0$ defined by (1.4).

Proof. This immediately follows upon integrating the first equation in (2.2) over $\Omega \times (0,t)$. □

The next feature of the second solution component will also play an important role in the sequel.

Lemma 2.3 *Let $\varepsilon \in (0,1)$. Then for the solution of (2.2),*

$$t \mapsto \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \qquad \textit{is nonincreasing in } [0,\infty).$$

Proof. Since $v_{\varepsilon t} \le \Delta v_\varepsilon$ due to the fact that F_ε and v_ε are nonnegative, the claim results upon an application of the maximum principle. □

A first – yet rather weak – indication for time decay of v is contained in the following.

Lemma 2.4 *For all $\varepsilon \in (0,1)$ we have*

$$\int_0^\infty \int_\Omega F_\varepsilon(u_\varepsilon) v_\varepsilon \le \int_\Omega v_0. \tag{2.6}$$

In particular, the limit couple (u,v) defined through (2.4) fulfils

$$\int_0^\infty \int_\Omega uv \le \int_\Omega v_0. \tag{2.7}$$

Proof. An integration of the second equation in (2.2) yields

$$\int_\Omega v_\varepsilon(\cdot,t) + \int_0^t \int_\Omega F_\varepsilon(u_\varepsilon) v_\varepsilon = \int_\Omega v_0 \qquad \text{for all } t>0.$$

Since $v_\varepsilon \ge 0$, this entails (2.6), whereas (2.7) results from (2.6) on an application of Fatou's lemma, because (2.4) and (2.3) assert that $F_\varepsilon(u_\varepsilon) \to u$ and $v_\varepsilon \to v$ a.e. in $\Omega \times (0,\infty)$. □

3 An energy inequality

In order to proceed further, let us recall from [21] a natural energy inequality associated with (1.1) and (2.2) (cf. also [5]).

Lemma 3.1 *For each $\varepsilon \in (0,1)$, the solution of (2.2) satisfies*

$$\frac{d}{dt}\Big\{ \int_\Omega u_\varepsilon \ln u_\varepsilon + 2\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2 \Big\} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \le 0 \tag{3.1}$$

for all $t>0$.

Proof. By straightforward computation (cf. [21, Lemma 3.2] for details), one verifies the identity

$$\begin{aligned}\frac{d}{dt}\Big\{ \int_\Omega u_\varepsilon \ln u_\varepsilon + 2\int_\Omega |\nabla\sqrt{v_\varepsilon}|^2 \Big\} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \ &+\ \frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\ &=\ \frac{1}{2}\int_{\partial\Omega} \frac{1}{v_\varepsilon}\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu}\end{aligned}$$

for $t>0$. Since the convexity of $\partial\Omega$ in conjunction with the boundary condition $\frac{\partial v_\varepsilon}{\partial\nu}=0$ on $\partial\Omega$ implies that $\frac{\partial|\nabla v_\varepsilon|^2}{\partial\nu} \le 0$ on $\partial\Omega$ ([4]), this immediately yields (3.1). □

We next collect some consequences of the above energy inequality which are convenient for our purpose.

Corollary 3.2 *There exists $C>0$ such that for all $\varepsilon \in (0,1)$ the solution of (2.2) satisfies*

$$\int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \le C, \tag{3.2}$$

$$\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 \le C \qquad \textit{for all } t>0, \tag{3.3}$$

$$\int_0^\infty \int_\Omega |D^2 v_\varepsilon|^2 \le C, \tag{3.4}$$

$$\int_0^\infty \int_\Omega |\nabla v_\varepsilon|^4 \le C \qquad \textit{and} \tag{3.5}$$

$$\int_0^\infty \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 \le C. \tag{3.6}$$

Proof. Integrating (3.1) over $t \in (0,\infty)$ we obtain

$$\begin{aligned} 2\int_\Omega |\nabla\sqrt{v_\varepsilon}(\cdot,t)|^2 \ + \ \int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_0^t \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2}\int_0^t \int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\ \le \ \int_\Omega u_0 \ln u_0 + 2\int_\Omega |\nabla\sqrt{v_0}|^2 - \int_\Omega u_\varepsilon(\cdot,t)\ln u_\varepsilon(\cdot,t) \end{aligned}$$

for all $t>0$ and $\varepsilon \in (0,1)$. Since $-\xi \ln \xi \le \frac{1}{e}$ for all $\xi>0$, and since $|\nabla\sqrt{v_\varepsilon}|^2 = \frac{|\nabla v_\varepsilon|^2}{4v_\varepsilon}$, this shows that

$$\begin{aligned} \frac{1}{2}\cdot \sup_{t>0}\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \ + \ \int_0^\infty \int_\Omega \frac{\nabla u_\varepsilon|^2}{u_\varepsilon} + \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 + \frac{1}{2}\int_0^\infty \int_\Omega F_\varepsilon(u_\varepsilon)\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon} \\ \le \ c_1 := \int_\Omega u_0 \ln u_0 + 2\int_\Omega |\nabla\sqrt{v_0}|^2 + \frac{|\Omega|}{e} \end{aligned} \tag{3.7}$$

for all $\varepsilon \in (0,1)$. Now by [21, Lemma 3.3] we have

$$\int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^4}{|v_\varepsilon|^3} \le (2+\sqrt{3})^2 \cdot \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 \tag{3.8}$$

for all $\varepsilon \in (0,1)$. Moreover, using that $(a-b)^2 \ge \frac{1}{2}a^2 - b^2$ for all $a,b \in \mathbb{R}$, we see that

$$\begin{aligned} \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2 &= \int_0^\infty \int_\Omega v_\varepsilon \cdot \sum_{k,l=1}^3 \Big| \frac{1}{v_\varepsilon}\cdot \frac{\partial^2 v_\varepsilon}{\partial x_k \partial x_l} - \frac{1}{v_\varepsilon^2}\cdot \frac{\partial v_\varepsilon}{\partial x_k}\cdot\frac{\partial v_\varepsilon}{\partial x_l}\Big|^2 \\ &\ge \frac{1}{2}\int_0^\infty \int_\Omega v_\varepsilon \cdot \sum_{k,l=1}^3 \Big| \frac{1}{v_\varepsilon}\cdot \frac{\partial^2 v_\varepsilon}{\partial x_k \partial x_l}\Big|^2 - \int_0^\infty \int_\Omega v_\varepsilon \cdot \sum_{k,l=1}^3 \Big| \frac{1}{v_\varepsilon^2}\cdot \frac{\partial v_\varepsilon}{\partial x_k}\cdot\frac{\partial v_\varepsilon}{\partial x_l}\Big|^2 \\ &= \frac{1}{2}\int_0^\infty \int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} - \int_0^\infty \int_\Omega \frac{|\nabla v_\varepsilon|^4}{v_\varepsilon^3}, \end{aligned}$$

so that (3.8) also implies that

$$\frac{1}{2}\int_0^\infty \int_\Omega \frac{|D^2 v_\varepsilon|^2}{v_\varepsilon} \le [(2+\sqrt{3})^2+1]\cdot \int_0^\infty \int_\Omega v_\varepsilon |D^2 \ln v_\varepsilon|^2. \tag{3.9}$$

Therefore (3.2)-(3.6) result from (3.7)-(3.9) upon recalling that $v_\varepsilon \le \|v_0\|_{L^\infty(\Omega)}$ in $\Omega \times (0,\infty)$ by Lemma 2.3. $\Box$

Without further comment we may state the following immediate consequences of the above estimates and (2.4).

Corollary 3.3 *The weak solution of (1.1) from Lemma 2.1 has the properties*

$$\int_0^\infty \int_\Omega \frac{|\nabla u|^2}{u} < \infty, \tag{3.10}$$

$$\sup_{t>0} \int_\Omega |\nabla v(\cdot,t)|^2 < \infty, \tag{3.11}$$

$$\int_0^\infty \int_\Omega |D^2 v|^2 < \infty, \tag{3.12}$$

$$\int_0^\infty \int_\Omega |\nabla v|^4 < \infty \textit{ and} \tag{3.13}$$

$$\int_0^\infty \int_\Omega u|\nabla v|^2 < \infty. \tag{3.14}$$

4 A weak stabilization result for u

As a first step on our way to (1.3), let us derive from Corollary 3.2 a provisional statement on convergence of $u_\varepsilon(\cdot,t)$ to $\overline{u}_0$ as $t \to \infty$.

Lemma 4.1 *There exists $C > 0$ such that for all $\varepsilon \in (0,1)$, the solution of (2.2) satisfies*

$$\int_0^\infty \|u_\varepsilon(\cdot,t) - \overline{u}_0\|^2_{L^{\frac{3}{2}}(\Omega)} dt \le C, \tag{4.1}$$

where $\overline{u}_0$ is as defined in (1.4). In particular, the weak solution of (1.1) gained from Lemma 2.1 has the property that

$$\int_0^\infty \|u(\cdot,t) - \overline{u}_0\|^2_{L^{\frac{3}{2}}(\Omega)} dt \le C. \tag{4.2}$$

PROOF. We apply the Cauchy-Schwarz inequality to (3.2) and recall (2.5) to obtain $c_1 > 0$ such that

$$\begin{aligned} \int_0^\infty \Big(\int_\Omega |\nabla u_\varepsilon|\Big)^2 &\le \int_0^\infty \Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\Big) \cdot \Big(\int_\Omega u_\varepsilon\Big) \\ &\le c_1 \qquad \text{for all } \varepsilon \in (0,1). \end{aligned} \tag{4.3}$$

Next, in view of the continuous embedding $W^{1,1}(\Omega) \hookrightarrow L^{\frac{3}{2}}(\Omega)$ ([2, 8.9]), a Poincaré-Sobolev inequality is available to yield $c_2 > 0$ such that

$$\|z\|_{L^{\frac{3}{2}}(\Omega)} \le c_2 \|\nabla z\|_{L^1(\Omega)} \qquad \text{for all } z \in W^{1,1}(\Omega) \text{ with } \int_\Omega z = 0.$$

Since $\int_\Omega (u_\varepsilon(\cdot,t) - \overline{u}_0) = 0$ for all $t > 0$ and $\varepsilon \in (0,1)$ by (2.5), we thus obtain from (4.3) that

$$\int_0^\infty \|u_\varepsilon(\cdot,t) - \overline{u}_0\|^2_{L^{\frac{3}{2}}(\Omega)} \le c_2^2 \int_0^\infty \|\nabla u_\varepsilon(\cdot,t)\|^2_{L^1(\Omega)} dt \le c_2^2 c_1 \qquad \text{for all } \varepsilon \in (0,1).$$

This proves (4.1), from which (4.2) immediately results due to Fatou's lemma and (2.4). $\square$

5 Uniform decay of v

The following auxiliary statement is elementary and thus we may omit a proof here.

Lemma 5.1 *Let*

$$K_1 := \sup_{\xi \ge 0} \frac{\xi}{(1+\xi)^2}, \quad K_2 := \sup_{\xi \ge 0} \frac{\xi}{(1+\xi)^2 \ln(1+\xi)} \quad \text{and} \quad K_3 := \sup_{\xi \ge 0} \frac{\xi^3}{(1+\xi)^6 \ln(1+\xi)}.$$

Then K_1, K_2 and K_3 are positive and finite.

We next make sure that the time derivative of v_ε decays in a natural integral sense.

Lemma 5.2 *There exists $C > 0$ such that for the solution of (2.2) we have*

$$\int_0^\infty \int_\Omega v_{\varepsilon t}^2 \le C \tag{5.1}$$

whenever $\varepsilon \in (0,1)$.

Proof. Testing the second PDE in (2.2) by $v_{\varepsilon t}$ we obtain

$$\begin{aligned} \int_\Omega v_{\varepsilon t}^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v_\varepsilon|^2 &= -\int_\Omega F_\varepsilon(u_\varepsilon) v_\varepsilon \cdot v_{\varepsilon t} \\ &= -\frac{1}{2}\int_\Omega F_\varepsilon(u_\varepsilon) \cdot (v_\varepsilon^2)_t \\ &= -\frac{1}{2}\frac{d}{dt}\int_\Omega F_\varepsilon(u_\varepsilon) \cdot v_\varepsilon^2 + \frac{1}{2}\int_\Omega F_\varepsilon'(u_\varepsilon) \cdot v_\varepsilon^2 u_{\varepsilon t} \qquad \text{for all } t > 0. \end{aligned} \tag{5.2}$$

Here we use the first equation in (2.2) and integrate by parts to see that

$$\begin{aligned} \frac{1}{2}\int_\Omega F_\varepsilon'(u_\varepsilon) \cdot v_\varepsilon^2 u_{\varepsilon t} &= -\frac{1}{2}\int_\Omega F_\varepsilon''(u_\varepsilon) v_\varepsilon^2 |\nabla u_\varepsilon|^2 - \int_\Omega F_\varepsilon'(u_\varepsilon) v_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\ &\quad + \frac{1}{2}\int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon) F_\varepsilon''(u_\varepsilon) v_\varepsilon^2 \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int_\Omega u_\varepsilon F_\varepsilon'(u_\varepsilon)^2 v_\varepsilon |\nabla v_\varepsilon|^2 \\ &=: I_1 + I_2 + I_3 + I_4 \qquad \text{for all } t > 0. \end{aligned} \tag{5.3}$$

Abbreviating $c := \|v_0\|_{L^\infty(\Omega)}$ and recalling the inequality $v_\varepsilon \le c$ provided by Lemma 2.3, we therefore obtain

$$\begin{aligned} I_1 &\le \frac{c^2}{2} \cdot \int_\Omega u_\varepsilon |F_\varepsilon''(u_\varepsilon)| \cdot \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \\ &\le \frac{c^2 K_1}{2} \cdot \int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \qquad \text{for all } t > 0, \end{aligned} \tag{5.4}$$

because from the definition (2.3) of F_ε and Lemma 5.1 we infer that

$$s \cdot |F_\varepsilon''(s)| = \frac{s \cdot \varepsilon}{(1+\varepsilon s)^2} \le K_1 \qquad \text{for all } s \ge 0 \text{ and } \varepsilon \in (0,1).$$

Next, in view of Young's inequality and, again, Lemma 2.3 we can estimate

$$\begin{aligned} I_2 &\le \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 + \frac{c^2}{2}\cdot\int_\Omega \frac{u_\varepsilon F_\varepsilon'^2(u_\varepsilon)}{F_\varepsilon(u_\varepsilon)}\cdot\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \\ &\le \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 + \frac{c^2K_2}{2}\cdot\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \qquad \text{for all } t>0, \end{aligned} \tag{5.5}$$

for by Lemma 5.1 we have

$$\frac{s\cdot F_\varepsilon'^2(s)}{F_\varepsilon(s)} = \frac{s\cdot\varepsilon}{(1+\varepsilon s)^2\ln(1+\varepsilon s)} \le K_2 \qquad \text{for all } s\ge 0 \text{ and } \varepsilon\in(0,1). \tag{5.6}$$

Similarly,

$$\begin{aligned} I_3 &\le \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 + \frac{c^4}{16}\cdot\int_\Omega \frac{u_\varepsilon^3 F_\varepsilon'^2(u_\varepsilon)F_\varepsilon''^2(u_\varepsilon)}{F_\varepsilon(u_\varepsilon)}\cdot\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \\ &\le \int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 + \frac{c^4K_3}{16}\cdot\int_\Omega \frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} \qquad \text{for all } t>0, \end{aligned} \tag{5.7}$$

since again in view of Lemma 5.1

$$\frac{s^3F_\varepsilon'^2(s)F_\varepsilon''^2(s)}{F_\varepsilon(s)} = \frac{s^3\cdot\varepsilon^3}{(1+\varepsilon s)^6\ln(1+\varepsilon s)} \le K_3 \qquad \text{for all } s\ge 0 \text{ and } \varepsilon\in(0,1).$$

Finally, once more using (5.6) we see that

$$\begin{aligned} I_4 &\le c\cdot\int_\Omega \frac{u_\varepsilon F_\varepsilon'^2(u_\varepsilon)}{F_\varepsilon(u_\varepsilon)}\cdot F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 \\ &\le cK_2\cdot\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 \qquad \text{for all } t>0. \end{aligned} \tag{5.8}$$

Combining (5.3)-(5.5), (5.7) and (5.8), integrating (5.2) in time we thus obtain on dropping nonnegative terms that

$$\begin{aligned} \int_0^t\int_\Omega v_{\varepsilon t}^2 &\le \frac{1}{2}\int_\Omega |\nabla v_0|^2 + \frac{1}{2}\int_\Omega F_\varepsilon(u_0)\cdot v_0^2 \\ &\quad +\Big(\frac{c^2K_1}{2}+\frac{c^2K_2}{4}+\frac{c^4K_3}{16}\Big)\cdot\int_0^t\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon} + (1+1+cK_2)\cdot\int_0^t\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2 \end{aligned}$$

holds for all $t>0$ and each $\varepsilon\in(0,1)$. Therefore (5.1) is a consequence of (3.2) and (3.6). □

One particular consequence of the above estimate is that actually v is continuous as an $L^2(\Omega)$-valued function. Inter alia, this will give a meaning to statements like '$v(\cdot,t)\to 0$ as $t\to\infty$'.

Corollary 5.3 *The function v defined by (2.4) satisfies*

$$v\in C^0([0,\infty);L^2(\Omega)). \tag{5.9}$$

Moreover, in (2.4) we may assume without loss of generality that as $\varepsilon=\varepsilon_j\searrow 0$ we have

$$v_\varepsilon\to v \qquad \text{in } L^2_{loc}([0,\infty);L^\infty(\Omega)), \tag{5.10}$$

$$v_\varepsilon(\cdot,t)\to v(\cdot,t) \qquad \text{in } L^\infty(\Omega) \qquad \text{for a.e. } t>0 \qquad \text{and} \tag{5.11}$$

$$v_\varepsilon\to v \qquad \text{in } L^\infty_{loc}([0,\infty);L^2(\Omega)). \tag{5.12}$$

Proof. Let $T > 0$. Then the boundedness of $(v_{\varepsilon t})_{\varepsilon\in(0,1)}$ in $L^2((0,T);L^2(\Omega))$ asserted by Lemma 5.2 implies that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $C^{\frac{1}{2}}([0,T];L^2(\Omega))$, because if $0 \le s < t \le T$ then thanks to the Hölder inequality we have

$$\|v_\varepsilon(\cdot,t) - v_\varepsilon(\cdot,s)\|^2_{L^2(\Omega)} = \int_\Omega \Big| \int_s^t v_{\varepsilon t}(x,\sigma)d\sigma \Big|^2 dx \le (t-s) \int_0^T \int_\Omega v_{\varepsilon t}^2.$$

Thus, since $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^\infty((0,T);W^{1,2}(\Omega))$ by (3.3) and Lemma 2.3, and since the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the Arzelà-Ascoli theorem says that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^\infty((0,T);L^2(\Omega))$. This establishes (5.12) which also implies (5.9).
Next, recalling (3.4) we know that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{2,2}(\Omega))$. Since $W^{2,2}(\Omega) \hookrightarrow\hookrightarrow W^{1,p}(\Omega)$ for each $p < 6$, we may combine this with the boundedness of $(v_{\varepsilon t})_{\varepsilon\in(0,1)}$ in $L^2((0,T);L^2(\Omega))$ to obtain from the Aubin-Lions lemma ([18]) that $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^2((0,T);W^{1,p}(\Omega))$ for any such p. In light of the fact that $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ for all $p > 3$, from this we easily deduce (5.10) and (5.11). □

Throughout the sequel, we fix any sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ such that both (2.4) and the conclusion of Corollary 5.3 hold.

We can now already prove part of the result claimed in Theorem 1.1.

Lemma 5.4 *The second component of the weak solution of (1.1) constructed in Lemma 2.1 satisfies*

$$v(\cdot,t) \to 0 \quad \textit{in } L^\infty(\Omega) \qquad \textit{as } t \to \infty. \tag{5.13}$$

Proof. Since v is bounded in $L^\infty(\Omega \times (0,\infty))$ by Lemma 2.3 and $\int_0^\infty \int_\Omega |\nabla v|^4 < \infty$ by Corollary 3.3, there exists a sequence of times $t_k \to \infty$ such that $t_k < t_{k+1} \le t_k + 1$ for all $k \in \mathbb{N}$ and $(v(\cdot,t_k))_{k\in\mathbb{N}}$ is bounded in $W^{1,4}(\Omega)$. Using that in the three-dimensional setting the space $W^{1,4}(\Omega)$ is compactly embedded into $L^\infty(\Omega)$, we may pass to a subsequence, not relabeled for convenience, along which

$$v(\cdot,t_k) \to v_\infty \quad \text{in } L^\infty(\Omega) \tag{5.14}$$

holds with some nonnegative $v_\infty \in L^\infty(\Omega)$. In order to relate this to an appropriate space-time integral using Lemma 5.2, we follow a standard reasoning (see [3], for instance) and use the Cauchy-Schwarz inequality to obtain

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} \int_\Omega |v_\varepsilon(x,t) - v_\varepsilon(x,t_k)|^2 dxdt &= \int_{t_k}^{t_{k+1}} \int_\Omega \Big(\int_{t_k}^t v_{\varepsilon t}(x,s)ds \Big)^2 dxdt \\
&\le \int_{t_k}^{t_{k+1}} \int_\Omega \Big(\int_{t_k}^t v_{\varepsilon t}^2(x,s)ds \Big) \cdot (t-t_k) dxdt \\
&\le \int_{t_k}^\infty \int_\Omega v_{\varepsilon t}^2 \qquad \text{for all } \varepsilon \in (0,1).
\end{aligned}$$

Since $\int_0^\infty \int_\Omega v_t^2 < \infty$ according to Corollary 3.3, in the limit $\varepsilon = \varepsilon_j \searrow 0$ this entails that

$$\int_{t_k}^{t_{k+1}} \int_\Omega |v(x,t) - v(x,t_k)|^2 dxdt \le \int_{t_k}^\infty \int_\Omega v_t^2 \to 0 \qquad \text{as } k \to \infty.$$

Since clearly (5.14) implies that

$$\int_\Omega |v(x,t_k)-v_\infty(x)|^2dx = \|v(\cdot,t_k)-v_\infty\|_{L^2(\Omega)}^2 \to 0 \qquad \text{as } k\to\infty,$$

this entails that

$$\int_{t_k}^{t_{k+1}} \|v(\cdot,t)-v_\infty\|_{L^2(\Omega)}^2 dt \to 0 \qquad \text{as } k\to\infty. \tag{5.15}$$

Next, recalling Lemma 4.1 we see that

$$\int_{t_k}^{t_{k+1}} \|u(\cdot,t)-\overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 dt \to 0 \qquad \text{as } k\to\infty \tag{5.16}$$

holds with $\overline{u}_0>0$ as in (1.4). Therefore, Hölder's inequality and Lemma 2.3 allow us to estimate

$$\begin{aligned}
\int_{t_k}^{t_{k+1}}\int_\Omega |u(x,t)v(x,t)-\overline{u}_0 v_\infty(x)|dxdt &\le \int_{t_k}^{t_{k+1}}\int_\Omega \Big|(u(x,t)-\overline{u}_0)\cdot v(x,t)\Big|dxdt \\
&\quad + \int_{t_k}^{t_{k+1}}\int_\Omega \Big|\overline{u}_0\cdot (v(x,t)-v_\infty(x))\Big|dxdt \\
&\le \int_{t_k}^{t_{k+1}} \|u(\cdot,t)-\overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}\cdot\|v(\cdot,t)\|_{L^3(\Omega)}dt \\
&\quad +\overline{u}_0\cdot|\Omega|^{\frac{1}{2}}\cdot\int_{t_k}^{t_{k+1}}\|v(\cdot,t)-v_\infty\|_{L^2(\Omega)}dt \\
&\le \|v_0\|_{L^\infty(\Omega)}\cdot|\Omega|^{\frac{1}{3}}\cdot\Big(\int_{t_k}^{t_{k+1}}\|u(\cdot,t)-\overline{u}_0\|_{L^{\frac{3}{2}}(\Omega)}^2 dt\Big)^{\frac{1}{2}} \\
&\quad +\overline{u}_0\cdot|\Omega|^{\frac{1}{2}}\cdot\Big(\int_{t_k}^{t_{k+1}}\|v(\cdot,t)-v_\infty\|_{L^2(\Omega)}^2\Big)^{\frac{1}{2}}
\end{aligned}$$

for all $k\in\mathbb{N}$, so that

$$\int_{t_k}^{t_{k+1}}\int_\Omega uv \to \overline{u}_0\cdot\int_\Omega v_\infty \qquad \text{as } k\to\infty.$$

Now if $v_\infty\not\equiv 0$, this would imply that

$$\sum_{k\in\mathbb{N}}\int_{t_k}^{t_{k+1}}\int_\Omega uv = +\infty,$$

because $\overline{u}_0$ was positive. On the other hand, since $t_{k+1}\le t_k+1$ for all $k\in\mathbb{N}$ we know from Lemma 2.4 that

$$\sum_{k\in\mathbb{N}}\int_{t_k}^{t_{k+1}}\int_\Omega uv \le \int_0^\infty\int_\Omega uv<\infty.$$

This contradiction shows that actually $v_\infty\equiv 0$, whence (5.14) becomes

$$v(\cdot,t_k)\to 0 \quad \text{in } L^\infty(\Omega) \qquad \text{as } k\to\infty.$$

Since $t\mapsto\|v(\cdot,t)\|_{L^\infty(\Omega)}$ is nonincreasing by Lemma 2.3, from this we conclude that indeed (5.13) is valid. $\square$

6 Eventual boundedness and regularity

Combining Lemma 5.4 with (5.11) and Lemma 2.3, we obtain that not only the limit v but also its approximations become conveniently small.

Lemma 6.1 *For any $\delta > 0$ there exist $t_0(\delta) > 0$ and $\varepsilon_0(\delta) \in (0,1)$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling $\varepsilon < \varepsilon_0(\delta)$, the solution of (2.2) satisfies*

$$v_\varepsilon \le \delta \qquad \textit{in } \Omega \times (t_0(\delta), \infty). \tag{6.1}$$

PROOF. Given $\delta > 0$, from Lemma 5.4 we obtain $\tilde{t}_0 > 0$ such that the limit v defined by (2.4) satisfies $v \le \frac{\delta}{2}$ in $\Omega \times (\tilde{t}_0, \infty)$. Now (5.11) ensures that we can find some $t_0 \in (\tilde{t}_0, \tilde{t}_0 + 1)$ such that $v_\varepsilon(\cdot, t_0) \to v(\cdot, t_0)$ in $L^\infty(\Omega)$ as $\varepsilon = \varepsilon_j \searrow 0$, so that in particular $v_\varepsilon(\cdot, t_0) \le \delta$ in Ω whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is sufficiently small. Since from Lemma 2.3 we know that $t \mapsto \|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ does not increase, we conclude that actually $v_\varepsilon \le \delta$ in $\Omega \times (t_0, \infty)$ for any such ε, as desired. $\square$

With the above result at hand, we can now perform an argument inspired by a similar reasoning in [17] which uses the smallness of v_ε to assert bounds for u_ε in $L^p(\Omega)$ for arbitrarily large p. Similar functionals have previously been used to derive regularity in [20].

Lemma 6.2 *Let $p \in (1,\infty)$. Then there exist $t_1(p) > 0, \varepsilon_1(p) \in (0,1)$ and $C(p) > 0$ such that the solution of (2.2) has the property*

$$\int_\Omega u_\varepsilon^p(\cdot,t) \le C(p) \qquad \textit{for all } t > t_1(p) \tag{6.2}$$

whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ is such that $\varepsilon < \varepsilon_1(p)$.

PROOF. Given $p \in (1,\infty)$, we can fix $q > 0$ such that $q < p - 1$. Then the function

$$\rho(\delta) := p - 1 - \frac{p}{4} \cdot \frac{4q^2 + (p-1)^2 \cdot (2\delta)}{q(q+1) - pq \cdot 2\delta}, \qquad \delta \in \Big(0, \frac{q+1}{2p}\Big),$$

satisfies

$$\rho(0) = p - 1 - \frac{p}{4} \cdot \frac{4q^2}{q(q+1)} = \frac{p-q-1}{q+1} > 0,$$

so that it is possible to pick $\delta \in (0, \frac{q+1}{2p})$ small such that still

$$c_1 := \rho(\delta) > 0. \tag{6.3}$$

We now let

$$\varphi(s) := (2\delta - s)^{-q}, \qquad s \in [0, 2\delta),$$

and take $t_0(\delta)$ and $\varepsilon_0(\delta)$ as provided by Lemma 6.1. Then for each $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ such that $\varepsilon < \varepsilon_0(\delta)$, the function $\varphi(v_\varepsilon)$ is smooth in $\bar\Omega \times (t_0(\delta), \infty)$ and we compute, using (2.2) and integrating by parts,

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u_\varepsilon^p \varphi(v_\varepsilon) &= p\int_\Omega u_\varepsilon^{p-1}\varphi(v_\varepsilon)\cdot\Big(\Delta u_\varepsilon - \nabla\cdot(u_\varepsilon F_\varepsilon'(u_\varepsilon)\nabla v_\varepsilon)\Big) \\
&\quad + \int_\Omega u_\varepsilon^p \varphi'(v_\varepsilon)\cdot(\Delta v_\varepsilon - F_\varepsilon(u_\varepsilon)v_\varepsilon) \\
&= -p(p-1)\int_\Omega u_\varepsilon^{p-2}\varphi(v_\varepsilon)|\nabla u_\varepsilon|^2 \\
&\quad - \int_\Omega u_\varepsilon^p\cdot[\varphi''(v_\varepsilon) - pF_\varepsilon'(u_\varepsilon)\varphi'(v_\varepsilon)]\cdot|\nabla v_\varepsilon|^2 \\
&\quad + p\int_\Omega u_\varepsilon^{p-1}\cdot[-2\varphi'(v_\varepsilon) + (p-1)F_\varepsilon'(u_\varepsilon)\varphi(v_\varepsilon)]\nabla u_\varepsilon\cdot\nabla v_\varepsilon \\
&\quad - \int_\Omega u_\varepsilon^p F_\varepsilon(u_\varepsilon)v_\varepsilon\varphi'(v_\varepsilon) \\
&=: I_1 + I_2 + I_3 + I_4 \qquad \text{for all } t > t_0(\delta).
\end{aligned} \tag{6.4}$$

Here we note that $I_4 \le 0$ since $F_\varepsilon \ge 0$, and using that $F_\varepsilon' \le 1$ we obtain

$$I_2 \le -\int_\Omega u_\varepsilon^p\cdot[\varphi''(v_\varepsilon) - p\varphi'(v_\varepsilon)]\cdot|\nabla v_\varepsilon|^2,$$

where

$$\begin{aligned}
\varphi''(v_\varepsilon) - p\varphi'(v_\varepsilon) &= (2\delta - v_\varepsilon)^{-q-2}\cdot[q(q+1) - pq\cdot(2\delta - v_\varepsilon)] \\
&\ge (2\delta - v_\varepsilon)^{-q-2}\cdot q(q+1-p\cdot 2\delta) \\
&> 0 \qquad \text{in } \Omega\times(t_0(\delta),\infty)
\end{aligned}$$

thanks to the fact that $\delta < \frac{q+1}{2p}$. We therefore may invoke Young's inequality to see that

$$I_3 \le -I_2 + \frac{p^2}{4}\cdot\int_\Omega u_\varepsilon^{p-2}\cdot\frac{[-2\varphi'(v_\varepsilon) + (p-1)F_\varepsilon'(u_\varepsilon)\varphi(v_\varepsilon)]^2}{\varphi''(v_\varepsilon) - p\varphi'(v_\varepsilon)}\cdot|\nabla u_\varepsilon|^2. \tag{6.5}$$

To estimate this, we use that $0 \le F_\varepsilon' \le 1$ to derive that

$$\begin{aligned}
J(x,t,s) &:= p(p-1)\varphi(s) - \frac{p^2}{4}\cdot\frac{[-2\varphi'(s) + (p-1)F_\varepsilon'(u_\varepsilon)\cdot\varphi(s)]^2}{\varphi''(s) - p\varphi'(s)}, \\
&\qquad\qquad (x,t)\in\Omega\times(t_0(\delta),\infty),\ \ s\in[0,2\delta),
\end{aligned}$$

satisfies

$$\begin{aligned}
J(x,t,s) &= p(p-1)\varphi(s) - \frac{p^2}{4}\cdot\frac{4\varphi'^2(s) - 4(p-1)F_\varepsilon'(u_\varepsilon)\varphi(s)\varphi'(s) + (p-1)^2F_\varepsilon'^2(u_\varepsilon)\varphi^2(s)}{\varphi''(s) - p\varphi'(s)} \\
&\ge p(p-1)\varphi(s) - \frac{p^2}{4}\cdot\frac{4\varphi'^2(s) + (p-1)^2\varphi^2(s)}{\varphi''(s) - p\varphi'(s)}
\end{aligned}$$

$$
\begin{aligned}
&= p(p-1)\varphi(s) - \frac{p^2}{4}\cdot\frac{4q^2(2\delta-s)^{-2q-2}+(p-1)^2(2\delta-s)^{-2q}}{q(q+1)(2\delta-s)^{-q-2}-pq(2\delta-s)^{-q-1}} \\
&= p(2\delta-s)^{-q}\cdot\Big\{p-1-\frac{p}{4}\cdot\frac{4q^2+(p-1)^2\cdot(2\delta-s)^2}{q(q+1)-pq\cdot(2\delta-s)}\Big\} \\
&\ge p(2\delta-s)^{-q}\cdot\rho(\delta) \qquad \text{for all } (x,t,s)\in\Omega\times(t_0(\delta),\infty)\times[0,2\delta)
\end{aligned}
$$

by definition of ρ. Recalling (6.3), we thus obtain that

$$
J(x,t,s)\ge c_2 := p\delta^{-q}c_1 \qquad \text{for all } (x,t,s)\in\Omega\times(t_0(\delta),\infty)\times[0,\delta].
$$

In view of (6.5) and the fact that $v_\varepsilon\le\delta$ in $\Omega\times(t_0(\delta),\infty)$ for $\varepsilon<\varepsilon_0(\delta)$ by Lemma 6.1, this entails that

$$
\begin{aligned}
I_3 &\le -I_2+\int_\Omega [p(p-1)\varphi(v_\varepsilon)-J(x,t,v_\varepsilon)]\cdot u_\varepsilon^{p-2}|\nabla v_\varepsilon|^2 \\
&\le -I_2-I_1-c_2\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 \qquad \text{for all } t>t_0(\delta).
\end{aligned}
$$

We thereby infer from (6.4) that

$$
\frac{d}{dt}\int_\Omega u_\varepsilon^p\varphi(v_\varepsilon)\le -c_2\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 = -\frac{4c_2}{p^2}\int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \tag{6.6}
$$

for all $t>t_0(\delta)$. Here we interpolate using the Gagliardo-Nirenberg inequality ([6]) and (2.5) to obtain $c_3>0$ and $c_4>0$ such that

$$
\begin{aligned}
\int_\Omega u_\varepsilon^p\varphi(v_\varepsilon) &\le \delta^{-q}\|u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \\
&\le c_3\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2a}\cdot\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{2(1-a)}+c_3\|u_\varepsilon^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^2 \\
&\le c_4\cdot\Big(\|\nabla u_\varepsilon^{\frac{p}{2}}\|_{L^2(\Omega)}^{2a}+1\Big) \qquad \text{for all } t>t_0(\delta)
\end{aligned}
$$

with $a:=\frac{3(p-1)}{3p-1}\in(0,1)$. Therefore (6.6) shows that $y_\varepsilon(t):=\int_\Omega u_\varepsilon^p\varphi(v_\varepsilon)(\cdot,t),\ t>t_0(\delta)$, satisfies

$$
y_\varepsilon'(t)\le -c_5\cdot(y_\varepsilon(t)-1)_+^{\frac{1}{a}} \qquad \text{for all } t>t_0(\delta)
$$

for some $c_5>0$, and hence an integration yields

$$
y_\varepsilon(t)\le 1+\Big(\frac{c_5(1-a)}{a}(t-t_0(\delta))\Big)^{-\frac{a}{1-a}} \qquad \text{for all } t>t_0(\delta).
$$

Since $\varphi(s)\ge(2\delta)^{-q}$ for all $s\in[0,2\delta)$, we thus conclude that

$$
\int_\Omega u_\varepsilon^p(\cdot,t)\le(2\delta)^q y_\varepsilon(t)\le(2\delta)^q\cdot\Big\{1+\Big(\frac{c_5(1-a)}{a}\Big)^{-\frac{a}{1-a}}\Big\} \qquad \text{for all } t\ge t_0(\delta)+1,
$$

provided that $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$ is sufficiently small. This proves (6.2) upon the choice $t_1(p):=t_0(p)+1$. $\square$

Now a straightforward reasoning involving standard bootstrap techniques yields eventual boundedness and smoothness of the weak solution in question.

Lemma 6.3 *There exist $T > 0$ and a subsequence $(\varepsilon_{j_i})_{i\in\mathbb{N}}$ of $(\varepsilon_j)_{j\in\mathbb{N}}$ such that for any $\varepsilon \in (\varepsilon_{j_i})_{i\in\mathbb{N}}$ we have*

$$\|u_\varepsilon(\cdot,t)\|_{C^2(\bar\Omega)} \le C \qquad \textit{for all } t \ge T, \tag{6.7}$$

and such that

$$u_\varepsilon \to u \quad \textit{and} \quad v_\varepsilon \to v \qquad \textit{in } C^{2,1}_{loc}(\bar\Omega\times[0,\infty)) \qquad \textit{as } \varepsilon=\varepsilon_{j_i}\searrow 0. \tag{6.8}$$

PROOF. The proof proceeds by standard regularity arguments (cf. e.g. [8] for details in quite a similar setting), and thus we may confine ourselves with an outline.
We fix any $p > 6$ and then obtain from Lemma 6.1 some $t_1 = t_1(p) > 0$, $c_1 = c_1(p) > 0$ and $\varepsilon_1(p) \in (0,1)$ such that

$$|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le c_1 \qquad \text{for all } t > t_1 \tag{6.9}$$

and any $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ fulfilling $\varepsilon < \varepsilon_1(p)$. Applying ∇ to both sides of the variation-of-constants formula for v_ε,

$$v_\varepsilon(\cdot,t) = e^{t(\Delta-1)}v_\varepsilon(\cdot,t_1) - \int_{t_1}^t e^{(t-s)(\Delta-1)}(F_\varepsilon(u_\varepsilon)-1)v_\varepsilon(\cdot,s)ds, \qquad t\ge t_1,$$

recalling that $|F_\varepsilon(u_\varepsilon)| \le u_\varepsilon$ and $v_\varepsilon \le \|v_0\|_{L^\infty(\Omega)}$ we therefore obtain $c_2>0$ and $c_3>0$ such that

$$\begin{aligned}\|\nabla v_\varepsilon(\cdot,t)\|_{L^p(\Omega)} &\le c_2\cdot\Big(\|v_\varepsilon(\cdot,t_1)\|_{L^\infty(\Omega)} + \int_{t_1}^t (t-s)^{-\frac12}e^{-(t-s)}(\|u_\varepsilon(\cdot,s)\|_{L^p(\Omega)}+1)ds\Big)\\ &\le c_3 \qquad \text{for all } t\ge t_2 := t_1+1.\end{aligned}$$

Since this implies that

$$\|u_\varepsilon(\cdot,t)\nabla v_\varepsilon(\cdot,t)\|_{L^{\frac p2}(\Omega)} \le c_1c_3 \qquad \text{for all } t\ge t_2,$$

the variation-of-constants formula for u_ε in the form

$$u_\varepsilon(\cdot,t) = e^{t\Delta}u_\varepsilon(\cdot,t_2) + \int_{t_2}^t e^{(t-s)\Delta}\nabla\cdot(u_\varepsilon F_\varepsilon'(u_\varepsilon)\nabla v_\varepsilon)(\cdot,s)ds, \qquad t\ge t_2,$$

along with (6.9) allows us to estimate

$$\|A^\theta u_\varepsilon(\cdot,t)\|_{L^q(\Omega)} \le c_4 \qquad \text{for all } t\ge t_3 := t_2+1 \tag{6.10}$$

and moreover

$$\|A^\theta u_\varepsilon(\cdot,t) - A^\theta u_\varepsilon(\cdot,s)\|_{L^q(\Omega)} \le c_4|t-s|^\eta \qquad \text{for all } t,s\ge t_3 \text{ such that } |t-s|\le 1 \tag{6.11}$$

with some $\eta\in(0,1), \theta\in(0,1)$ and $q>1$ large enough such that $2\theta-\frac3q>0$, where A^θ denotes the fractional power of the realization of $-\Delta+1$ in $L^q(\Omega)$ under homogeneous Neumann boundary conditions.
Along with the fact that the domain of definition of A^θ satisfies $D(A^\theta)\hookrightarrow C^\beta(\bar\Omega)$ for all $\beta\in(0,2\theta-\frac3q)$ ([7]), the estimates (6.10) and (6.11) show that $(u_\varepsilon)_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ is bounded in both $L^\infty(\Omega\times(t_3,\infty))$ and in

$C^{\beta,\frac{\beta}{2}}_{loc}(\bar\Omega\times[t_3,\infty))$ for some $\beta\in(0,1)$. Therefore standard parabolic Schauder estimates ([14]) applied to the second equation in (2.2) yield boundedness of $(v_\varepsilon)_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ in both $L^\infty((t_4,\infty);C^{2+\beta}(\bar\Omega))$ and in $C^{2+\beta,1+\frac{\beta}{2}}_{loc}(\bar\Omega\times[t_4,\infty))$ for $t_4:=t_3+1$. This in turn, by a similar argument, entails boundedness of $(u_\varepsilon)_{\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}}$ in both $L^\infty((t_5,\infty);C^{2+\beta'}(\bar\Omega))$ and in $C^{2+\beta',1+\frac{\beta'}{2}}_{loc}(\bar\Omega\times[t_5,\infty))$ for some $\beta'\in(0,1)$ and $t_5:=t_4+1$. An application of the Arzelà-Ascoli theorem completes the proof. $\Box$

7 Large time behavior of u

We now aim at improving the rather weak stabilization result for u warranted by Lemma 4.1. As a preparation, we assert that u_t decays at least in some weak sense in the large time limit.

Lemma 7.1 *There exists $C>0$ such that for all $\varepsilon\in(0,1)$ the solution of (2.2) satisfies*

$$\int_0^\infty \|u^2_{\varepsilon t}(\cdot,t)\|^2_{(W^{3,2}(\Omega))^\star}dt\le C. \tag{7.1}$$

Consequently,

$$\int_0^\infty \|u^2_t(\cdot,t)\|^2_{(W^{3,2}(\Omega))^\star}dt<\infty. \tag{7.2}$$

Proof. We fix $\psi\in W^{3,2}(\Omega)$ and test the first equation in (2.2) against ψ to obtain

$$\begin{aligned}\int_\Omega u_{\varepsilon t}\psi &= \int_\Omega\Delta u_\varepsilon\psi-\int_\Omega\nabla\cdot(u_\varepsilon F'_\varepsilon(u_\varepsilon)\nabla v_\varepsilon)\psi\\ &= -\int_\Omega\nabla u_\varepsilon\cdot\nabla\psi+\int_\Omega u_\varepsilon F'_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla\psi\qquad\text{for all } t>0.\end{aligned}\tag{7.3}$$

Here by the Hölder inequality and (2.5),

$$\begin{aligned}\Big|-\int_\Omega\nabla u_\varepsilon\cdot\nabla\psi\Big| &\le \Big(\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\Big)^{\frac12}\cdot\Big(\int_\Omega u_\varepsilon|\nabla\psi|^2\Big)^{\frac12}\\ &\le \Big(\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}\Big)^{\frac12}\cdot\overline{u}_0^{\frac12}\|\nabla\psi\|_{L^\infty(\Omega)}.\end{aligned}$$

By the same tokens and Lemma 5.1,

$$\begin{aligned}\Big|\int_\Omega u_\varepsilon F'_\varepsilon(u_\varepsilon)\nabla v_\varepsilon\cdot\nabla\psi\Big| &\le \Big(\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\Big)^{\frac12}\cdot\Big(\int_\Omega\frac{u_\varepsilon^2F'^2_\varepsilon(u_\varepsilon)}{F_\varepsilon(u_\varepsilon)}|\nabla\psi|^2\Big)^{\frac12}\\ &\le \Big(\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\Big)^{\frac12}\cdot K_2^{\frac12}\overline{u}_0^{\frac12}\|\nabla\psi\|_{L^\infty(\Omega)}.\end{aligned}$$

Since $W^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ and hence $\|\nabla z\|_{L^\infty(\Omega)}\le c_1\|z\|_{W^{3,2}(\Omega)}$ for all $z\in W^{3,2}(\Omega)$ and some $c_1>0$, (7.3) therefore shows that

$$\begin{aligned}\|u_{\varepsilon t}(\cdot,t)\|^2_{(W^{3,2}(\Omega))^\star} &= \sup_{\psi\in W^{3,2}(\Omega),\ \|\psi\|_{W^{3,2}(\Omega)}\le1}\Big|\int_\Omega u_{\varepsilon t}(\cdot,t)\psi\Big|\\ &\le c_1\overline{u}_0\cdot\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon}+c_1K_1\overline{u}_0\cdot\int_\Omega F_\varepsilon(u_\varepsilon)|\nabla v_\varepsilon|^2\qquad\text{for all } t>0.\end{aligned}$$

Hence, in view of (3.2) and (3.6), an integration over $t \in (0, \infty)$ yields (7.1), whereas (7.2) again results from lower semicontinuity of the norm in the Hilbert space $L^2((0,\infty);(W^{3,2}(\Omega))^\star)$ with respect to weak convergence. $\Box$

We are now in the position to prove a convergence result in the flavor of Theorem 1.1 also for u.

Lemma 7.2 *The weak solution of (1.1) from lemma 2.1 satisfies*

$$u(\cdot,t) \to \overline{u}_0 \quad in\ L^\infty(\Omega) \qquad as\ t \to \infty, \tag{7.4}$$

where $\overline{u}_0$ is given by (1.4).

Proof. Let us suppose on the contrary that (7.4) be false. Then we can find a sequence of times $t_k \to \infty$ such that

$$\inf_{k\in\mathbb{N}} \|u(\cdot,t_k) - \overline{u}_0\|_{L^\infty(\Omega)} > 0, \tag{7.5}$$

where we may assume without loss of generality that $t_k > T$ for all $k \in \mathbb{N}$ with T as provided by Lemma 6.3. Since then $(u(\cdot,t_k))_{k\in\mathbb{N}}$ is relatively compact in $L^\infty(\Omega)$ according to (6.7) and the Arzelà-Ascoli theorem, we can extract a subsequence, again denoted by $(t_k)_{k\in\mathbb{N}}$, such that

$$u(\cdot,t_k) \to u_\infty \quad \text{in } L^\infty(\Omega) \qquad \text{as } k \to \infty \tag{7.6}$$

is valid with some nonnegative $u_\infty \in L^\infty(\Omega)$.
Now performing a variant of an argument from Lemma 5.4, by the Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} \|u_\varepsilon(\cdot,t) - u_\varepsilon(\cdot,t_k)\|^2_{(W^{3,2}(\Omega))^\star} dt &= \int_{t_k}^{t_{k+1}} \Big\| \int_{t_k}^{t} u_{\varepsilon t}(\cdot,s)ds \Big\|^2_{(W^{3,2}(\Omega))^\star} dt \\
&\le \int_{t_k}^{t_{k+1}} \Big(\int_{t_k}^{t} \|u_{\varepsilon t}(\cdot,s)\|^2_{(W^{3,2}(\Omega))^\star} ds \Big) \cdot (t-t_k)dt \\
&\le \int_{t_k}^{\infty} \|u_{\varepsilon t}(\cdot,s)\|^2_{(W^{3,2}(\Omega))^\star} ds \qquad \text{for all } \varepsilon \in (0,1)
\end{aligned}$$

and hence

$$\begin{aligned}
\int_{t_k}^{t_{k+1}} \|u(\cdot,t) - u(\cdot,t_k)\|^2_{(W^{3,2}(\Omega))^\star} dt &\le \int_{t_k}^{\infty} \|u_t(\cdot,s)\|^2_{(W^{3,2}(\Omega))^\star} ds \\
&\to 0 \qquad \text{as } k \to \infty
\end{aligned}$$

according to Lemma 7.1. Since (7.6) entails that $u(\cdot,t_k) \to u_\infty$ in $(W^{3,2}(\Omega))^\star$ due to the fact that $L^\infty(\Omega) \hookrightarrow (W^{3,2}(\Omega))^\star$, this ensures that

$$\int_{t_k}^{t_{k+1}} \|u(\cdot,t) - u_\infty\|^2_{(W^{3,2}(\Omega))^\star} dt \to 0 \qquad \text{as } k \to \infty. \tag{7.7}$$

On the other hand, since also $L^{\frac{3}{2}}(\Omega) \hookrightarrow (W^{3,2}(\Omega))^\star$, Lemma 4.1 asserts that

$$\int_0^\infty \|u(\cdot,t) - \overline{u}_0\|^2_{(W^{3,2}(\Omega))^\star} dt < \infty$$

and thus in particular

$$\int_{t_k}^{t_{k+1}} \|u(\cdot,t)-\overline{u}_0\|^2_{(W^{3,2}(\Omega))^\star} dt \to 0 \qquad \text{as } k\to\infty. \tag{7.8}$$

Clearly, (7.7) and (7.8) are possible only if $u_\infty \equiv \overline{u}_0$, which contradicts (7.5) and (7.6). □

Our main result can now be obtained by simply collecting what we have found so far.

PROOF of Theorem 1.1. The statement on eventual boundedness and regularity immediately results from Lemma 6.3 and (2.4). The convergence properties in (1.3) have already been proved in Lemma 5.4 and Lemma 7.2. □

8 Global boundedness and convergence in the case $N=2$

In this section we plan to prove Proposition 1.2, and correspondingly we shall assume throughout that Ω is a bounded convex domain in $\mathbb{R}^2$ with smooth boundary. Since $v_t \le \Delta v$ due to the fact that u and v are nonnegative, the inequality $v \le \|v_0\|_{L^\infty(\Omega)}$ in $\Omega\times(0,\infty)$ again results upon an application of the maximum principle. In proving our global boundedness result for u we shall once more rely on the natural energy inequality associated with (1.1) (cf. Lemma 3.1).

Lemma 8.1 *The solution of (1.1) satisfies*

$$\frac{d}{dt}\Big\{\int_\Omega u\ln u + 2\int_\Omega |\nabla\sqrt{v}|^2\Big\} + \int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega v|D^2\ln v|^2 + \frac{1}{2}\int_\Omega u\frac{|\nabla v|^2}{v} \le 0 \tag{8.1}$$

for all $t>0$.

Among the numerous consequences collected in Corollary 3.2 for the three-dimensional case, we now shall need the analogue of only three.

Corollary 8.2 *We have*

$$\int_0^\infty\int_\Omega \frac{|\nabla u|^2}{u} < \infty, \tag{8.2}$$

$$\int_0^\infty\int_\Omega |\nabla v|^4 < \infty \qquad \textit{and} \tag{8.3}$$

$$\int_0^\infty\int_\Omega u|\nabla v|^2 < \infty. \tag{8.4}$$

We can now assert uniform boundedness of $u(\cdot,t)$ in $L^p(\Omega)$ for any finite p.

Lemma 8.3 *For all $p>1$ there exists $C>0$ such that for the solution of (1.1) the inequality*

$$\int_\Omega u^p(x,t)dx \le C \qquad \textit{for all } t>0. \tag{8.5}$$

holds.

PROOF. With the estimate (8.3) at hand, (8.5) can be proved as in [21, Lemma 4.4]. For convenience, let us recall the main ideas. Multiplying the first equation in (1.1) by u^{p-1} and applying Young's inequality we find $c_1 > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 \le c_1 \int_\Omega u^p |\nabla v|^2 \qquad \text{for all } t > 0. \tag{8.6}$$

By the Hölder inequality,

$$\int_\Omega u^p|\nabla v|^2 \le \Big(\int_\Omega u^{2p}\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega |\nabla v|^4\Big)^{\frac{1}{2}},$$

and now the Gagliardo-Nirenberg inequality provides $c_2 > 0$ such that

$$\Big(\int_\Omega u^{2p}\Big)^{\frac{1}{2}} = \|u^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le c_2\Big(\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)} \cdot \|u^{\frac{p}{2}}\|_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)}\Big),$$

where we have used the fact that $N = 2$. Since $\|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)} = (\int_\Omega u)^{\frac{p}{2}} \equiv (\int_\Omega u_0)^{\frac{p}{2}}$ upon integration of the first equation in (1.1) over Ω , we can thus pick $c_3 > 0$ such that

$$c_1\int_\Omega u^p|\nabla v|^2 \le \frac{p-1}{2}\int_\Omega u^{p-2}|\nabla u|^2 + c_3\Big(\int_\Omega |\nabla v|^4\Big)\cdot\Big(\int_\Omega u^p + 1\Big),$$

so that from (8.3) we obtain that $y(t) := \int_\Omega u^p(x,t)dx,\ t \in [0,T)$, satisfies the ODI

$$y'(t) \le c_4\Big(\int_\Omega |\nabla v|^4\Big)\cdot(y(t)+1) \qquad \text{for all } t > 0$$

with some $c_4 > 0$. On integration we infer that

$$y(t) + 1 \le (y(0)+1)\cdot e^{c_4\int_0^T\int_\Omega |\nabla v|^4} \qquad \text{for all } t > 0,$$

whereupon recalling (8.3) we can complete the proof. □

Lemma 8.4 *There exists $C > 0$ such that for the solution of (1.1) we have*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{8.7}$$

PROOF. With a bound for u in $L^p(\Omega)$ for arbitrarily large p, a straightforward reasoning involving the well-known Moser-Alikakos iteration procedure ([1]) yields (8.7). □

In what follows we shall give a simple proof of the convergence properties for the solution of (1.1) in two dimensions, which is much easier than that in three dimensions. We first assert the decay of u_t at least in some weak sense.

Lemma 8.5 *The solution of (1.1) satisfies*

$$\int_0^\infty \|u_t^2(\cdot,t)\|^2_{(W^{3,2}(\Omega))^\star}dt < \infty. \tag{8.8}$$

Proof. Using the estimates in Corollary 8.2 and proceeding as in the proof of Lemma 7.1, we directly obtain (8.8). □

We are now in the position to assert the desired convergence result for u.

Lemma 8.6 *The classical solution of (1.1) satisfies*

$$u(\cdot,t) \longrightarrow \overline{u}_0 \qquad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty, \tag{8.9}$$

where $\overline{u}_0$ is given by (1.4).

Proof. Building on the estimate (8.8), the proof of (8.9) is similar to that of Lemma 7.2. □

Finally, we can give a simple proof of the corresponding stabilization result for v.

Lemma 8.7 *The classical solution of (1.1) satisfies*

$$v(\cdot,t) \longrightarrow 0 \qquad \text{in } L^\infty(\Omega) \quad \text{as } t \to \infty. \tag{8.10}$$

Proof. From (1.2) we first obtain that

$$\overline{u}_0 > 0. \tag{8.11}$$

Using (8.9) and (8.11) and noting that (u,v) is a classical solution we find $T > 0$ such that

$$u(\cdot,t) \ge \frac{\overline{u}_0}{2} \qquad \text{for any } t \ge T. \tag{8.12}$$

This, along with the second equation in (1.1) and the positivity of v, yields

$$v_t \le \Delta v - \frac{\overline{u}_0}{2} v, \qquad x \in \Omega,\ t > T.$$

By comparison we infer that

$$0 < v(\cdot,t) \le \|v(\cdot,T)\|_{L^\infty(\Omega)} e^{-\frac{\overline{u}_0}{2}(t-T)} \qquad \text{for any } t > T.$$

This proves (8.10). □

Proof of Proposition 1.2. Proposition 1.2 is a consequence of Lemma 8.4, Lemma 8.6 and Lemma 8.7. □

Acknowledgment. Y. Tao was supported by the National Natural Science Foundation of China.

References

[1] Alikakos, N.D.: *L^p bounds of solutions of reaction-diffusion equations.* Comm. Partial Differential Equations **4**, 827-868 (1979)

[2] Alt, H.W.: *Lineare Funktionalanalysis (Fifth Ed.).* Springer, Berlin/Heidelberg 2006

[3] Aronson, D.G.: *The porous medium equation.* Nonlinear diffusion problems, Lect. 2nd 1985 Sess. C.I.M.E.. Montecatini Terme/Italy 1985, Lect. Notes Math. **1224**, 1-46 (1986)

[4] Dal Passo, R., Garcke, H., Grün, G.: *On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions.* SIAM J. Math. Anal. **29** (2), 321-342 (1998)

[5] Duan, R.J., Lorz, A., Markowich, P.A.: *Global solutions to the coupled chemotaxis-fluid equations.* Comm. Part. Differ. Eq. **35**, 1635-1673 (2010)

[6] Friedman, A.: *Partial Differential Equations.* Holt, Rinehart & Winston, New York, 1969

[7] Henry, D.: *Geometric Theory of Semilinear Parabolic Equations.* Springer-Verlag, Berlin, 1981

[8] Horstmann, D., Winkler, M.: *Boundedness vs. blow-up in a chemotaxis system.* J. Differential Equations **215** (1), 52-107 (2005)

[9] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instaility.* J. Theor. Biol. **26**, 399-415 (1970)

[10] Herrero, M.A., Velázquez, J.L.L.: *A blow-up mechanism for a chemotaxis model.* Ann. Sc. Norm. Super. Pisa Cl. Sci. **24** (4), 633-683 (1997)

[11] Hillen, Th., Painter, K.: *A users' guide to PDE models for chemotaxis.* J. Math. Biol. **58** (1-2), 183-217 (2009)

[12] Jäger, W., Luckhaus, S.: *On Explosions of Solutions to a System of Partial Differential Equations Modelling Chemotaxis.* Trans. Amer. Math. Soc. **329** (2), 819-824 (1992)

[13] Liu, J.-G., Lorz, A.: *A Coupled Chemotaxis-Fluid Model: Global Existence.* Annales de l'Institute Henri Poincaŕe. Analyse NonLinéaire, doi: 10.1016/j.anihpc.2011.04.005

[14] Ladyzenskaja, O.A., Solonnikov, V.A., Ural'ceva, N.N.: Linear and Quasi-linear Equations of Parabolic Type. AMS, Providence, 1968

[15] Rappel, W.-J., Loomis, W.F.: *Eukaryotic Chemotaxis.* Wiley Interdiscip. Rev. Syst. Biol. Med. **1** (1), 141-149 (2009)

[16] Rosen, G.: *Steady-state distribution of bacteria chemotactic toward oxygen.* Bull. Math. Biol. **40**, 641-674 (1978)

[17] Tao, Y.: *Boundedness in a chemotaxis model with oxygen consumption by bacteria.* J. Math. Anal. Appl. **381**, 521-529 (2011)

[18] Teman, R.: *Navier-Stokes equations. Theory and numerical analysis. Studies in Mathematics and its Applications. Vol. 2.* North-Holland, Amsterdam, 1977

[19] Tuval, I., Cisneros, L., Dombrowski, C., Wolgemuth, C.W., Kessler, J.O., Goldstein, R.E.: *Bacterial swimming and oxygen transport near contact lines.* Proc. Nat. Acad. Sci. USA **102**, 2277-2282 (2005)

[20] Winkler, M.: *Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity.* Math. Nachrichten **283** (11), 1664-1673 (2010)

[21] Winkler, M.: *Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops.* Comm. Part. Differ. Equations, to appear