Global existence and boundedness in a Keller-Segel-Stokes model with arbitrary porous medium diffusion

Youshan Tao*
Department of Applied Mathematics, Dong Hua University,
Shanghai 200051, P.R. China

Michael Winkler#
Institut für Mathematik, Universität Paderborn,
33098 Paderborn, Germany

Abstract

This paper deals with a boundary-value problem in two-dimensional smoothly bounded domains for the coupled chemotaxis-fluid model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n^m - \nabla \cdot (n\chi(c)\nabla c) \\ c_t + u \cdot \nabla c = \Delta c - nf(c) \\ u_t + \nabla P - \eta \Delta u + n\nabla\phi = 0 \\ \nabla \cdot u = 0, \end{cases}$$

which describes the motion of oxygen-driven swimming bacteria in an incompressible fluid. The given functions χ and f are supposed to be sufficiently smooth and such that $f(0) = 0$.
It is proved that global bounded weak solutions exist whenever $m > 1$ and the initial data (n_0, c_0, u_0) are sufficiently regular satisfying $n_0 \ge 0$ and $c_0 \ge 0$. This extends a recent result by Di Francesco, Lorz and Markowich (*Discrete Cont. Dyn. Syst. A* **28** (2010)) which asserts global existence of weak solutions under the constraint $m \in (\frac{3}{2}, 2]$.

Key words: chemotaxis, Stokes, porous medium diffusion, global existence, boundedness
AMS Classification: 35K55, 35Q92, 35Q35, 92C17

*taoys@dhu.edu.cn
#michael.winkler@uni-due.de

1 Introduction

We consider a mathematical model for the motion of oxygen-driven swimming bacteria in an incompressible fluid. In addition to random diffusion, bacteria often swim upwards an oxygen gradient to survive, a process which may be referred to as *oxygentaxis*. On the other hand, bacteria often live in a viscous fluid so that they are also transported with the fluid. The motion of the fluid is under the influence of gravitational force exerted from aggregating bacteria onto the fluid. Typically, the motion of the fluid is modeled by incompressible Navier-Stokes equations or Stokes equations. The oxygen also diffuses and is transported by the fluid. Unlike in the classical Keller-Segel model for chemotactic movement ([7]), in the present context the oxygen is consumed, rather than produced, by the bacteria.

Taking into account all these processes, to describe the above biological phenomena the authors in [23] proposed the model

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \\ u_t + u \cdot \nabla u + \nabla P - \eta \Delta u + n\nabla\phi = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

where n and c denote the bacterium density and the oxygen concentration, respectively, and u represents the velocity field of the fluid subject to an incompressible Navier-Stokes equation with pressure P and viscosity η and a gravitational force $\nabla\phi$. The function $\chi(c)$ measures the chemotactic sensitivity, $f(c)$ is the consumption rate of the oxygen by the bacteria, and ϕ is a given potential function. In [23] the authors numerically studied the model (1.1) and performed experiments showing large scale convection patterns. In [13] the author proved local existence of solutions to (1.1), whereas in [4] the authors proved global existence of classical solutions near constant states in three space dimensions.

When the fluid motion is slow, we can use the Stokes equations instead of the Navier-Stokes equations. The accordingly simplified chemotaxis-fluid model takes the form

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (n\chi(c)\nabla c), \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \\ u_t + \nabla P - \eta \Delta u + n\nabla\phi = 0, \\ \nabla \cdot u = 0, \end{cases} \tag{1.2}$$

where compared with (1.1), the nonlinear convective term $u \cdot \nabla u$ is ignored in the u-equation of (1.2). In [4] the authors addressed the two-dimensional version of (1.2) and proved global existence of certain weak solutions under suitable smallness assumptions on either ϕ or $c(x, 0)$. To the best of our knowledge, the question of global solvability of (1.2) with large data is still open.

Since the diffusion of bacteria (or, more generally, of cells) in a viscous fluid is more like movement in a porous medium (see the discussions in [24], [20], [2] and [8], for instance), the authors in [3] extended the model (1.2) to one with a porous medium-type diffusion of bacteria according to

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n^m - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + \nabla P - \eta\Delta u + n\nabla\phi = 0, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0. \end{cases} \tag{1.3}$$

We shall subsequently consider this system along with the boundary conditions

$$\partial_\nu n^m(x,t) = \partial_\nu c(x,t) = 0 \quad \text{and} \quad u(x,t)=0, \quad x\in\partial\Omega,\ t>0, \tag{1.4}$$

and the initial conditions

$$n(x,0)=n_0(x)\ge 0, \quad c(x,0)=c_0(x)\ge 0, \quad u(x,0)=u_0(x), \quad x\in\Omega, \tag{1.5}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, ν is the outward normal unit vector to $\partial\Omega$, $m>1$ is a constant, and P is the Lagrangian multiplier associated to $\nabla\cdot u=0$. Throughout this paper we shall assume that

$$\begin{cases} \chi\in C^1([0,+\infty)) \quad \text{is nonnegative,} \\ f\in C^1([0,\infty)) \quad \text{satisfies } f(0)=0 \text{ and } f(c)>0 \text{ for all } c>0, \qquad \text{and} \\ \phi\in W^{1,\infty}(\Omega). \end{cases} \tag{1.6}$$

Under the assumptions that $3/2<m\le 2$ and that $n_0^m\in W^{1,2}(\Omega), c_0\in W^{1,2}(\Omega)\cap L^\infty(\Omega)$ and $u_0\in W^{1,2}(\Omega)$, the authors in [3] proved existence of a global weak solution to problem (1.3)-(1.5) (see Definition 2.1 below). Their proof is crucially based on a free-energy inequality. By using a different method involving more general entropy-like functionals (cf. Section 2), the present paper will extend the above result so as to cover the entire range $m>1$, and apart from that we shall assert global boundedness of the solutions obtained. More precisely, let us assume henceforth that

$$\begin{cases} n_0\in L^\infty(\Omega) \text{ and } c_0\in W^{1,\infty}(\Omega) \text{ are nonnegative, and that} \\ u_0\in D(A^\theta) \qquad \text{for some } \theta>\frac{1}{2}, \end{cases} \tag{1.7}$$

where A^θ denotes the – possibly fractional – power of the usual Stokes operator A in the Hilbert space $L^2_\sigma(\Omega) := \{u\in L^2(\Omega) \mid \nabla\cdot u = 0 \text{ in } \mathcal{D}'(\Omega)\}$ of all solenoidal vector fields over Ω, with domain $D(A)=W^{2,2}(\Omega)\cap W^{1,2}_0(\Omega)\cap L^2_\sigma(\Omega)$ ([18]). Under these assumptions, we shall derive the following.

Theorem 1.1 *Suppose that $m>1$, and that the triple (n_0,c_0,u_0) satisfies (1.7). Then there exists at least one global weak solution (n,c,u,P) of (1.3)-(1.5) such that (n,c,u) is bounded in $(L^\infty(\Omega\times(0,\infty)))^4$ and such that $n\ge 0$ and $c\ge 0$ in $\Omega\times(0,\infty)$.*

Since it is well-known that porous medium-type diffusion in general does not allow for classical solvability ([24]), the above statement on weak solvability seems to be the best available. Thereby the picture in space dimension two becomes complete in that it shows that there does *not* exist a critical exponent $m^\star > 1$ such that blow-up occurs when $m < m^\star$, as suspected in [3].

From a mathematical point of view, it seems worthwhile observing that there are several results in the literature that address the interaction of nonlinear diffusive movement of cells with the destabilizing mechanism of cross-diffusion. For instance, consider the standard Keller-Segel system in bounded domains $\Omega \subset \mathbb{R}^N$ with nonlinear diffusion and nonlinear cross-diffusion,

$$\begin{cases} n_t = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (S(n)\nabla c), \\ c_t = \Delta c - c + n, \end{cases}$$

under the assumption that $D(n)$ does not decay faster than algebraically as $n \to \infty$. Then known results say that solutions remain bounded whenever the (self-) difusivity D is large enough, as related to S, at large densities: Namely, if $\frac{S(n)}{D(n)} \le cn^{\frac{2}{N}-\varepsilon}$ holds for some $c > 0, \varepsilon > 0$ and all large n, then all solutions are global in time and bounded ([16], [22], [6], [9]), whereas if $\frac{S(n)}{D(n)} \ge cn^{\frac{2}{N}+\varepsilon}$ for some $c > 0, \varepsilon > 0$ and large n, then there exist blow-up solutions ([25], [26]). Similar results and more detailed information about the behavior of blow-up solutions are available in the corresponding Cauchy problem when $\Omega = \mathbb{R}^N$ and D and S are of exact power type (see [19] and the references therein).
As to a related system with a more involved interplay between its components, recently in [21] the question of global solvability for a chemotaxis-haptotaxis model of cancer invasion was studied, with the corresponding self-diffusivity $D(n)$ of cells growing like n^m as $n \to \infty$ for some $m > 1$. Again, the conlcusion is that large values of m seem to enhance the tendency towards global solvability.

The crucial technical difference between our approach and that in [3] appears to consist of the fact that we will not address the regularity questions for n, c and u separately. Indeed, we shall rather chain n to c in deriving a differential inequality for the functional

$$\int_\Omega n^\gamma(x,t)dx + \int_\Omega |\nabla c|^2(x,t)dx, \qquad t > 0,$$

with suitably large $\gamma > 0$ (see Lemma 2.6 below). This can be achieved in a way inspired by that pursued in [6] for a pure chemotaxis system, because it turns out that the coupling to the Stokes equation is mild enough for this purpose (cf. Lemma 2.3). Once this crucial step has been accomplished, the remaining part is a straightforward rearrangement of standard arguments ([3]): A Moser-type iteration will allow to pass to a time-independent a priori estimate for n in $L^\infty(\Omega)$. This will provide sufficient regularity to pass to the limit in some sequence of conveniently regularized problems to finally obtain a global and bounded weak solution.

2 A priori estimates

From [3] we adopt the following solution concept.

Definition 2.1 *(weak solution). Let $T \in (0,\infty)$. A quadruple (n,c,u,P) is said to be a weak solution to problem (1.3)-(1.5) in $\Omega \times (0,T)$ if*
(1) $n \in L^\infty(\Omega \times (0,T)), \nabla n^m \in L^2((0,T); L^2(\Omega))$ and $n_t \in L^2((0,T); (W^{1,2}(\Omega))')$,
(2) $c \in L^\infty(\Omega \times (0,T)) \cap L^2((0,T); W^{2,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$,
(3) $u \in L^2((0,T); W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega))$,
(4) The identities

$$\begin{aligned}
&\int_0^T \int_\Omega n_t\psi - \int_0^T \int_\Omega \nabla\psi \cdot un + \int_0^T \int_\Omega \nabla n^m \cdot \nabla\psi = \int_0^T \int_\Omega n\chi(c)\nabla c \cdot \nabla\psi, \\
&\int_0^T \int_\Omega c_t\psi - \int_0^T \int_\Omega \nabla\psi \cdot uc + \int_0^T \int_\Omega \nabla c \cdot \nabla\psi = -\int_0^T \int_\Omega nf(c)\psi, \\
&-\int_0^T \int_\Omega \tilde\psi_t \cdot u + \int_\Omega \tilde\psi \cdot u_0 - \eta \int_0^T \int_\Omega u \cdot \Delta\tilde\psi + \int_0^T \int_\Omega n\nabla\phi \cdot \tilde\psi = 0
\end{aligned}$$

hold for all $\psi \in L^2((0,T); W^{1,2}(\Omega))$ and any $\tilde\psi \in L^2((0,T); W^{2,2}(\Omega)) \cap W^{1,2}((0,T); L^2(\Omega))$ with values in $\mathbb{R}^2$, $\nabla \cdot \tilde\psi = 0$ and $\tilde\psi|_{\partial\Omega} = 0$. If (n,c,u,P) is a weak solution of (1.3)-(1.5) in $\Omega \times (0,T)$ for any $T \in (0,\infty)$, then we call (n,c,u,P) a global weak solution.

In view of the regularization of (1.3)-(1.5) we have in mind (cf. (3.1) below), the a priori estimates in this section will be derived for the slightly more general system given by

$$\begin{cases}
n_t + u \cdot \nabla n = \nabla \cdot (D(n)\nabla n) - \nabla \cdot (n\chi(c)\nabla c), \qquad x \in \Omega,\ t > 0, \\
c_t + u \cdot \nabla c = \Delta c - nf(c), \qquad x \in \Omega,\ t > 0, \\
u_t + \nabla P - \eta\Delta u + n\nabla\phi = 0, \qquad x \in \Omega,\ t > 0, \\
\nabla \cdot u = 0, \qquad x \in \Omega,\ t > 0, \\
D(n)\partial_\nu n(x,t) = \partial_\nu c(x,t) = 0 \quad \text{and} \quad u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0, \\
n(x,0) = n_0(x), \quad c(x,0) = c_0(x) \quad \text{and} \quad u(x,0) = u_0(x), \qquad x \in \Omega,
\end{cases} \tag{2.1}$$

where D satisfies

$$D \in C^1([0,\infty)) \qquad \text{and} \qquad D(s) \geq D_0 s^{m-1} \quad \text{for all } s \geq 0 \tag{2.2}$$

with some constants $D_0 > 0$ and $m > 1$. In order to be able to deal with classical approximate solutions, we require the initial data to be such that

$$\begin{cases}
n_0 \in C^1(\bar\Omega) \text{ and } c_0 \in C^1(\bar\Omega) \text{ are nonnegative, and that} \\
u_0 \in D(A),
\end{cases} \tag{2.3}$$

and that

$$\|n_0\|_{L^\infty(\Omega)} \leq K, \quad \|c_0\|_{W^{1,\infty}(\Omega)} \leq K \quad \text{and} \quad \|A^\theta u_0\|_{L^2(\Omega)} \leq K \tag{2.4}$$

for some $K > 0$, where $\theta > 0$ is as in (1.7).

Let us first state two basic estimates concerning n and c.

Lemma 2.1 *Suppose that (2.3) and (2.2) hold with some $D_0 > 0$ and $m > 1$, and that (n, c, u, P) is a classical solution of (2.1) in $\Omega \times (0, T)$ for some $T \in (0, \infty]$. Then $n \geq 0$ and $c \geq 0$ in $\Omega \times (0, T)$ and*

$$\|n(t)\|_{L^1(\Omega)} = \|n_0\|_{L^1(\Omega)} \qquad \textit{for all } t \in (0, T) \tag{2.5}$$

and

$$\|c(t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)}, \qquad \textit{for all } t \in (0, T). \tag{2.6}$$

Proof. Parabolic comparison immediately yields nonnegativity of both n and c as well as the inequality $c \leq \|c_0\|_{L^\infty(\Omega)}$ in $\Omega \times (0, T)$. Thereupon, the identity (2.5) results from an integration of the first equation in (2.1) in space. □

2.1 Separate differential inequalities for $\int_\Omega n^\gamma$ and $\int_\Omega |\nabla c|^2$

We proceed to derive a differential inequality for $t \mapsto \int_\Omega n^\gamma(\cdot, t)$ for sufficiently large γ.

Lemma 2.2 *Given any $m > 1$, $D_0 > 0$, $K > 0$, $p > 1$ and $\gamma > \max\{1, m - 1 + \frac{1}{p}\}$, there exists $C > 0$ with the following property. If D and (n_0, c_0, u_0) satisfy (2.2), (2.3) and (2.4), and if (n, c, u, P) is a classical solution of (2.1) in $\Omega \times (0, T)$ for some $T \in (0, \infty]$, then*

$$\begin{aligned} \frac{d}{dt}\int_\Omega n^\gamma + \frac{1}{C}\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 & \\ \leq \ C\Big\{\Big(\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2\Big)^{\frac{-m+\gamma+1-\frac{1}{p}}{m+\gamma-1}} + 1\Big\} \cdot \Big\{\Big(\int_\Omega |\Delta c|^2\Big)^{\frac{1}{p}} + 1\Big\} + C & \end{aligned} \tag{2.7}$$

for all $t \in (0, T)$.

Proof. We fix $p > 1$ and $\gamma > \max\{1, m - 1 + \frac{1}{p}\}$. Then testing the first equation in (2.1) against $n^{\gamma-1}$ and using (2.2) yields

$$\begin{aligned} \frac{1}{\gamma}\frac{d}{dt}\int_\Omega n^\gamma &= -(\gamma-1)\int_\Omega n^{\gamma-2}D(n)|\nabla n|^2 + (\gamma-1)\int_\Omega n^{\gamma-1}\chi(c)\nabla n \cdot \nabla c - \frac{1}{\gamma}\int_\Omega u \cdot \nabla n^\gamma \\ &\leq -D_0(\gamma-1)\int_\Omega n^{m+\gamma-3}|\nabla n|^2 + (\gamma-1)\int_\Omega n^{\gamma-1}\chi(c)\nabla n \cdot \nabla c \end{aligned}$$

for all $t \in (0, T)$, because $\nabla \cdot u \equiv 0$ implies that $\int_\Omega u \cdot \nabla n^\gamma = 0$. By Young's inequality, we can estimate

$$\begin{aligned} (\gamma-1)\int_\Omega n^{\gamma-1}\chi(c)\nabla n \cdot \nabla c &\leq \frac{D_0(\gamma-1)}{2}\int_\Omega n^{m+\gamma-3}|\nabla n|^2 \\ &\quad + \frac{(\gamma-1)\|\chi\|_{L^\infty(\Omega)}}{2D_0}\int_\Omega n^{-m+\gamma+1}|\nabla c|^2 \end{aligned}$$

and thus obtain

$$\frac{d}{dt}\int_\Omega n^\gamma + C_1\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 \leq C_2\int_\Omega n^{-m+\gamma+1}|\nabla c|^2 \qquad \text{for all } t \in (0, T), \tag{2.8}$$

where, as throughout the rest of the proof, $C_1, C_2, ...$ denote positive constants depending on $m, D_0, p, \gamma, \Omega$ and the initial data only.
Now the Hölder inequality asserts that

$$\begin{aligned}\int_\Omega n^{-m+\gamma+1}|\nabla c|^2 &\leq \Big(\int_\Omega n^{(-m+\gamma+1)\cdot p}\Big)^{\frac{1}{p}} \cdot \Big(\int_\Omega |\nabla c|^{2p'}\Big)^{\frac{1}{p'}} \\ &= \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2(-m+\gamma+1)p}{m+\gamma-1}}(\Omega)}^{\frac{2(-m+\gamma+1)}{m+\gamma-1}} \cdot \|\nabla c\|_{L^{2p'}(\Omega)}^2, \qquad (2.9)\end{aligned}$$

where $p' := \frac{p}{p-1}$. Here, the Gagliardo-Nirenberg inequality ([14]) yields $C_3 > 0$ such that

$$\begin{aligned}\|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2(-m+\gamma+1)p}{m+\gamma-1}}(\Omega)}^{\frac{2(-m+\gamma+1)}{m+\gamma-1}} &\leq C_3\|\nabla n^{\frac{m+\gamma-1}{2}}\|_{L^2(\Omega)}^{\frac{2(-m+\gamma+1)}{m+\gamma-1}\cdot a} \cdot \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2}{m+\gamma-1}}(\Omega)}^{\frac{2(-m+\gamma+1)}{m+\gamma-1}\cdot(1-a)} \\ &\quad +C_3\|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2}{m+\gamma-1}}(\Omega)}^{\frac{2(-m+\gamma+1)}{m+\gamma-1}},\end{aligned}$$

where

$$a = \frac{2\cdot\frac{m+\gamma-1}{2}\cdot(1-\frac{1}{(-m+\gamma+1)p})}{1-\frac{2}{2}+\frac{2(m+\gamma-1)}{2}} = 1 - \frac{1}{(-m+\gamma+1)p}$$

satisfies $0 < a < 1$ since $\gamma > m-1+\frac{1}{p}$. Thus, recalling (2.5) we see that for some $C_4 > 0$,

$$\|n^{\frac{m+\gamma-1}{2}}(\cdot,t)\|_{L^{\frac{2(-m+\gamma+1)p}{m+\gamma-1}}(\Omega)}^{\frac{2(-m+\gamma+1)}{m+\gamma-1}} \leq C_4\|\nabla n^{\frac{m+\gamma-1}{2}}(\cdot,t)\|_{L^2(\Omega)}^{\frac{2(-m+\gamma+1)-\frac{2}{p}}{m+\gamma-1}} + C_4 \qquad (2.10)$$

for all $t \in (0,T)$. Similarly, we use a Gagliardo-Nirenberg inequality ([5]) to obtain

$$\|\nabla c\|_{L^{2p'}(\Omega)}^2 \leq C_5\|\Delta c\|_{L^2(\Omega)}^{2b} \cdot \|c\|_{L^\infty(\Omega)}^{2(1-b)} + C_5\|c\|_{L^\infty(\Omega)}^2$$

with some $C_5 > 0$ and

$$b = \frac{1-\frac{2}{2p'}}{2-\frac{2}{2}} = 1-\frac{1}{p'} = \frac{1}{p},$$

whence by (2.6) we infer that there exists $C_6 > 0$ such that

$$\|\nabla c(\cdot,t)\|_{L^{2p'}(\Omega)}^2 \leq C_6\|\Delta c(\cdot,t)\|_{L^2(\Omega)}^{\frac{2}{p}} + C_6 \qquad \text{for all } t \in (0,T).$$

Combined with (2.8), (2.9) and (2.10), this yields (2.7). □

In order to cope with the term containing c on the right of (2.7), in Lemma 2.4 below we shall use the second PDE in (2.1) to derive an ODI of a similar flavor. Since unlike in the previous lemma there will remain a term containing the fluid component u (cf. (2.14) and (2.16)), let us first establish an estimate for u in $L^r(\Omega)$, $r < \infty$, in terms of a convenient norm of n.

Lemma 2.3 *Assume that (2.3) holds, and that $r \in [1,\infty]$. Then there exists $C > 0$ with the property that whenever (2.2) holds with some $D_0 > 0$ and $m > 1$ and (n,c,u,P) is a classical solution of (2.1) in $\Omega \times (0,T)$ for some $T \in (0,\infty]$, we have*

$$\|u(\cdot,t)\|_{L^r(\Omega)} + \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \le C + C \cdot \sup_{s\in(0,T)} \|n(\cdot,s)\|_{L^2(\Omega)} \qquad \text{for all } t \in (0,T). \quad (2.11)$$

Proof. It is well-known (cf. [15, pp. 114] or [18], for instance) that the Stokes operator $A = -\eta\mathcal{P}\Delta$, with $\mathcal{P}$ denoting the Helmholtz projection in $L^2(\Omega)$, is sectorial and generates a contraction semigroup $(e^{-tA})_{t\ge0}$ in $L^2(\Omega)$ with its operator norm bounded according to

$$\|e^{-tA}\| \le C_1 e^{-\mu t} \qquad \text{for all } t \ge 0$$

with some $C_1 > 0$ and $\mu > 0$. We now pick θ as in (1.7) and apply the fractional power A^θ to the variation-of-constants formula

$$u(\cdot,t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}(n(\cdot,s)\nabla\phi)ds, \qquad t \in (0,T),$$

to find $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned} \|A^\theta u(\cdot,t)\|_{L^2(\Omega)} &\le C_2 \cdot \Big(1 + \int_0^t (t-s)^{-\theta}e^{-\mu(t-s)}\|n(\cdot,s)\|_{L^2(\Omega)}ds\Big) \\ &\le C_2 \cdot \Big(1 + \sup_{s\in(0,T)} \|n(\cdot,s)\|_{L^2(\Omega)} \cdot \int_0^\infty \sigma^{-\theta}e^{-\mu\sigma}d\sigma\Big) \\ &\le C_3 + C_3 \cdot \sup_{s\in(0,T)} \|n(\cdot,s)\|_{L^2(\Omega)} \qquad \text{for all } t \in (0,T). \quad (2.12) \end{aligned}$$

Since $D(A^\theta) \hookrightarrow W^{1,q}(\Omega)$ for any $q \ge 2$ satisfying $2\theta + \frac{2}{q} \ge 2$ (see [18, Lemma 2.4.3]), it follows that $D(A^\theta)$ is continuously embedded into both $W^{1,2}(\Omega)$ and $L^\infty(\Omega)$. Therefore (2.11) results from (2.12). □

We can now estimate ∇c appropriately.

Lemma 2.4 *For all $m > 1$, $D_0 > 0$, $K > 0$ and $\gamma > 2$ we can find $C > 0$ and $\kappa \in (0,1)$ such that the following holds. If D and (n_0,c_0,u_0) satisfy (2.2), (2.3) and (2.4), and if (n,c,u,P) is a classical solution of (2.1) in $\Omega \times (0,T)$ for some $T \in (0,\infty]$, then*

$$\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 \le C \cdot \Big(\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2\Big)^{\frac{1}{m+\gamma-1}} + C \cdot \Big(\sup_{s\in(0,T)} \int_\Omega n^\gamma(\cdot,s)\Big)^\kappa + C \quad (2.13)$$

for all $t \in (0,T)$.

Proof. Using $-\Delta c$ as a test function in the second equation in (2.1), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 = \int_\Omega nf(c)\Delta c + \int_\Omega (u\cdot\nabla c)\Delta c \qquad \text{for all } t \in (0,T), \quad (2.14)$$

where Young's inequality and the boundedness of c asserted by Lemma 2.1 yield

$$\int_\Omega nf(c)\Delta c \le \frac{1}{6}\int_\Omega |\Delta c|^2 + C_1\int_\Omega n^2 \tag{2.15}$$

with some $C_1 > 0$. Also by Young's inequality, we find $C_2 > 0$ such that

$$\int_\Omega (u\cdot\nabla c)\Delta c \le \frac{1}{6}\int_\Omega |\Delta c|^2 + C_2\int_\Omega |u|^2|\nabla c|^2 \qquad \text{for all } t\in(0,T). \tag{2.16}$$

We now pick any number $q > \frac{2(\gamma-1)}{\gamma-2}$ and thereby obtain that

$$\kappa := \frac{q}{(q-2)(\gamma-1)}$$

satisfies $0 < \kappa < 1$. According to the Hölder inequality,

$$C_2\int_\Omega |u|^2|\nabla c|^2 \le C_2\|u\|^2_{L^q(\Omega)}\cdot\|\nabla c\|^2_{L^{\frac{2q}{q-2}}(\Omega)} \qquad \text{for all } t\in(0,T),$$

where an application of the Gagliardo-Nirenberg inequality yields

$$\begin{aligned}
\|\nabla c\|^2_{L^{\frac{2q}{q-2}}(\Omega)} &\le C_3\|\Delta c\|^{\frac{4}{q}}_{L^2(\Omega)}\cdot\|c\|^{\frac{2(q-2)}{q}}_{L^\infty(\Omega)} + C_4\|c\|^2_{L^\infty(\Omega)} \\
&\le C_4\|\Delta c\|^{\frac{4}{q}}_{L^2(\Omega)} + C_4 \qquad \text{for all } t\in(0,T)
\end{aligned}$$

for some $C_3 > 0$ and $C_4 > 0$ in view of (2.6). Since $q > 2$, by Young's inequality we therefore find $C_5 > 0$ fulfilling

$$C_2\int_\Omega |u|^2|\nabla c|^2 \le \frac{1}{6}\int_\Omega |\Delta c|^2 + C_5\|u\|^{\frac{2q}{q-2}}_{L^q(\Omega)} + C_5 \qquad \text{for all } t\in(0,T). \tag{2.17}$$

To estimate the term containing u, we recall Lemma 2.3 to gain $C_6 > 0$ such that

$$C_5\|u\|^{\frac{2q}{q-2}}_{L^q(\Omega)} \le C_6\cdot\Big(1+\sup_{s\in(0,T)}\|n(\cdot,s)\|^{\frac{2q}{q-2}}_{L^2(\Omega)}\Big) \qquad \text{for all } t\in(0,T). \tag{2.18}$$

Here, using the Hölder inequality and (2.5) we can interpolate to infer that with some $C_7 > 0$,

$$\begin{aligned}
\|n(\cdot,s)\|^{\frac{2q}{q-2}}_{L^2(\Omega)} &\le \|n(\cdot,s)\|^{\frac{q\gamma}{(q-2)(\gamma-1)}}_{L^\gamma(\Omega)}\cdot\|n(\cdot,s)\|^{\frac{q(\gamma-2)}{(q-2)(\gamma-1)}}_{L^1(\Omega)} \\
&\le C_7\cdot\Big(\sup_{s\in(0,T)}\int_\Omega n^\gamma(\cdot,s)\Big)^\kappa \qquad \text{for all } s\in(0,T)
\end{aligned}$$

according to our definition of κ. From (2.14)-(2.18) we thus see that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \frac{1}{2}\int_\Omega |\Delta c|^2 \le C_1\int_\Omega n^2 + C_5 + C_6 + C_7\cdot\Big(\sup_{s\in(0,T)}\int_\Omega n^\gamma(\cdot,s)\Big)^\kappa \tag{2.19}$$

for all $t \in (0,T)$. Finally, again by the Gagliardo-Nirenberg inequality we find $C_8 > 0$ satisfying

$$\begin{aligned}
\int_\Omega n^2 &= \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{4}{m+\gamma-1}}(\Omega)}^{\frac{4}{m+\gamma-1}} \\
&\le C_8 \|\nabla n^{\frac{m+\gamma-1}{2}}\|_{L^2(\Omega)}^{\frac{2}{m+\gamma-1}} \cdot \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2}{m+\gamma-1}}(\Omega)}^{\frac{2}{m+\gamma-1}} + C_8 \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2}{m+\gamma-1}}(\Omega)}^{\frac{4}{m+\gamma-1}} \qquad (2.20)
\end{aligned}$$

Hence, (2.5) entails that

$$\int_\Omega n^2 \le C_9 \|\nabla n^{\frac{m+\gamma-1}{2}}\|_{L^2(\Omega)}^{\frac{2}{m+\gamma-1}} + C_9 \qquad \text{for all } s \in (0,T), \qquad (2.21)$$

which combined with (2.19) yields (2.13). □

2.2 A combined entropy-type estimate

The following elementary ingredient for Lemma 2.6 can be proved by twice applying Young's inequality.

Lemma 2.5 *Let $\alpha > 0$ and $\beta > 0$ be such that $\alpha + \beta < 1$. Then for all $\varepsilon > 0$ there exists $C > 0$ such that*

$$x^\alpha y^\beta \le \varepsilon \cdot (x+y) + C \qquad \textit{for all } x \ge 0 \textit{ and } y \ge 0.$$

Now a combination of Lemma 2.2 and Lemma 2.4 yields a differential inequality for $t \mapsto \int_\Omega n^\gamma(\cdot,t) + \int_\Omega |\nabla c(\cdot,t)|^2$ which turns out to be favorable for our purpose due to the fact that $m > 1$.

Lemma 2.6 *Let $m > 1$, $D_0 > 0$, $K > 0$ and $\gamma > \max\{2, m-1\}$. Then there exist $C > 0$ and $\kappa \in (0,1)$ such that if D and (n_0, c_0, u_0) satisfy (2.2), (2.3) and (2.4), and if (n,c,u,P) is a classical solution of (2.1) in $\Omega \times (0,T)$ for some $T \in (0,\infty]$, we have*

$$\begin{aligned}
\frac{d}{dt}\Big\{ \int_\Omega n^\gamma + \int_\Omega |\nabla c|^2 \Big\} &+ \frac{1}{C} \cdot \Big\{ \int_\Omega n^\gamma + \int_\Omega |\nabla c|^2 \Big\} \\
&+ \frac{1}{C} \cdot \Big\{ \int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 + \int_\Omega |\Delta c|^2 \Big\} \\
&\le C + C \cdot \Big(\sup_{s \in (0,T)} \int_\Omega n^\gamma(\cdot,s) \Big)^\kappa \qquad (2.22)
\end{aligned}$$

for all $t \in (0,T)$.

Proof. Given $\gamma > \max\{2, m-1\}$, we let

$$\alpha(\xi) := \frac{-m+\gamma+1-\xi}{m+\gamma-1} \quad \text{and} \quad \beta(\xi) := \xi \qquad \text{for } \xi \ge 0.$$

Then using our assumption $m > 1$, we see that

$$\alpha(0) + \beta(0) = \frac{-m+\gamma+1}{m+\gamma-1} < 1,$$

whence by a continuity argument we can pick $\xi_0 \in (0,1)$ such that

$$\alpha(\xi) + \beta(\xi) < 1 \qquad \text{for all } \xi \in (0, \xi_0).$$

Now choosing a number $p > 1$ such that $p > \frac{1}{\xi_0}$ and $\gamma > m - 1 + \frac{1}{p}$, for $\alpha := \alpha(\frac{1}{p})$ and $\beta := \beta(\frac{1}{p})$ we thus have

$$\alpha + \beta < 1, \tag{2.23}$$

and moreover Lemma 2.2 becomes applicable. Combined with Lemma 2.4, it provides $C_1 > 0$, $C_2 > 0$ and $\kappa \in (0,1)$ such that

$$\begin{aligned}
\frac{d}{dt}\Big\{ \int_\Omega n^\gamma + \int_\Omega |\nabla c|^2 \Big\} \quad &+ \quad C_1 \Big\{ \int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 + \int_\Omega |\Delta c|^2 \Big\} \\
&\le \quad C_2 \Big(\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 \Big)^\alpha \cdot \Big(\int_\Omega |\Delta c|^2 \Big)^\beta \\
&\quad\ + C_2 \Big(\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 \Big)^{\frac{1}{m+\gamma-1}} \\
&\quad\ + C_2 \Big(\int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 \Big)^\alpha + C_2 \Big(\int_\Omega |\Delta c|^2 \Big)^\beta \\
&\quad\ + C_2 \Big(\sup_{s\in(0,T)} \int_\Omega n^\gamma(\cdot,s) \Big)^\kappa + C_2
\end{aligned}$$

for all $t \in (0,T)$. In view of the fact that $\frac{1}{m+\gamma-1} < 1$, and since clearly $\alpha < 1$ and $\beta < 1$, Lemma 2.5 and Young's inequality yield $C_3 > 0$ such that

$$\frac{d}{dt}\Big\{ \int_\Omega n^\gamma + \int_\Omega |\nabla c|^2 \Big\} + \frac{C_1}{2} \cdot \Big\{ \int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 + \int_\Omega |\Delta c|^2 \Big\} \le C_3 + C_2 \Big(\sup_{s\in(0,T)} \int_\Omega n^\gamma(\cdot,s) \Big)^\kappa \tag{2.24}$$

for all $t \in (0,T)$. Now using the Gagliardo-Nirenberg and Young's inequality in quite the same way as in (2.20) and (2.21), thanks to (2.5) we find $C_4 > 0$, $C_5 > 0$ and $C_6 > 0$ fulfilling

$$\begin{aligned}
\int_\Omega n^\gamma \quad &= \quad \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2\gamma}{m+\gamma-1}}(\Omega)}^{\frac{2\gamma}{m+\gamma-1}} \\
&\le \quad C_4 \|\nabla n^{\frac{m+\gamma-1}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma-1)}{m+\gamma-1}} \cdot \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2}{m+\gamma-1}}(\Omega)}^{\frac{2}{m+\gamma-1}} + C_4 \|n^{\frac{m+\gamma-1}{2}}\|_{L^{\frac{2}{m+\gamma-1}}(\Omega)}^{\frac{2\gamma}{m+\gamma-1}} \\
&\le \quad C_5 \|\nabla n^{\frac{m+\gamma-1}{2}}\|_{L^2(\Omega)}^{\frac{2(\gamma-1)}{m+\gamma-1}} + C_5 \\
&\le \quad C_6 \int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 + C_6,
\end{aligned}$$

because $\frac{2(\gamma-1)}{m+\gamma-1} < 2$ due to the fact that $m > 0$. Proceeding similarly, from (2.6) we derive the existence of $C_7 > 0$ such that

$$\int_\Omega |\nabla c|^2 \le C_7 \int_\Omega |\Delta c|^2 + C_7.$$

Therefore, (2.24) shows that

$$\begin{aligned} \frac{d}{dt}\Big\{ \int_\Omega n^\gamma + \int_\Omega |\nabla c|^2 \Big\} &+ \frac{C_1}{4C_6} \int_\Omega n^\gamma + \frac{C_1}{4C_7} \int_\Omega |\nabla c|^2 \\ &+ \frac{C_1}{4} \cdot \Big\{ \int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 + \int_\Omega |\Delta c|^2 \Big\} \\ &\le \frac{C_1}{2} + C_3 + C_2 \Big(\sup_{s\in(0,T)} \int_\Omega n^\gamma(\cdot,s) \Big)^\kappa \end{aligned}$$

for all $t \in (0,T)$, from which (2.22) easily follows. $\square$

2.3 Boundedness properties of n and ∇c

Integrating the inequality from Lemma 2.6 and using the fact that $\kappa < 1$ in (2.22) yields a bound for n in $L^\infty((0,T); L^\gamma(\Omega))$ for all $\gamma < \infty$.

Corollary 2.7 *For each $m > 1$, $D_0 > 0$, $K > 0$ and $\gamma > \max\{2, m-1\}$ there exists $C > 0$ with the property that if D and (n_0, c_0, u_0) satisfy (2.2), (2.3) and (2.4), and if (n, c, u, P) is a classical solution of (2.1) in $\Omega \times (0,T)$ for some $T \in (0,\infty]$, then*

$$\int_\Omega n^\gamma(x,t)dx \le C \quad \textit{and} \quad \int_\Omega |\nabla c(x,t)|^2 dx \le C \qquad \textit{for all } t \in (0,T), \tag{2.25}$$

and moreover

$$\int_0^t \int_\Omega |\nabla n^{\frac{m+\gamma-1}{2}}|^2 \le C(1+t) \quad \textit{for all } t \in (0,T). \tag{2.26}$$

Proof. We apply Lemma 2.6 which in particular states that $y(t) := \int_\Omega n^\gamma(\cdot,t) + \int_\Omega |\nabla c(\cdot,t)|^2,\ t \in [0,T)$, satisfies the ODI

$$y'(t) \le -C_1 y(t) + C_2 M^\kappa + C_3 \qquad \text{for all } t \in (0,T)$$

with some constants $C_1 > 0, C_2 > 0, C_3 > 0, \kappa \in (0,1)$ and $M := \sup_{t\in(0,T)} y(t)$. A straightforward ODE comparison argument shows that

$$y(t) \le \max\Big\{ y(0), \frac{C_2 M^\kappa + C_3}{C_1} \Big\} \qquad \text{for all } t \in (0,T),$$

which upon taking the supremum over $t \in (0,T)$ implies that

$$M \le \max\Big\{ \Big(\frac{C_1 y_0}{C_2}\Big)^{\frac{1}{\kappa}}, \Big(\frac{C_2+1}{C_1}\Big)^{\frac{1}{1-\kappa}}, C_3^{\frac{1}{\kappa}} \Big\}.$$

This immediately leads to (2.25). Inserted into (2.22), this easily yields (2.26). $\square$

Combining (2.11) and (2.25), we obtain

$$\int_\Omega |u(x,t)\cdot\nabla c(x,t)|^2 \le C \qquad \text{for all } t\in(0,T).$$

This, along with (2.6), the first estimate in (2.25) and the second equation in (2.1), yields (cf. [9, Lemma 1] and [6, Lemma 4.1], for instance)

$$\int_\Omega |\nabla c(x,t)|^r \le C \qquad \text{for all } t\in(0,T) \text{ and any } r\in[2,\infty).$$

Then, by a straightforward iteration procedure of Moser-Alikakos type (cf. [1] and [22] for details), we finally arrive at the following L^∞ estimate for n.

Corollary 2.8 *For all $m>1$, $D_0>0$ and $K>0$ we can pick $C>0$ such that whenever D and (n_0,c_0,u_0) satisfy (2.2), (2.3) and (2.4), and whenever (n,c,u,P) is a classical solution of (2.1) in $\Omega\times(0,T)$ for some $T\in(0,\infty]$, we have*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t\in(0,T). \tag{2.27}$$

3 Proof of Theorem 1.1

We are now in the position to prove our main result.

Proof of Theorem 1.1. For $\varepsilon\in(0,1)$, we consider the approximate problems given by

$$\begin{cases} n_{\varepsilon t}+u_\varepsilon\cdot\nabla n_\varepsilon=\nabla\cdot(D_\varepsilon(n_\varepsilon)\nabla n_\varepsilon)-\nabla\cdot(n_\varepsilon\chi(c_\varepsilon)\nabla c_\varepsilon), \qquad x\in\Omega,\ t>0,\\ c_{\varepsilon t}+u_\varepsilon\cdot\nabla c_\varepsilon=\Delta c_\varepsilon-n_\varepsilon f(c_\varepsilon), \qquad x\in\Omega,\ t>0,\\ u_{\varepsilon t}+\nabla P_\varepsilon-\eta\Delta u_\varepsilon+n_\varepsilon\nabla\phi=0, \qquad x\in\Omega,\ t>0,\\ \nabla\cdot u_\varepsilon=0, \qquad x\in\Omega,\ t>0,\\ \partial_\nu n_\varepsilon(x,t)=\partial_\nu c_\varepsilon(x,t)=0 \quad \text{and} \quad u_\varepsilon(x,t)=0, \qquad x\in\partial\Omega,\ t>0,\\ n_\varepsilon(x,0)=n_{0\varepsilon}(x), \quad c_\varepsilon(x,0)=c_{0\varepsilon}(x) \quad \text{and} \quad u_\varepsilon(x,0)=u_{0\varepsilon}(x), \qquad x\in\Omega, \end{cases} \tag{3.1}$$

where $D_\varepsilon(s):=m(s+\varepsilon)^{m-1}$ for $s\ge 0$. The initial data $(n_{0\varepsilon},c_{0\varepsilon},u_{0\varepsilon})$ are supposed to be smooth approximations of (n_0,c_0,u_0) in the sense that for each fixed $\varepsilon\in(0,1)$ they satisfy (2.3), and that

$$\begin{cases} n_{0\varepsilon}\overset{\star}{\rightharpoonup} n_0 \text{ in } L^\infty(\Omega),\\ c_{0\varepsilon}\overset{\star}{\rightharpoonup} c_0 \text{ in } W^{1,\infty}(\Omega), \qquad \text{and}\\ u_{0\varepsilon}\rightharpoonup u_0 \text{ in } W_0^{1,2}(\Omega)\cap W^{1+\theta,2}(\Omega) \end{cases} \tag{3.2}$$

as $\varepsilon\searrow 0$, where $\theta>0$ is as in (1.7). In particular, (3.2) entails that the hypothesis (2.4) holds for $(n_{0\varepsilon},c_{0\varepsilon},u_{0\varepsilon})$ with some $K>0$ independent of $\varepsilon\in(0,1)$. Therefore, Corollary 2.8, Corollary 2.7 and Lemma 2.3 say that for all $\gamma>\max\{2,m-1\}$ we can find a constant $C_1(\gamma) > 0$ such that if for some $\varepsilon \in (0,1)$ and $T \in (0,\infty]$ we are given a solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ of (3.1), then

$$\begin{cases} \|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \quad \text{for all } t \in (0,T) \quad \text{and} \\ \int_0^t \int_\Omega n_\varepsilon^{m+\gamma-3} |\nabla n_\varepsilon|^2 \le C_1(1+t) \quad \text{for all } t \in (0,T), \\ \|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \quad \text{and} \quad \|\nabla c_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_1 \quad \text{for all } t \in (0,T), \\ \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 \quad \text{and} \quad \|\nabla u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \le C_1 \quad \text{for all } t \in (0,T) \end{cases} \tag{3.3}$$

are valid. As done in [13, Section 2], we first use Schauder's fixed point theorem to conclude that for each $\varepsilon \in (0,1)$, the problem (3.1) has at least one local-in-time weak solution. Then, in view of regularity theory for parabolic equations and the Stokes equation ([11] and [10]), the local weak solution is actually classical. Next, according to the uniform estimate (3.3), regularity theory for parabolic equations and the Stokes equation ([11] and [10]), a bootstrap argument can extend the above local solution to any given time interval $(0,T)$. Since the issue of global existence for the non-degenerate approximate problem (3.1) has already been addressed by Di Franceso, Lorz and Markowich in [3], we may refrain from repeating the details of the corresponding procedure here.

In order to achieve a strong precompactness property of $(n_\varepsilon)_{\varepsilon\in(0,1)}$, let us fix $\alpha > \max\{m+1, 2m-2\}$ and multiply the first equation in (3.1) by $n_\varepsilon^{\alpha-1}\zeta(x)$, where $\zeta \in C_0^\infty(\Omega)$. On integrating by parts, we thereby obtain

$$\begin{aligned} \frac{1}{\alpha}\int_\Omega (n_\varepsilon^\alpha)_t \cdot \zeta \quad = \quad & -(\alpha-1)m\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{\alpha-2}|\nabla n_\varepsilon|^2\zeta - m\int_\Omega (n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{\alpha-1}\nabla n_\varepsilon\cdot\nabla\zeta \\ & +(\alpha-1)\int_\Omega n_\varepsilon^{\alpha-1}\nabla n_\varepsilon\cdot\chi(c_\varepsilon)\nabla c_\varepsilon\zeta + \int_\Omega n_\varepsilon^\alpha \chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\zeta \\ & +\frac{1}{\alpha}\int_\Omega n_\varepsilon^\alpha u_\varepsilon\cdot\nabla\zeta \end{aligned} \tag{3.4}$$

for $t > 0$. Here we estimate

$$\left| \int_\Omega (n_\varepsilon+\varepsilon)^{m-1} n_\varepsilon^{\alpha-2}|\nabla n_\varepsilon|^2\zeta \right| \le (\|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}+1)^{m-1}\cdot\Big(\int_\Omega n_\varepsilon^{\alpha-2}|\nabla n_\varepsilon|^2\Big)\cdot\|\zeta\|_{L^\infty(\Omega)}$$

and continue this in a straightforward manner for the remaining terms on the right of (3.4). As a result, we see that since $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ due to the fact that $n = 2$, there exists $C_2 > 0$ such that

$$\begin{aligned} \sup_{\zeta\in C_0^\infty(\Omega), \|\zeta\|_{W^{2,2}(\Omega)}\le 1} \left|\int_\Omega (n_\varepsilon^\alpha)_t\cdot\zeta\right| \quad \le \quad & C_2\cdot(\|n_\varepsilon\|_{L^\infty(\Omega\times(0,T))}+1)^{m-1}\times \\ & \times\Big\{1+\int_\Omega n_\varepsilon^{\alpha-2}|\nabla n_\varepsilon|^2 + \int_\Omega n_\varepsilon^{2\alpha-2}|\nabla n_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^2\Big\} \end{aligned}$$

for all $t > 0$. According to our restriction on α, we can pick $\gamma > \max\{2, m-1\}$ such that $m+\gamma-3 \le \alpha-2$ and hence deduce from (3.3) that for any such α there exists $C_3 > 0$ such that

$$\|(n_\varepsilon^\alpha)_t\|_{L^1\left((0,t);(W_0^{2,2}(\Omega))^\star\right)} \le C_3(1+t) \qquad \text{for all } t \in (0,T).$$

Similarly, we can find $C_4 > 0$ fulfilling

$$\|c_{\varepsilon t}\|_{L^2\left((0,t);(W_0^{1,2}(\Omega))^\star\right)} \le C_4(1+t) \qquad \text{for all } t > 0.$$

In conjunction with (3.3) and the Aubin-Lions compactness lemma ([12, Ch. IV] and [17]), we thus infer the existence of a sequence of numbers $\varepsilon = \varepsilon_j \searrow 0$ along which

$$\begin{cases} n_\varepsilon \to n & \text{a.e. in } \Omega \times (0,\infty), \\ n_\varepsilon \overset{\star}{\rightharpoonup} n & \text{in } L^\infty(\Omega \times (0,\infty)), \\ \nabla n_\varepsilon^m \rightharpoonup \nabla n^m & \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)), \\ c_\varepsilon \to c & \text{a.e. in } \Omega \times (0,\infty), \\ \nabla c_\varepsilon \rightharpoonup \nabla c & \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)), \\ u_\varepsilon \rightharpoonup u & \text{in } L^2_{loc}([0,\infty); W_0^{1,2}(\Omega)) \end{cases}$$

holds for some limit $(n,c,u) \in (L^\infty(\Omega \times (0,\infty)))^4$ with nonnegative n and c. Due to these convergence properties, applying standard arguments we may take $\varepsilon = \varepsilon_j \searrow 0$ in each term of the natural weak formulation of (3.1) separately to verify that in fact (n,c,u) can be complemented by some pressure function P in such a way that (n,c,u,P) is a weak solution of (1.3)-(1.5).
Finally, the boundedness statement is a straightforward consequence of Corollary 2.8, Lemma 2.1 and Lemma 2.3. □

Acknowledgment. Y. Tao was supported by the National Natural Science Foundation of China.

References

[1] N. D. Alikakos, *L^p bounds of solutions of reaction-diffusion equations*, Comm. Partial Differential Equations 4 (1979), 827-868.

[2] V. Calvez and J. A. Carrillo, *Volume effects in the Keller-Segel model: energy estimates preventing blow-up*, J. Math. Pures Appl. (9), 86 (2006), 155-175.

[3] M. Di Francesco, A. Lorz and P. Markowich, *chemotaxis-fluid coupled model for swimming bacteria with nonlinear diffusion: global existence and asymptotic behavior*, Discrete Cont. Dyns. S-A 28 (2010), 1437-1453.

[4] R. J. Duan, A. Lorz and P. Markowich, *Global solutions to the coupled chemotaxis-fluid equations*, Comm. Partial Differential Equations 35 (2010), 1635-1673.

[5] A. Friedman, *Partial Differential Equations*, Holt, Rinehart & Winston, New York, 1969.

[6] D. Horstmann and M. Winkler, *Boundedness vs. blow-up in a chemotaxis system*, J. Diff. Eqns. 215 (2005), 52-107.

[7] E. F. Keller and L. A. Segel, *Initiation of slime mold aggregation viewed as an instaility*, J. Theor. Biol. 26 (1970), 399-415.

[8] R. Kowalczyk, *Preventing blow-up in a chemotaxis model*, J. Math. Anal. Appl. 305 (2005), 566-585.

[9] R. Kowalczyk and Z. Szymańska, *On the global existence of solutions to an aggregation model*, J. Math. Anal. Appl. 343 (2008), 379-398.

[10] O. A. Ladyzenskaja, *Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach Science Publishers Inc., 1968.

[11] O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and Quasi-Linear Equations of Parabolic Type,* Amer. Math. Soc. Transl., Vol. 23, Providence, RI, 1968.

[12] J. L. Lions, *Équations différentielles opérationnelles et problémes aux limites*, Die Grundlehren der mathematischen Wissenschaften, Springer, 1961.

[13] A. Lorz, *Coupled chemotaxis fluid model*, Math. Mod. Meth. Appl. Sci. 20 (2010), 987-1004.

[14] L. Nirenberg, *An extended interpolation inequality*, Ann. Scuola Norm. Sup. Pisa (3) 20 (1966), 733-737.

[15] G. R. Sell and Y. You, *Dynamics of Evolutionary Equations*, Applied Mathematical Sciences, Vol. 143, Springer, New York, 2002.

[16] T. Senba and T. Suzuki, *quasi-linear system of chemotaxis*, Abstr. Appl. Analysis (2006), 1-21.

[17] J. Simon, *Compact sets in the space $L^p(0,T;B)$*, Ann. Math. Pura Appl., 146 (4) (1987), 65-96.

[18] H. Sohr, *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser, Basel, 2001.

[19] Y. Sugiyama, *On ε-regularity theorem and asymptotic behaviors of solutions for Keller-Segel systems*, SIAM J. Math. Anal. 41 (2009), 1664-1692.

[20] Z. Szymańska, C. Morales-Rodrigo, M. Lachowicz and M. Chaplain, *Mathematical modelling of cancer invasion of tissue: The role and effect of nonlocal interactions*, Math. Mod. Meth. Appl. Sci. 19 (2009), 257-281.

[21] Y. Tao and M. Winkler, *A chemotaxis-haptotaxis model: the roles of porous medium diffusion and logistic source*, SIAM J. Math. Anal. 43 (2011), 685-704.

[22] Y. Tao and M. Winkler, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations, to appear.

[23] I. Tuval, L. Cisnerous, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler and R. E. Goldstein, *Bacterial swimming and oxygen transport near contact lines*, Proc. Natl. Acad. Sci. USA 102 (2005), 2277-2282.

[24] J. L. Vázquez, *The Porous Medium Equations*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.

[25] M. Winkler, *Does a 'volume-filling effect' always prevent chemotactic collapse?*, Mathematical Methods in the Applied Sciences 33 (2010), 12-24.

[26] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Differential Equations 248 (2010), 2889-2905.