Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity

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Abstract

We consider the quasilinear parabolic-parabolic Keller-Segel system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a convex smooth bounded domain $\Omega \subset \mathbb{R}^n$ with $n \geq 1$.

It is proved that if $\frac{S(u)}{D(u)} \leq cu^\alpha$ with $\alpha < \frac{2}{n}$ and some constant $c > 0$ for all $u > 1$, then the classical solutions to the above system are uniformly-in-time bounded, provided that $D(u)$ satisfies some technical conditions such as algebraic upper and lower growth (resp. decay) estimates as $u \to \infty$. This boundedness result is optimal according to a recent result by the second author (*Math. Meth. Appl. Sci.* **33** (2010), 12-24), which says that if $\frac{S(u)}{D(u)} \geq cu^\alpha$ for $u > 1$ with $c > 0$ and some $\alpha > \frac{2}{n}$, $n \geq 2$, then for each mass $M > 0$ there exist blow-up solutions with mass $\int_\Omega u_0 = M$.

In addition, this paper also proves a general boundedness result for quasilinear non-uniformly parabolic equations by modifying the iterative technique of Moser-Alikakos (Alikakos, *Comm. PDE* **4** (1979), 827-868).

Key words: chemotaxis; boundedness; prevention of blow-up
AMS Classification: 92C17, 35K55, 35B35, 35B40

Introduction

This work is concerned with the initial-boundary value problem

$$
\begin{cases}
u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \\
\partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{0.1}
$$

for the unknown $u = u(x,t), v = v(x,t)$, where Ω is a bounded convex domain in $\mathbb{R}^n$ with smooth boundary, and $n \geq 1$. The initial distributions u_0 and v_0 are assumed to be nonnegative functions subject to the inclusions $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$, respectively.

Chemotaxis, the biased movement of cells (or organisms) in response to chemical gradients, plays an important role coordinating cell migration in many biological phenomena (cf. the review article [HP09]). In (0.1), u denotes the cell density and v describes the concentration of the chemical signal secreted by cells. In addition to diffusion, cells move towards higher signal concentration, whereas the chemical signal undergoes random diffusion and decay. An important variant of the quasilinear chemotaxis model (0.1) was initially proposed by Painter and Hillen [PH]. Their approach assumes the presence of a so-called *volume-filling effect*: The movement of cells is inhibited near points where the cells are densely packed. Painter and Hillen [PH] derived their model via a random walk approach and they found a functional link between the diffusivity $D(u)$ and the chemotactic sensitivity $S(u)$ that, in a non-dimensionalized version, takes the form

$$
D(u) = Q(u) - uQ'(u), \qquad S(u) = uQ(u)
\tag{0.2}
$$

where $Q(u)$ denotes the density-dependent probability for a cell to find space somewhere in its neighboring location. Since this probability is basically unknown, different choices for Q are conceivable.

If $Q(u) \equiv 1$ we arrive at the classical Keller-Segel model ([KSe]),

$$
\begin{cases}
u_t = \Delta u - \nabla \cdot (u\nabla v), & x \in \Omega, \ t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \\
\partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \ t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{0.3}
$$

which has been investigated quite thoroughly during the past three decades. In view of the biologically meaningful question whether or not cell populations spontaneously form aggregates, most mathematical studies focused on whether solutions remain bounded or blow up. If $n = 1$, then all solutions of (0.3) are global in time and bounded ([OY]); if $n = 2$ and $\int_\Omega u_0 < 4\pi$, then the solution will be global and bounded ([NSY]); if $n \geq 3$ and, for any $\delta > 0$, the quantities $\|u_0\|_{L^{n/2+\delta}(\Omega)}$ and $\|\nabla v_0\|_{L^{n+\delta}(\Omega)}$ are small, then the solution is global and bounded ([Wi2]). On the other hand, if $n = 2$ then for almost every $M > 4\pi$ there exist smooth initial data (u_0, v_0) with $\int_\Omega u_0 = M$ such that corresponding solution

of (0.3) blows up either in finite or infinite time provided Ω is simply connected ([HWa]); in the particular framework of radially symmetric solutions in a planar disk, solutions may even blow up in finite time ([HV]); if $n \geq 3$ and Ω is a ball, then for all $M > 0$ there exist initial data with $\int_\Omega u_0 = M$ such that the solution will become unbounded either in finite or infinite time ([Wi2]).

In [HP01] the authors analyze (0.1) upon the particular choices $D(u) \equiv 1$ and $S(u) = u(1 - u)_+$. This corresponds to the case of the compactly supported probability $Q(u) = (1 - u)_+$ in the volume-filling model, in particular meaning that the chemotactic movement is entirely stopped when the cell density reaches the critical level $u = 1$. The resulting system admits global bounded solutions only ([HP01]). Furthermore, Wrzosek ([Wr2], [Wr1]) studied the dynamical properties such as instability of constant steady states or the existence of attractors.

The focus of this paper is to provide some further step towards understanding in more detail the interaction of the competing nonlinear mechanisms of diffusion and cross-diffusion in (0.1), allowing for rather general choices of $D(u)$ and $S(u)$. Here we concentrate on the particular phenomenon of blow-up, and observe that in this respect, previous results indicate that the asymptotic behavior of the ratio $\frac{S(u)}{D(u)}$ for large values of u seems to be decisive: Namely, in [Wi1] it has been shown that

$$
\begin{aligned}
&\text{if} \quad \frac{S(u)}{D(u)} \geq cu^{\frac{2}{n}+\varepsilon} \quad \text{for all } u > 1 \text{ and some } c > 0 \text{ and } \varepsilon > 0, \\
&\qquad\qquad \text{then there exist smooth solutions of (0.1) which blow up}
\end{aligned}
\tag{0.4}
$$

either in finite or infinite time, provided that Ω is a ball. However, to the best of our knowledge the existing literature leaves open the question in how far this growth condition is critical in respect of blow-up.

It is the purpose of the present work to close this gap, and correspondigly we shall suppose throughout that D and S, besides

$$
D \in C^2([0, \infty)) \qquad \text{and} \qquad S \in C^2([0, \infty)) \quad \text{with } S(0) = 0,
\tag{0.5}
$$

are such that their ratio satisfies the growth condition

$$
\frac{S(u)}{D(u)} \leq K(u + 1)^\alpha \qquad \text{for all } u \geq 0
\tag{0.6}
$$

with some $K > 0$ and $\alpha > 0$. Moreover, our approach will require the further technical assumptions that

$$
D(u) \geq K_0(u + 1)^{m-1} \quad \text{for all } u \geq 0
\tag{0.7}
$$

and

$$
D(u) \leq K_1(u + 1)^{M-1} \quad \text{for all } u \geq 0
\tag{0.8}
$$

are valid with some constants $m \in \mathbb{R}, M \in \mathbb{R}, K_0 > 0$ and $K_1 > 0$.

Under these hypotheses, our main result reads as follows.

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Theorem 0.1 *Suppose that $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded convex domain with smooth boundary. Assume that D and S satisfy (0.5), (0.6), (0.7) and (0.8) with some $m \in \mathbb{R}, M \in \mathbb{R}$ and positive constants K, K_0, K_1 and*

$$\alpha < \frac{2}{n}.$$

Then for any nonnegative $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$, there exists a couple (u, v) of nonnegative bounded functions belonging to $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$ which solve (0.1) classically.

In conjunction with (0.4), this provides an essentially complete picture on the dichotomy *boundedness vs. blow-up* in (0.1), provided that the (self-)diffusivity $D(u)$ has an asymptotically algebraic behavior. It is an interesting open question that unfortunately has to be left open here whether the above boundedness statement is also valid when $D(u)$ is allowed to grow or decay exponentially, for instance.

Let us mention some further previous contributions in this direction. In the particular case $D(u) \equiv 1$, the criticality of $\frac{S(u)}{D(u)} \simeq u^{\frac{2}{n}}$ was already revealed in [HWi], where global boundedness of solutions was shown when $S(u) \leq cu^{\frac{2}{n} - \varepsilon}$ for all $u > 1$ and some $c > 0$ and $\varepsilon > 0$, and where some radial blow-up solutions were constructed if $S(u) \geq cu^{\frac{2}{n} + \varepsilon}$ for $u > 1$ with $c > 0$ and $\varepsilon > 0$, and if some further technical restrictions hold.

As to the special case when $S(u) = u$, Kowalczyk and Szymańska ([KSz]) proved that solutions remain bounded under the condition that $D(u) \geq cu^{2 - \frac{4}{n} + \varepsilon}$ for all $u > 0$ with some $c > 0$ and $\varepsilon > 0$. In view of the above results, this is optimal for non-degenerate diffusion (with $D > 0$ on $[0, \infty)$) if and only if $n = 2$. For the same choice of $S(u)$ and $D > 0$ on $[0, \infty)$, Senba and Suzuki ([SeSu]) reached the critical exponent by showing boundedness under the hypothesis that $D(u) \geq cu^{\frac{n-2}{n} + \varepsilon}$ be valid for $u > 1$ with some $c > 0$ and $\varepsilon > 0$. For more general $D(u)$ and $S(u)$ satisfying some technical assumptions, Cieślak ([C2]) asserted boundedness of solutions when either $n = 2$ and $\frac{S(u)}{D(u)} \leq cu^{\frac{1}{2} - \varepsilon}$, or $n = 3$ and $\frac{S(u)}{D(u)} \leq cu^{-\varepsilon}$ for all $u > 1$ and some $c > 0$ and $\varepsilon > 0$ (cf. also [C1] for related results).

When the diffusion of the chemical signal is considered to occur much faster than that of cells, by the approach of quasi-steady-state approximation (cf. [JL] or [P]), the parabolic-parabolic chemotaxis model (0.1) can be reduced to simplified parabolic-elliptic models where the second PDE in (0.1) is replaced with either $0 = \Delta v - v + u$, or with $0 = \Delta v - M + u$, where $M := \int_\Omega u_0$ denotes the total mass of cells. For the former model, if $n = 2$, $S(u) = u$ and $D(u) \geq c(1 + u)^{1 + \varepsilon}$ with $c > 0$ and $\varepsilon > 0$, boundedness of solutions was proved in [K], and the same conclusion was found in [CM-R] for more general $D(u)$ and $S(u)$ with the property that for some $c > 0$ and $\varepsilon > 0$ we have $\frac{S(u)}{D(u)} \leq cu^{-\varepsilon}$ when $n = 2$, and $\frac{S(u)}{D(u)} \leq cu^{-1-\varepsilon}$ when $n = 3$.

As to the latter simplification, the knowledge appears to be rather complete and consistent with the results for the parabolic-parabolic case if $D(u) \simeq u^{-\gamma}$ and $S(u) \simeq u^\alpha$ for large u with some $\gamma \geq 0$ and $\alpha \in \mathbb{R}$: Solutions remain bounded if $\alpha + \gamma < \frac{2}{n}$, whereas blow-up

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may occur if $\alpha + \gamma > \frac{2}{n}$ ([DW], cf. also [CW] for a precedent addresing the special case $S(u) = u$). Moreover, if $S(u) = u$, then even the critical case $D(u) \simeq u^{\frac{n-2}{n}}$ can be analyzed, and Cieślak and Laurençot have shown it to belong to the blow-up regime ([CL]). Refined conditions ensuring boundedness in two-dimensional parabolic-elliptic Keller-Segel models can be found in [CC]. For results in the whole space $\mathbb{R}^n$ with $D(u)$ and $S(u)$ being exact powers of u (thus involving porous medium-type or fast diffusion), we refer to [S] and the references therein.

The proof of our main results will be based on a priori estimates in spatial Lebesgue spaces for u and ∇v. Due to the careful adjustment of some parameters (cf. Section 2), our technique of deriving integral bounds (see Section 3) does not need any iterative argument to establish bounds for $u(\cdot, t)$ in $L^p(\Omega)$ for any finite p, as required in some previous approaches (cf. [HWi], for instance). Only in a final step an iteration is needed in order to turn this into a bound in $L^\infty(\Omega)$ by means of a Moser-Alikakos-type procedure (cf. the appendix).

1 Local existence

The following statement concerning local existence of classical solution can be proved by well-established methods involving standard parabolic regularity theory and an appropriate fixed point framework (for details see [HWi], [Wr1] or also [C1], for instance).

Lemma 1.1 *Let D and S satisfy (0.5), (0.7) and (0.8) with some $m \in \mathbb{R}, M \in \mathbb{R}, K_0 > 0$ and $K_1 > 0$, and assume that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in C^1(\bar{\Omega})$ are nonnegative. Then there exist $T_{max} \in (0, \infty]$ and a pair (u, v) of functions from $C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ solving (0.1) classically in $\Omega \times (0, T_{max})$. These functions satisfy the inequalities*

$$u \geq 0 \qquad and \qquad v \geq 0 \qquad in \ \Omega \times (0, T_{max}),$$

and moreover

$$either \ T_{max} = \infty, \quad or \ \limsup_{t \nearrow T_{max}} \left(\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{L^\infty(\Omega)} \right) = \infty. \qquad (1.1)$$

The following properties of solutions of (0.1) are well-known.

Lemma 1.2 *i) The first component u of the solution of (0.1) satisfies the mass conservation property*

$$\|u(t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \qquad for \ all \ t \in (0, T_{max}). \qquad (1.2)$$

ii) For all $s \in [1, \frac{n}{n-1})$ there exists $c > 0$ such that

$$\|v(t)\|_{W^{1,s}(\Omega)} \leq c \qquad for \ all \ t \in (0, T_{max}) \qquad (1.3)$$

holds.

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PROOF. Integrating with respect to $x \in \Omega$, we see that $\frac{d}{dt} \int_\Omega u \equiv 0$, and that $\frac{d}{dt} \int_\Omega v = -\int_\Omega v + \int_\Omega u$ for $t \in (0, T_{max})$. This yields (1.2) and moreover shows that v is bounded in $L^\infty((0, T_{max}); L^1(\Omega))$. Now this implies (1.3) upon a standard regularity argument involving the variation-of-constants formula for v and $L^p - L^q$ estimates for the heat semigroup (see [HWi, Lemma 4.1], for instance). $\qquad\square$

2 Adjusting some parameters

We now make sure that when the parameter α in (0.6) indeed satisfies $\alpha < \frac{2}{n}$, we can choose certain parameters, to be used in Lemma 3.3 below, appropriately.

Lemma 2.1 Let $n \geq 2, m \in \mathbb{R}, \alpha \in (0, \frac{2}{n}), \bar{p} \geq 1$ and $\bar{q} \geq 2$. Then there exist numbers $p \geq \bar{p}, q \geq \bar{q}, s \in [1, \frac{n}{n-1}), \theta > 1$ and $\mu > 1$ such that

$$p > \max\left\{ 4 - m, \frac{n(1-m)}{2} \right\}, \tag{2.1}$$

$$\frac{n-2}{n} \cdot \frac{m+p+2\alpha-3}{m+p-1} < \frac{1}{\theta}, \tag{2.2}$$

$$\frac{n-2}{n} \cdot \frac{2}{m+p-1} < \frac{1}{\mu}, \tag{2.3}$$

$$\frac{1}{\theta} < 1 - \frac{n-2}{n} \cdot \frac{1}{q} \qquad \text{and} \tag{2.4}$$

$$\frac{1}{\mu} < \frac{2}{n} + \frac{n-2}{n} \cdot \frac{1}{q}, \tag{2.5}$$

and such that moreover

$$\frac{m+p+2\alpha-3-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{2}{s}-1+\frac{1}{\theta}}{1-\frac{n}{2}+\frac{nq}{s}} < \frac{2}{n} \tag{2.6}$$

as well as

$$\frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{2(q-1)}{s}-1+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{nq}{s}} < \frac{2}{n} \tag{2.7}$$

hold.

PROOF. Let us first fix numbers $\theta > 1$ and $\mu > 1$ such that

$$(n-2)\theta < n \tag{2.8}$$

and

$$\mu > \frac{n}{2}, \tag{2.9}$$

and let

$$q_0(p) := \frac{n(m+p-1)}{2(n-1)} \qquad \text{for } p \geq 1.$$

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Then we can easily find some large $p \geq \bar{p}$ fulfilling

$$q_0(p) > \bar{q}, \tag{2.10}$$

and such that (2.1), (2.2) and (2.3) hold as well as

$$\frac{1}{\theta} < 1 - \frac{n-2}{n} \cdot \frac{1}{q_0(p)} \tag{2.11}$$

and

$$\frac{1}{\mu} < \frac{2}{n} + \frac{n-2}{n} \cdot \frac{1}{q_0(p)}. \tag{2.12}$$

Here we note that (2.8) asserts that (2.2) is true for all sufficiently large p, whereas the fact that $q_0(p) \to +\infty$ as $p \to \infty$ along with the inequality $\theta > 1$ and (2.9) guarantees the validity of (2.11) and (2.12) for appropriately large p.
We next let

$$f(q,s) := \frac{m+p+2\alpha-3-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{2}{s}-1+\frac{1}{\theta}}{1-\frac{n}{2}+\frac{nq}{s}} \qquad \text{for } q \geq 2 \text{ and } s \in [1, \frac{n}{n-1}], \tag{2.13}$$

and

$$g(q,s) := \frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{\frac{2(q-1)}{s}-1+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{nq}{s}} \qquad \text{for } q \geq 2 \text{ and } s \in [1, \frac{n}{n-1}]. \tag{2.14}$$

Then

$$\begin{aligned}
g\left(q_0(p), \frac{n}{n-1}\right) &= \frac{2-\frac{1}{\mu}+\frac{2(n-1)}{n}\cdot\left(\frac{n(m+p-1)}{2(n-1)}-1\right)-1+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&= \frac{1+(m+p-1)-\frac{2(n-1)}{n}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&= \frac{\frac{2}{n}-1+(m+p-1)}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&= \frac{2}{n}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{\partial g}{\partial q}\left(q, \frac{n}{n-1}\right) &= \frac{\frac{2(n-1)}{n}\cdot\left[1-\frac{n}{2}+(n-1)q\right]-\left[\frac{2(n-1)}{n}\cdot(q-1)-1+\frac{1}{\mu}\right]\cdot(n-1)}{\left[1-\frac{n}{2}+(n-1)q\right]^2} \\
&= (n-1)\cdot\frac{\frac{2}{n}-1+\frac{2(n-1)q}{n}-\frac{2(n-1)q}{n}+\frac{2(n-1)}{n}+1-\frac{1}{\mu}}{\left[1-\frac{n}{2}+(n-1)q\right]^2} \\
&= (n-1)\cdot\frac{2-\frac{1}{\mu}}{\left[1-\frac{n}{2}+(n-1)q\right]^2} \\
&> 0 \qquad \text{for all } q > 2,
\end{aligned}$$

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this implies

$$g\left(q, \frac{n}{n-1}\right) < \frac{2}{n} \qquad \text{for all } q \in (2, q_0(p)). \tag{2.15}$$

Moreover, our assumption $\alpha < \frac{2}{n}$ entails that

$$\begin{aligned}
f\left(q_0(p), \frac{n}{n-1}\right) &= \frac{\left(m+p+2\alpha-3-\frac{1}{\theta}\right) + \left(1-\frac{2}{n}+\frac{1}{\theta}\right)}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&= \frac{m+p+2\alpha-2-\frac{2}{n}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&< \frac{m+p-2+\frac{2}{n}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&= \frac{2}{n} \cdot \frac{\frac{n(m+p-1)}{2}+1-\frac{n}{2}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} \\
&= \frac{2}{n}.
\end{aligned}$$

Therefore by a continuity argument using (2.10) we can now fix $q > \bar{q}$ fulfilling

$$q < q_0(p) \tag{2.16}$$

and

$$f\left(q, \frac{n}{n-1}\right) < \frac{2}{n} \tag{2.17}$$

and such that furthermore (2.4) and (2.5) hold, where the latter two can be achieved on choosing q close enough to $q_0(p)$ according to (2.11) and (2.12). We observe that by (2.16) and (2.15) we also have

$$g\left(q, \frac{n}{n-1}\right) < \frac{2}{n},$$

so that, again by continuity, we can finally find $s \in [1, \frac{n}{n-1})$ close to $\frac{n}{n-1}$ such that with q as fixed above we still have

$$f(q, s) < \frac{2}{n} \qquad \text{and} \qquad g(q, s) < \frac{2}{n}.$$

In view of the definitions (2.13) and (2.14) of f and g, these two inequalities are equivalent to (2.6) and (2.7). $\qquad\square$

3   Proof of the main results

The following preparation is a direct consequence of Young's inequality.

Lemma 3.1 *Let $\beta > 0$ and $\gamma > 0$ be such that $\beta + \gamma < 1$. Then for all $\varepsilon > 0$ there exists $c > 0$ such that*

$$a^\beta b^\gamma \leq \varepsilon(a + b) + c \qquad \text{for all } a \geq 0 \text{ and } b \geq 0.$$

Our approach strongly relies on the following favorable property of functions satisfying a homogeneous Neumann boundary condition on *convex* domains. Its proof is implicitly contained in [DalPGG, Appendix], but we include an elementary proof here for completeness.

Lemma 3.2 *Assume that Ω is convex, and that $w \in C^2(\bar{\Omega})$ satisfies $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$. Then*

$$\frac{\partial |\nabla w|^2}{\partial \nu} \leq 0 \qquad \text{on } \partial\Omega.$$

PROOF. Let us fix $x_0 \in \partial\Omega$. Since $\partial\Omega$ belongs to the class C^2 and hence can be represented locally as the graph of a C^2 function, upon a translation and a rotation we may assume that $x_0 = 0$, and that there exist neighborhoods $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^{n-1}$ of the origins in $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$, respectively, such that $\partial\Omega \cap U = \{(x', \psi(x')) \mid x' \in V\}$ and $\Omega \cap U = \{(x', x_n) \mid x_n < \psi(x'), x' \in V\}$ for some $\psi \in C^2(V)$. Here since Ω is convex, we may moreover assume that

$$\psi(x') \leq 0 = \psi(0) \qquad \text{for all } x' \in V \tag{3.1}$$

and hence

$$\nabla_{x'}\psi(0) = 0. \tag{3.2}$$

Now if $x \in \partial\Omega \cap U$ then $\nu(x)$ is parallel to $(-\nabla_{x'}\psi(x'), 1)$, and therefore our hypothesis $\frac{\partial w}{\partial \nu} = 0$ on $\partial\Omega$ entails that

$$-\sum_{i=1}^{n-1} \psi_{x_i} w_{x_i} + w_{x_n} = 0 \qquad \text{for all } x' \in V, \tag{3.3}$$

where for convenience we drop the arguments x' of ψ and $(x', \psi(x'))$ of w. Differentiating (3.3) with respect to $x_j, j \in \{1, ..., n-1\}$, yields

$$w_{x_j x_n} = \sum_{i=1}^{n-1} \psi_{x_i x_j} w_{x_i} + \sum_{i=1}^{n-1} \psi_{x_i} w_{x_i x_j} \qquad \text{for all } x' \in V$$

and thus, by (3.2),

$$w_{x_j x_n} = \sum_{i=1}^{n-1} \psi_{x_i x_j} w_{x_i} \qquad \text{at } x = 0 \text{ for } j \in \{1, ..., n-1\}.$$

At $x = 0$ we consequently obtain, using that $w_{x_n} = 0$ at this point by (3.3) and (3.2),

$$
\begin{aligned}
\frac{\partial |\nabla w|^2}{\partial \nu} &= \frac{\partial |\nabla w|^2}{\partial x_n} = 2\sum_{j=1}^{n} w_{x_j} w_{x_j x_n} = 2\sum_{j=1}^{n-1} w_{x_j} w_{x_j x_n} = 2\sum_{j=1}^{n-1} w_{x_j} \cdot \sum_{i=1}^{n-1} \psi_{x_i x_j} w_{x_i} \\
&= 2\nabla_{x'} w \cdot (D_{x'}^2 \psi \cdot \nabla_{x'} w).
\end{aligned}
$$

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Since the Hessian $D^2_{x'}\psi$ is negative semidefinite at $x = 0$ according to (3.1), this shows that $\frac{\partial|\nabla w|^2}{\partial\nu} \leq 0$ at this point and thereby completes the proof. □

We proceed to establish the main step towards our boundedness proof.

Lemma 3.3 *Suppose that Ω is convex, and that (0.6), (0.7) and (0.8) hold with some $K > 0$, $K_0 > 0$, $K_1 > 0$, $m \in \mathbb{R}$, $M \in \mathbb{R}$ and some positive*

$$\alpha < \frac{2}{n}.$$

Then for all $p \in [1, \infty)$ and each $q \in [1, \infty)$ there exists $c > 0$ such that

$$\|u(t)\|_{L^p(\Omega)} \leq c \qquad \text{for all } t \in (0, T_{max}) \tag{3.4}$$

and

$$\|\nabla v(t)\|_{L^{2q}(\Omega)} \leq c \qquad \text{for all } t \in (0, T_{max}) \tag{3.5}$$

PROOF.   We only consider the case $n \geq 2$ and remark that upon straightforward modifications, the proof in the one-dimensional case can be can be carried out along the same lines.

It is evidently sufficient to prove that for any $p_0 > 1$ and $q_0 > 2$ we can find some $p > p_0$ and $q > q_0$ such that (3.5) and

$$\|u(t)\|_{L^{p+m-M}(\Omega)} \leq c \qquad \text{for all } t \in (0, T_{max}) \tag{3.6}$$

hold with some $c > 0$, where m and M are taken from (0.7) and (0.8), respectively. To achieve this, given such p_0 and q_0 let us set $\bar{p} := p_0 + M - m$ and $\bar{q} := q_0$ and then fix $p > \bar{p}, q > \bar{q}, s \in [1, \frac{n}{n-1}), \theta > 1$ and $\mu > 1$ as provided by Lemma 2.1. Then by (0.7),

$$\phi(r) := \int_0^r \int_0^\rho \frac{(\sigma+1)^{m+p-3}}{D(\sigma)} d\sigma d\rho \qquad \text{for } r \geq 0. \tag{3.7}$$

is finite and positive for all $r \geq 0$ with

$$\phi(r) \leq \frac{1}{K_0} \cdot \int_0^r \int_0^\rho (\sigma+1)^{p-2} d\sigma d\rho \leq \frac{1}{p(p-1)K_0} \cdot (r+1)^p \qquad \text{for all } r \geq 0, \tag{3.8}$$

and furthermore due to (0.8) we have

$$\phi(r) \geq c_0(r+1)^{p+m-M} \qquad \text{for all } r \geq 0 \tag{3.9}$$

with some $c_0 > 0$. Since moreover ϕ is smooth on $(0, \infty)$ and u is positive in $\Omega \times (0, T_{max})$ by the strong maximum principle, we may use $\phi'(u)$ as a test function for the first equation in (0.1). Integrating by parts we thereby see that

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega \phi(u) &= \int_\Omega \phi'(u)\nabla \cdot (D(u)\nabla u) - \int_\Omega \phi'(u)\nabla \cdot (S(u)\nabla v) \\
&= -\int_\Omega \phi''(u)D(u)|\nabla u|^2 + \int_\Omega \phi''(u)S(u)\nabla u \cdot \nabla v \\
&= -\int_\Omega (u+1)^{m+p-3}|\nabla u|^2 + \int_\Omega (u+1)^{m+p-3}\frac{S(u)}{D(u)}\nabla u \cdot \nabla v \tag{3.10}
\end{aligned}
$$

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for all $t \in (0, T_{max})$, where thanks to Young's inequality and (0.6),

$$\int_\Omega (u+1)^{m+p-3} \frac{S(u)}{D(u)} \nabla u \cdot \nabla v \leq \frac{1}{2} \int_\Omega (u+1)^{m+p-3} |\nabla u|^2 + \frac{K^2}{2} \int_\Omega (u+1)^{m+p+2\alpha-3} |\nabla v|^2.$$
(3.11)

We next differentiate the second equation in (0.1) to obtain

$$(|\nabla v|^2)_t = 2\nabla v \cdot \nabla \Delta v - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v$$

and hence, recalling the identity $\Delta |\nabla v|^2 = 2\nabla v \cdot \nabla \Delta v + 2|D^2 v|^2$,

$$(|\nabla v|^2)_t = \Delta |\nabla v|^2 - 2|D^2 v|^2 - 2|\nabla v|^2 + 2\nabla u \cdot \nabla v$$

for all $x \in \Omega$ and $t \in (0, T_{max})$. Testing this against $|\nabla v|^{2q-2}$ yields

$$\frac{1}{q} \frac{d}{dt} \int_\Omega |\nabla v|^{2q} \quad + \quad (q-1) \int_\Omega |\nabla v|^{2q-4} \Big| \nabla |\nabla v|^2 \Big|^2 + 2 \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 + 2 \int_\Omega |\nabla v|^{2q}$$

$$\leq \quad 2 \int_\Omega |\nabla v|^{2q-2} \nabla u \cdot \nabla v \qquad \text{for all } t \in (0, T_{max}), \tag{3.12}$$

where we have used that $\frac{\partial |\nabla v|^2}{\partial \nu} \leq 0$ on $\partial \Omega$ by Lemma 3.2. On the right of (3.12) we integrate by parts and use Young's inequality to find

$$2 \int_\Omega |\nabla v|^{2q-2} \nabla u \cdot \nabla v \quad = \quad -2(q-1) \int_\Omega u |\nabla v|^{2q-4} \nabla v \cdot \nabla |\nabla v|^2 - 2 \int_\Omega u |\nabla v|^{2q-2} \Delta v$$

$$\leq \quad \frac{q-1}{2} \int_\Omega |\nabla v|^{2q-4} \Big| \nabla |\nabla v|^2 \Big|^2 + 2(q-1) \int_\Omega u^2 |\nabla v|^{2q-2}$$

$$+ \frac{2}{n} \int_\Omega |\nabla v|^{2q-2} |\Delta v|^2 + \frac{n}{2} \int_\Omega u^2 |\nabla v|^{2q-2}, \tag{3.13}$$

where

$$\frac{2}{n} \int_\Omega |\nabla v|^{2q-2} |\Delta v|^2 \leq 2 \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2$$

in view of the pointwise inequality $|\Delta v|^2 \leq n |D^2 v|^2$. We thus infer from (3.10)-(3.13) that there exists $c_1 > 0$ such that

$$\frac{d}{dt} \left\{ \int_\Omega \phi(u) + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \right\} \quad + \quad \frac{2}{(m+p-1)^2} \int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 + \frac{2(q-1)}{q^2} \int_\Omega \Big| \nabla |\nabla v|^q \Big|^2$$

$$\leq \quad c_1 \int_\Omega (u+1)^{m+p+2\alpha-3} |\nabla v|^2 + c_1 \int_\Omega (u+1)^2 |\nabla v|^{2q-2} \tag{3.14}$$

for all $t \in (0, T_{max})$. Here we use the Hölder inequality to estimate the integrals on the right according to

$$\int_\Omega (u+1)^{m+p+2\alpha-3} |\nabla v|^2 \leq \left(\int_\Omega (u+1)^{(m+p+2\alpha-3)\theta} \right)^{\frac{1}{\theta}} \cdot \left(\int_\Omega |\nabla v|^{2\theta'} \right)^{\frac{1}{\theta'}} \tag{3.15}$$

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and

$$\int_\Omega (u+1)^2 |\nabla v|^{2q-2} \le \left(\int_\Omega (u+1)^{2\mu} \right)^{\frac{1}{\mu}} \cdot \left(\int_\Omega |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}} \tag{3.16}$$

with $\theta' := \frac{\theta}{\theta-1}$ and $\mu' := \frac{\mu}{\mu-1}$. Now since (2.1) in conjunction with the positivity of α and the fact that $\theta > 1$ implies that

$$\frac{2(m+p+2\alpha-3)\theta}{m+p-1} > \frac{2}{m+p-1},$$

and since (2.2) asserts that

$$\frac{2(m+p+2\alpha-3)\theta}{m+p-1} < \frac{2n}{n-2},$$

we may invoke the Gagliardo-Nirenberg inequality to estimate

$$
\left(\int_\Omega (u+1)^{(m+p+2\alpha-3)\theta} \right)^{\frac{1}{\theta}} = \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2(m+p+2\alpha-3)\theta}{m+p-1}}(\Omega)}^{\frac{2(m+p+2\alpha-3)}{m+p-1}}
$$

$$
\le c_2 \left\| \nabla (u+1)^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{\frac{2(m+p+2\alpha-3)}{m+p-1} \cdot a} \cdot \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2(m+p+2\alpha-3)}{m+p-1} \cdot (1-a)}
$$

$$
+ c_2 \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2(m+p+2\alpha-3)}{m+p-1}} \qquad \text{for all } t \in (0, T_{max}), \tag{3.17}
$$

with some $c_2 > 0$ and $a \in (0,1)$ determined by

$$-\frac{n(m+p-1)}{2(m+p+2\alpha-3)\theta} = \left(1 - \frac{n}{2}\right) \cdot a - \frac{n(m+p-1)}{2} \cdot (1-a).$$

Thus,

$$a = \frac{\frac{n(m+p-1)}{2} \cdot \left(1 - \frac{1}{(m+p+2\alpha-3)\theta}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}}$$

and hence

$$\frac{2(m+p+2\alpha-3)}{m+p-1} \cdot a = n \cdot \frac{m+p+2\alpha-3-\frac{1}{\theta}}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}},$$

so that (3.17) yields

$$\left(\int_\Omega (u+1)^{(m+p+2\alpha-3)\theta} \right)^{\frac{1}{\theta}} \le c_3 \left(\int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 \right)^{\frac{n}{2} \cdot \frac{m+p+2\alpha-3-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}} + c_3 \tag{3.18}$$

for all $t \in (0, T_{max})$ with some $c_3 > 0$, because (1.2) states boundedness of $(u+1)^{\frac{m+p-1}{2}}$ in $L^\infty((0,T_{max}); L^{\frac{2}{m+p-1}}(\Omega))$.

Similarly, using that $\mu > 1$ implies

$$\frac{4\mu}{m+p-1} > \frac{2}{m+p-1},$$

and that (2.3) entails

$$\frac{4\mu}{m+p-1} < \frac{2n}{n-2},$$

we interpolate

$$
\begin{aligned}
\left(\int_\Omega (u+1)^{2\mu}\right)^{\frac{1}{\mu}} &= \left\|(u+1)^{\frac{m+p-1}{2}}\right\|_{L^{\frac{4\mu}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}} \\
&\leq c_4 \left\|\nabla(u+1)^{\frac{m+p-1}{2}}\right\|_{L^2(\Omega)}^{\frac{4}{m+p-1}\cdot b} \cdot \left\|(u+1)^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}\cdot(1-b)} \\
&\quad + c_4 \left\|(u+1)^{\frac{m+p-1}{2}}\right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{4}{m+p-1}} \qquad \text{for all } t \in (0, T_{max})
\end{aligned}
$$

with some $c_4 > 0$ and

$$b = \frac{\frac{n(m+p-1)}{2}\left(1 - \frac{1}{2\mu}\right)}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}} \in (0,1).$$

Again in view of (1.2), we therefore obtain $c_5 > 0$ such that

$$\left(\int_\Omega (u+1)^{2\mu}\right)^{\frac{1}{\mu}} \leq c_5 \left(\int_\Omega \left|\nabla(u+1)^{\frac{m+p-1}{2}}\right|^2\right)^{\frac{n}{2}\cdot\frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}}} + c_5 \qquad \text{for all } t \in (0, T_{max}). \tag{3.19}$$

As to the integrals in (3.15) and (3.16) involving ∇v, we proceed in quite the same manner, relying on (1.3) rather than on (1.2). First, we note that

$$\frac{2\theta'}{q} > \frac{s}{q}, \tag{3.20}$$

because $\theta' > 1$ and $s < \frac{n}{n-1} \leq 2$ whenever $n \geq 2$. Moreover, we know that

$$\frac{2\theta'}{q} < \frac{2n}{n-2}, \tag{3.21}$$

for (2.4) says that

$$\frac{1}{\theta'} = 1 - \frac{1}{\theta} > \frac{n-2}{n}\cdot\frac{1}{q}.$$

Now (3.20) and (3.21) allow for an application of the Gagliardo-Nirenberg inequality which ensures the existence of $c_6 > 0$ fulfilling

$$
\begin{aligned}
\left(\int_\Omega |\nabla v|^{2\theta'}\right)^{\frac{1}{\theta'}} &= \left\||\nabla v|^q\right\|_{L^{\frac{2\theta'}{q}}(\Omega)}^{\frac{2}{q}} \\
&\leq c_6 \left\|\nabla|\nabla v|^q\right\|_{L^2(\Omega)}^{\frac{2}{q}\cdot c} \cdot \left\||\nabla v|^q\right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2}{q}(1-c)} + c_6 \left\||\nabla v|^q\right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2}{q}}
\end{aligned}
$$

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with

$$c = \frac{nq(\frac{1}{s} - \frac{1}{2\theta'})}{1 - \frac{n}{2} + \frac{nq}{s}} \in (0,1).$$

By means of (1.3), we thus find $c_7 > 0$ such that

$$\left(\int_\Omega |\nabla v|^{2\theta'} \right)^{\frac{1}{\theta'}} \le c_7 \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\frac{n}{2} \cdot \frac{\frac{2}{s} - \frac{1}{\theta'}}{1 - \frac{n}{2} + \frac{nq}{s}}} + c_7 \qquad \text{for all } t \in (0, T_{max}). \qquad (3.22)$$

As to the corresponding term in (3.16), we similarly observe that

$$\frac{2(q-1)\mu'}{q} > \frac{s}{q}, \qquad (3.23)$$

which immediately follows from the inequalities $\mu' > 1$ and $q > \bar{q} \ge 2$ and our assumption $n \ge 2$. We furthermore have

$$\frac{2(q-1)\mu'}{q} < \frac{2n}{n-2}, \qquad (3.24)$$

because (2.5) asserts that

$$\frac{1}{\mu'} = 1 - \frac{1}{\mu} > 1 - \frac{2}{n} - \frac{n-2}{n} \cdot \frac{1}{q} = \frac{n-2}{n} \cdot \frac{q-1}{q}.$$

Thanks to (3.23), (3.24) and the Gagliardo-Nirenberg inequality, we can find $c_8 > 0$ satisfying

$$\begin{aligned}
\left(\int_\Omega |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}} &= \left\| |\nabla v|^q \right\|_{L^{\frac{2(q-1)\mu'}{q}}(\Omega)}^{\frac{2(q-1)}{q}} \\
&\le c_8 \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{2(q-1)}{q} \cdot d} \cdot \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q} \cdot (1-d)} + c_8 \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{2(q-1)}{q}}
\end{aligned}$$

with

$$d = \frac{nq \cdot (\frac{1}{s} - \frac{1}{2(q-1)\mu'})}{1 - \frac{n}{2} + \frac{nq}{s}} \in (0,1).$$

Consequently, once again recalling (1.3) we have

$$\left(\int_\Omega |\nabla v|^{2(q-1)\mu'} \right)^{\frac{1}{\mu'}} \le c_9 \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\frac{n}{2} \cdot \frac{\frac{2(q-1)}{s} - \frac{1}{\mu'}}{1 - \frac{n}{2} + \frac{nq}{s}}} + c_9 \qquad \text{for all } t \in (0, T_{max}) \quad (3.25)$$

for some positive constant c_9.

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Now collecting (3.18), (3.19), (3.22) and (3.25), from (3.15) and (3.16) we obtain

$$c_1 \int_\Omega (u+1)^{m+p+2\alpha-3} |\nabla v|^2 + c_1 \int_\Omega (u+1)^2 |\nabla v|^{2q-2}$$

$$\leq c_{10} \left(\int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 \right)^{\beta_1} \cdot \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\gamma_1}$$

$$+ c_{10} \left(\int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 \right)^{\beta_2} \cdot \left(\int_\Omega \left| \nabla |\nabla v|^q \right|^2 \right)^{\gamma_2}$$

$$+ c_{10} \qquad \text{for all } t \in (0, T_{max}) \tag{3.26}$$

with some $c_{10} > 0$ and positive numbers $\beta_1, \beta_2, \gamma_1$ and γ_2 satisfying

$$\beta_1 + \gamma_1 = \frac{n}{2} \cdot \frac{m+p+2\alpha-3-\frac{1}{\theta}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{n}{2} \cdot \frac{\frac{2}{s}-1+\frac{1}{\theta}}{1-\frac{n}{2}+\frac{nq}{s}}$$

$$< 1$$

according to (2.6), and

$$\beta_2 + \gamma_2 = \frac{n}{2} \cdot \frac{2-\frac{1}{\mu}}{1-\frac{n}{2}+\frac{n(m+p-1)}{2}} + \frac{n}{2} \cdot \frac{\frac{2(q-1)}{s}-1+\frac{1}{\mu}}{1-\frac{n}{2}+\frac{nq}{s}}$$

$$< 1$$

by (2.7). Therefore Lemma 3.1 states that for some $c_{11} > 0$ we have

$$c_1 \int_\Omega (u+1)^{m+p+2\alpha-3} |\nabla v|^2 + c_1 \int_\Omega (u+1)^2 |\nabla v|^{2q-2}$$

$$\leq \frac{1}{(m+p-1)^2} \int_\Omega |\nabla (u+1)^{\frac{m+p-1}{2}}|^2 + \frac{q-1}{q^2} \int_\Omega \left| \nabla |\nabla v|^q \right|^2 + c_{11} \tag{3.27}$$

for all $t \in (0, T_{max})$. Here we once more employ the Gagliardo-Nirenberg inequality to estimate

$$\int_\Omega (u+1)^p = \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2p}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}}$$

$$\leq c_{12} \left\| \nabla (u+1)^{\frac{m+p-1}{2}} \right\|_{L^2(\Omega)}^{\frac{2p}{m+p-1} \cdot \kappa_1} \cdot \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1} \cdot (1-\kappa_1)}$$

$$+ c_{12} \left\| (u+1)^{\frac{m+p-1}{2}} \right\|_{L^{\frac{2}{m+p-1}}(\Omega)}^{\frac{2p}{m+p-1}} \tag{3.28}$$

and

$$\int_\Omega |\nabla v|^{2q} = \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^2$$

$$\leq c_{12} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{2\kappa_2} \cdot \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{2(1-\kappa_2)} + c_{12} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^2 \tag{3.29}$$

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with some $c_{12} > 0$ and

$$\kappa_1 = \frac{\frac{n(m+p-1)}{2}(1 - \frac{1}{p})}{1 - \frac{n}{2} + \frac{n(m+p-1)}{2}} \qquad \text{and} \qquad \kappa_2 = \frac{\frac{nq}{s} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{nq}{s}},$$

where we note that $\frac{2p}{m+p-1} < \frac{2n}{n-2}$ by (2.1) and $\frac{s}{q} < 2$ since $q > \bar{q} > 2$ and $s < \frac{n}{n-1} \leq 2$. As a consequence of (3.27), (3.28), (3.29), (1.2) and (1.3), (3.14) can be turned into the inequality

$$\frac{d}{dt}\left(\int_\Omega \phi(u) + \frac{1}{q} \int_\Omega |\nabla v|^{2q} \right) + c_{13}\left(\int_\Omega (u+1)^p \right)^{\frac{m+p-1}{p\kappa_1}} + c_{13}\left(\int_\Omega |\nabla v|^{2q} \right)^{\frac{1}{\kappa_2}} \leq c_{14}$$

for all $t \in (0, T_{max})$ and positive constants c_{13} and c_{14}. In view of (3.8), we infer that the function

$$y(t) := \int_\Omega \phi(u(t)) + \frac{1}{q} \int_\Omega |\nabla v(t)|^{2q}, \qquad t \in [0, T_{max}),$$

satisfies

$$y'(t) + c_{15} y^\kappa(t) \leq c_{16} \qquad \text{for all } t \in (0, T_{max})$$

with certain positive constants κ, c_{15} and c_{16}. Upon an ODE comparison argument this entails that

$$y(t) \leq c_{17} := \max\left\{ y_0, \left(\frac{c_{16}}{c_{15}}\right)^{\frac{1}{\kappa}} \right\} \qquad \text{for all } t \in (0, T_{max}).$$

Thus, in view of (3.9) we arrive at the inequalities

$$\int_\Omega (u+1)^{p+m-M}(t) \leq \frac{c_{17}}{c_0} \qquad \text{and} \qquad \int_\Omega |\nabla v(t)|^{2q} \leq q c_{17} \qquad \text{for all } t \in (0, T_{max})$$

and conclude. $\qquad\square$

Now we can immediately pass to our main result.

PROOF (of Theorem 0.1)   The proof is an evident consequence of Lemma 3.3, Lemma 4.1 below and the extendibility criterion provided by Lemma 1.1. $\qquad\square$

4   Appendix: A general boundedness result for quasilinear non-uniformly parabolic equations

In this concluding section, which might be of interest of its own, we derive uniform bounds for nonnegative subsolutions of some quasilinear problems which need not necessarily be uniformly parabolic. More precisely, we consider functions u fulfilling

$$\begin{cases} u_t \leq \nabla \cdot (D(x,t,u)\nabla u) + \nabla \cdot f(x,t) + g(x,t), & x \in \Omega,\ t \in (0,T), \\ \partial_\nu u(x,t) \leq 0, & x \in \partial\Omega,\ t \in (0,T), \end{cases} \qquad (4.1)$$

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in the classical sense, where we allow the diffusion to be degenerate in the sense that we require that

$$D \in C^1(\bar{\Omega} \times [0, T) \times [0, \infty)) \qquad \text{and} \qquad D \geq 0, \tag{4.2}$$

and that there exist $m \in \mathbb{R}$, $s_0 \geq 1$ and $\delta > 0$ such that

$$D(x, t, s) \geq \delta s^{m-1} \qquad \text{for all } x \in \Omega, \, t \in (0, T) \text{ and } s \geq s_0. \tag{4.3}$$

Our goal is to derive estimates in $L^\infty(\Omega \times (0, T))$ under the assumptions that

$$f \in C^0((0, T); C^0(\bar{\Omega}) \cap C^1(\Omega)) \qquad \text{and} \qquad g \in C^0(\Omega \times (0, T)) \tag{4.4}$$

with

$$f \cdot \nu \leq 0 \qquad \text{on } \partial\Omega \times (0, T), \tag{4.5}$$

that

$$f \in L^\infty((0, T); L^{q_1}(\Omega)) \qquad \text{and} \qquad g \in L^\infty((0, T); L^{q_2}(\Omega)), \tag{4.6}$$

and that

$$u \in L^\infty((0, T); L^{p_0}(\Omega)) \tag{4.7}$$

be valid with suitably large q_1, q_2 and p_0.

The derivation of the following statement follows a well-established iterative technique (see [A] for an application in a similar framework). Since we could not find a precise reference covering our situation, and since some major modifications to the original procedure are necessary, we include a full proof here for the sake of completeness.

Lemma 4.1 *Suppose that $T \in (0, \infty]$, that $\Omega \subset \mathbb{R}^n$, $n \geq 1$, is a bounded domain, and that D, f and g comply with (4.2), (4.4) and (4.5). Moreover, assume that (4.3) and (4.6) hold for some $\delta > 0$, $m \in \mathbb{R}$ and $s_0 \geq 1$, and some $q_1 > n + 2$ and $q_2 > \frac{n+2}{2}$. Then if $u \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\bar{\Omega} \times (0, T))$ is a nonnegative function satisfying (4.1), and if (4.7) is valid for some $p_0 \geq 1$ fulfilling*

$$p_0 > 1 - m \cdot \frac{(n+1)q_1 - (n+2)}{q_1 - (n+2)} \tag{4.8}$$

and

$$p_0 > 1 - \frac{m}{1 - \frac{n}{n+2}\frac{q_2}{q_2-1}} \tag{4.9}$$

as well as

$$p_0 > \frac{n(1 - m)}{2}, \tag{4.10}$$

then there exists $C > 0$, only depending on $m, \delta, \Omega, \|f\|_{L^\infty((0,T);L^{q_1}(\Omega))}, \|g\|_{L^\infty((0,T);L^{q_2}(\Omega))}, \|u\|_{L^\infty((0,T);L^{p_0}(\Omega))}$ and $\|u(0)\|_{L^\infty(\Omega)}$, such that

$$\|u(t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t \in (0, T).$$

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PROOF. We evidently may assume that $m \leq 0$, and then fix $r \in (2, \frac{2(n+2)}{n})$ close enough to $\frac{2(n+2)}{n}$ such that writing $\theta(\rho) := \frac{\rho}{2} \cdot \frac{m+p_0-1}{-m+p_0-1}$ and $\mu(\rho) := \frac{\rho}{2} \cdot \frac{m+p_0-1}{p_0-1}$ we have $\theta(r) \geq \frac{q_1}{q_1-2}$ and $\mu(r) \geq \frac{q_2}{q_2-1}$. Indeed, this is possible since our assumption (4.8) on p_0 ensures that

$$
\begin{aligned}
\theta\Big(\frac{2(n+2)}{n}\Big) &= \frac{n+2}{n} \cdot \Big(1 + \frac{2m}{-m+p_0-1}\Big) \\
&> \frac{n+2}{n} \cdot \Big(1 + \frac{2m}{-m+[1-m\cdot\frac{(n+1)q_1-(n+2)}{q_1-(n+2)}]-1}\Big) \\
&= \frac{n+2}{n} \cdot \frac{nq_1}{(n+2)(q_1-2)} \\
&= \frac{q_1}{q_1-2}
\end{aligned}
$$

due to the fact that $q_1 > n+2$, and since (4.9) entails

$$
\begin{aligned}
\mu\Big(\frac{2(n+2)}{n}\Big) &= \frac{n+2}{n} \cdot \Big(1 + \frac{m}{p_0-1}\Big) \\
&> \frac{n+2}{n} \cdot \Big(1 + \frac{m}{\big[1-\frac{m}{1-\frac{n}{n+2}\cdot\frac{q_2}{q_2-1}}\big]-1}\Big) \\
&= \frac{q_2}{q_2-1}.
\end{aligned}
$$

We can now pick $s \in (0,2)$ sufficiently close to 2 fulfilling

$$ r < \frac{2(n+s)}{n} \tag{4.11} $$

and such that

$$ \frac{\frac{nr}{s}-n}{\frac{2q_1}{q_1-2}\cdot(1-\frac{n}{2}+\frac{n}{s})} < 1, \tag{4.12} $$

where the latter can be achieved due to the fact that as $s \to 2$, the expression on the left tends to

$$ \frac{\frac{nr}{2}-n}{\frac{2q_1}{q_1-2}} < \frac{\frac{n}{2}\cdot\frac{2(n+2)}{n}-n}{\frac{2q_1}{q_1-2}} = 1 - \frac{2}{q_1} < 1. $$

We now recursively define

$$ p_k := \frac{2}{s} \cdot p_{k-1} + 1 - m, \qquad k \geq 1, \tag{4.13} $$

and note that $(p_k)_{k\in\mathbb{N}}$ increases and

$$ c_1 \cdot \Big(\frac{2}{s}\Big)^k \leq p_k \leq c_2 \cdot \Big(\frac{2}{s}\Big)^k \qquad \text{for all } k \in \mathbb{N} \tag{4.14} $$

holds with positive c_1 and c_2 which, as all constants $c_3, c_4, \dots$ appearing below, are independent of k. Writing

$$\theta_k := \frac{r}{2} \cdot \frac{m + p_k - 1}{-m + p_k - 1}, \qquad k \in \mathbb{N}, \tag{4.15}$$

since $m \leq 0$ we see that also $(\theta_k)_{k \in \mathbb{N}}$ is increasing with $\theta_k \geq \theta_0 = \theta(r) \geq \frac{q_1}{q_1 - 2}$, and hence $\theta'_k := \frac{\theta_k}{\theta_k - 1}$ satisfies

$$1 < \theta'_k \leq \frac{q_1}{2} \qquad \text{for all } k \in \mathbb{N}. \tag{4.16}$$

Similarly,

$$\mu_k := \frac{r}{2} \cdot \frac{m + p_k - 1}{p_k - 1}, \qquad k \in \mathbb{N}, \tag{4.17}$$

defines an increasing sequence of numbers such that $\mu_k \geq \mu_0 = \mu(r) \geq \frac{q_2}{q_2 - 1}$, and such that for $\mu'_k := \frac{\mu_k}{\mu_k - 1}$ we have

$$1 < \mu'_k \leq q_2 \qquad \text{for all } k \in \mathbb{N}. \tag{4.18}$$

Our goal is to derive a recursive inequality for

$$M_k := \sup_{t \in (0,T)} \int_\Omega \hat{u}^{p_k}(x,t) dx, \qquad k \in \mathbb{N}, \tag{4.19}$$

where $\hat{u}(x,t) := \max\{u(x,t), s_0\}$ for $x \in \bar{\Omega}$ and $t \in [0,T]$. To this end, we note that by a standard approximation procedure we may use $p_k \hat{u}^{p_k - 1}$ as a test function in (4.1) to obtain for $k \geq 1$

$$\frac{d}{dt} \int_\Omega \hat{u}^{p_k} \;+\; p_k(p_k - 1) \int_\Omega D(x,t,u) \hat{u}^{p_k - 2} |\nabla \hat{u}|^2$$
$$\leq \;\; -p_k(p_k - 1) \int_\Omega \hat{u}^{p_k - 2} f \cdot \nabla \hat{u} + p_k \int_\Omega \hat{u}^{p_k - 1} g$$

for all $t \in (0,T)$, where we have used our assumptions that $f \cdot \nu \leq 0$ and $\partial_\nu u \leq 0$ on $\partial\Omega$. We now employ Young's inequality in estimating

$$-p_k(p_k - 1) \int_\Omega \hat{u}^{p_k - 2} f \cdot \nabla \hat{u} \;\; \leq \;\; \frac{p_k(p_k - 1)\delta}{2} \int_\Omega \hat{u}^{m + p_k - 3} |\nabla \hat{u}|^2$$
$$+ \frac{p_k(p_k - 1)}{2\delta} \int_\Omega \hat{u}^{-m + p_k - 1} |f|^2,$$

recall (4.3) and observe that $D(x,t,u) = D(x,t,\hat{u})$ wherever $u \geq s_0$, to find $c_3 > 0$ and $c_4 > 0$ such that

$$\frac{d}{dt} \int_\Omega \hat{u}^{p_k} + c_3 \int_\Omega \left| \nabla \hat{u}^{\frac{m + p_k - 1}{2}} \right|^2 \;\; \leq \;\; c_4 p_k^2 \int_\Omega \hat{u}^{-m + p_k - 1} |f|^2$$
$$+ p_k \int_\Omega \hat{u}^{p_k - 1} g \tag{4.20}$$

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for all $t \in (0, T)$. Here, by the Hölder inequality, (4.6) and (4.16), there exists $c_5 > 0$ such that

$$\int_\Omega \hat{u}^{-m+p_k-1}|f|^2 \le \left(\int_\Omega \hat{u}^{(-m+p_k-1)\theta_k} \right)^{\frac{1}{\theta_k}} \cdot \left(\int_\Omega |f|^{q_1} \right)^{\frac{2}{q_1}} \cdot |\Omega|^{\frac{q_1-2\theta_k'}{q_1\theta_k'}}$$

$$\le c_5 \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^{\frac{2(-m+p_k-1)\theta_k}{m+p_k-1}}(\Omega)}^{\frac{2(-m+p_k-1)}{m+p_k-1}}$$

$$= c_5 \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{r}{\theta_k}} \qquad \text{for all } t \in (0, T)$$

due to (4.15). Similarly, thanks to (4.18) there exists $c_6 > 0$ such that

$$\int_\Omega \hat{u}^{p_k-1}g \le \left(\int_\Omega \hat{u}^{(p_k-1)\mu_k} \right)^{\frac{1}{\mu_k}} \cdot \left(\int_\Omega |g|^{q_2} \right)^{\frac{1}{q_2}} \cdot |\Omega|^{\frac{q_2-\mu_k'}{q_2\mu_k'}}$$

$$\le c_6 \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{r}{\mu_k}}$$

$$= c_6 \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{r}{\theta_k}} \cdot \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{2m}{m+p_k-1}} \qquad \text{for all } t \in (0, T),$$

where we have recalled the definitions (4.15) and (4.17) of μ_k and θ_k. Now since $\hat{u} \ge s_0 \ge 1$ and $m < 0$, the latter factor can be estimated from above according to

$$\left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{2m}{m+p_k-1}} \le |\Omega|^{\frac{2m}{r(m+p_k-1)}} \to 1 \qquad \text{as } k \to \infty.$$

Therefore, using that $p_k \ge 1$ for $k \ge 1$, from (4.20) we thus see that

$$\frac{d}{dt} \int_\Omega \hat{u}^{p_k} + c_3 \int_\Omega \left| \nabla \hat{u}^{\frac{m+p_k-1}{2}} \right|^2 \le c_7 p_k^2 \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{r}{\theta_k}} \qquad \text{for all } t \in (0, T) \qquad (4.21)$$

is valid with some $c_7 > 0$.

Now invoking the Gagliardo-Nirenberg inequality ([F]) we find $c_8 > 0$, by (4.16) yet independent of k, such that

$$\left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{r}{\theta_k}} \le c_8 \left\| \nabla \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^2(\Omega)}^{\frac{ra}{\theta_k}} \cdot \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^s(\Omega)}^{\frac{r(1-a)}{\theta_k}}$$

$$+ c_8 \left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^s(\Omega)}^{\frac{r}{\theta_k}},$$

whence observing that $\frac{(m+p_k-1)s}{2} = p_{k-1}$ by (4.13), from (4.19) we obtain

$$\left\| \hat{u}^{\frac{m+p_k-1}{2}} \right\|_{L^r(\Omega)}^{\frac{r}{\theta_k}} \le c_8 M_{k-1}^{\frac{r(1-a)}{\theta_k s}} \cdot \left(\int_\Omega \left| \nabla \hat{u}^{\frac{m+p_k-1}{2}} \right|^2 \right)^{\frac{ra}{2\theta_k}} + c_8 M_{k-1}^{\frac{r}{\theta_k s}}$$

for all $t \in (0, T)$, with

$$a = \frac{\frac{n}{s} - \frac{n}{r}}{1 - \frac{n}{2} + \frac{n}{s}} \in (0, 1). \qquad (4.22)$$

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Upon an application of Young's inequality, (4.21) thus yields

$$\frac{d}{dt}\int_\Omega \hat{u}^{p_k} + \frac{c_3}{2}\int_\Omega \left|\nabla \hat{u}^{\frac{m+p_k-1}{2}}\right|^2 \le c_9\left(p_k^2 M_{k-1}^{\frac{r(1-a)}{\theta_k s}}\right)^{\frac{2\theta_k}{2\theta_k-ra}} + c_9 p_k^2 M_{k-1}^{\frac{r}{\theta_k s}} \tag{4.23}$$

for all $t \in (0,T)$ and some $c_9 > 0$, where we made use of the fact that (4.12) entails that

$$\frac{ra}{2\theta_k} \le \frac{ra}{2\theta_0} \le \frac{ra}{\frac{2q_1}{q_1-2}} = \frac{\frac{nr}{s}-n}{\frac{2q_1}{q_1-2}\cdot\left(1-\frac{n}{2}+\frac{n}{s}\right)} < 1 \qquad \text{for all } k \in \mathbb{N}.$$

Next, since $p_k > \frac{n(1-m)}{2}$ for all $k \ge 1$ by (4.10), we can pick $\lambda \in (2, \frac{2n}{(n-2)_+})$ such that $\frac{2p_k}{m+p_k-1} \le \lambda$ for all $k \ge 1$. Thus, by the Hölder inequality,

$$\int_\Omega \hat{u}^{p_k} = \left\|\hat{u}^{\frac{m+p_k-1}{2}}\right\|_{L^{\frac{2p_k}{m+p_k-1}}(\Omega)}^{\frac{2p_k}{m+p_k-1}}$$

$$\le |\Omega|^{1-\frac{2p_k}{\lambda(m+p_k-1)}} \cdot \left\|\hat{u}^{\frac{m+p_k-1}{2}}\right\|_{L^\lambda(\Omega)}^{\frac{2p_k}{m+p_k-1}}$$

$$\le c_{10}\left\|\hat{u}^{\frac{m+p_k-1}{2}}\right\|_{L^\lambda(\Omega)}^{\frac{2p_k}{m+p_k-1}} \qquad \text{for all } t \in (0,T)$$

with some $c_{10} > 0$, and therefore applying the Poincaré inequality in the form

$$\|\varphi\|_{L^\lambda(\Omega)}^2 \le c_{11}\left(\|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^s(\Omega)}^2\right) \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

we infer that

$$\int_\Omega \hat{u}^{p_k} \le c_{12}\cdot\left(\left\|\nabla\hat{u}^{\frac{m+p_k-1}{2}}\right\|_{L^2(\Omega)}^2 + \left\|\hat{u}^{\frac{m+p_k-1}{2}}\right\|_{L^s(\Omega)}^2\right)^{\frac{p_k}{m+p_k-1}}$$

holds for all $t \in (0,T)$ with a certain $c_{12} > 0$. In consequence, writing $c_{13} := \inf_{k\ge 1} c_{12}^{-\frac{m+p_k-1}{p_k}} > 0$, we have

$$\int_\Omega \left|\nabla\hat{u}^{\frac{m+p_k-1}{2}}\right|^2 \ge c_{13}\left(\int_\Omega \hat{u}^{p_k}\right)^{\frac{m+p_k-1}{p_k}} - M_{k-1}^{\frac{2}{s}} \qquad \text{for all } t \in (0,T).$$

Combined with (4.23), this gives the inequality

$$\frac{d}{dt}\int_\Omega \hat{u}^{p_k} \le -\frac{c_3}{2}\cdot c_{13}\cdot\left(\int_\Omega \hat{u}^{p_k}\right)^{\frac{m+p_k-1}{p_k}}$$

$$+c_9 p_k^{\frac{4\theta_k}{2\theta_k-ra}}\cdot M_{k-1}^{\frac{2r(1-a)}{s(2\theta_k-ra)}} + c_9 p_k^2 M_{k-1}^{\frac{r}{\theta_k s}} + \frac{c_3}{2}M_{k-1}^{\frac{2}{s}} \tag{4.24}$$

for all $t \in (0,T)$ and $k \ge 1$. To simplify this, we observe that

$$\frac{2r(1-a)}{s(2\theta_k-ra)} \ge \max\left\{\frac{r}{\theta_k s}, \frac{2}{s}\right\} \qquad \text{for all } k \ge 1,$$

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because (4.15) guarantees that $\theta_k \leq \frac{r}{2}$. Since furthermore clearly

$$2 < \frac{4\theta_k}{2\theta_k - ra} \leq \frac{4\theta_0}{2\theta_0 - ra} \qquad \text{for all } k \geq 1,$$

from (4.24) and (4.14) we obtain

$$\frac{d}{dt} \int_\Omega \hat{u}^{p_k} \leq -c_{14} \left(\int_\Omega \hat{u}^{p_k} \right)^{\frac{m+p_k-1}{p_k}} + c_{15} \cdot \tilde{b}^k \cdot M_{k-1}^{\frac{2r(1-a)}{s(2\theta_k-ra)}}$$

for all $t \in (0, T)$ and $k \geq 1$, suitable $c_{14} > 0$ and $c_{15} > 0$ and $\tilde{b} := (\frac{2}{s})^{\frac{4\theta_0}{2\theta_0-ra}} > 1$.
An integration of this ODI provides $c_{16} > 0$ such that

$$\begin{aligned}
M_k &\leq \max\left\{ \int_\Omega \hat{u}_0^{p_k}, \left[\frac{c_{15}}{c_{14}} \cdot \tilde{b}^k \cdot M_{k-1}^{\frac{2r(1-a)}{s(2\theta_k-ra)}} \right]^{\frac{p_k}{m+p_k-1}} \right\} \\
&\leq \max\left\{ \int_\Omega \hat{u}_0^{p_k}, c_{16} b^k M_{k-1}^{\kappa_k} \right\} \qquad \text{for all } k \geq 1,
\end{aligned} \qquad (4.25)$$

where $\hat{u}_0(x) := \hat{u}(x, 0)$ for $x \in \Omega$, $\kappa_k := \frac{2r(1-a)}{s(2\theta_k-ra)} \cdot \frac{p_k}{m+p_k-1}$ and $b := \tilde{b}^{\frac{p_0}{m+p_0-1}}$, and where we have used that $\frac{p_k}{m+p_k-1} \leq \frac{p_0}{m+p_0-1}$ for all $k \geq 1$. Writing

$$\kappa_k = \frac{2}{s} \cdot \left(1 + \frac{1 - \frac{2\theta_k}{r}}{\frac{2\theta_k}{r} - a} \right) \cdot \left(1 + \frac{1 - m}{m + p_k - 1} \right),$$

we easily infer from (4.13), (4.15) and (4.14) that

$$\kappa_k = \frac{2}{s} \cdot (1 + \varepsilon_k), \qquad k \geq 1, \qquad (4.26)$$

holds with some $\varepsilon_k \geq 0$ satisfying

$$\varepsilon_k \leq \frac{c_{17}}{p_k} \leq c_{18} \cdot \left(\frac{s}{2} \right)^k \qquad (4.27)$$

for all $k \geq 1$ and appropriately large $c_{17} > 0$ and $c_{18} > 0$.
Therefore, in the case when $c_{16} b^k M_{k-1}^{\kappa_k} < \int_\Omega \hat{u}_0^{p_k}$ holds for infinitely many $k \geq 1$, we obtain

$$\sup_{t \in (0,T)} \left(\int_\Omega \hat{u}^{p_{k-1}} \right)^{\frac{1}{p_{k-1}}} \leq \left(\frac{1}{c_{16} b^k} \int_\Omega \hat{u}_0^{p_k} \right)^{\frac{1}{\kappa_k p_{k-1}}}$$

for all such k, and hence conclude that

$$\|\hat{u}(t)\|_{L^\infty(\Omega)} \leq \|\hat{u}_0\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (0, T),$$

because $\frac{p_k}{\kappa_k p_{k-1}} \to 1$ as $k \to \infty$ according to (4.13), (4.26) and (4.27).
In the opposite case, upon enlarging c_{16} if necessary we may assume that

$$M_k \leq c_{16} b^k M_{k-1}^{\kappa_k} \qquad \text{for all } k \geq 1.$$

By a straightforward induction, this yields

$$M_k \leq c_{16}^{1+\sum\limits_{j=0}^{k-2}\prod\limits_{i=k-j}^{k}\kappa_i} \cdot b^{k+\sum\limits_{j=0}^{k-2}(k-j-1)\cdot\prod\limits_{i=k-j}^{k}\kappa_i} \cdot M_0^{\prod\limits_{i=1}^{k}\kappa_i}$$

for all $k \geq 2$, and hence in view of (4.26) and (4.14) we obtain

$$M_k^{\frac{1}{p_k}} \quad \leq \quad c_{16}^{\frac{1}{c_1}(\frac{s}{2})^k + \frac{1}{c_1}\cdot\sum\limits_{j=0}^{k-2}(\frac{s}{2})^{k-j-1}\cdot\prod\limits_{i=k-j}^{k}(1+\varepsilon_i)}$$

$$\times b^{\frac{1}{c_1}k(\frac{s}{2})^k + \frac{1}{c_1}\cdot\sum\limits_{j=0}^{k-2}(k-j-1)\cdot(\frac{s}{2})^{k-j-1}\cdot\prod\limits_{i=k-j}^{k}(1+\varepsilon_i)}$$

$$\times M_0^{\frac{1}{c_1}\cdot\prod\limits_{i=1}^{k}(1+\varepsilon_i)}$$

for $k \geq 2$. Since $\ln(1+z) \leq z$ for $z \geq 0$, from (4.27) and the fact that $s < 2$ we gain

$$\ln\Big(\prod_{i=1}^{k}(1+\varepsilon_i)\Big) = \sum_{i=1}^{k}\varepsilon_i \leq \frac{c_{18}}{1-\frac{s}{2}},$$

so that using $\sum\limits_{j=0}^{k-2}(k-j-1)\cdot(\frac{s}{2})^{k-j-1} \leq \sum\limits_{l=1}^{\infty}l(\frac{s}{2})^l < \infty$,

from this we conclude that also in this case $\|\hat{u}(t)\|_{L^\infty(\Omega)}$ is bounded from above by a constant independent of $t \in (0,T)$. This clearly proves the lemma. $\qquad\square$

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