# Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops

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**Abstract**

In the modeling of collective effects arising in bacterial suspensions in fluid drops, coupled chemotaxis-(Navier-)Stokes systems generalizing the prototype

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c - nc, & x\in\Omega,\ t>0,\\ u_t &= \Delta u + \kappa(u\cdot\nabla)u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u &= 0, & x\in\Omega,\ t>0,\end{cases}$$

have been proposed to describe the spontaneous emergence of patterns in populations of oxygen-driven swimming bacteria. Here, $\kappa\in\mathbb{R}$ and the gravitational potential $\phi$ are given and $\Omega\subset\mathbb{R}^N$ is a bounded convex domain with smooth boundary.
Under the boundary conditions $\frac{\partial n}{\partial\nu}=\frac{\partial c}{\partial\nu}=0$ and $u=0$ on $\partial\Omega$, it is shown in this paper that suitable regularity assumptions on the initial data entail the following:

- If $N=2$, then the full chemotaxis-Navier-Stokes system (with any $\kappa\in\mathbb{R}$) admits a unique global classical solution.
- If $N=3$, then the simplified chemotaxis-Stokes system (with $\kappa=0$) possesses at least one global weak solution.

In particular, no smallness condition on either $\phi$ or on the initial data needs to be fulfilled here, as required in a related recent work by Duan/Lorz/Markowich (*Comm. PDE* **35**, 2010).

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## 1 Introduction

When bacteria of the species *Bacillus subtilis* are suspended in water, it can be observed experimentally that spatial patterns may spontaneously emerge from initially almost homogeneous distributions of bacteria ([3]). A mathematical model for such processes was proposed in [24], where it is assumed that the essentially responsible mechanisms are a chemotactic movement of bacteria towards oxygen which they consume, a gravitational effect on the motion of the fluid by the heavier bacteria, and a convective transport of both cells and oxygen through the water (cf. also [3] and [15]). This leads to a PDE model of the form

$$\begin{cases} n_t + u\cdot\nabla n &=& \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &=& \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t &=& \Delta u + \kappa(u\cdot\nabla)u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u &=& 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

for the unknown bacterial density $n$, the oxygen concentration $c$, the fluid velocity field $u$ and the associated pressure $P$ in the physical domain $\Omega\subset\mathbb{R}^N$. The gravitational potential $\phi$, the chemotactic sensitivity $\chi(c)$ and the per-capita oxygen consumption rate $f(c)$ are supposed to be sufficiently smooth given functions. A simple model case can be obtained upon the choices

$$\nabla\phi(x)\equiv const., \qquad \chi(c)\equiv const., \qquad f(c)=c, \tag{1.2}$$

for instance, which in view of standard assumptions in the modeling of chemotactic cell migration ([11]) and oxygen consumption ([18]) might be considered prototypical. However, different functional forms of $\chi$ and $f$ are meaningful as well, according to various conceivable threshold effects and saturation mechanisms. For instance, in [24] both $\chi$ and $f$ were proposed as step-type functions vanishing for small positive $c$ below a threshold, beyond which both attain positive constants. It can also be reasonable to assume that at large oxygen concentrations chemotaxis is inhibited in the sense that $\chi(c)\to 0$ as $c\to\infty$ ([13], [11]).
When the fixed number $\kappa\in\mathbb{R}$ in (1.1) is nonzero, the fluid motion is governed by the full Navier-Stokes equations involving nonlinear convection, whereas if $\kappa=0$ we consider the simplified Stokes evolution for $u$ which appears to be justified if the fluid flow remains small ([15]).

As to the mathematical analysis of (1.1), only few results seem to be available so far, and they concentrate on the natural first question of local and global solvability of corresponding initial-value problems in either bounded or unbounded domains $\Omega$. In [15], certain local-in-time weak solutions were constructed for a boundary-value problem for (1.1) in the three-dimensional setting under the assumptions that $\chi\equiv const.$ and $f$ be nondecreasing such that $f(0)=0$. In [5], the Cauchy problem for (1.1) has been studied on the basis of a priori estimates involving energy-type functionals. It is asserted there that when $\Omega=\mathbb{R}^2$, appropriate smallness assumptions on either $\nabla\phi$ or the initial data for $c$ ensure global existence of weak solutions to the chemotaxis-Stokes system (1.1) with $\kappa=0$, provided that some further technical conditions are satisfied. structural conditions on $\chi$ and $f$ are satisfied. In the recent preprint [14], an improvement of this a priori estimation technique allows for the construction of global weak solutions to the Navier-Stokes version of (1.1) with $\kappa=-1$ and arbitrarily large initial data in $\Omega=\mathbb{R}^2$, under basically the same assumptions on $\chi$ and $f$ as made in [5].

In the three-dimensional case, the problem of global existence seems to be more delicate: Indeed, here the only result that we are aware of is presented in [5], where global classical solutions near constant steady states are constructed for the full chemotaxis-Navier-Stokes system (1.1) with $\kappa = -1$ and $\Omega = \mathbb{R}^3$. To the best of our knowledge, the question of global solvability of (1.1) for large initial data is still open.

In the present paper we shall suppose that $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, is a smoothly bounded convex domain, and we shall consider (1.1) along with the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{for } x \in \partial\Omega \text{ and } t > 0, \tag{1.3}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.4}$$

We will assume throughout that the initial data satisfy the technically motivated regularity and positivity requirements

$$\begin{cases} n_0 \in C^0(\bar\Omega), \quad n_0 > 0 \quad \text{in } \bar\Omega, \\ c_0 \in W^{1,q}(\Omega) \text{ for some } q > N, \quad c_0 > 0 \quad \text{in } \bar\Omega, \\ u_0 \in D(A^\alpha) \text{ for some } \alpha \in (\frac{N}{4}, 1), \end{cases} \tag{1.5}$$

where $A$ denotes the realization of the Stokes operator in the solenoidal subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0\}$ of $L^2(\Omega)$ ([20]).
As to the parameter functions in (1.1), we require that they are sufficiently regular so as to satisfy

$$\begin{cases} \chi \in C^2([0,\infty)), \quad \chi > 0 \quad \text{in } [0,\infty), \\ f \in C^2([0,\infty)), \quad f(0) = 0, \quad f > 0 \quad \text{in } (0,\infty), \\ \phi \in C^2(\bar\Omega), \end{cases} \tag{1.6}$$

and that they generalize those in (1.2) by fulfilling the structural hypotheses

$$\Big(\frac{f}{\chi}\Big)' > 0 \qquad \text{on } [0,\infty) \tag{1.7}$$

as well as

$$\Big(\frac{f}{\chi}\Big)'' \le 0 \qquad \text{on } [0,\infty) \tag{1.8}$$

and

$$(\chi \cdot f)' \ge 0 \qquad \text{on } [0,\infty). \tag{1.9}$$

Our main results on global solvability of (1.1)-(1.3) then read as follows.

**Theorem 1.1** *Let $N \in \{2,3\}$ and $\Omega \subset \mathbb{R}^N$ be a bounded convex domain with smooth boundary, and let $\kappa \in \mathbb{R}$. Assume that $\chi$ and $f$ satisfy (1.6)-(1.9), and suppose that $n_0, c_0$ and $u_0$ comply with (1.5).*
*i) If $N = 2$, then (1.1), (1.3), (1.4) possesses a classical solution which is global in time. This solution*

*is unique, up to addition of constants to the pressure $P$, within the class of functions which for all $T \in (0,\infty)$ enjoy the regularity properties*

$$\begin{cases} n \in C^0([0,T); L^2(\Omega)) \cap L^\infty((0,T); C^0(\bar\Omega)) \cap C^{2,1}(\bar\Omega \times (0,T)), \\ c \in C^0([0,T); L^2(\Omega)) \cap L^\infty((0,T); W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0,T)), \\ u \in C^0([0,T); L^2(\Omega)) \cap L^\infty((0,T); D(A^\alpha)) \cap C^{2,1}(\bar\Omega \times (0,T)), \\ P \in L^1((0,T); W^{1,2}(\Omega)). \end{cases} \tag{1.10}$$

*ii) If $N = 3$ and $\kappa = 0$, then there exists at least one global weak solution of (1.1), (1.3), (1.4) in the sense of Definition 5.1 below.*

Let us underline in which respects these statements go beyond what is known in the existing literature.

**Large-data solutions in the three-dimensional case.** First, none of our results requires any smallness condition on the parameter functions $\chi, f$ or $\phi$, nor on the size of the initial data. In particular, according to the remarks above, Theorem 1.1 ii) seems to provide *the first global existence result for large-data solutions of (1.1) in the three-dimensional framework*. Unfortunately we are not able to cover the full chemotaxis-Navier-Stokes system with $\kappa \neq 0$ here; it would be interesting to see whether certain generalized solutions in $\Omega \times (0,\infty)$ can be constructed for this problem with large data as well.

**Including the prototype case (1.2).** Next, our assumptions (1.7)-(1.9) are weaker than those imposed in [5] and [14]. Indeed, the approaches in both these references seem to rely in an essential way on the requirement that the inequalities in (1.8) and (1.9) be strict on $[0,\infty)$, because then, roughly speaking, a certain weighted spatial $L^4$ norm of $\nabla c$ is dissipated (cf. (3.1) below). Pursuing a refined analysis of a second-order energy-like functional, we shall be able to include the borderline cases when equality holds in (1.8) and (1.9). This extension is not marginal, because *it covers the prototypical choices $\chi \equiv const.$ and $f(c) = c$ as in (1.2)*, which is evidently not possible under the stronger assumptions in [5] and [14]. We remark that by a slight refinement of our analysis it is even possible to further relax the hypothesis (1.8) so as to require $(f/\chi)''(c) \le \varepsilon$ only, where $\varepsilon$ is a positive number which may depend on $\|c_0\|_{L^\infty(\Omega)}$ (cf. the remark following Lemma 3.4). Therefore, the present approach is even able to cover the case when we allow for at least rough approximations of the step-type functional choices for $\chi$ and $f$ as mentioned above.

**Uniqueness.** We finally point out that our uniqueness statement for $N = 2$, which is actually a by-product of our local existence theory, appears to be the first result of this type for (1.1). In the case $N = 3$ we have to leave open the uniqueness question, which is essentially due to the fact that we are not able to derive appropriate regularity properties of our weak solution.

Before going into details, let us mention the papers [4], [22] and [14] where a closely related variant of (1.1) is considered in which the self-diffusive term in the first PDE is replaced by the porous medium-type expression $\Delta n^m$ with $m > 1$. From a mathematical point of view, choosing $m$ large should enhance the balancing effect of (self-)diffusion in the first equation in (1.1), so that solutions should more likely remain bounded and hence be global in time; in fact, heuristic arguments of this type could be rigorously justified for Keller-Segel systems without fluid interaction (see [19], [21] and [1], for instance). A first result of this flavor concerning (1.1) has been found in [4], where global existence of weak solutions is asserted for bounded domains $\Omega \subset \mathbb{R}^2$ when $m \in (\frac{3}{2}, 2]$, $\kappa = 0$ and $f$ is increasing with $f(0) = 0$. This global existence result has recently been extended in [22] so as to cover the whole range $m \in (1, \infty)$, and moreover it has been shown there that all solutions evolving from sufficiently regular initial data even remain bounded in $\Omega \times (0, \infty)$. The paper [14] addresses the three-dimensional analogue thereof, and establishes global existence of weak solutions for $\kappa = 0$ and the precise value $m = \frac{4}{3}$ under some additional assumptions on $\chi$ and $f$. This complements a corresponding result in [4] which shows global weak solvability in the three-dimensional case for any $m \in [\frac{7+\sqrt{217}}{12}, 2]$. In the case $N = 3$, a complete classification of all $m > 1$ which allow for global solutions is still lacking; however, Theorem 1.1 ii) gives rise to the conjecture that for any $m > 1$ global weak solutions exist, at least in the simplified chemotaxis-Stokes system when $\kappa = 0$.

We finally note that all our results remain valid if we replace $\Omega$ with the whole space $\mathbb{R}^N$, provided that we additionally complement (1.5) by some convenient conditions on the spatial decay of the initial data as $|x| \to \infty$. Since passing to this alternative case is possible by straightforward arguments, we refrain from giving details here and rather refer to a corresponding construction in [5], for instance.

The paper is organized as follows. In Section 2 we shall apply a standard fixed point procedure to ensure local existence and uniqueness of smooth solutions, and derive a suitable extensibility criterion for such solutions. Section 3 will provide an energy identity in the style of that already used in [5] and [14]. A novel way to exploit this identity, involving a weighted integral inequality (Lemma 3.3) along with interpolation arguments, will lead to an energy-type inequality (Lemma 3.4) which will form the cornerstone of our subsequent analysis. This inequality will be used to establish our main results in the cases $N = 2$ and $N = 3$ in Sections 4 and 5, respectively.
In order to avoid redundantly specifying assumptions, let us tacitly assume throughout the sequel that $\Omega$ is a bounded convex domain in $\mathbb{R}^N$ with smooth boundary, and only mention here that all statements before Lemma 3.4 remain valid if the convexity assumption is dropped.

## 2 Local existence of classical solutions

We first assert local solvability by means of a straighforward fixed point reasoning. Procedures of this type are well-established in the context of both the Navier-Stokes equations (see [7] or also the survey [25]) and various types of chemotaxis equations (cf. [28] or [29], for instance). However, since the argument leading to the proof of our main result will essentially rely on the extensibility criterion (2.3), for the reader's convenience we include a proof. Upon slight modifications, it can be carried over to any space dimension $N \geq 2$, but in order to keep the presentation as simple as possible, we restrict ourselves to the physically relevant cases $N = 2$ and $N = 3$.

In view of an approximation procedure we intend to apply in the three-dimensional case (cf. Section 5 below), let us here consider a problem slightly more general than (1.1), (1.3), (1.4), namely

$$\begin{cases} n_t + u \cdot \nabla n = \Delta n - \nabla \cdot (nF'(n)\chi(c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c = \Delta c - F(n)f(c), & x \in \Omega,\ t > 0, \\ u_t = \Delta u + \kappa(u \cdot \nabla)u + \nabla P + n\nabla\phi, & x \in \Omega,\ t > 0, \\ \nabla \cdot u = 0, & x \in \Omega,\ t > 0, \\ \frac{\partial n(x,t)}{\partial \nu} = \frac{\partial c(x,t)}{\partial \nu} = 0, \quad u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where

$$F \in C^2([0,\infty)) \quad \text{is nonnegative and satisfies} \quad 0 \le F'(n) \le 1 \text{ for all } n \ge 0. \tag{2.2}$$

Here it will turn out that if we choose $F(n) := n$ if $N = 2$ and $F(n) = F_\varepsilon(n) = \frac{1}{\varepsilon}\ln(1+\varepsilon n)$ for $\varepsilon > 0$ in the case $N = 3$, then (2.1) will admit a unique global classical solution. In the two-dimensional setting this will evidently prove our main result concerning (1.1), whereas if $N = 3$ we shall need to take $\varepsilon \searrow 0$ appropriately to end up with a global object that will solve (1.1) at least in some generalized sense (cf. Definition 5.1).

**Lemma 2.1** *Let $N \in \{2,3\}$ and $\kappa \in \mathbb{R}$. Suppose that (1.6) and (2.2) hold, and that $n_0$, $c_0$ and $u_0$ satisfy (1.5). Then there exist $T_{max} \in (0,\infty]$ and a classical solution $(n,c,u,P)$ of (2.1) in $\Omega \times (0,T_{max})$. Moreover, we have $n > 0$ and $c > 0$ in $\bar\Omega \times [0,T_{max})$, and*

$$\text{if } T_{max} < \infty, \text{ then } \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \to \infty \qquad \text{as } t \nearrow T_{max}. \tag{2.3}$$

*For any $T \in (0,T_{max})$, this solution is unique, up to addition of constants to $P$, among all functions satisfying (1.10).*

Proof. <u>Existence.</u> With $R > 0$ and $T \in (0,1)$ to be specified below, in the Banach space

$$X := L^\infty((0,T); C^0(\bar\Omega) \times W^{1,q}(\Omega) \times D(A^\alpha)),$$

we consider the closed set

$$S := \Big\{ (n,c,u) \in X \;\Big|\; \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,q}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le R \qquad \text{for a.e. } t \in (0,T) \Big\}$$

and introduce a mapping $\Phi = (\Phi_1,\Phi_2,\Phi_3)$ on $S$ by defining

$$\begin{aligned}
\Phi_1(n,c,u)(\cdot,t) &:= e^{t\Delta}n_0 - \int_0^t e^{(t-s)\Delta}\Big\{\nabla\cdot(nF'(n)\chi(c)\nabla c) + u\cdot\nabla n\Big\}(\cdot,s)ds,\\
\Phi_2(n,c,u)(\cdot,t) &:= e^{t\Delta}c_0 - \int_0^t e^{(t-s)\Delta}\Big\{F(n)f(c) + u\cdot\nabla c\Big\}(\cdot,s)ds \qquad \text{and}\\
\Phi_3(n,c,u)(\cdot,t) &:= e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}\Big\{\kappa(u\cdot\nabla)u + n\nabla\phi\Big\}(\cdot,s)ds
\end{aligned}$$

for $(n,c,u) \in S$ and $t \in (0,T)$. Here and below, $(e^{t\Delta})_{t\ge0}$, $(e^{-tA})_{t\ge0}$ and $\mathcal{P}$ denote the Neumann heat semigroup, the Stokes semigroup with Dirichlet boundary data, and the Helmholtz projection in $L^2(\Omega)$, respectively ([20]).
Then since $q > N$, we can pick $\beta \in (0,1)$ such that $\frac{N}{2q} < \beta < \frac12$, so that in particular $D(B^\beta) \hookrightarrow C^0(\bar\Omega)$, where $B$ stands for the sectorial operator $-\Delta + 1$ in $L^q(\Omega)$ with homogeneous Neumann boundary conditions ([9]). Using that $\nabla\cdot(nu) = u\cdot\nabla n$ due to the fact that $\nabla\cdot u = 0$, applying Lemma 1.3 (iv) in [27] and recalling (2.2) we can thus find positive constants $c_1, c_2$ and $c_3(R)$ such that

$$\begin{aligned}
\|\Phi_1(n,c,u)(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{t\Delta}n_0\|_{L^\infty(\Omega)} + c_1\int_0^t \|B^\beta e^{-(t-s)(B-1)}\nabla\cdot(nF'(n)\chi(c)\nabla c + nu)(\cdot,s)\|_{L^q(\Omega)}ds\\
&\le \|n_0\|_{L^\infty(\Omega)} + c_2\int_0^t (t-s)^{-\beta-\frac12}\|(nF'(n)\chi(c)\nabla c + nu)(\cdot,s)\|_{L^q(\Omega)}ds\\
&\le \|n_0\|_{L^\infty(\Omega)} + c_3(R)\cdot T^{\frac12-\beta} \qquad \text{for all } t \in (0,T),
\end{aligned} \tag{2.4}$$

where we note that $q > N$ and $\alpha > \frac{N}{4}$ imply that both $W^{1,q}(\Omega)$ and $D(A^\alpha)$ are continuously embedded into $C^0(\bar{\Omega})$ ([9], [20]). Proceeding similarly, we fix any $\gamma \in (\frac{1}{2}, 1)$ and estimate

$$\begin{aligned}
\|\Phi_2(n,c,u)(\cdot,t)\|_{W^{1,q}(\Omega)} &\le \|e^{t\Delta}c_0\|_{W^{1,q}(\Omega)} + c_4 \int_0^t \|B^\gamma e^{-(t-s)(B-1)}(F(n)f(c) + u\cdot\nabla c)(\cdot,s)\|_{L^q(\Omega)} ds \\
&\le c_5\|c_0\|_{W^{1,q}(\Omega)} + c_5 \int_0^t (t-s)^{-\gamma} \|(F(n)f(c) + u\cdot\nabla c)(\cdot,s)\|_{L^q(\Omega)} ds \\
&\le c_5\|c_0\|_{W^{1,q}(\Omega)} + c_6(R)\cdot T^{1-\gamma} \qquad \text{for all } t \in (0,T) \qquad (2.5)
\end{aligned}$$

for some $c_4 > 0, c_5 > 0$ and $c_6(R) > 0$. Finally, we can find $c_7 > 0$ fulfilling

$$\begin{aligned}
\|A^\alpha \Phi_3(n,c,u)(\cdot,t)\|_{L^2(\Omega)} &\le \|A^\alpha e^{-tA}u_0\|_{L^2(\Omega)} \\
&\quad + c_7 \int_0^t (t-s)^{-\alpha} \cdot \Big\{ \|(u\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)} + \|n(\cdot,s)\nabla\phi\|_{L^2(\Omega)} \Big\} ds
\end{aligned}$$

for all $t \in (0,T)$. Here, since $\alpha > \frac{N}{4} \ge \frac{1}{2}$, we can estimate

$$\|(u\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)} \le \|u(\cdot,s)\|_{L^\infty(\Omega)} \cdot \|\nabla u(\cdot,s)\|_{L^2(\Omega)} \le c_8 \|A^\alpha u(\cdot,s)\|^2_{L^2(\Omega)}$$

with some $c_8 > 0$, so that from the boundedness of $\nabla\phi$ in $L^\infty$ we obtain $c_9(R) > 0$ such that

$$\|A^\alpha \Phi_3(n,c,u)(\cdot,t)\|_{L^2(\Omega)} \le \|A^\alpha u_0\|_{L^2(\Omega)} + c_9(R)\cdot T^{1-\alpha} \qquad \text{for all } t \in (0,T).$$

Combined with (2.4) and (2.5), this proves that $\Phi$ maps $S$ into itself if we first pick $R > 0$ large enough (depending on $\|n_0\|_{L^\infty(\Omega)}, \|c_0\|_{W^{1,q}(\Omega)}$ and $\|A^\alpha u_0\|_{L^2(\Omega)}$) and then $T > 0$ sufficiently small.
By a straightforward adaptation of the above reasoning, using the local Lipschitz continuity of $\chi$ and $f$, one can easily deduce that if $T$ is further diminished then $\Phi$ in fact becomes a contraction on $S$. For such $R$ and $T$ we therefore infer from the Banach fixed point theorem that there exists $(n,c,u) \in S$ such that $\Phi(n,c,u) = (n,c,u)$. According to standard bootstrap arguments involving the regularity theories for parabolic equations and the Stokes semigroup ([10], [8]), $(n,c,u)$ actually enjoys the smoothness properties listed in (1.10), from which it follows ([20]) that there exists a smooth function $P$ such that $(n,c,u,P)$ solves (1.1) classically in $\Omega \times (0,T)$. Now the conclusion (2.3) is an immediate consequence of the observation that our above choice of $T$ actually depends on $\|n_0\|_{L^\infty(\Omega)}, \|c_0\|_{W^{1,q}(\Omega)}$ and $\|A^\alpha u_0\|_{L^2(\Omega)}$ only.

Positivity. Since $f(0) = 0$, by comparison we obtain $c \ge 0$ and also $n \ge 0$. Since both $n$ and $c$ are classical solutions of their respective equations, we now even may apply the strong maximum principle to obtain that both functions are strictly positive in $\bar{\Omega} \times [0, T_{max})$.

Uniqueness. In order to demonstrate uniqueness within the indicated class, we suppose that $(n,c,u,P)$ and $(\hat{n},\hat{c},\hat{u},\hat{P})$ are two solutions of (1.1) in $\Omega \times (0,T)$ for some $T > 0$, satisfying (1.10). We fix $T_0 \in (0,T)$ and multiply the difference of the PDEs satisfied by $n$ and $\hat{n}$ by $(n - \hat{n})$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (n-\hat{n})^2 + \int_\Omega |\nabla(n-\hat{n})|^2 &= \int_\Omega (n-\hat{n})F'(n)\chi(c)\nabla c\cdot\nabla(n-\hat{n}) \\
&\quad + \int_\Omega \hat{n}(F'(n) - F'(\hat{n}))\chi(c)\nabla c\cdot\nabla(n-\hat{n})
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_\Omega \hat{n} F(\hat{n}) (\chi(c) - \chi(\hat{c})) \nabla c \cdot \nabla (n - \hat{n}) \\
&\quad + \int_\Omega \hat{n} F'(\hat{n}) \chi(\hat{c}) \nabla (c - \hat{c}) \cdot \nabla (n - \hat{n}) \\
&\quad - \int_\Omega (u - \hat{u}) \cdot \nabla n (n - \hat{n}) - \int_\Omega \hat{u} \cdot \nabla (n - \hat{n})(n - \hat{n}) \\
=: &\quad I_1 + \ldots + I_6 \qquad \text{for all } t \in (0, T_0).
\end{aligned}
$$

Since $T_0 < T$, there exists $c_{10} > 0$ such that

$$
\|\hat{n}(\cdot,t)\|_{L^\infty(\Omega)} + \|n(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla c(\cdot,t)\|_{L^q(\Omega)} + \|\hat{u}(\cdot,t)\|_{L^\infty(\Omega)} \le c_{10} \qquad \text{for all } t \in (0, T_0),
$$

the latter again resulting from the embedding $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ asserted by our assumption $\alpha > \frac{N}{4}$. Now using Young's and Hölder's inequalities, (2.2) and the local Lipschitz continuity of $\chi$ on $[0, \infty)$, we find $c_{11} > 0$ and $c_{12} > 0$ such that

$$
\begin{aligned}
I_3 &\le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{11} \int_\Omega \hat{n}^2 (c - \hat{c})^2 |\nabla c|^2 \\
&\le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{11} c_{10}^4 \|c - \hat{c}\|^2_{L^{\frac{2q}{q-2}}(\Omega)} \\
&\le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + \int_\Omega |\nabla (c - \hat{c})|^2 + c_{12} \int_\Omega (c - \hat{c})^2 \qquad \text{for all } t \in (0, T_0)
\end{aligned}
$$

according to Ehrling's lemma and the compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2q}{q-2}}(\Omega)$. By similar arguments involving the Lipschitz continuity of $F'$ on $[0, c_{10}]$, for some positive $c_i$, $i = 13, ..., 19$, we have

$$
I_1 + I_2 \le \frac{1}{20} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{13} \int_\Omega (n - \hat{n})^2 |\nabla c|^2 \le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{14} \int_\Omega (n - \hat{n})^2
$$

and

$$
I_4 \le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{15} \int_\Omega |\nabla (c - \hat{c})|^2
$$

as well as

$$
I_6 \le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{16} \int_\Omega |\hat{u}|^2 (n - \hat{n})^2 \le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{17} \int_\Omega (n - \hat{n})^2
$$

and

$$
\begin{aligned}
I_5 &= \int_\Omega n (u - \hat{u}) \cdot \nabla (n - \hat{n}) \\
&\le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{18} \int_\Omega n^2 |u - \hat{u}|^2 \\
&\le \frac{1}{10} \int_\Omega |\nabla (n - \hat{n})|^2 + c_{19} \int_\Omega |u - \hat{u}|^2
\end{aligned}
$$

for all $t \in (0, T_0)$, where we have used that $\nabla \cdot u = \nabla \cdot \hat{u} = 0$. Altogether,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (n-\hat{n})^2 + \frac{1}{2}\int_\Omega |\nabla(n-\hat{n})|^2 &\le (c_{15}+1)\int_\Omega |\nabla(c-\hat{c})|^2 \\ &\quad +(c_{14}+c_{17})\int_\Omega (n-\hat{n})^2 + c_{12}\int_\Omega (c-\hat{c})^2 + c_{19}\int_\Omega |u-\hat{u}|^2 \qquad (2.6)\end{aligned}$$

for all $t \in (0, T_0)$. Proceeding similarly, we obtain $c_{20} > 0$ such that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega (c-\hat{c})^2 + \int_\Omega |\nabla(c-\hat{c})|^2 &= -\int_\Omega (F(n)-F(\hat{n}))f(c)(c-\hat{c}) - \int_\Omega F(\hat{n})(f(c)-f(\hat{c}))(c-\hat{c}) \\ &\quad -\int_\Omega (u-\hat{u})\cdot\nabla c(c-\hat{c}) - \int_\Omega \hat{u}\cdot\nabla(c-\hat{c})(c-\hat{c}) \\ &\le \frac{1}{2}\int_\Omega |\nabla(c-\hat{c})|^2 \\ &\quad +c_{20}\int_\Omega (n-\hat{n})^2 + c_{20}\int_\Omega (c-\hat{c})^2 + c_{20}\int_\Omega |u-\hat{u}|^2 \qquad (2.7)\end{aligned}$$

for all $t \in (0, T_0)$. Finally, integrating by parts and once more using that $\nabla \cdot u = \nabla \cdot \hat{u} = 0$, we find $c_{21} > 0$ fulfilling

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |u-\hat{u}|^2 + \int_\Omega |\nabla(u-\hat{u})|^2 &= -\kappa\int_\Omega ((u-\hat{u})\cdot\nabla)(u-\hat{u})\cdot u + \kappa\int_\Omega (\hat{u}\cdot\nabla)(u-\hat{u})\cdot(u-\hat{u}) \\ &\quad +\int_\Omega (n-\hat{n})\nabla\phi\cdot(u-\hat{u}) \\ &\le \frac{1}{2}\int_\Omega |\nabla(u-\hat{u})|^2 + c_{21}\int_\Omega (n-\hat{n})^2 + c_{21}\int_\Omega |u-\hat{u}|^2 \qquad (2.8)\end{aligned}$$

for all $t \in (0, T_0)$. All in all, from (2.6)-(2.8) we infer that $y(t) := \frac{1}{2}\int_\Omega (n-\hat{n})^2 + \frac{c_{15}+1}{2}\int_\Omega (c-\hat{c})^2 + \frac{1}{2}\int_\Omega |u-\hat{u}|^2$ satisfies $y'(t) \le c_{22}y(t)$ for all $t \in (0, T_0)$ for some $c_{22} > 0$ depending on $T_0$ only. On integration, this yields $y \equiv 0$ in $(0, T_0)$ and thereby proves the claim, for $T_0 \in (0, T)$ was arbitrary. $\Box$

The following two basic properties immediately result from an integration of the first PDE in (2.1), and from the maximum principle, because $f \ge 0$ and $F \ge 0$.

**Lemma 2.2** *Under the assumptions of Lemma 2.1, the solution of (2.1) satisfies*

$$\int_\Omega n(x,t)dx = \int_\Omega n_0 \qquad \textit{for all } t \in (0, T_{max}) \qquad (2.9)$$

*and*

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t \in (0, T_{max}). \qquad (2.10)$$

# 3 An energy inequality

In this section we shall establish an energy-type identity associated with the first two PDEs in our system (see Lemma 3.2) and turn this into a useful inequality when $\Omega$ is convex and (1.7)-(1.9) and

(2.2) are satisfied (Lemma 3.4). The former has basically been discovered for $\Omega = \mathbb{R}^2$ in [5] already, but our conclusion seems to go beyond the one drawn there.
As underlined in [5], the main technical difficulty stems from the fact that this identity indeed may provide a genuinely dissipated quantity when $u \equiv 0$ (which is possible when $\phi \equiv const.$), but in the general case $u \not\equiv 0$ the drift terms in (1.1) give rise to extra, apparently unsigned, terms (see (3.1) below). We shall see, however, that this obstacle can be overcome by taking full advantage of some second-order dissipative term which in the prototype case $\chi \equiv 1$, $f(c) = c$ and $F(n) = n$ becomes $\int_\Omega c|D^2 \ln c|^2$, where $D^2\varphi$ denotes the Hessian of $\varphi$. The crucial tool in this context seems to be an integral inequality to be stated in Lemma 3.3, which ensures that this dissipative term dominates an integral of $|\nabla c|^4$ with a singular weight; in the latter special situation, for instance, it reads

$$\int_\Omega \frac{|\nabla c|^4}{c^3} \le (2+\sqrt{N})^2 \int_\Omega c|D^2 \ln c|^2.$$

We note that the results from this section basically apply to any space dimension $N \ge 2$ (provided that a local solution exists, cf. the comment before Lemma 2.1).

To begin with, let us recall a higher-dimensional analogue of the formula $\int_\Omega h'(\varphi)\varphi_x^2\varphi_{xx} = -\frac{1}{3}\int_\Omega h''(\varphi)\varphi_x^4$ that holds for bounded intervals $\Omega \subset \mathbb{R}$ and any smooth $h$ and $\varphi$ satisfying $\varphi_x = 0$ on $\partial\Omega$. For a proof based on integration by parts, we refer to [2].

**Lemma 3.1** *Suppose that $h \in C^2(\mathbb{R})$. Then for all $\varphi \in C^2(\bar{\Omega})$ fulfilling $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega$ we have*

$$\begin{aligned}\int_\Omega h'(\varphi)|\nabla\varphi|^2\Delta\varphi &= -\frac{2}{3}\int_\Omega h(\varphi)|\Delta\varphi|^2 + \frac{2}{3}\int_\Omega h(\varphi)|D^2\varphi|^2 - \frac{1}{3}\int_\Omega h''(\varphi)|\nabla\varphi|^4 \\ &\quad -\frac{1}{3}\int_{\partial\Omega} h(\varphi)\frac{\partial}{\partial\nu}|\nabla\varphi|^2.\end{aligned}$$

With this lemma at hand, a series of straightforward integrations by parts will lead to the following energy-type equality which, up to the boundary integral, was already used in [5] in the case $\Omega = \mathbb{R}^2$. However, since the sign of this boundary integral needs to be controlled in the sequel, we prefer to present a full derivation here.

**Lemma 3.2** *Assume that requirements of Lemma 2.1 are met. Then the solution $(n, c, u, P)$ of (2.1) satisfies the identity*

$$\begin{aligned}\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac{1}{2}\int_\Omega |\nabla\psi(c)|^2\Big\} &+ \int_\Omega \frac{|\nabla n|^2}{n} + \int_\Omega g(c)|D^2\rho(c)|^2 \\ &= -\frac{1}{2}\int_\Omega \frac{g'(c)}{g^2(c)}\cdot|\nabla c|^2\cdot(u\cdot\nabla c) + \int_\Omega \frac{1}{g(c)}\cdot\Delta c\cdot(u\cdot\nabla c) \\ &\quad + \int_\Omega F(n)\cdot\Big(\frac{f(c)g'(c)}{2g^2(c)} - \frac{f'(c)}{g(c)}\Big)\cdot|\nabla c|^2 \\ &\quad + \frac{1}{2}\int_\Omega \frac{g''(c)}{g^2(c)}\cdot|\nabla c|^4 + \frac{1}{2}\int_{\partial\Omega}\frac{1}{g(c)}\cdot\frac{\partial}{\partial\nu}|\nabla c|^2 \qquad (3.1)\end{aligned}$$

*for all $t \in (0, T_{max})$, where we have set*

$$g(s) := \frac{f(s)}{\chi(s)}, \qquad \psi(s) := \int_1^s \frac{d\sigma}{\sqrt{g(\sigma)}} \qquad \textit{and} \qquad \rho(s) := \int_1^s \frac{d\sigma}{g(\sigma)} \qquad \textit{for } s > 0. \qquad (3.2)$$

PROOF. First, testing the first equation in (2.1) by $\ln n$ yields

$$\frac{d}{dt}\int_\Omega n\ln n + \int_\Omega \frac{|\nabla n|^2}{n} = \int_\Omega F'(n)\chi(c)\nabla c\cdot\nabla n = \int_\Omega \chi(c)\nabla c\cdot\nabla F(n) \qquad \text{for all } t\in(0,T_{max}), \tag{3.3}$$

and by a straightforward differentiation and integration by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi(c)|^2 = -\int_\Omega \psi'(c)\psi''(c)|\nabla c|^2 c_t - \int_\Omega \psi'^2(c)\Delta c\cdot c_t.$$

Replacing $c_t = \Delta c - F(n)f(c) - u\cdot\nabla c$ and once more integrating by parts, we arrive at the relation

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi(c)|^2 &= -\int_\Omega \psi'^2(c)|\Delta c|^2 - \int_\Omega \psi'(c)\psi''(c)|\nabla c|^2\Delta c\\ &\quad -\int_\Omega F(n)f(c)\psi'(c)\psi''(c)|\nabla c|^2 - \int_\Omega F(n)f'(c)\psi'^2(c)|\nabla c|^2\\ &\quad -\int_\Omega f(c)\psi'^2(c)\nabla c\cdot\nabla F(n)\\ &\quad +\int_\Omega \psi'(c)\psi''(c)|\nabla c|^2(u\cdot\nabla c) + \int_\Omega \psi'^2(c)\Delta c(u\cdot\nabla c)\end{aligned}$$

for $t\in(0,T_{max})$. Here we substitute $\psi' = \frac{1}{\sqrt{g}} = \sqrt{\frac{\chi}{f}}$ and $\psi'' = -\frac{g'}{2\sqrt{g^3}}$ to see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\psi(c)|^2 &= -\int_\Omega \frac{1}{g(c)}|\Delta c|^2 + \int_\Omega \frac{g'(c)}{2g^2(c)}|\nabla c|^2\Delta c\\ &\quad +\int_\Omega F(n)\cdot\Big(\frac{f(c)g'(c)}{2g^2(c)} - \frac{f'(c)}{g(c)}\Big)|\nabla c|^2\\ &\quad -\int_\Omega \chi(c)\nabla c\cdot\nabla F(n)\\ &\quad -\int_\Omega \frac{g'(c)}{2g^2(c)}|\nabla c|^2(u\cdot\nabla c) + \int_\Omega \frac{1}{g(c)}\Delta c(u\cdot\nabla c) \quad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{3.4}$$

In order to rearrange the first two terms on the right appropriately, we apply Lemma 3.1 to obtain

$$\begin{aligned}-\int_\Omega \frac{1}{g(c)}|\Delta c|^2 &= -\int_\Omega \frac{1}{g(c)}|D^2c|^2 - \frac{3}{2}\int_\Omega \frac{g'(c)}{g^2(c)}|\nabla c|^2\Delta c + \int_\Omega \frac{g'^2(c)}{g^3(c)}|\nabla c|^4\\ &\quad -\frac{1}{2}\int_\Omega \frac{g''(c)}{g^2(c)}|\nabla c|^4 + \frac{1}{2}\int_{\partial\Omega} \frac{1}{g(c)}\cdot\frac{\partial}{\partial\nu}|\nabla c|^2.\end{aligned} \tag{3.5}$$

On the other hand, a direct computation using $\rho' = \frac{1}{g}$ shows that

$$\begin{aligned}\int_\Omega g(c)|D^2\rho(c)|^2 &= \int_\Omega g(c)\rho'^2(c)|D^2c|^2 + 2\int_\Omega g(c)\rho'(c)\rho''(c)(D^2c\cdot\nabla c)\cdot\nabla c + \int_\Omega g(c)\rho''^2(c)|\nabla c|^4\\ &= \int_\Omega \frac{1}{g(c)}|D^2c|^2 - 2\int_\Omega \frac{g'(c)}{g^2(c)}(D^2c\cdot\nabla c)\cdot\nabla c + \int_\Omega \frac{g'^2(c)}{g^3(c)}|\nabla c|^4,\end{aligned}$$

where integrating by parts we find

$$
\begin{aligned}
-2\int_\Omega \frac{g'(c)}{g^2(c)}(D^2c\cdot\nabla c)\cdot\nabla c &= -\int_\Omega \frac{g'(c)}{g^2(c)}\nabla|\nabla c|^2\cdot\nabla c \\
&= \int_\Omega \frac{g'(c)}{g^2(c)}|\nabla c|^2\Delta c + \int_\Omega \Big(\frac{g''(c)}{g^2(c)} - \frac{2g'^2(c)}{g^3(c)}\Big)|\nabla c|^4,
\end{aligned}
$$

again because $\frac{\partial c}{\partial\nu}=0$ on $\partial\Omega$. We thus obtain

$$
\int_\Omega g(c)|D^2\rho(c)|^2 = \int_\Omega \frac{1}{g(c)}|D^2c|^2 + \int_\Omega \frac{g'(c)}{g^2(c)}|\nabla c|^2\Delta c - \int_\Omega \frac{g'^2(c)}{g^3(c)}|\nabla c|^4 + \int_\Omega \frac{g''(c)}{g^2(c)}|\nabla c|^4,
$$

whence (3.4) and (3.5) entail that

$$
\begin{aligned}
\frac12\frac{d}{dt}\int_\Omega|\nabla\psi(c)|^2 &= -\int_\Omega g(c)|D^2\rho(c)|^2 + \frac12\int_\Omega\frac{g''(c)}{g^2(c)}|\nabla c|^4 + \frac12\int_{\partial\Omega}\frac{1}{g(c)}\cdot\frac{\partial}{\partial\nu}|\nabla c|^2 \\
&\quad + \int_\Omega F(n)\cdot\Big(\frac{f(c)g'(c)}{2g^2(c)} - \frac{f'(c)}{g(c)}\Big)|\nabla c|^2 \\
&\quad - \int_\Omega \chi(c)\nabla c\cdot\nabla F(n) \\
&\quad - \int_\Omega \frac{g'(c)}{2g^2(c)}|\nabla c|^2(u\cdot\nabla c) + \int_\Omega\frac{1}{g(c)}\Delta c(u\cdot\nabla c) \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}
$$

Added to (3.3) this proves (3.1) □

In deriving a useful estimate from this, we shall make use of one further preparation.

**Lemma 3.3** *Let $h\in C^1((0,\infty))$ be positive, and let $\Theta(s):=\int_1^s\frac{d\sigma}{h(\sigma)}$ for $s>0$. Then for all positive $\varphi\in C^2(\bar\Omega)$ fulfilling $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, the inequality*

$$
\int_\Omega\frac{h'(\varphi)}{h^3(\varphi)}|\nabla\varphi|^4 \le (2+\sqrt{N})^2\int_\Omega\frac{h(\varphi)}{h'(\varphi)}|D^2\Theta(\varphi)|^2 \tag{3.6}
$$

*holds.*

PROOF. Using that $\Theta'=\frac1h$, we can rewrite

$$
\int_\Omega\frac{h'(\varphi)}{h^3(\varphi)}|\nabla\varphi|^4 = \int_\Omega|\nabla\Theta(\varphi)|^2\nabla\Theta(\varphi)\cdot\nabla h(\varphi). \tag{3.7}
$$

Since $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, an integration by parts gives

$$
\begin{aligned}
\int_\Omega|\nabla\Theta(\varphi)|^2\nabla\Theta(\varphi)\cdot\nabla h(\varphi) &= -\int_\Omega h(\varphi)\nabla|\nabla\Theta(\varphi)|^2\cdot\nabla\Theta(\varphi) - \int_\Omega h(\varphi)|\nabla\Theta(\varphi)|^2\Delta\Theta(\varphi) \\
&= -2\int_\Omega\frac{1}{h(\varphi)}(D^2\Theta(\varphi)\cdot\nabla\varphi)\cdot\nabla\varphi - \int_\Omega\frac{1}{h(\varphi)}|\nabla\varphi|^2\Delta\Theta(\varphi), \qquad (3.8)
\end{aligned}
$$

because $\nabla|\nabla z|^2 = 2D^2z \cdot \nabla z$ for $z \in C^2(\bar{\Omega})$, and again because $\theta' = \frac{1}{h}$. Now by the Hölder inequality we obtain

$$-2\int_\Omega \frac{1}{h(\varphi)}(D^2\Theta(\varphi)\cdot\nabla\varphi)\cdot\nabla\varphi \le 2\Big(\int_\Omega \frac{h'(\varphi)}{h^3(\varphi)}|\nabla\varphi|^4\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega \frac{h(\varphi)}{h'(\varphi)}|D^2\Theta(\varphi)|^2\Big)^{\frac{1}{2}} \tag{3.9}$$

and

$$-\int_\Omega \frac{1}{h(\varphi)}|\nabla\varphi|^2\Delta\Theta(\varphi) \le \Big(\int_\Omega \frac{h'(\varphi)}{h^3(\varphi)}|\nabla\varphi|^4\Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega \frac{h(\varphi)}{h'(\varphi)}|\Delta\Theta(\varphi)|^2\Big)^{\frac{1}{2}}. \tag{3.10}$$

In view of the pointwise inequality $|\Delta z|^2 \le N|D^2z|^2$ for $z \in C^2(\bar{\Omega})$, combining (3.7)-(3.10) and dividing by $(\int_\Omega \frac{h'(\varphi)}{h^3(\varphi)}|\nabla\varphi|^4)^{\frac{1}{2}}$ we easily arrive at (3.6). □

We are now in the position to state the announced energy-type inequality. Its derivation is the only place in this work where our assumption that $\Omega$ be convex will explicitly be needed.

**Lemma 3.4** *Suppose that the assumptions of Theorem 1.1 hold, and let $g, \psi$ and $\rho$ be as defined in (3.2). Then there exists $C > 0$ such that the solution of (2.1) satisfies*

$$\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac{1}{2}\int_\Omega |\nabla\psi(c)|^2\Big\} + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{4}\int_\Omega g(c)|D^2\rho(c)|^2 \le C\int_\Omega |u|^4 \qquad \textit{for all } t \in (0, T_{max}). \tag{3.11}$$

Proof. Since $\Omega$ is convex and $\frac{\partial c}{\partial\nu} = 0$ on $\partial\Omega$, it follows that $\frac{\partial}{\partial\nu}|\nabla c|^2 \le 0$ on $\partial\Omega$ ([2]). Moreover, (1.8) means that $g'' \le 0$ on $[0,\infty)$, whereas (1.9) ensures that

$$\frac{fg'}{2g^2} - \frac{f'}{g} = -\frac{(\chi f)'}{2f} \le 0 \qquad \text{on } (0,\infty).$$

From Lemma 3.2 we thus infer the inequality

$$\begin{aligned}\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac{1}{2}\int_\Omega |\nabla\psi(c)|^2\Big\} \quad &+ \quad \int_\Omega \frac{|\nabla n|^2}{n} + \int_\Omega g(c)|D^2\rho(c)|^2 \\ &\le \quad -\frac{1}{2}\int_\Omega \frac{g'(c)}{g^2(c)}|\nabla c|^2(u\cdot\nabla c) + \int_\Omega \frac{1}{g(c)}\Delta c(u\cdot\nabla c) \\ &=: \quad I_1 + I_2 \qquad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.12}$$

Here, since

$$\Delta\rho(c) = \rho'(c)\Delta c + \rho''(c)|\nabla c|^2 = \frac{1}{g(c)}\Delta c - \frac{g'(c)}{g^2(c)}|\nabla c|^2,$$

we have

$$\Delta c = g(c)\Delta\rho(c) + \frac{g'(c)}{g(c)}|\nabla c|^2$$

and hence

$$\begin{aligned}
I_1+I_2 &= I_1+\int_\Omega \Delta\rho(c)(u\cdot\nabla c)+\int_\Omega \frac{g'(c)}{g^2(c)}|\nabla c|^2(u\cdot\nabla c)\\
&= \frac{1}{2}\int_\Omega \frac{g'(c)}{g^2(c)}|\nabla c|^2(u\cdot\nabla c)+\int_\Omega \Delta\rho(c)(u\cdot\nabla c)\\
&=: J_1+J_2. \qquad (3.13)
\end{aligned}$$

According to Young's inequality and, again, the fact that $|\Delta z|^2\le N|D^2z|^2$ for $z\in C^2(\bar\Omega)$, we have

$$\begin{aligned}
|J_2| &\le \frac{1}{4N}\int_\Omega g(c)|\Delta\rho(c)|^2+N\int_\Omega \frac{1}{g(c)}|\nabla c|^2\cdot|u|^2\\
&\le \frac{1}{4}\int_\Omega g(c)|D^2\rho(c)|^2+N\int_\Omega \frac{1}{g(c)}|\nabla c|^2\cdot|u|^2. \qquad (3.14)
\end{aligned}$$

As to $J_1$, we once more use that $\rho'=\frac{1}{g}$ to rewrite

$$J_1=\frac{1}{2}\int_\Omega |\nabla\rho(c)|^2(u\cdot\nabla g(c)),$$

so that an integration by parts yields

$$J_1=-\frac{1}{2}\int_\Omega g(c)\nabla|\nabla\rho(c)|^2\cdot u=-\int_\Omega g(c)(D^2\rho(c)\cdot\nabla\rho(c))\cdot u,$$

because $u=0$ on $\partial\Omega$ and $\nabla\cdot u\equiv 0$. In this form, we may estimate $J_1$ by means of Young's inequality to find that

$$\begin{aligned}
|J_1| &\le \frac{1}{4}\int_\Omega g(c)|D^2\rho(c)|^2+\int_\Omega g(c)|\nabla\rho(c)|^2\cdot|u|^2\\
&= \frac{1}{4}\int_\Omega g(c)|D^2\rho(c)|^2+\int_\Omega \frac{1}{g(c)}|\nabla c|^2\cdot|u|^2,
\end{aligned}$$

so that from (3.13) and (3.14) we infer that

$$|I_1+I_2|\le \frac{1}{2}\int_\Omega g(c)|D^2\rho(c)|^2+(N+1)\int_\Omega \frac{1}{g(c)}|\nabla c|^2\cdot|u|^2. \qquad (3.15)$$

Now since $g'>0$ on $[0,\infty)$ by (1.7), and since $0\le c\le K:=\|c_0\|_{L^\infty(\Omega)}$ by Lemma 2.1 and (2.10), we know that $c_1:=\inf_{s\in(0,K)}g'(s)$ is positive and satisfies $g'(c)\ge c_1$ in $\Omega\times(0,T_{max})$. Using this along with Young's inequality, from Lemma 3.3 applied to $h:=g$ and $\varphi:=c$ we obtain that

$$\begin{aligned}
(N+1)\int_\Omega \frac{1}{g(c)}|\nabla c|^2\cdot|u|^2 &\le \frac{c_1}{4(2+\sqrt{N})^2}\int_\Omega \frac{g'(c)}{g^3(c)}|\nabla c|^4+\frac{(N+1)^2(2+\sqrt{N})^2}{c_1}\int_\Omega \frac{g(c)}{g'(c)}|u|^4\\
&\le \frac{c_1}{4}\int_\Omega \frac{g(c)}{g'(c)}|D^2\rho(c)|^2+\frac{(N+1)^2(2+\sqrt{N})^2}{c_1}\int_\Omega \frac{g(c)}{g'(c)}|u|^4\\
&\le \frac{1}{4}\int_\Omega g(c)|D^2\rho(c)|^2+\frac{(N+1)^2(2+\sqrt{N})^2c_2}{c_1^2}\int_\Omega |u|^4
\end{aligned}$$

is valid with $c_2 := \sup_{s\in(0,K)} g(s)$. Inserting the latter inequality into (3.15), going back to (3.12) we easily deduce (3.11) on choosing $C := \frac{(N+1)^2(2+\sqrt{N})^2c_2}{c_1^2}$. $\square$

**Remark.** A straightforward modification of the above proof shows that given $c_0$, the inequality (3.11) continues to hold if instead of (1.8) we merely require that

$$\frac{(\frac{f}{\chi})\cdot(\frac{f}{\chi})''}{(\frac{f}{\chi})'} < \frac{2c_1}{(2+\sqrt{N})^2} \qquad \text{on } [0,K], \qquad \text{where } K := \|c_0\|_{L^\infty(\Omega)} \text{ and } c_1 := \inf_{s\in(0,K)} \Big(\frac{f}{\chi}\Big)'(s). \quad (3.16)$$

This is apparently weaker than (1.8), and in particular (3.16) holds if, for instance,

$$\Big(\frac{f}{\chi}\Big)'' < \frac{2c_1^2}{(2+\sqrt{N})^2c_2} \qquad \text{on } [0,K] \qquad \text{with } c_2 := \sup_{s\in(0,K)} \Big(\frac{f}{\chi}\Big)(s).$$

We note, however, that both these relaxed conditions depend on the size of $c_0$.

# 4 The case $N = 2$

The goal of this section is to prove the first part of Theorem 1.1. To achieve this, in the statements of the preceding section we choose

$$F(n) := n, \quad n \geq 0,$$

which is clearly consistent with (2.2) and such that the auxiliary problem (2.1) coincides with the original system (1.1). Therefore, Lemma 2.1 provides a unique local-in-time solution of (1.1) which can be extended up to a maximal time $T_{max} \leq \infty$, and in order to prove its global existence we only need to establish suitable a priori estimates which allow for an application of (2.3) to rule out the case $T_{max} < \infty$. For this purpose we shall exploit the inequality provided by Lemma 3.4 which now applies directly to the original problem (1.1).
Throughout this section we shall tacitly assume that all the requirements of Theorem 1.1 are fulfilled.

## 4.1 A priori estimates

In view of (3.11), a natural next step is to find a bound for $\int_0^T \int_\Omega |u|^4$, which will be accomplished in Lemma 4.3. As a preparation for this, let us use the mass identity (2.9) and an interpolation to provide a control of the source term in the PDE for $u$ in terms of some part of the left-hand side of (3.11). In view of a later application in the three-dimensional setting, and since it might be of interest also in other applications, we formulate the following statement for general $N \geq 1$.

**Lemma 4.1** *Suppose that $N \geq 1$ and that $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary Let $p > 1$ and $r \geq 1$ be such that*

$$p \leq \frac{N}{(N-2)_+} \tag{4.1}$$

*and*

$$r \leq \frac{2p}{N(p-1)}. \tag{4.2}$$

*Then for all $T>0$ and each $M>0$ there exists $C(T,M)>0$ such that if $\varphi\in L^2((0,T);W^{1,2}(\Omega))$ is nonnegative with*

$$\int_\Omega \varphi(\cdot,t)\le M \qquad \textit{for all } t\in(0,T), \tag{4.3}$$

*then the estimate*

$$\int_0^T \|\varphi\|_{L^p(\Omega)}^r dt \le C(T,M)\cdot\Big\{\int_0^T\int_\Omega \frac{|\nabla\varphi|^2}{\varphi}+1\Big\}^{\frac{N(p-1)r}{2p}} \tag{4.4}$$

*holds.*

PROOF. Since $W^{1,2}(\Omega)\hookrightarrow L^{2p}(\Omega)$ due to (4.1), the Gagliardo-Nirenberg inequality ([6]) provides $c_1>0$ such that

$$\|\varphi^{\frac12}(\cdot,t)\|_{L^{2p}(\Omega)} \le c_1\|\nabla\varphi^{\frac12}(\cdot,t)\|_{L^2(\Omega)}^{\frac{N(p-1)}{2p}}\cdot\|\varphi^{\frac12}(\cdot,t)\|_{L^2(\Omega)}^{1-\frac{N(p-1)}{2p}}+c_1\|\varphi^{\frac12}(\cdot,t)\|_{L^2(\Omega)} \qquad \text{for all } t\in(0,T).$$

In view of (4.3), this means that we can find $c_2(M)>0$ such that

$$\|\varphi^{\frac12}(\cdot,t)\|_{L^{2p}(\Omega)} \le c_2\Big(\int_\Omega\frac{|\nabla\varphi|^2}{\varphi}+1\Big)^{\frac{N(p-1)}{4p}} \qquad \text{for all } t\in(0,T).$$

Thus, an integration in time and an application of the Hölder inequality show that

$$\begin{aligned}\int_0^T \|\varphi(\cdot,t)\|_{L^p(\Omega)}^r dt &= \int_0^T \|\varphi^{\frac12}(\cdot,t)\|_{L^{2p}(\Omega)}^{2r}dt\\ &\le c_2^{2r}(M)\int_0^T\Big(\int_\Omega\frac{|\nabla\varphi|^2}{\varphi}+1\Big)^{\frac{N(p-1)r}{2p}}\\ &\le c_2^{2r}(M)T^{1-\frac{N(p-1)r}{2p}}\cdot\Big\{\int_0^T\int_\Omega\frac{|\nabla\varphi|^2}{\varphi}+T\Big\}^{\frac{N(p-1)r}{2p}},\end{aligned}$$

because $r\le\frac{2p}{N(p-1)}$. This proves (4.5). $\square$

In view of (2.9), in the two-dimensional setting this immediately implies the following.

**Corollary 4.2** *Suppose that $N=2$. Let $T_0>0$, $p>1$ and $r\in[1,\frac{p}{p-1}]$. Then there exists $C>0$ such that for the solution of (1.1), (1.3), (1.4) we have*

$$\int_0^T \|n(\cdot,t)\|_{L^p(\Omega)}^r dt\le C\cdot\Big\{\int_0^T\int_\Omega\frac{|\nabla n|^2}{n}+1\Big\}^{\frac{(p-1)r}{p}} \tag{4.5}$$

*with $T:=\min\{T_0,T_{max}\}$.*

Invoking the standard energy inequality associated to the Navier-Stokes equations, we can now estimate the desired space-time integral against an arbitrarily small power of $\int_0^T\int_\Omega\frac{|\nabla n|^2}{n}$.

**Lemma 4.3** *Let $N = 2$, and assume that $\theta \in (0,1)$ and $T_0 > 0$. Then there exists $C > 0$ such that the solution of (1.1), (1.3), (1.4) satisfies*

$$\int_0^T \int_\Omega |u|^4 \le C \cdot \left\{ \int_0^T \int_\Omega \frac{|\nabla n|^2}{n} + 1 \right\}^\theta \tag{4.6}$$

*with $T := \min\{T_0, T_{max}\}$.*

Proof. Testing the third equation in (1.1) with $u$ we obtain (cf. also [23, Ch. 3])

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega n\nabla\phi \cdot u \qquad \text{for all } t \in (0,T), \tag{4.7}$$

because $\int_\Omega (u\cdot\nabla)u = 0$ due to the fact that $u = 0$ on $\partial\Omega$ and $\nabla \cdot u \equiv 0$. Now given $\theta \in (0,1)$, we apply Corollary 4.2 to $p := \frac{4}{4-\theta} > 1$ and $r := 2$ which is possible because $\frac{p}{p-1} = \frac{4}{\theta} > 2$. We thereby obtain $c_1 > 0$ such that

$$\int_0^T \|n(\cdot,t)\|^2_{L^p(\Omega)} \le c_1 \cdot K^{\frac{2(p-1)}{p}} = c_1 K^{\frac{\theta}{2}}, \tag{4.8}$$

where $K := \int_0^T \int_\Omega \frac{|\nabla n|^2}{n} + 1$. Next, writing $p' := \frac{p}{p-1}$, since $N = 2$ we have the embedding $W^{1,2}(\Omega) \hookrightarrow L^{p'}(\Omega)$, so that a Poincaré-Sobolev inequality yields $c_2 > 0$ such that

$$\|\varphi\|^2_{L^{p'}(\Omega)} \le c_2 \|\nabla\varphi\|^2_{L^2(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega).$$

Therefore, in view of the boundedness of $\nabla\phi$ in $L^\infty(\Omega)$ and Hölder's and Young's inequalities we can estimate

$$\begin{aligned}
\int_\Omega n\nabla\phi\cdot u &\le c_3 \|u(\cdot,t)\|_{L^{p'}(\Omega)} \cdot \|n(\cdot,t)\|_{L^p(\Omega)} \\
&\le \frac{1}{2c_2}\|u(\cdot,t)\|^2_{L^{p'}(\Omega)} + \frac{c_2 c_3^2}{2}\|n(\cdot,t)\|^2_{L^p(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |\nabla u(\cdot,t)|^2 + \frac{c_2 c_3^2}{2}\|n(\cdot,t)\|^2_{L^p(\Omega)} \qquad \text{for all } t \in (0,T).
\end{aligned}$$

Hence, (4.7) becomes

$$\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 \le c_2 c_3^2 \|n(\cdot,t)\|^2_{L^p(\Omega)} \qquad \text{for all } t \in (0,T),$$

which upon integration gives

$$\int_\Omega |u(\cdot,t)|^2 + \int_0^t \int_\Omega |\nabla u|^2 \le \int_\Omega |u_0|^2 + c_2 c_3^2 \cdot c_1 K^{\frac{\theta}{2}} \qquad \text{for all } t \in (0,T).$$

Since $K \ge 1$, this means that for some $c_4 > 0$ we have

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_4 K^{\frac{\theta}{2}} \quad \text{for all } t \in (0,T) \qquad \text{and} \qquad \int_0^T \|\nabla u(\cdot,t)\|^2_{L^2(\Omega)} dt \le c_4 K^{\frac{\theta}{2}}.$$

Interpolating this yields (4.6): Indeed, according to the Gagliardo-Nirenberg inequality we can find $c_5 > 0$ such that

$$\int_0^T \int_\Omega |u|^4 = \int_0^T \|u(\cdot,t)\|_{L^4(\Omega)}^4 dt \le c_5 \int_0^T \|\nabla u(\cdot,t)\|_{L^2(\Omega)}^2 \cdot \|u(\cdot,t)\|_{L^2(\Omega)}^2 dt$$

and hence conclude that

$$\int_0^T \int_\Omega |u|^4 \le c_4^2 c_5 K^\theta,$$

which yields (4.6). $\square$

Combining the latter assertion with Lemma 3.4, among several conceivable consequences we state the following one which will be needed below.

**Corollary 4.4** *Let $N = 2$. Then for each $T_0 > 0$ there exists $C > 0$ such that for the solution of (1.1), (1.3), (1.4) we have*

$$\int_0^T \int_\Omega |\nabla c|^4 \le C, \tag{4.9}$$

*where $T := \min\{T_0, T_{max}\}$.*

Proof. From Lemma 3.4 we know that there exists $c_1 > 0$ such that with $g, \psi$ and $\rho$ as in (3.2) we have

$$\begin{aligned}\int_\Omega n \ln n + \frac{1}{2}\int_\Omega |\nabla\psi(c)|^2 \quad &+ \quad \int_0^T \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{4}\int_0^T \int_\Omega g(c)|D^2\rho(c)|^2 \\ &\le \quad \int_\Omega n_0 \ln n_0 + \frac{1}{2}\int_\Omega |\nabla\psi(c_0)|^2 + c_1 \int_0^T \int_\Omega |u|^4 \qquad \text{for all } t \in (0,T).\end{aligned}$$

Since $g$ is positive on $(0,\infty)$ and $c_0 > 0$ in $\bar\Omega$, there exists $c_2 > 0$ such that $|\nabla\psi(c_0)| \le c_2$ in $\Omega$. We thus can pick $c_3 > 0$ fulfilling

$$\int_0^T \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{4}\int_0^T \int_\Omega g(c)|D^2\rho(c)|^2 \le c_3 + c_1 \int_0^T \int_\Omega |u|^4. \tag{4.10}$$

But if we now fix any $\theta \in (0,1)$ and apply Lemma 4.3, we obtain $c_4 > 0$ such that

$$\int_0^T \int_\Omega |u|^4 \le c_4 \cdot \left\{ \int_0^T \int_\Omega \frac{|\nabla n|^2}{n} + 1 \right\}^\theta. \tag{4.11}$$

Inserted into (4.10), this implies that for some $c_5 > 0$ we have

$$\int_0^T \int_\Omega \frac{|\nabla n|^2}{n} \le c_5,$$

because $\theta < 1$. Therefore, (4.11) and (4.10) show that

$$\int_0^T \int_\Omega g(c)|D^2\rho(c)|^2 \le c_6$$

with a certain $c_6 > 0$. Since thanks to the boundedness of $c$ we know that $g'(c) \ge c_7$ and $g(c) \le c_8$ throughout $\Omega \times (0,T)$ with positive constants $c_7$ and $c_8$, recalling Lemma 3.3 we see that

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla c|^4 &\le \frac{c_8^3}{c_7} \int_0^T \int_\Omega \frac{g'(c)}{g^3(c)} |\nabla c|^4 \\
&\le \frac{c_8^3 (2+\sqrt{2})^2}{c_7} \int_0^T \int_\Omega \frac{g(c)}{g'(c)} |D^2 \rho(c)|^2 \\
&\le \frac{c_8^3 (2+\sqrt{2})^2}{c_7^2} \int_0^T \int_\Omega g(c) |D^2 \rho(c)|^2
\end{aligned}$$

and hence conclude. □

In the spatially two-dimensional framework, the latter result is sufficient to guarantee boundedness of $n$ in $L^\infty((0,T);L^p(\Omega))$ for any $p < \infty$. This follows from a standard regularity argument which relies on testing the first equation in (1.1) by powers of $n$. Fortunately, due to the fact that $\nabla \cdot u \equiv 0$, at this stage this step does not require any regularity property of $u$.

**Lemma 4.5** *Let $N = 2$. Then for all $T_0 > 0$ and any $p > 1$ there exists $C > 0$ such that for the solution of (1.1), (1.3), (1.4) the inequality*

$$\int_\Omega n^p(x,t)dx \le C \qquad \textit{for all } t \in (0,T) \tag{4.12}$$

*holds with $T := \min\{T_0, T_{max}\}$.*

PROOF. We multiply the first equation in (1.1) by $n^{p-1}$ to obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega n^p + (p-1)\int_\Omega n^{p-2}|\nabla n|^2 = (p-1)\int_\Omega \chi(c) n^{p-1} \nabla n \cdot \nabla c \qquad \text{for all } t \in (0,T),$$

because $\nabla \cdot u \equiv 0$. Since $c$ and hence also $\chi(c)$ is bounded, we may apply Young's inequality to find $c_1 > 0$ such that

$$(p-1)\int_\Omega \chi(c) n^{p-1}\nabla n \cdot \nabla c \le \frac{p-1}{2}\int_\Omega n^{p-2}|\nabla n|^2 + c_1 \int_\Omega n^p |\nabla c|^2. \tag{4.13}$$

By the Hölder inequality,

$$\int_\Omega n^p |\nabla c|^2 \le \Big(\int_\Omega n^{2p}\Big)^{\frac12} \cdot \Big(\int_\Omega |\nabla c|^4\Big)^{\frac12},$$

and now the Gagliardo-Nirenberg inequality (see e.g. [26] for a version involving $L^r$ spaces with $r < 1$) provides $c_2 > 0$ such that

$$\Big(\int_\Omega n^{2p}\Big)^{\frac12} = \|n^{\frac{p}{2}}\|^2_{L^4(\Omega)} \le c_2 \Big( \|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)} \cdot \|n^{\frac{p}{2}}\|_{L^2(\Omega)} + \|n^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)} \Big).$$

Since $\|n^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}$ is bounded by (2.9), we can thus pick $c_3 > 0$ such that

$$c_1 \int_\Omega n^p |\nabla c|^2 \le \frac{p-1}{2}\int_\Omega n^{p-2}|\nabla n|^2 + c_3 \Big(\int_\Omega |\nabla c|^4\Big) \cdot \Big(\int_\Omega n^p + 1\Big),$$

so that from (4.13) we obtain that $y(t) := \int_\Omega n^p(x,t)dx,\ t \in [0,T)$, satisfies the ODI

$$y'(t) \le c_4\Big(\int_\Omega |\nabla c|^4\Big) \cdot (y(t)+1) \qquad \text{for all } t \in (0,T)$$

with some $c_4 > 0$. On integration we infer that

$$y(t)+1 \le (y(0)+1) \cdot e^{c_4 \int_0^T \int_\Omega |\nabla c|^4} \qquad \text{for all } t \in (0,T),$$

whereupon an application of Corollary 4.4 completes the proof. □

## 4.2 Proof of the main result for $N = 2$

We are now in the position to prove global solvability which, in view of Lemma 2.1, amounts to establishing a priori bounds for $n, c$ and $u$ in $L^\infty((0,T_{max}); L^\infty(\Omega))$, $L^\infty((0,T_{max}); W^{1,q}(\Omega))$ and $L^\infty((0,T_{max}); D(A^\alpha))$, respectively. These will be obtained by a bootstrap procedure connecting a series of regularity arguments which are quite well-established in the theories of the Navier-Stokes equations and chemotaxis equations ([23], [20], [12], [16]).

Proof (of Theorem 1.1). Let $(n,c,u,P)$ denote the classical solution of (1.1), (1.3), (1.4) in $\Omega \times (0,T_{max})$ provided by Lemma 2.1. We only need to make sure that $T_{max} = \infty$. To this end, we assume on the contrary that $T_{max} < \infty$, and proceed to derive a contradiction to (2.3). For this purpose, according to Lemma 4.5, given any $p \in (1,\infty)$ we can pick $c_1(p) > 0$ such that

$$\int_\Omega n^p(x,t)dx \le c_1(p) \qquad \text{for all } t \in (0,T_{max}). \tag{4.14}$$

Hence, testing the third equation in (1.1) by $u$ gives

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega n\nabla\phi \cdot u \le \int_\Omega |u|^2 + \frac{c_1(2)\|\nabla\phi\|^2_{L^\infty(\Omega)}}{4} \qquad \text{for all } t \in (0,T_{max}),$$

which proves that

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le c_2 \quad \text{for all } t \in (0,T_{max}) \qquad \text{and} \qquad \int_0^{T_{max}} \int_\Omega |\nabla u|^2 \le c_2 \tag{4.15}$$

for some $c_2 > 0$. We next apply $\mathcal{P}$ to both sides of the same equation and multiply the resulting identity by $Au$. Using Young's inequality and the projection property $\|\mathcal{P}\varphi\|_{L^2(\Omega)} \le \|\varphi\|_{L^2(\Omega)}$ for $\varphi \in L^2(\Omega)$, we thereby obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u|^2 + \int_\Omega |Au|^2 &= + \int_\Omega \mathcal{P}n\nabla\phi Au + \int_\Omega \mathcal{P}((u\cdot\nabla)u)Au \\ &\le \frac{1}{2}\int_\Omega |Au|^2 + c_1(2)\|\nabla\phi\|^2_{L^\infty(\Omega)} + \int_\Omega |(u\cdot\nabla)u|^2 \text{ for all } t \in (0,T_{max}).\end{aligned} \tag{4.16}$$

Now by the Gagliardo-Nirenberg inequality and (4.15) we can pick $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned}
\int_\Omega |(u\cdot\nabla)u|^2 &\le \|u\|^2_{L^\infty(\Omega)} \cdot \|\nabla u\|^2_{L^2(\Omega)} \\
&\le c_3\|u\|_{W^{2,2}(\Omega)} \cdot \|u\|_{L^2(\Omega)} \cdot \|\nabla u\|^2_{L^2(\Omega)} \\
&\le c_2c_3\|u\|_{W^{2,2}(\Omega)} \cdot \|\nabla u\|^2_{L^2(\Omega)} \\
&\le \frac{1}{4}\|Au\|^2_{L^2(\Omega)} + c_4\|\nabla u\|^4_{L^2(\Omega)} \quad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

where we have employed Young's inequality and the well-known fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ ([20, Theorem 2.1.1]). Hence, recalling that $A = -\mathcal{P}\Delta$ and hence $\|A^{\frac{1}{2}}u\|_{L^2(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$, we see that (4.16) provides $c_5 > 0$ such that

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 + \int_\Omega |\Delta u|^2 \le c_5\Big(\int_\Omega |\nabla u|^2 + 1\Big)^2 \qquad \text{for all } t \in (0, T_{max}). \tag{4.17}$$

In conjunction with (4.15) this means that there exists $c_6 > 0$ with the property that

$$\int_\Omega |\nabla u|^2(x,t)dx \le c_6 \quad \text{for all } t \in (0, T_{max}) \qquad \text{and} \qquad \int_0^{T_{max}}\int_\Omega |\Delta u|^2 \le c_6, \tag{4.18}$$

because if $t \mapsto \int_\Omega |\nabla u|^2(x,t)dx$ blew up at some $T \in (0, T_{max}]$,(4.17) would entail that $\int_\Omega |\nabla u|^2(x,t)dx \ge \frac{1}{c_5(T-t)} - 1$ for all $t \in (0,T)$ which is incompatible with (4.15).
Now (4.18) along with (4.14) yields

$$\sup_{t\in(0,T_{max})} \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} < \infty \tag{4.19}$$

with $\alpha$ taken from the hypothesis of the theorem: Indeed, from the variation-of-constants formula for $u$ and the contractivity of the Stokes semigroup in $L^2(\Omega)$ we know that

$$\begin{aligned}
\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} &\le \|A^\alpha u_0\|_{L^2(\Omega)} + \int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}n(\cdot,s)\nabla\phi\|_{L^2(\Omega)}ds \\
&\quad + \int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}(u\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)}ds \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{4.20}$$

where thanks to (4.14) there exists $c_7 > 0$ fulfilling

$$\|A^\alpha e^{-(t-s)A}\mathcal{P}n(\cdot,s)\nabla\phi\|_{L^2(\Omega)} \le c_7(t-s)^{-\alpha} \qquad \text{whenever } 0 < s < t < T_{max}. \tag{4.21}$$

Moreover, since $\alpha < 1$ we can find $p > 2$ large such that $p' := \frac{p}{p-1}$ satisfies $p'\alpha < 1$, and use Hölder's inequality to estimate

$$\begin{aligned}
\int_0^t \|A^\alpha e^{-(t-s)A}\mathcal{P}(u\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)}ds &\le c_8\int_0^t (t-s)^{-\alpha}\|(u\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)}ds \\
&\le c_8\Big(\int_0^t (t-s)^{-p'\alpha}\Big)^{\frac{1}{p'}} \cdot \Big(\int_0^t \|(u\cdot\nabla)u(\cdot,s)\|^p_{L^2(\Omega)}ds\Big)^{\frac{1}{p}}
\end{aligned} \tag{4.22}$$

for all $t \in (0, T_{max})$ with some $c_8 > 0$. Here we note that the Hölder and the Gagliardo-Nirenberg inequality and the fact that $N = 2$ asserts the embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ imply that there exist positive $c_9$ and $c_{10}$ such that

$$\begin{aligned}
\int_0^{T_{max}} \|(u\cdot\nabla)u(\cdot,s)\|^p_{L^2(\Omega)}ds &\le \int_0^{T_{max}} \|u(\cdot,s)\|^p_{L^p(\Omega)} \cdot \|\nabla u(\cdot,s)\|^p_{L^{\frac{2p}{p-2}}(\Omega)} ds \\
&\le c_9 \int_0^{T} \|\nabla u(\cdot,s)\|^p_{L^2(\Omega)} \cdot \|\nabla u(\cdot,s)\|^p_{L^{\frac{2p}{p-2}}(\Omega)} ds \\
&\le c_{10} \int_0^{T_{max}} \|\nabla u(\cdot,s)\|^p_{L^2(\Omega)} \cdot \|\Delta u(\cdot,s)\|^2_{L^2(\Omega)} \cdot \|\nabla u(\cdot,s)\|^{p-2}_{L^2(\Omega)} ds \\
&\le c_{10} \cdot c_6^p
\end{aligned}$$

by (4.18). Hence, collecting (4.20)-(4.22) we easily arrive at (4.19). In particular, since $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ due to the fact that $\alpha > \frac{1}{2}$ ([20]), we infer that

$$|u| \le c_{11} \qquad \text{in } \Omega \times (0, T_{max}) \tag{4.23}$$

for some $c_{11} > 0$. Therefore Lemma 3.4 provides $c_{12} > 0$ satisfying

$$\int_\Omega |\nabla c|^2(x,t)dx \le c_{12} \qquad \text{for all } t \in (0, T_{max}), \tag{4.24}$$

because according to the boundedness of $c$ and hence of $g(c) = \frac{f(c)}{\chi(c)}$ we know that $\psi'^2(c) = \frac{1}{g(c)}$ is bounded from below by a positive constant in $\Omega \times (0, T_{max})$. As a consequence of (4.14), (4.23), (4.24) and the inclusion $c_0 \in W^{1,q}(\Omega)$, the variation-of-constants formula and well-known smoothing estimates for the Neumann heat semigroup ([27, Lemma 1.3]) yield the estimate

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^q(\Omega)} &\le c_{13}\|\nabla c_0\|_{L^q(\Omega)} + c_{13}\int_0^t (t-s)^{-\frac{1}{2}-(\frac{1}{2}-\frac{1}{q})} \|(nf(c) + u\cdot\nabla c)(\cdot,s)\|_{L^2(\Omega)}ds \\
&\le c_{14}\Big(1 + \int_0^t (t-s)^{-1+\frac{1}{q}} \cdot \Big(c_1(2) + c_{10}\cdot c_{11}^{\frac{1}{2}}\Big)ds\Big) \\
&\le c_{15} \qquad \text{for all } t \in (0, T_{max})
\end{aligned} \tag{4.25}$$

with certain positive $c_{13}, c_{14}$ and $c_{15}$. Finally, since $q > N = 2$, we can fix a number $\beta \in (\frac{1}{q}, \frac{1}{2})$ and then $r \in (\frac{1}{\beta}, q)$ and apply $B^\beta$ to the variation-of-constants representation of $n$, where $B$ denotes the realization of $-\Delta + 1$ with homogeneous Neumann boundary conditions in $L^r(\Omega)$. Due to the embedding $D(B^\beta) \hookrightarrow L^\infty(\Omega)$ and the maximum principle, we thereby find $c_{16} > 0$ and $c_{17} > 0$ such that

$$\begin{aligned}
\|n(\cdot,t)\|_{L^\infty(\Omega)} &\le \|n_0\|_{L^\infty(\Omega)} + c_{16}\int_0^t \|B^\beta e^{-(t-s)(B-1)}\nabla\cdot(\chi(c)n\nabla c + nu)(\cdot,s)\|_{L^r(\Omega)}ds \\
&\le c_{17}\Big(1 + \int_0^t (t-s)^{-\frac{1}{2}-\beta}\|(\chi(c)n\nabla c + nu)(\cdot,s)\|_{L^r(\Omega)}ds\Big)
\end{aligned} \tag{4.26}$$

for all $t \in (0, T_{max})$. Now using that $\chi(c)$ is bounded in $\Omega \times (0, T_{max})$, that

$$\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^r(\Omega)} \le \|n(\cdot,s)\|_{L^{\frac{qr}{q-r}}(\Omega)} \cdot \|\nabla c(\cdot,s)\|_{L^q(\Omega)} \le c_1(\tfrac{qr}{q-r}) \cdot c_{15} \qquad \text{for all } s \in (0, T_{max}).$$

and that

$$\|n(\cdot,s)u(\cdot,s)\|_{L^r(\Omega)} \le c_1(r)c_{11} \qquad \text{for all } s \in (0,T_{max})$$

by (4.14), (4.25) and (4.23), from (4.26) in conjunction with (4.19) and (4.25) we infer that (2.3) does not hold, which is is absurd. We therefore conclude that $T_{max} = \infty$, that is, that $(n,c,u,P)$ is global in time. $\Box$

## 5 The chemotaxis-Stokes system in the case $N = 3$

We now consider the case when in (1.1) we have $\kappa = 0$ and the domain $\Omega$ is a convex subset of $\mathbb{R}^3$. Since we will not be able to prove global *classical* solvability of (1.1), we first rather deal with the family of approximate problems obtained upon the choice $F(n) = \frac{1}{\varepsilon}\ln(1+\varepsilon n)$ in (2.1). More precisely, for $\varepsilon \in (0,1)$ let us consider

$$\begin{cases} n_{\varepsilon t} + u_\varepsilon \cdot \nabla n_\varepsilon = \Delta n_\varepsilon - \nabla \cdot \Big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\chi(c_\varepsilon)\nabla c_\varepsilon\Big), & x \in \Omega,\ t>0, \\ c_{\varepsilon t} + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - \frac{1}{\varepsilon}\ln(1+\varepsilon n_\varepsilon) f(c_\varepsilon), & x \in \Omega,\ t>0, \\ u_{\varepsilon t} = \Delta u_\varepsilon + \nabla P_\varepsilon + n_\varepsilon \nabla\phi, & x \in \Omega,\ t>0, \\ \nabla \cdot u_\varepsilon = 0, & x \in \Omega,\ t>0, \\ \frac{\partial n_\varepsilon}{\partial\nu} = \frac{\partial c_\varepsilon}{\partial\nu} = 0,\ u_\varepsilon = 0, & x \in \partial\Omega,\ t>0, \\ n(x,0) = n_0(x),\ c(x,0) = c_0(x), u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{5.1}$$

and as before we assume throughout that the assumptions of Theorem 1.1 are satisfied. Then Lemma 2.1 again asserts the existence of a unique local-in-time solution $(n_\varepsilon, c_\varepsilon, u_\varepsilon, P_\varepsilon)$ up to a maximal time $T_{max,\varepsilon} \le \infty$. Our goal is to show that actually $T_{max,\varepsilon} = \infty$ for all $\varepsilon \in (0,1)$, and that the solutions of (5.1) approach a weak solution of (1.1)-(1.3) as $\varepsilon \searrow 0$. Here our notion of weak solution is a rather natural one, consisting of carrying over derivatives to test functions and suppressing the pressure variable $P$ by projecting the third equation in (1.1) to the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega)$.

**Definition 5.1** *By a* global weak solution *of the chemotaxis-Stokes system (1.1), (1.3), (1.4) with $\kappa = 0$ we mean a triple $(n,c,u)$ of functions*

$$n \in L^1_{loc}([0,\infty); L^1(\Omega)), \qquad c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \qquad u \in L^1_{loc}([0,\infty); W^{1,1}_0(\Omega) \cap L^2_\sigma(\Omega))$$

*such that*

$$nf(c),\ n\chi(c)\nabla c,\ nu \ \text{ and } \ cu \quad \text{belong to } L^1_{loc}([0,\infty); L^1(\Omega)),$$

*and such that the identities*

$$\begin{aligned} -\int_0^\infty\int_\Omega n\partial_t\zeta_1 - \int_\Omega n_0\zeta_1(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla n\cdot\nabla\zeta_1 + \int_0^\infty\int_\Omega n\chi(c)\nabla c\cdot\nabla\zeta_1 + \int_0^\infty\int_\Omega nu\cdot\nabla\zeta_1, \\ -\int_0^\infty\int_\Omega c\partial_t\zeta_2 - \int_\Omega c_0\zeta_2(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\zeta_2 - \int_0^\infty\int_\Omega nf(c)\zeta_2 + \int_0^\infty\int_\Omega cu\cdot\nabla\zeta_2 \quad \text{and} \\ -\int_0^\infty\int_\Omega u\cdot\partial_t\zeta_3 - \int_\Omega u_0\cdot\zeta_3(\cdot,0) &= -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\zeta_3 + \int_0^\infty\int_\Omega n\nabla\phi\cdot\zeta_3 \end{aligned} \tag{5.2}$$

*hold for all $\zeta_1 \in C_0^\infty(\bar\Omega\times[0,\infty)), \zeta_2 \in C_0^\infty(\bar\Omega\times[0,\infty))$ and $\zeta_3 \in C_0^\infty(\Omega\times[0,\infty))$ fulfilling $\nabla\cdot\zeta_3 \equiv 0$ in $\Omega\times(0,\infty)$.*

## 5.1 $\varepsilon$-independent a priori estimates for the chemotaxis-Stokes system

Our first step towards establishing $\varepsilon$-independent bounds for the solutions of (5.1) consists of the following analogue of Corollary 4.2, which again is a direct consequence of Lemma 4.1.

**Corollary 5.1** *Suppose that $N = 3$. Then for all $T_0 > 0$, $p \in (1, 3]$ and $r \in [1, \frac{2p}{3(p-1)}]$ there exists $C > 0$ such that for all $\varepsilon \in (0, 1)$, the solution of (5.1) satisfies*

$$\int_0^T \|n_\varepsilon(\cdot,t)\|_{L^p(\Omega)}^r dt \le C \cdot \left\{ \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + 1 \right\}^{\frac{3(p-1)r}{2p}} \tag{5.3}$$

*with $T := \min\{T_0, T_{max,\varepsilon}\}$.*

We next derive a counterpart of Lemma 4.3. It relies on maximal Sobolev regularity estimates for the inhomogeneous linear Stokes evolution equation and thereby essentially uses the fact that the convective term $(u \cdot \nabla)u$ is absent here.

**Lemma 5.2** *Let $N = 3$. Then for all $T_0 > 0$ there exists $C > 0$ such that whenever $\varepsilon \in (0, 1)$, for the solution of (5.1) we have*

$$\int_0^T \int_\Omega |u_\varepsilon|^4 \le C \cdot \left\{ \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + 1 \right\}^{\frac{1}{2}} \tag{5.4}$$

*and*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^{\frac{5}{2}} \le C \cdot \left\{ \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + 1 \right\} \tag{5.5}$$

*with $T := \min\{T_0, T_{max,\varepsilon}\}$.*

Proof. With $K_\varepsilon := \int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} + 1$, for any $p \in (1, 3]$ and $r \in [1, \frac{2p}{3(p-1)}]$, Corollary 5.1 yields $c_1(p, r) > 0$ such that $\|n_\varepsilon\|_{L^r((0,T);L^p(\Omega))}^r \le c_1 K^{\frac{3(p-1)r}{2p}}$ for all $\varepsilon \in (0, 1)$. Hence, known results on maximal Sobolev regularity for the Stokes semigroup ([8, Theorem 2.7]) imply that for any such $p$ and $r$ we can find $c_2(p, r) > 0$ such that

$$\int_0^T \|u_\varepsilon(\cdot,t)\|_{W^{2,p}(\Omega)}^r \le c_2 K^{\frac{3(p-1)r}{2p}} \tag{5.6}$$

for all $\varepsilon \in (0, 1)$. Here we first pick $p := \frac{12}{11}$ and $r := 4 = \frac{1}{2} \cdot \frac{2p}{3(p-1)}$. Since then $W^{2,p}(\Omega) \hookrightarrow L^4(\Omega)$ by a Sobolev embedding, (5.6) entails (5.4). Similarly, (5.5) results from (5.6) upon the choices $p := \frac{15}{11}$ and $r := \frac{5}{2} = \frac{2p}{3(p-1)}$, because then $W^{2,p}(\Omega) \hookrightarrow W^{1,\frac{5}{2}}(\Omega)$. $\square$

Since the right-hand side of (5.4) again grows at most like a multiple of a strictly sublinear power of $\int_0^T \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}$, we may repeat the reasoning from Corollary 4.4 to end up with the following collection of a priori bounds.

**Corollary 5.3** *Let $N = 3$. Then for each $T_0 > 0$ there exists $C > 0$ such that for any $\varepsilon \in (0, 1)$, the solution of (5.1) satisfies*

$$\int_0^T \int_\Omega n_\varepsilon^{\frac{5}{3}} \le C \qquad \text{and} \qquad \int_0^T \int_\Omega |u_\varepsilon|^4 \le C \tag{5.7}$$

*as well as*

$$\int_0^T\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon} \le C, \qquad \int_0^T\int_\Omega |\nabla c_\varepsilon|^4 \le C, \qquad \textit{and} \qquad \int_0^T\int_\Omega |\nabla u_\varepsilon|^{\frac{5}{2}} \le C, \tag{5.8}$$

*and moreover we have*

$$\int_\Omega |\nabla c_\varepsilon|^2(\cdot,t) \le C \qquad \textit{for all } t\in(0,T), \tag{5.9}$$

*where* $T := \min\{T_0, T_{max,\varepsilon}\}$.

PROOF. Proceeding in a similar way as in Corollary 4.4, from (5.4) and Lemma 3.4 we derive the first two inequalities in (5.8) as well as (5.9, again using the fact that in view of (2.10), $\psi'^2(c_\varepsilon) = \frac{\chi(c_\varepsilon)}{f(c_\varepsilon)}$ is bounded from below by an $\varepsilon$-independent positive constant in $\Omega\times(0,T_{max,\varepsilon})$. From this, however, the third immediately results in view of (5.5), whereas the estimates in (5.7) now are consequences of (5.4) and of Corollary 5.1 applied to $p := \frac{5}{3}$ and $r := \frac{5}{3}$. □

### 5.2 Global solvability of the approximate problems

With Corollary 5.3 at hand, we can proceed to show that our approximate solutions are actually global in time.

**Lemma 5.4** *For each $\varepsilon\in(0,1)$, we have $T_{max,\varepsilon}=\infty$; that is, all the solutions of (5.1) are global in time.*

PROOF. Let us assume on the contrary that $T_{max,\varepsilon}<\infty$ for some $\varepsilon\in(0,1)$, and then pick any $\tau\in(0,T_{max,\varepsilon})$. Multiplying the first equation in (5.1) by $n_\varepsilon^3$, integrating by parts and using Young's inequality, we obtain $c_1>0$, as all constants in this proof possibly depending on $\varepsilon$ but not on $t\in(0,T_{max,\varepsilon})$, such that

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega n_\varepsilon^4 + 3\int_\Omega n_\varepsilon^2|\nabla n_\varepsilon|^2 &= 3\int_\Omega \frac{n_\varepsilon^3}{1+\varepsilon n_\varepsilon}\chi(c_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \\
&\le \int_\Omega n_\varepsilon^2|\nabla n_\varepsilon|^2 + \int_\Omega |\nabla c_\varepsilon|^4 + c_1\int_\Omega n_\varepsilon^4 \qquad \text{for all } t\in(0,T_{max,\varepsilon}).
\end{aligned}$$

Upon a time integration, in view of (5.8) this yields $c_2>0$ such that

$$\int_\Omega n_\varepsilon^4(\cdot,t) \le c_2 \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{5.10}$$

We next pick $\alpha'\in(\frac{3}{4},1)$ and thus have $D(A^{\alpha'})\hookrightarrow L^\infty(\Omega)$ ([20]), so that we can find positive $c_3, c_4$ and $c_5$ such that

$$\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_3\|A^{\alpha'}u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} &= c_3\Big\|A^{\alpha'}e^{-tA}u_0 + \int_0^t e^{-(t-s)A}\mathcal{P}n_\varepsilon(\cdot,s)\nabla\phi ds\Big\|_{L^2(\Omega)} \\
&\le c_4\Big(1+\int_0^t (t-s)^{-\alpha'}\|n_\varepsilon(\cdot,s)\|_{L^2(\Omega)}ds\Big) \le c_5 \qquad (5.11)
\end{aligned}$$

for all $t \in (\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon})$. In particular, this entails that

$$\Big\|\frac{1}{\varepsilon}\ln(1+\varepsilon n_\varepsilon)f(c_\varepsilon) + u_\varepsilon \cdot \nabla c_\varepsilon\Big\|_{L^2(\Omega)} \le c_6 \qquad \text{for all } t \in \Big(\frac{1}{2}T_{max,\varepsilon}, T_{max,\varepsilon}\Big)$$

because of (5.10), (5.9) and (2.10). Therefore, arguing as in (4.25) we obtain $c_7 > 0$ and $c_8 > 0$ fulfilling

$$\begin{aligned}
\|\nabla c_\varepsilon(\cdot,t)\|_{L^4(\Omega)} &\le c_7\Bigg(\Big\|\nabla e^{t\Delta}c_\varepsilon\Big(\frac{1}{2}T_{max,\varepsilon}\Big)\Big\|_{L^4(\Omega)} \\
&\quad + \int_{\frac{1}{2}T_{max,\varepsilon}}^t (t-s)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{2}-\frac{1}{4})}\Big\|\Big(\frac{1}{\varepsilon}\ln(1+\varepsilon n_\varepsilon)f(c_\varepsilon)+u_\varepsilon\cdot\nabla c_\varepsilon\Big)(\cdot,s)\Big\|_{L^2(\Omega)}ds\Bigg) \\
&\le c_8 \qquad \text{for all } t \in \Big(\frac{3}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\Big).
\end{aligned} \tag{5.12}$$

According to (5.11), for some $c_9 > 0$ we thus have

$$\Big\|\Big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\chi(c_\varepsilon)\nabla c_\varepsilon + n_\varepsilon u_\varepsilon\Big)(\cdot,s)\Big\|_{L^4(\Omega)} \le c_9 \qquad \text{for all } t \in \Big(\frac{3}{4}T_{max,\varepsilon}, T_{max,\varepsilon}\Big).$$

Now taking any $\beta \in (\frac{3}{8}, \frac{1}{2})$ and letting $B$ denote the operator $-\Delta + 1$ in $L^4(\Omega)$ with homogeneous Neumann data, we have $D(B^\beta) \hookrightarrow L^\infty(\Omega)$ and hence we find constants $c_{10}, c_{11}$ and $c_{12}$ such that

$$\begin{aligned}
\|n_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le \Big\|n_\varepsilon\Big(\cdot, \frac{3}{4}T_{max,\varepsilon}\Big)\Big\|_{L^\infty(\Omega)} \\
&\quad + c_{10}\int_{\frac{3}{4}T_{max,\varepsilon}}^t \Big\|B^\beta e^{-(t-s)(B-1)}\nabla\cdot\Big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\chi(c_\varepsilon)\nabla c_\varepsilon + n_\varepsilon u_\varepsilon\Big)(\cdot,s)\Big\|_{L^4(\Omega)}ds \\
&\le c_{11}\Bigg(1 + \int_{\frac{3}{4}T_{max,\varepsilon}}^t (t-s)^{-\frac{1}{2}-\beta}\Big\|\Big(\frac{n_\varepsilon}{1+\varepsilon n_\varepsilon}\chi(c_\varepsilon)\nabla c_\varepsilon + n_\varepsilon u_\varepsilon\Big)(\cdot,s)\Big\|_{L^4(\Omega)}ds\Bigg) \\
&\le c_{12} \qquad \text{for all } t \in \Big(\frac{7}{8}T_{max,\varepsilon}, T_{max,\varepsilon}\Big).
\end{aligned}$$

Combined with (5.11) and (5.12), this contradicts (2.3) and thereby proves that $T_{max,\varepsilon} = \infty$. $\square$

## 5.3 Proof of the main result for $N = 3$

As a last preparation for the proof of Theorem 1.1 ii), we use the estimates gained in Corollary 5.3 to derive strong compactness properties by means of the Aubin-Lions lemma ([23]).

**Corollary 5.5** *Let $N = 3$. Then for each $T > 0$, the solutions of (5.1) have the properties that*

$$\begin{aligned}
&(n_\varepsilon)_{\varepsilon\in(0,1)} \quad \textit{is strongly precompact in } L^1((0,T);L^1(\Omega)), \\
&(c_\varepsilon)_{\varepsilon\in(0,1)} \quad \textit{is strongly precompact in } L^1((0,T);W^{1,1}(\Omega)) \qquad \textit{and} \\
&(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \textit{is strongly precompact in } L^1((0,T);L^1(\Omega)).
\end{aligned} \tag{5.13}$$

PROOF. We fix $\xi \in C_0^\infty(\Omega)$ and test the first equation in (5.1) by $\frac{1}{2}(n_\varepsilon+1)^{-\frac{1}{2}}\xi$ to see that

$$\begin{aligned}\int_\Omega \partial_t (n_\varepsilon+1)^{\frac{1}{2}} \cdot \xi &= \int_\Omega (n_\varepsilon+1)^{\frac{1}{2}}\Delta\xi + \frac{1}{4}\int_\Omega (n_\varepsilon+1)^{-\frac{3}{2}}|\nabla n_\varepsilon|^2\xi \\ &\quad -\frac{1}{4}\int_\Omega (n_\varepsilon+1)^{-\frac{3}{2}} n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla n_\varepsilon\cdot\nabla c_\varepsilon \xi + \frac{1}{2}\int_\Omega (n_\varepsilon+1)^{-\frac{1}{2}} n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\xi \\ &\quad + \int_\Omega (n_\varepsilon+1)^{\frac{1}{2}} - u_\varepsilon\cdot\nabla\xi \\ &=: I_1+I_2+I_3+I_4+I_5 \qquad \text{for all } t\in(0,T), \end{aligned} \tag{5.14}$$

where $F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1+\varepsilon s)$ for $s \ge 0$. By straightforward application of Hölder's and Young's inequalities, in view of (2.9), (2.10) and the fact that $0 \le F_\varepsilon' \le 1$ we find $c_1>0$ such that

$$|I_1| \le c_1\|\Delta\xi\|_{L^2(\Omega)}$$

and

$$|I_2| \le \frac{1}{4}\Big(\int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big)\cdot\|\xi\|_{L^\infty(\Omega)}$$

as well as

$$\begin{aligned}|I_3| &\le \frac{1}{4}\|\chi\|_{L^\infty((0,\|c_0\|_{L^\infty(\Omega)}))}\cdot\Big(\int_\Omega n_\varepsilon(n_\varepsilon+1)^{-\frac{3}{2}}|\nabla n_\varepsilon\cdot\nabla c_\varepsilon|\Big)\cdot\|\xi\|_{L^\infty(\Omega)} \\ &\le c_1\Big(\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega n_\varepsilon^3(n_\varepsilon+1)^{-3}\cdot\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big)\cdot\|\xi\|_{L^\infty(\Omega)} \\ &\le c_1\Big(\int_\Omega |\nabla c_\varepsilon|^2 + \int_\Omega \frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}\Big)\cdot\|\xi\|_{L^\infty(\Omega)}\end{aligned}$$

for all $t\in(0,T)$. Similarly, for some $c_2>0$ we have

$$|I_4| \le c_2\Big(1+\int_\Omega |\nabla c_\varepsilon|^2\Big)\cdot\|\nabla\xi\|_{L^\infty(\Omega)}$$

and

$$|I_5| \le c_2\Big(1+\int_\Omega |\nabla u_\varepsilon|^2\Big)\cdot\|\nabla\xi\|_{L^\infty(\Omega)}$$

for all $t\in(0,T)$. Thus, noting that $W_0^{3,2}(\Omega)\hookrightarrow W^{1,\infty}(\Omega)$ due to the fact that $N=3$, from (5.14) we infer that there exists $c_3>0$ such that

$$\int_0^T \|\partial_t(n_\varepsilon+1)^{\frac{1}{2}}(\cdot,t)\|_{(W_0^{3,2}(\Omega))^\star}dt \le c_3\int_0^T\Big(1+\int_\Omega\frac{|\nabla n_\varepsilon|^2}{n_\varepsilon}+\int_\Omega|\nabla c_\varepsilon|^2+\int_\Omega|\nabla u_\varepsilon|^2\Big).$$

In view of the estimates collected in Corollary 5.3, this proves that $(\partial_t(n_\varepsilon+1)^{\frac{1}{2}})_{\varepsilon\in(0,1)}$ is bounded in $L^1((0,T);(W_0^{3,2}(\Omega))^\star)$. Since moreover $((n_\varepsilon+1)^{\frac{1}{2}})_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{1,2}(\Omega))$ according to (5.8) and (2.9), we conclude using the Aubin-Lions lemma [23, Remark 2.1 in Ch.III.3] that $((n_\varepsilon + 1)^{\frac{1}{2}})_{\varepsilon\in(0,1)}$ is relatively compact in $L^2((0,T);L^2(\Omega))$, which imples the first statement in (5.13).

In order to verify the claimed compactness property of $(c_\varepsilon)_{\varepsilon\in(0,1)}$, let us write $a_\varepsilon(x,t) := F_\varepsilon(n_\varepsilon)f(c_\varepsilon) + u_\varepsilon\cdot\nabla c_\varepsilon$. Then since $0 \le F_\varepsilon(n_\varepsilon) \le n_\varepsilon$, in view of (2.10) and Corollary 5.3 we can pick positive constants $c_4, c_5, c_6$ and $c_7$ such that

$$\int_0^T \int_\Omega |F_\varepsilon(n_\varepsilon)f(c_\varepsilon)|^{\frac{5}{3}} \le c_4 \int_0^T \int_\Omega n_\varepsilon^{\frac{5}{3}} \le c_5$$

and

$$\int_0^T \int_\Omega |u_\varepsilon\cdot\nabla c_\varepsilon|^2 \le c_6\Big(\int_0^T \int_\Omega |u_\varepsilon|^4\Big)^{\frac{1}{2}} \cdot \Big(\int_0^T \int_\Omega |\nabla c_\varepsilon|^4\Big)^{\frac{1}{2}} \le c_7$$

for all $\varepsilon \in (0,1)$. This shows that $(a_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{5}{3}}(\Omega\times(0,T))$, so that since $c_{\varepsilon t} - \Delta c_\varepsilon = -a_\varepsilon$, standard results on maximal Sobolev regularity for the heat equation ([10]) assert boundedness of both $(c_{\varepsilon t})_{\varepsilon\in(0,1)}$ in $L^{\frac{5}{3}}(\Omega\times(0,T))$ and of $(c_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^{\frac{5}{3}}((0,T);W^{2,\frac{5}{3}}(\Omega))$. Again by the Aubin-Lions lemma, this shows that $(c_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^{\frac{5}{3}}((0,T);W^{1,\frac{5}{3}}(\Omega))$ and thereby also in $L^1((0,T);W^{1,1}(\Omega))$.

Finally, from (5.7) and maximal Sobolev regularity properties of the Stokes evolution equation ([8]) we infer that $(u_{\varepsilon t})_{\varepsilon\in(0,1)}$ is bounded in $L^{\frac{5}{3}}(\Omega\times(0,T))$. Along with (5.8) and again the Aubin-Lions lemma this implies the claimed compactness property of $(u_\varepsilon)_{\varepsilon\in(0,1)}$. $\square$

We can now pass to the proof of our main result in the three-dimensional case.

Proof (of Theorem 1.1 ii)). According to Corollary 5.5, it is possible to pick a sequence of numbers $(0,1) \ni \varepsilon_j \searrow 0$ such that as $\varepsilon = \varepsilon_j \searrow 0$, the solutions of (5.1) satisfy

$$n_\varepsilon \to n \quad \text{and} \quad c_\varepsilon \to c \qquad \text{in } L^1_{loc}(\bar\Omega\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty), \tag{5.15}$$

$$\nabla c_\varepsilon \to \nabla c \qquad \text{in } L^1_{loc}(\bar\Omega\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty) \qquad \text{and} \tag{5.16}$$

$$u_\varepsilon \to u \qquad \text{in } L^1_{loc}(\bar\Omega\times[0,\infty)) \text{ and a.e. in } \Omega\times(0,\infty) \tag{5.17}$$

for some limit function $(n,c,u)$. To see that $(n,c,u)$ is a weak solution of (1.1) in the sense of Definition 5.1, we fix $\zeta_1 \in C_0^\infty(\bar\Omega\times[0,\infty))$ and first assume that $\frac{\partial\zeta_1}{\partial\nu} = 0$ on $\partial\Omega$. Multiplying the first equation in (5.1) by $\zeta_1$, on integrating by parts we obtain

$$-\int_0^\infty n_\varepsilon\partial_t\zeta_1 - \int_\Omega n_0\zeta_1(\cdot,0) \;=\; \int_0^\infty\int_\Omega n_\varepsilon\Delta\zeta_1 + \int_0^\infty\int_\Omega n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon\cdot\nabla\zeta_1 + \int_0^\infty\int_\Omega n_\varepsilon u_\varepsilon\cdot\nabla\zeta_1,$$

where $F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1+\varepsilon s)$ for $s \ge 0$.
In view of (5.7), (5.8), (2.10) and the fact that $0 \le F_\varepsilon' \le 1$, using the Hölder inequality we can find $c_1 > 0$ and, given $T > 0$, $c_2(T) > 0$ such that $w_\varepsilon := n_\varepsilon F_\varepsilon'(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon$ satisfies

$$\int_0^T\int_\Omega |w_\varepsilon|^{\frac{20}{17}} \le c_1\int_0^T\int_\Omega |n_\varepsilon\nabla c_\varepsilon|^{\frac{20}{17}} \le c_1\Big(\int_0^T\int_\Omega n_\varepsilon^{\frac{5}{3}}\Big)^{\frac{12}{17}}\Big(\int_0^T\int_\Omega|\nabla c_\varepsilon|^4\Big)^{\frac{5}{17}} \le c_2 \qquad \text{for all } \varepsilon\in(0,1).$$

Hence, passing to a subsequence if necessary we may assume that $w_\varepsilon \rightharpoonup w$ in $L^{\frac{20}{17}}_{loc}(\bar\Omega \times (0,T))$ as $\varepsilon = \varepsilon_j \searrow 0$ for a certain limit $w$. Since we already know from (5.15), (5.16) and the fact that $F'_\varepsilon(s) \to 1$ for all $s \ge 0$ that

$$w_\varepsilon \to n\chi(c)\nabla c \quad \text{a.e. in } \Omega \times (0,\infty) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0,$$

we may identify $w = n\chi(c)\nabla c$ in $\Omega \times (0,\infty)$ by Egorov's theorem and hence conclude that

$$\int_0^\infty \int_\Omega n_\varepsilon F'_\varepsilon(n_\varepsilon)\chi(c_\varepsilon)\nabla c_\varepsilon \cdot \nabla\zeta_1 \to \int_0^\infty \int_\Omega n\chi(c)\nabla c \cdot \nabla\zeta_1 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Next, defining $z_\varepsilon := n_\varepsilon u_\varepsilon$, using (5.7) we see that for all $T > 0$ there exists $c_3(T) > 0$ such that

$$\int_0^T \int_\Omega |z_\varepsilon|^{\frac{20}{17}} \le \Big(\int_0^T \int_\Omega n_\varepsilon^{\frac{5}{3}}\Big)^{\frac{12}{17}} \Big(\int_0^T \int_\Omega |u_\varepsilon|^4\Big)^{\frac{5}{17}} \le c_3(T) \qquad \text{for all } \varepsilon \in (0,1).$$

Since (5.15) and (5.17) ensure that $z_\varepsilon \to nu$ a.e. in $\Omega \times (0,\infty)$, we infer that $z_\varepsilon \rightharpoonup nu$ in $L^{\frac{20}{17}}_{loc}(\bar\Omega \times [0,\infty))$ and thus

$$\int_0^\infty \int_\Omega n_\varepsilon u_\varepsilon \cdot \nabla\zeta_1 \to \int_0^\infty \int_\Omega nu \cdot \nabla\zeta_1 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Finally, from (5.15) we immediately obtain

$$-\int_0^\infty \int_\Omega n_\varepsilon \partial_t\zeta_1 \to -\int_0^\infty \int_\Omega n\partial_t\zeta_1 \quad \text{and} \quad \int_0^\infty \int_\Omega n_\varepsilon \Delta\zeta_1 \to \int_0^\infty \int_\Omega n\Delta\zeta_1 \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

Moreover, since from (5.8), (2.9) and (5.15) it follows that $\frac{|\nabla n|^2}{n} \in L^1_{loc}(\bar\Omega \times [0,\infty))$ and $n \in L^\infty((0,\infty); L^1(\Omega))$, by the Cauchy-Schwarz inequality we see that for any $T > 0$ we can find $c_4(T) > 0$ fulfilling

$$\int_0^T \int_\Omega |\nabla n| \le \Big(\int_0^T \int_\Omega \frac{|\nabla n|^2}{n}\Big)^{\frac{1}{2}} \Big(\int_0^T \int_\Omega n\Big)^{\frac{1}{2}} \le c_4(T).$$

We therefore may integrate by parts to obtain

$$\int_0^\infty \int_\Omega n\Delta\zeta_1 = -\int_0^\infty \int_\Omega \nabla n \cdot \nabla\zeta_1$$

and thus conclude that the first identity in (5.2) is satisfied.
In the case when $\zeta_1 \in C_0^\infty(\bar\Omega \times [0,\infty))$ has arbitrary behavior at $\partial\Omega$, we can easily construct a sequence of functions $\zeta_{1,j} \in C_0^\infty(\bar\Omega \times [0,\infty))$ satisfying $\frac{\partial\zeta_{1,j}}{\partial\nu} = 0$ on $\partial\Omega$ and $\zeta_{1,j} \to \zeta_1$ in $L^\infty_{loc}(\bar\Omega \times [0,\infty))$ as well as $\nabla\zeta_{1,j} \overset{\star}{\rightharpoonup} \nabla\zeta_1$ and $\partial_t\zeta_{1,j} \overset{\star}{\rightharpoonup} \partial_t\zeta_1$ in $L^\infty_{loc}(\bar\Omega \times [0,\infty))$ as $j \to \infty$. Since by what we have just shown we know that the first in (5.2) is valid for all $j \in \mathbb{N}$, taking $j \to \infty$ we see that the desired identity holds for arbitrary $\zeta_1$.

The verification of the second and third equations in (5.2) can be run along the same lines, the only major issue being the convergence

$$F_\varepsilon(n_\varepsilon) \cdot f(c_\varepsilon) \equiv \frac{1}{\varepsilon}\ln(1 + \varepsilon n_\varepsilon) \cdot f(c_\varepsilon) \rightharpoonup nf(c) \quad \text{in } L^{\frac{5}{3}}_{loc}(\bar\Omega \times [0,\infty)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0.$$

To justify this, besides (5.15) we only need to observe that for all $T > 0$ we can fix $c_5(T) > 0$ such that

$$\int_0^T \int_\Omega \Big| \frac{1}{\varepsilon} \ln(1+\varepsilon n_\varepsilon) \cdot f(c_\varepsilon) \Big|^{\frac{5}{3}} \le \|f\|^{\frac{5}{3}}_{L^\infty((0,\|c_0\|_{L^\infty(\Omega)}))} \cdot \int_0^T \int_\Omega n_\varepsilon^{\frac{5}{3}} \le c_5(T) \qquad \text{for all } \varepsilon \in (0,1)$$

according to (2.10), (5.7) and the inequality $0 \le \ln(1+\xi) \le \xi$ for $\xi \ge 0$. Thereby the proof can easily be completed. □

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