Infinite-time gradient blow-up in a degenerate parabolic equation

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Abstract

This work deals with the Dirichlet problem for the degenerate parabolic equation $u_t = u^p u_{xx} + u^q$ in a bounded interval $\Omega \subset \mathbb{R}$, where $p \geq 2$ and $q \in [1, p-1]$.

It is shown that whenever the initial data u_0 belong to $W^{1,\infty}(\Omega)$, are nonnegative and vanish on $\partial\Omega$, the so-called maximal solution u undergoes an *infinite-time gradient blow-up*. That is, the function $u(\cdot, t)$ belongs to $W^{1,\infty}(\Omega)$ for all $t \in [0, \infty)$, but we have $\|u_x(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to \infty$. Moreover, it is shown that if $q < p-1$ then for sufficiently large $m > 1$, even the functional $\int_\Omega u^\alpha |u_x|^m$ blows up for some $\alpha = \alpha(m) \geq 0$. Finally, by providing explicit upper estimates for the growth of u_x with respect to time, it is shown that the rate of gradient blow-up in any of the integral norms considered above is not faster than algebraic, provided that $q > 1$. In the special case when $u_0(x) \geq c\,\mathrm{dist}(x, \partial\Omega)$ for all $x \in \Omega$ and some $c > 0$, the same is valid for the norm of u_x in $L^\infty(\Omega)$.

Key words: degenerate diffusion, gradient blow-up, blow-up rate, singularity formation
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Introduction

We consider nonnegative solutions of the Dirichlet problem

$$\begin{cases} u_t = u^p u_{xx} + u^q & \text{in } \Omega \times (0, \infty), \\ u|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0, \end{cases} \tag{0.1}$$

in a bounded interval $\Omega = (0, L) \subset \mathbb{R}$, where $p > 0$ and $q \geq 1$ and the initial data u_0 are assumed to satisfy

$$u_0 \in W^{1,\infty}(\Omega), \qquad u_0 \geq 0 \quad \text{in } \Omega \qquad \text{and} \quad u_0|_{\partial\Omega} = 0. \tag{0.2}$$

Equations with degeneracies of this type are used in various applications such as, for instance, in electromagnetism, differential geometry and population dynamics ([2], [3], [15]), where usually $p \geq 1$. In the case $p \in (0, 1)$, the PDE in (0.1) can be transformed into the forced porous medium equation $v_t = (v^m)_{xx} + v^\sigma$ with $m = \frac{1}{1-p}$ and $\sigma = \frac{q-p}{1-p}$, which is essentially well-understood ([11]).

The existing literature provides a number of examples showing that in the case $p \geq 1$ the properties of the diffusion operator in (0.1) are significantly different from those in the range $p \in (0, 1)$. First results in this direction revealed that weak solutions need not be unique, and that families of uniformly bounded smooth solutions need not be locally uniformly continuous ([16], [4], [5]). More recently, further peculiar phenomena were detected. For instance, classical solutions of $u_t = u^p \Delta u$ exist for which $u(\cdot, t)$ belongs to $C_0^\infty(\mathbb{R})$ for all times ([22]). In the case $p \geq 3$ and $q = p + 1$, (0.1) possesses positive classical solutions which heavily oscillate in time in the sense that $u(\cdot, t_j) \to 0$ and $u(\cdot, t_{j+1}) \to +\infty$ along some sequences $t_j, t_{j+1} \to \infty$ ([24]). A more subtle result concerning nonconvergent trajectories states that if $p > 1$ and $q \in (p-1, p+1)$ is such that $q \geq 3 - p$, then (0.1) allows for positive solutions for which $(u(\cdot, t))_{t \geq 0}$ is bounded in $C^1(\bar\Omega)$, but for which $u(\cdot, t)$ does not converge in $C^1(\bar\Omega)$ as $t \to \infty$ ([25]).

The present paper focuses on the related phenomenon of *gradient blow-up*, which is said to occur when a solution itself remains bounded in $L^\infty(\Omega)$, but has a spatial gradient that becomes unbounded either in finite or infinite

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time. Such effects are ruled by classical parabolic regularity results in any semilinear diffusion equation with source terms depending on x and u only, or growing at most quadratically with respect to u_x ([13]). Accordingly, phenomena of this type have been detected quite rarely so far, and most examples of gradient blow-up available in the literature concentrate on equations of type $u_t = \Delta u + f(u, \nabla u)$ (see [1], [12], [9], [8], [14] and the references in the latter, for instance).

As to the quasilinear problem (0.1), it was shown in [18] that when

$$p > 2 \qquad \text{and} \qquad 1 \le q \le p - 1,$$

under the additional assumption that u_0 is smooth in $\bar{\Omega}$ and satisfies $u_0(x) \ge c\,\mathrm{dist}(x, \partial\Omega)$ for all $x \in \Omega$ and some $c > 0$, the problem (0.1) has a positive classical solution u for which $u(\cdot, t)$ belongs to $C^1(\bar{\Omega})$ for all $t \ge 0$ but $\|u(\cdot, t)\|_{W^{1,\infty}(\Omega)} \to \infty$ as $t \to \infty$.

One goal of this work is to clarify whether this phenomenon indeed relies on this additional lower estimate for u_0. In fact, in view of the mentioned lack of regularity in (0.1) it is a priori conceivable that oscillatory behavior of u_0 near some of its zeros may result in more irregular behavior such as finite-time or even immediate gradient blow-up. Our main result in this direction states that

- if $p \ge 2$ and $q \in [1, p-1]$, then for *all* (maximal) solutions of (0.1) we have $u_x \in L^\infty(\Omega \times (0, T))$ for all $T > 0$, but $\|u_x(\cdot, t)\|_{L^\infty(\Omega)} \to \infty$ as $t \to \infty$ (cf. Theorem 2.5 and Lemma 3.2, and see Section 1 for a precise definition of our solution concept).

We moreover ask whether gradient blow-up also occurs in some weaker sense, involving norms of u_x measured only in $L^m(\Omega)$ for some finite m. Indeed, we shall find that this is true for any sufficiently large m:

- If $p > 2$, $q \in [1, p-1)$ and $\alpha \ge 0$ and $m > 1$ are such that $m \ge \frac{p+1-q+2\alpha}{p-q-1}$, then

$$\int_\Omega u^\alpha(x, t)|u_x(x, t)|^m dx \to \infty \qquad \text{as } t \to \infty$$

(Lemma 3.3).

A natural next step in the study of this type of singularity formation appears to consist of determining the rate at which the gradient blows up. In this respect, we shall prove that if $q > 1$ then this blow-up occurs at a rate not faster than algebraic. Namely,

- if $p > 2$ and $q \in (1, p-1]$, then for all sufficiently large $m > 2$ there exists $C_m > 0$ such that for all $t \ge 1$,

$$\|u_x(\cdot, t)\|_{L^m(\Omega)} \le Ct^{\frac{1}{2}} \qquad \qquad \text{if } q \ge \tfrac{p}{2},$$
$$\|(u^{\frac{p+2-2q}{2}})_x\|_{L^m(\Omega)} \le Ct^{\frac{(p-2q)(m-1)}{2p}} \qquad \text{if } q < \tfrac{p}{2}.$$

and the respective expressions on the left tend to $+\infty$ as $t \to \infty$ (see Corollary 4.3 and also Corollary 4.2 for a slightly more general result).

Finally, under the positivity assumption from [18] we can derive an algebraic upper bound for the blow-up rate in $W^{1,\infty}(\Omega)$ when $q > 1$:

- If $p > 2$ and $q \in (1, p-1]$ and $u_0(x) \ge c\,\mathrm{dist}(x, \partial\Omega)$ for all $x \in \Omega$ and some $c > 0$, then for all $\nu > 0$ there exists $C(\nu) > 0$ such that

$$\|u_x(\cdot, t)\|_{L^\infty(\Omega)} \le C(\nu) \cdot t^{\frac{p-q-1}{2(q-1)}+\nu} \qquad \text{for all } t \ge 1$$

(Theorem 4.5).

We do not know whether the above estimates are optimal, and it is an interesting open problem to find any lower estimate for the rate of gradient blow-up in any of the spaces considered above. Also, we do not know whether the range of q considered here is the maximal one within which gradient blow-up occurs. Finally, it is worth mentioning that all of our results refer to the maximal among several possible weak solutions only. We are not aware of any example of a weak solution u for which u_x blows up before $t = \infty$, but unfortunately we cannot rule out such a possibility.

1 Preliminaries

A natural way of approximating solutions of (0.1) consists of solving the regularized problem

$$\begin{cases} u_{\varepsilon t} = u_\varepsilon^p u_{\varepsilon xx} + u_\varepsilon^q & \text{in } \Omega \times (0, T), \\ \tilde{u}_{\eta\varepsilon}|_{\partial\Omega} = \varepsilon, \\ \tilde{u}_{\eta\varepsilon}|_{t=0} = u_0 + \varepsilon \end{cases} \tag{1.1}$$

for $\varepsilon \in (0, 1)$. Indeed, due to the a priori lower bound $u_\varepsilon \geq \varepsilon$ this problem is actually non-degenerate, and it can be seen by standard arguments that (1.1) has a classical solution u_ε defined up to a maximal existence time $T_{max,\varepsilon} \in (0, \infty]$. Moreover, $T_{max,\varepsilon}$ increases and u_ε decreases with ε, so that $T_{max} := \lim_{\varepsilon \searrow 0} T_{max,\varepsilon} \in (0, \infty]$ exists and

$$u(x, t) := \lim_{\varepsilon \searrow 0} u_\varepsilon(x, t)$$

defines a nonnegative function in $\bar{\Omega} \times [0, T_{max})$ ([20], [16]). Throughout the sequel we shall call this limit u the *maximal solution* of (0.1), ignoring here the question in which (pointwise or integral) sense u actually solves (0.1). We only remark without proof that by the methods presented in [16] and [4] it can be seen that u indeed solves (0.1) in the natural weak sense, but that weak solutions are not unique; however, u in fact is maximal among all weak solutions.

Being interested in gradient estimates, we should notice that unlike the limit problem (0.1), for each fixed $\varepsilon \in (0, 1)$ the problem (1.1) contains a source term approximately equal to the positive constant ε^q near $\partial\Omega$. Since this might inconveniently distort the actual gradient behavior of u near $\partial\Omega$ (and, more generally, wherever u is small), in the sequel we shall rely on a different regularization with a slightly weaker source term. Moreover, we shall introduce a second regularization parameter which will enable us to separate technical difficulties arising from possible zeros in the interior of Ω from those stemming from the enforced behavior near $\partial\Omega$.

To be precise, for $\eta \in (0, 1)$ and $\varepsilon \in (0, 1)$, we consider

$$\begin{cases} \tilde{u}_{\eta\varepsilon t} = (\tilde{u}_{\eta\varepsilon} + \varepsilon)^p \tilde{u}_{\eta\varepsilon xx} + (\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^q & \text{in } \Omega \times (0, T), \\ \tilde{u}_{\eta\varepsilon}|_{\partial\Omega} = 0, \\ \tilde{u}_{\eta\varepsilon}|_{t=0} = \tilde{u}_{0\eta}, \end{cases} \tag{1.2}$$

where

$$\tilde{u}_{0\eta}(x) := u_0(x) + \eta \sin \frac{\pi x}{L}, \qquad x \in (0, L),$$

and κ is a fixed number satisfying

$$\kappa > 1, \quad \kappa > \frac{p}{2q} \quad \text{and} \quad \kappa < \frac{p+1}{q}, \tag{1.3}$$

which is possible whenever $0 < q < p + 1$. Again, (1.2) has a unique positive classical solution $\tilde{u}_{\eta\varepsilon}$ defined up to a maximal existence time $T_{max,\eta\varepsilon} \in (0, \infty]$. We shall see in Lemma 1.4 and Lemma 1.5 below that actually $T_{max,\eta\varepsilon} = \infty$, and that by this procedure after taking $\varepsilon \searrow 0$ and then $\eta \searrow 0$ we again rediscover the maximal solution u defined above.

We prepare this by three statements concerning steady states of (0.1) as well as stationary sub- and supersolutions of (1.2) that will be used in several places in the sequel.

To begin with, let us consider the – possibly singular – elliptic boundary-value problem

$$\begin{cases} -w_{xx} = w^{-\beta} & \text{in } (a, b), \\ w(a) = w(b) = 0, \end{cases} \tag{1.4}$$

where $\beta \geq 0$. A list of usefuly properties of (1.4) is provided by the following lemma which can be proved in quite an elementary way using a straightforward integration of the ODE in (1.4) (cf. [21], [20] and [7], where details can be found in the general n-dimensional setting).

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Lemma 1.1 *Suppose that $-\infty < a < b < \infty$. Then for all $\beta \geq 0$, the problem (1.4) admits a unique classical solution $w = w_{a,b,\beta} \in C^0([a,b]) \cap C^2((a,b))$ which is positive in (a,b). This solution satisfies*

$$c \cdot \rho_\beta(x) \leq w(x) \leq C \cdot \rho_\beta(x) \qquad \text{for all } x \in (a,b) \tag{1.5}$$

and

$$c \cdot \rho_\beta(x) \leq \operatorname{dist}(x, \partial\Omega) \cdot |w_x(x)| \leq C \cdot \rho_\beta(x) \qquad \text{for all } x \in \left(a, \frac{3a+b}{4}\right) \cup \left(\frac{a+3b}{4}, b\right), \tag{1.6}$$

with certain positive constants c and C, where

$$\rho_\beta(x) := \begin{cases} (\operatorname{dist}(x, \partial\Omega))^{\frac{2}{\beta+1}} & \text{if } \beta > 1, \\ \operatorname{dist}(x, \partial\Omega) \cdot \sqrt{\ln \frac{K}{\operatorname{dist}(x, \partial\Omega)}} & \text{if } \beta = 1, \\ \operatorname{dist}(x, \partial\Omega) & \text{if } 0 \leq \beta < 1 \end{cases}$$

with some conveniently large $K > 0$.

Moreover, if $\underline{w}$ and $\overline{w}$ belong to $C^0([a,b]) \cap C^2((a,b))$ and are positive in (a,b) with $\underline{w}(a) = \underline{w}(b) = \overline{w}(a) = \overline{w}(b) = 0$ and $\underline{w}_{xx} - \underline{w}^{-\beta} \leq 0 \leq -\overline{w}_{xx} - \overline{w}^{-\beta}$ in (a,b), then $\underline{w} \leq \overline{w}$.

Finally, if $(a_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ and $(b_j)_{j\in\mathbb{N}} \subset \mathbb{R}$ are such that $a \swarrow a_j < b_j \nearrow b$ as $j \to \infty$, then the corresponding solutions in (a_j, b_j) satisfy $w_{a_j, b_j, \beta} \to w_{a,b,\beta}$ in $C^0_{loc}((a,b))$ as $j \to \infty$.

In respect of boundedness of solutions in $L^\infty(\Omega)$, it is a convenient feature of (1.2) that large multiples of steady states of (0.1) are supersolutions of (1.2).

Lemma 1.2 *Let $q \in [1, p]$ and $-\infty < a < b < \infty$. Then the solution $w := w_{a,b,p-q}$ of (1.4) has the property that for all $C > 1$ and $\varepsilon \in (0,1)$,*

$$v(x) := C \cdot w(x), \qquad x \in [a,b],$$

satisfies

$$(v+\varepsilon)^p v_{xx} + (v+\varepsilon^\kappa)^q \leq 0 \qquad \text{in } (a,b).$$

PROOF.   Since $\kappa > 1$ by (1.3), and since $\varepsilon < 1$, we have $(Cw+\varepsilon^\kappa)^q \leq (Cw+\varepsilon)^q$ and hence using (1.4) we find

$$\frac{-(v+\varepsilon)^p v_{xx}}{(v+\varepsilon^\kappa)^q} = \frac{(Cw+\varepsilon)^{p-q} \cdot Cw^{q-p}}{(Cw+\varepsilon^\kappa)^q} \geq (Cw+\varepsilon)^{p-q} \cdot Cw^{q-p} \geq (Cw)^{p-q} \cdot Cw^{q-p} = C^{p+1-q} \geq 1,$$

because $p \geq q$ and $C \geq 1$. ////

Thanks to the source term in (1.2), we can also identify some arbitrarily small stationary subsolutions of (1.2).

Lemma 1.3 *Suppose that $q \in [1, p+1)$, and that $-\infty < a < b < \infty$. Then there exist $c_0 > 0$ and $\varepsilon_0 \in (0,1)$ such that for all $c \in (0, c_0]$ and $\varepsilon \in (0, \varepsilon_0)$,*

$$v(x) := c \cdot \sin \frac{\pi(x-a)}{b-a}, \qquad x \in [a,b],$$

satisfies

$$(v+\varepsilon)^p v_{xx} + (v+\varepsilon^\kappa)^q \geq 0 \qquad \text{in } [a,b]. \tag{1.7}$$

PROOF.   Writing $\lambda_1 := \left(\frac{\pi}{b-a}\right)^2$, we let $c_0 := (2^p \lambda_1)^{-\frac{1}{p+1-q}}$ and fix $\varepsilon_0 \in (0,1)$ small enough fulfilling $\varepsilon_0^{p+1-q\kappa} < (2^p \lambda_1)^{-1}$, which is possible since $p+1-q\kappa > 0$ by (1.3). Then given $\varepsilon \in (0, \varepsilon_0)$, if x is such that $v(x) \in (0, \varepsilon)$ we have

$$\frac{\lambda_1 v (v+\varepsilon)^p}{(v+\varepsilon^\kappa)^q} \leq \frac{\lambda_1 \varepsilon \cdot (2\varepsilon)^p}{\varepsilon^{q\kappa}} \leq 2^p \lambda_1 \cdot \varepsilon^{p+1-q\kappa} < 1,$$

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whereas if $\varepsilon \leq v(x) \leq c_0$, then

$$\frac{\lambda_1 v(v+\varepsilon)^p}{(v+\varepsilon^\kappa)^q} \leq \frac{\lambda_1 v \cdot (2v)^p}{v^q} \leq 2^p \lambda_1 \cdot v^{p+1-q} \leq 1.$$

As $v_{xx} = -\lambda_1 v$ in $[a,b]$, this proves (1.7) for all $c \in (0, c_0]$ and $\varepsilon \in (0, \varepsilon_0)$. ////

As a aconsequence of Lemma 1.2 we obtain global solvability of (1.2) and some pointwise estimates from above and below which are essentially independent of η and ε.

Lemma 1.4 *Let $p \geq 1$ and $q \in [1, p]$. Then there exist $c > 0$, $C > 0$ and $\varepsilon_0 \in (0, 1)$ such that for all $\eta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_0)$, the solution $\tilde{u}_{\eta\varepsilon}$ of (1.2) exists globally in time and satisfies*

$$c \min \left\{ 1, \eta + \inf_{y \in \Omega} \frac{u_0(y)}{\text{dist}(y, \partial\Omega)} \right\} \text{dist}(x, \partial\Omega) \leq \tilde{u}_{\eta\varepsilon}(x, t) \leq \begin{cases} C(\text{dist}(x, \partial\Omega))^{\frac{2}{p+1-q}} & \text{if } q < p-1 \\ C\text{dist}(x, \partial\Omega)\sqrt{\ln \frac{C}{\text{dist}(x\partial\Omega)}} & \text{if } q = p-1, \\ C\text{dist}(x, \partial\Omega) & \text{if } q > p-1 \end{cases} \quad (1.8)$$

for all $(x, t) \in \Omega \times (0, \infty)$.

PROOF.   Letting $w = w_{0,L,p-q}$ denote the solution of (1.4), thanks to (1.5) and the fact that $\sup_{\eta \in (0,1)} \|\tilde{u}_{0\eta x}\|_{L^\infty(\Omega)}$ is finite, we can find $c_1 \geq 1$ independent of η and ε such that $\tilde{u}_{0\eta} \leq c_1 \cdot w$ in Ω. According to Lemma 1.2, $(x, t) \mapsto c_1 \cdot w(x)$ is a supersolution of (1.2) dominating $\tilde{u}_{\eta\varepsilon}$ on the parabolic boundary of $\Omega \times (0, T_{max,\eta\varepsilon})$, so that the comparison principle ensures that $\tilde{u}_{\eta\varepsilon} \leq c_1 \cdot w$ in $\Omega \times (0, T_{max,\eta\varepsilon})$. By parabolic regularity theory, this entails that actually $T_{max,\eta\varepsilon} = \infty$, and in view of (1.5) this moreover establishes the right inequality in (1.8). To see the left one, we fix c_0 and ε_0 as in Lemma 1.3 and let $c_2 := \inf_{x \in \Omega} \frac{\text{dist}(x, \partial\Omega)}{\Theta(x)}$ and $c_3 := \inf_{x \in \Omega} \frac{u_0(x)}{\text{dist}(x, \partial\Omega)}$, where $\Theta(x) := \sin \frac{\pi x}{L}$. Given $\eta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_0)$, we then have the inequality

$$\tilde{u}_{0\eta}(x) \geq (\eta + c_2 c_3) \cdot \Theta(x) \qquad \text{for all } x \in \Omega.$$

By Lemma 1.3 and the comparison principle, from this we infer that

$$\tilde{u}_{\eta\varepsilon}(x, t) \geq \min\{c_0, \eta + c_2 c_3\} \cdot \Theta(x) \qquad \text{for all } (x, t) \in \Omega \times (0, \infty)$$

and thereby easily complete the proof of (1.8). ////

We can now make sure that the approximation procedure (1.2) indeed leads to the same result as (1.1).

Lemma 1.5 *Let $p \geq 1$ and $q \in [1, p]$. Then for each $\eta \in (0, 1)$, the limit $\bar{u}_\eta := \lim_{\varepsilon \searrow 0} \tilde{u}_{\eta\varepsilon}$ exists in the pointwise sense, and $\bar{u}_\eta$ is the unique positive classical solution of (0.1) with initial data $\tilde{u}_{0\eta}$. Moreover, we have $\bar{u}_\eta \searrow u$ in $\bar{\Omega} \times [0, \infty)$ as $\eta \searrow 0$, where u denotes the maximal solution of (0.1).*

PROOF.   According to the two-sided estimates in (1.8), standard parabolic regularity theory ([13]) and the Arzelá-Ascoli theorem ensure that for fixed $\eta \in (0, 1)$ the set $(\tilde{u}_{\eta\varepsilon})_{\varepsilon \in (0, \varepsilon_0)}$ is relatively compact in $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\Omega \times (0, \infty))$. Since positive classical solutions of (0.1) are unique ([20]), this entails that $\tilde{u}_{\eta\varepsilon} \to \bar{u}_\eta$ in $C^0_{loc}(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}_{loc}(\Omega \times (0, \infty))$, and that $\bar{u}_\eta$ has the claimed solution property.
Next, by comparison ([20]), $\bar{u}_\eta$ decreases to some $\bar{u}$ as $\eta \searrow 0$. Clearly, $\bar{u} \leq u$, because $\tilde{u}_{0\eta} \leq u_0 + \eta$ and hence $\tilde{u}_{\eta\varepsilon} \leq u_\varepsilon$ whenever $\varepsilon \geq \eta$. To see the opposite inequality, we consider the solutions $\hat{u}_{\eta\varepsilon}$ of (1.1) with initial data $\hat{u}_{\eta\varepsilon}|_{t=0} = \tilde{u}_{0\eta} + \varepsilon$. By comparison, these satisfy $\hat{u}_{\eta\varepsilon} \geq u_\varepsilon$, but according to a similar limit procedure they also decrease to a positive classical solution of (0.1) with initial data $\tilde{u}_{0\eta}$ ([20]). Again by uniqueness of positive classical solutions of (0.1), we infer that $\hat{u}_{\eta\varepsilon} \to \bar{u}_\eta$ as $\varepsilon \searrow 0$, so that taking $\eta \searrow 0$ we obtain $\bar{u} \geq u$. ////

As a last preliminary, we shall need the following adaptation to (1.2) of the one-sided estimate $\frac{u_t}{u} \geq -\frac{1}{pt}$ that can easily be derived for (0.1) at a formal level. Semi-convexity inequalities of this type play a considerable role in the qualitative study of a number of related degenerate equations such as the classical porous medium and fast diffusion equations (see [19], for instance).

Lemma 1.6 *Let $p > 0$ and $q \in [1, p+1)$, and let $B := (b+1)(\frac{q}{p+1-q})^{q-1}$. Then for any $\eta \in (0,1)$ and $\varepsilon \in (0,1)$ the solution $\tilde{u}_{\eta\varepsilon}$ of (1.2) satisfies*

$$(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1} \tilde{u}_{\eta\varepsilon xx} + \frac{(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^q}{\tilde{u}_{\eta\varepsilon} + \varepsilon} \geq -z_\varepsilon(t) \qquad \text{for all } (x,t) \in \Omega \times (0,\infty), \tag{1.9}$$

where z_ε deotes the solution of $z'_\varepsilon = -pz_\varepsilon^2 + B\varepsilon^{q-1} z_\varepsilon$ for $t > 0$ with $\lim_{t \searrow 0} z_\varepsilon(t) = +\infty$ given by

$$z_\varepsilon(t) := \frac{B\varepsilon^{q-1}}{p \cdot (1 - e^{-B\varepsilon^{q-1}t})}, \qquad t > 0. \tag{1.10}$$

PROOF. We first observe that according to parabolic regularity results, for each $\tau > 0$ the function v defined by $v(x,t) := \tilde{u}_{\eta\varepsilon}(x, \tau + t) + \varepsilon$ is smooth in $\bar\Omega \times [0,\infty)$ and satisfies $v_t = v^p v_{xx} + (v-\delta)^q$ in $\Omega \times (0,\infty)$ with $\delta := \varepsilon - \varepsilon^\kappa > 0$ according to (1.3). Since $v \geq \varepsilon$, we thus see that also $V := \frac{v_t}{v} \equiv v^{p-1} v_{xx} + \frac{(v-\delta)^q}{v}$ is smooth in $\bar\Omega \times [0,\infty)$, and a straightforward computation reveals that

$$0 = \mathcal{P}V := V_t - v^p V_{xx} - 2v^{p-1} v_x V_x - pV^2 - I(x,t)V \qquad \text{in } \Omega \times (0,\infty) \tag{1.11}$$

with $I(x,t) := \frac{(v-\delta)^{q-1}}{v} \cdot \left\{ (p+1)\delta - (p+1-q)v \right\}$. Clearly, $I \leq 0$ holds at each point where $v \geq \frac{(p+1)\delta}{p+1-q}$. However, if $v < \frac{(p+1)\delta}{p+1-q}$, then since $v \geq \varepsilon \geq \delta$ and $q \geq 1$, we have

$$
\begin{aligned}
I(x,t) &\leq \frac{(v-\delta)^{q-1}}{v} \cdot (p+1)\delta \leq \frac{(v-\delta)^{q-1}}{\varepsilon} \cdot (p+1)\delta \\
&\leq \frac{(p+1)(\frac{p+1}{p+1-q} - 1)^{q-1} \cdot \delta^q}{\varepsilon} \leq (p+1)\left(\frac{q}{p+1-q}\right)^{q-1} \cdot \varepsilon^{q-1},
\end{aligned}
$$

and therefore it follows that $I \leq B\varepsilon^{q-1}$ in $\Omega \times (0,\infty)$. Since z_ε is positive, this implies that $(x,t) \mapsto -z_\varepsilon(t)$ is a subsolution of the parabolic equation in (1.11). Using that $-z_\varepsilon(t) \to -\infty$ as $t \searrow 0$, upon a comparison argument we conclude that $V \geq -z_\varepsilon(t)$ in $\Omega \times (0,\infty)$. As $\tau > 0$ was arbitrary, this shows that $\frac{\tilde{u}_{\eta\varepsilon t}}{\tilde{u}_{\eta\varepsilon}} \geq -z_\varepsilon(t)$ in $\Omega \times (0,\infty)$ and thereby completes the proof. ////

2 Estimates for the large time behavior near $\partial\Omega$. Absence of gradient blow up in finite time

In this section our goal is to find estimates for the quotient $\frac{\tilde{u}_{\eta\varepsilon}(x,t)}{\operatorname{dist}(x, \partial\Omega)}$ which are essentially independent of η and ε and thus allow for a control of $\tilde{u}_{\eta\varepsilon x}(\cdot, t)$ on $\partial\Omega$. As a preparation, we prove an elementary calculus lemma.

Lemma 2.1 *For all $\gamma \in (0,1)$ there exists $\sigma > 0$ with the property that for all $k > 0$ and each $\mu \in (0, \sigma \cdot k^{\frac{1}{1-\gamma}})$ we have*

$$\xi + \mu \leq k\xi^\gamma \qquad \text{for all } \xi \in \left[2\left(\frac{\mu}{k}\right)^{\frac{1}{\gamma}}, \frac{1}{2} \cdot k^{\frac{1}{1-\gamma}} \right].$$

PROOF. We let $\varphi_\mu(\xi) := \xi + \mu - k\xi^\gamma$ for $\xi > 0$ and $\mu > 0$. Then

$$\varphi_\mu\left(2\left(\frac{\mu}{k}\right)^{\frac{1}{\gamma}} \right) = 2\left(\frac{\mu}{k}\right)^{\frac{1}{\gamma}} + \mu - 2^\gamma \mu = \left(2k^{-\frac{1}{\gamma}} \mu^{\frac{1-\gamma}{\gamma}} + 1 - 2^\gamma \right) \cdot \mu < 0$$

for all $\mu < \mu_1 := \left(\frac{2^\gamma - 1}{2}\right)^{\frac{\gamma}{1-\gamma}} \cdot k^{\frac{1}{1-\gamma}}$, and moreover

$$\varphi_\mu(\xi)\left(\frac{1}{2} \cdot k^{\frac{1}{1-\gamma}} \right) = \frac{1}{2} k^{\frac{1}{1-\gamma}} + \mu - k \cdot \frac{1}{2^\gamma} \cdot k^{\frac{\gamma}{1-\gamma}} = \mu - \frac{1 - 2^{\gamma-1}}{2^\gamma} \cdot k^{\frac{1}{1-\gamma}} < 0$$

holds whenever $\mu < \mu_2 := \frac{1-2^{\gamma-1}}{2^\gamma} \cdot k^{\frac{1}{1-\gamma}}$. Since $\frac{d^2}{d\xi^2}\varphi_\mu(\xi) = \gamma(1-\gamma)k\xi^{\gamma-2}$ is positive on $(0,\infty)$, this means that φ_μ is negative throughout $[2(\frac{\mu}{k})^{\frac{1}{\gamma}}, \frac{1}{2}k^{\frac{1}{1-\gamma}}]$, provided that $\mu < \min\{\mu_1, \mu_2\}$. This proves the claim upon an evident choice of σ. ////

We can now establish the desired boundary estimate in the case $q > 1$.

Lemma 2.2 *Let $p > 2$ and $q \in (1, p-1]$. Then for all $\nu > 0$ there exists $C(\nu) > 0$ with the property that for all $T > 0$ one can find $\varepsilon_0(T) \in (0,1)$ such that for each $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_0(T))$ the solution $\tilde{u}_{\eta\varepsilon}$ of (1.2) satisfies*

$$\tilde{u}_{\eta\varepsilon}(x,t) \le C(\nu) \cdot (t+1)^{\frac{p-q-1}{2(q-1)}+\nu} \cdot \operatorname{dist}(x, \partial\Omega) \qquad \text{for all } (x,t) \in \Omega \times (0,T). \tag{2.1}$$

PROOF.   Since $\nu > 0$, we can fix $\beta \in (0,1)$ close enough to 1 such that

$$\delta := \frac{p-q-\beta}{(\beta+1)(q-1)} \le \frac{p-q-1}{2(q-1)} + \nu$$

and, according to (1.3), that also

$$\kappa > \frac{p}{(\beta+1)q}. \tag{2.2}$$

Then by Lemma 1.1, the solution w of (1.4) in Ω satisfies

$$c_1 \operatorname{dist}(x, \partial\Omega) \le w(x) \le c_2 \operatorname{dist}(x, \partial\Omega) \qquad \text{for all } x \in \Omega \tag{2.3}$$

with certain positive constants c_1 and c_2. We abbreviate $c_3 := \frac{1}{2} \cdot \delta^{\frac{1}{q-1}}$ and let

$$y_0 := \max\left\{ \frac{1}{c_1} \cdot \sup_{\eta \in (0,1)} \|\tilde{u}_{0\eta x}\|_{L^\infty(\Omega)}, \; c_3^{-\frac{p-q-\beta}{\beta+1}} \right\}.$$

We next fix $T > 0$ and define a comparison function v in $\bar\Omega \times [0,T]$ by

$$v(x,t) := y(t) \cdot w(x), \qquad \text{where} \quad y(t) := y_0 \cdot (t+1)^\delta, \qquad (x,t) \in \bar\Omega \times [0,T].$$

Then since

$$\frac{\tilde{u}_{0\eta}(x)}{w(x)} \le \frac{\tilde{u}_{0\eta}(x)}{c_1 \operatorname{dist}(x, \partial\Omega)} \le \frac{1}{c_1} \cdot \|\tilde{u}_{0\eta x}\|_{L^\infty(\Omega)} \le y_0 \qquad \text{for all } x \in \Omega,$$

if follows that $v \ge \tilde{u}_{\eta\varepsilon}$ holds on the parabolic boundary of $\Omega \times (0,T)$ for any $\eta \in (0,1)$ and $\varepsilon \in (0,1)$. In view of (2.3) and the comparison principle, it is thus sufficient for the proof of (2.1) to show that there exists $\varepsilon_0(T) \in (0,1)$ such that

$$\mathcal{P}_\varepsilon v := v_t - (v+\varepsilon)^p v_{xx} - (v+\varepsilon^\kappa)^q$$

satisfies

$$\mathcal{P}_\varepsilon v \ge 0 \qquad \text{in } \Omega \times (0,T) \tag{2.4}$$

whenever $\varepsilon \in (0, \varepsilon_0(T))$. To achieve this, we compute

$$\mathcal{P}_\varepsilon v = y' \cdot w + (yw+\varepsilon)^p \cdot yw^{-\beta} - (yw+\varepsilon^\kappa)^q \qquad \text{in } \Omega \times (0,T) \tag{2.5}$$

and decompose $Q := \Omega \times (0,T)$ into $Q = Q_1 \cup Q_2 \cup Q_3$, where

$$Q_1 := \left\{ (x,t) \in Q \mid w(x) > c_3 y_0^{\frac{1}{(q-1)\delta}} y^{-1-\frac{1}{(q-1)\delta}}(t) \right\},$$

$$Q_2 := \left\{ (x,t) \in Q \mid w(x) < c_4 \varepsilon^{q\kappa} y_0^{-\frac{1}{\delta}} y^{-1+\frac{1}{\delta}}(t) \right\} \qquad \text{and}$$

$$Q_3 := Q \setminus (Q_1 \cup Q_2)$$

with $c_4 := \frac{2}{\delta}$.

We first claim that there exists $\varepsilon_1(T) \in (0,1)$ such that

$$\mathcal{P}_\varepsilon v \geq 0 \quad \text{in } Q_3 \qquad \text{whenever } \varepsilon \in (0, \varepsilon_1(T)). \tag{2.6}$$

Indeed, applying Lemma 2.1 to $\xi := y \cdot w, \gamma := \frac{1}{q}, \mu := \varepsilon^\kappa$ and $k := \delta^{\frac{1}{q}} \cdot (\frac{y_0}{y})^{\frac{1}{q\delta}}$, we obtain that for all points $(x,t) \in Q$ at which

$$2 \cdot \left\{ \frac{\varepsilon^\kappa}{\delta^{\frac{1}{q}} \cdot (\frac{y_0}{y(t)})^{\frac{1}{q\delta}}} \right\}^q \leq y(t)w(x) \leq \frac{1}{2} \cdot \left\{ \delta^{\frac{1}{q}} \cdot \left(\frac{y_0}{y(t)} \right)^{\frac{1}{q\delta}} \right\}^{\frac{1}{1-\frac{1}{q}}}, \tag{2.7}$$

the inequality

$$y(t)w(x) + \varepsilon^\kappa \leq \delta^{\frac{1}{q}} \cdot \left(\frac{y_0}{y(t)} \right)^{\frac{1}{q\delta}} \cdot \left(y(t)w(x) \right)^{\frac{1}{q}} \tag{2.8}$$

is valid, provided that

$$\varepsilon^\kappa < \sigma \cdot \left\{ \delta^{\frac{1}{q}} \cdot \left(\frac{y_0}{y(t)} \right)^{\frac{1}{q\delta}} \right\}^{\frac{1}{1-\frac{1}{q}}} \equiv \sigma \cdot \delta^{\frac{1}{q-1}} \cdot (t+1)^{-\frac{1}{q-1}} \tag{2.9}$$

with σ as given by Lemma 2.1. Since $(x,t) \in Q$ can easily be seen to satisfy (2.7) if and only if it belongs to Q_3, we conclude upon rearrangng (2.8) and (2.9) that if

$$\varepsilon < \varepsilon_1(T) := \left\{ \sigma \cdot \delta^{\frac{1}{q-1}} \cdot (T+1)^{-\frac{1}{q-1}} \right\}^{\frac{1}{\kappa}},$$

then for all $(x,t) \in Q_3$ we have

$$\left(y(t)w(x) + \varepsilon^\kappa \right)^q \leq \delta \cdot \left(\frac{y_0}{y(t)} \right)^{\frac{1}{\delta}} \cdot y(t)w(x) \equiv y'(t)w(x)$$

and hence, by (2.5), arrive at (2.6).

In order to prove a similar statement in Q_2, let us consider the auxiliary function φ defined by

$$\varphi(\xi) := \frac{(y\xi + \varepsilon)^p \cdot y}{(y\xi + \varepsilon^\kappa)^q \xi^\beta}, \qquad \xi > 0,$$

where $y = y(t)$ and $t \in (0,T)$ and $\varepsilon \in (0,1)$ are fixed. We calculate

$$\frac{1}{y} \cdot \varphi'(\xi) = \frac{py(y\xi + \varepsilon)^{p-1} \cdot (y\xi + \varepsilon^\kappa)^q \cdot \xi^\beta - (y\xi + \varepsilon)^p \cdot [qy(y\xi + \varepsilon^\kappa)^{q-1} \cdot \xi^\beta + \beta(y\xi + \varepsilon^\kappa)^q \cdot \xi^{\beta-1}]}{(y\xi + \varepsilon^\kappa)^{2q} \cdot s^{2\beta}}$$

for $\xi > 0$ to see that $\varphi'(\xi) \leq 0$ if and only if

$$py(y\xi + \varepsilon^\kappa) \cdot \xi \leq (y\xi + \varepsilon) \cdot [qy\xi + \beta(y\xi + \varepsilon^\kappa)],$$

that is, if

$$(p - q - \beta) \cdot (y\xi)^2 \leq [(q + \beta)\varepsilon - (p - \beta)\varepsilon^\kappa] \cdot (y\xi) + \beta\varepsilon^{\kappa+1}. \tag{2.10}$$

Since $\kappa > 1$, we have $(p - \beta)\varepsilon^\kappa \leq \frac{1}{2}(q + \beta)\varepsilon$ for all $\varepsilon \in (0, \varepsilon_2)$ with $\varepsilon_2 := (\frac{q+\beta}{2(p-\beta)})^{\frac{1}{\kappa-1}}$, and hence (2.10) implies that

$$\varphi'(\xi) \leq 0 \qquad \text{for all } \xi \leq c_5 \cdot \frac{\varepsilon}{y} \tag{2.11}$$

holds with $c_5 := \frac{q+\beta}{2(p-q-\beta)}$.

Now suppose that $(x,t) \in Q_2$. Then

$$\frac{w(x)}{c_5 \cdot \frac{\varepsilon}{y}} < \frac{c_4 \varepsilon^{q\kappa} y_0^{-\frac{1}{\delta}} y^{-1+\frac{1}{\delta}}}{c_5 \cdot \frac{\varepsilon}{y}} = \frac{c_4}{c_5} \cdot \varepsilon^{q\kappa-1} \cdot \left(\frac{y}{y_0} \right)^{\frac{1}{\delta}} \leq \frac{c_4}{c_5} \cdot \varepsilon^{q\kappa-1} \cdot (1+T) \leq 1,$$

provided that $0 < \varepsilon < \varepsilon_3(T) := (\frac{c_5}{c_4(T+1)})^{\frac{1}{q\kappa-1}}$. Hence, if in addition $\varepsilon < \varepsilon_2$ then the monotonicity property (2.11) yields

$$\frac{(yw+\varepsilon)^p y}{(yw+\varepsilon^\kappa)^q \cdot w\beta} = \varphi(w) \geq \varphi\left(c_4\varepsilon^{q\kappa}y_0^{-\frac{1}{\delta}}y^{-1+\frac{1}{\delta}}\right) = \frac{\left(c_4\varepsilon^{q\kappa}y_0^{-\frac{1}{\delta}}y^{\frac{1}{\delta}}+\varepsilon\right)^p \cdot y}{\left(c_4\varepsilon^{q\kappa}y_0^{-\frac{1}{\delta}}y^{\frac{1}{\delta}}+\varepsilon^\kappa\right)^q \cdot \left(c_4\varepsilon^{q\kappa}y_0^{-\frac{1}{\delta}}y^{-1+\frac{1}{\delta}}\right)^\beta}$$

for such (x,t). Since $q > 1$ and thus

$$\frac{c_4\varepsilon^{q\kappa}y_0^{-\frac{1}{\delta}}y^{\frac{1}{\delta}}}{\varepsilon^\kappa} \leq c_4\varepsilon^{(q-1)\kappa} \cdot (T+1) \leq 1 \qquad \text{for all } t \in (0,T)$$

if $0 < \varepsilon < \varepsilon_4(T) := [c_4(T+1)]^{-\frac{1}{(q-1)\kappa}}$, from this and (2.2) we infer that

$$\begin{aligned}
\frac{(yw+\varepsilon)^p y}{(yw+\varepsilon^\kappa)^q \cdot w\beta} &\geq \frac{\varepsilon^p y}{(2\varepsilon^\kappa)^q \cdot \left(c_3\varepsilon^{q\kappa}y_0^{-\frac{1}{\delta}}y^{-1+\frac{1}{\delta}}\right)^\beta} \\
&= \frac{1}{2^q c_4^\beta} \cdot \varepsilon^{-[(\beta+1)q\kappa-p]} \cdot y_0^{\frac{\beta}{\delta}} \cdot y^{\beta-\frac{\beta}{\delta}+1} \\
&= \frac{1}{2^q c_4^\beta} \cdot \varepsilon^{-[(\beta+1)q\kappa-p]} \cdot y_0^{\beta+1} \cdot (t+1)^{(\beta+1)\delta-\beta} \\
&\geq \frac{1}{2^q c_4^\beta} \cdot \min\left\{1, (T+1)^{(\beta+1)\delta-\beta}\right\} \cdot \varepsilon^{-[(\beta+1)q\kappa-p]} \cdot y_0^{\beta+1} \\
&\geq 1 \qquad \text{for all } (x,t) \in Q_2
\end{aligned}$$

for any $\varepsilon \in (0, \varepsilon_5(T))$, where $\varepsilon_5(T) \in (0,1)$ is such that $\varepsilon_5(T) \leq \min\{\varepsilon_2, \varepsilon_3(T), \varepsilon_4(T)\}$ and

$$\varepsilon_5(T) \leq \left\{\frac{y_0^{\beta+1}}{2^q c_4^\beta} \cdot \min\left\{1, (T+1)^{(\beta+1)\delta-\beta}\right\}\right\}^{\frac{1}{(\beta+1)q\kappa-p}}.$$

Consequently, (2.5) ensures that

$$\mathcal{P}_\varepsilon v \geq 0 \quad \text{in } Q_2 \qquad \text{whenever } \varepsilon \in (0, \varepsilon_5(T)). \tag{2.12}$$

Finally, if $(x,t) \in Q_1$ then $w(x) > c_3 y_0^{\frac{1}{(q-1)\delta}} y^{-1-\frac{1}{(q-1)\delta}}(t)$, and therefore using that $\varepsilon^\kappa < \varepsilon$ and $p > q + \beta > q$ we may estimate

$$\begin{aligned}
\frac{(yw+\varepsilon)^p y}{(yw+\varepsilon^\kappa)^q \cdot w\beta} &\geq \frac{(yw+\varepsilon)^{p-q}y}{w^\beta} \geq \frac{(yw)^{p-q}y}{w^\beta} = y^{p+1-q}w^{p-q-\beta} \\
&> c_3^{p-q-\beta}y_0^{\frac{p-q-\beta}{(q-1)\delta}} \cdot y^{p+1-q+(-1-\frac{1}{(q-1)\delta})\cdot(p-q-\beta)} \qquad \text{for all } (x,t) \in Q_1.
\end{aligned}$$

Since according to our choice of δ we have

$$\frac{p-q-\beta}{(q-1)\delta} = \beta+1 \qquad \text{and} \qquad \left(-1-\frac{1}{(q-1)\delta}\right) \cdot (p-q-\beta) = 0,$$

this means that

$$\frac{(yw+\varepsilon)^p y}{(yw+\varepsilon^\kappa)^q \cdot w\beta} > c_3^{p-q-\beta}y_0^{\beta+1} \geq 1 \qquad \text{for all } (x,t) \in Q_1$$

in view of our definition of y_0. Hence, (2.5) results in the inequality

$$\mathcal{P}_\varepsilon v \geq 0 \quad \text{in } Q_1 \qquad \text{for all } \varepsilon \in (0,1),$$

and thus recalling (2.6) and (2.12) we conclude that indeed (2.4) is valid, whereby the proof is completed. ////

The above estimate is obviously no longer meaningful when $q = 1$. Correspondingly, in this borderline case we only obtain an exponential upper bound:

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Lemma 2.3 *Let $p \geq 2$ and $q = 1$. Then for all $\theta > 1$ there exists $C > 0$ such that for all $\eta \in (0,1)$ and $\varepsilon \in (0,1)$ the solution $\tilde{u}_{\eta\varepsilon}$ of (1.2) satisfies*

$$\tilde{u}_{\eta\varepsilon}(x,t) \leq C\, e^{\theta t} \cdot \operatorname{dist}(x, \partial\Omega) \qquad \text{for all } (x,t) \in \Omega \times (0,\infty). \tag{2.13}$$

PROOF.   According to (1.3), we can fix $\beta \in (0,1)$ such that $(\beta+1)\kappa \geq p$, and let w denote the solution of (1.4) in Ω. Then thanks to (1.5) there exists $c_1 > 0$ such that $(\theta - 1)c_1^{\frac{\beta+1}{\beta}} \geq 1$ and $\tilde{u}_{0\eta}(x) \leq c_1 \cdot w(x)$ for all $x \in \Omega$ and each $\eta \in (0,1)$. Thus, for arbitrary $\eta \in (0,1)$ and $\varepsilon \in (0,1)$, the function $v(x,t) := y(t) \cdot w(x)$, $(x,t) \in \bar{\Omega} \times [0,\infty)$, with $y(t) := c_1 e^{\theta t}, t \geq 0$, satisfies $v \geq \tilde{u}_{\eta\varepsilon}$ on $\partial\Omega$ and at $t = 0$. Moreover,

$$\mathcal{P}v := v_t - (v+\varepsilon)^p v_{xx} - (v+\varepsilon^\kappa)^q = (\theta-1)yw + (yw+\varepsilon)^p \cdot yw^{-\beta} - \varepsilon^\kappa. \tag{2.14}$$

Here, at each point where $w(x) \leq \varepsilon^{\frac{p-\kappa}{\beta}} y^{\frac{1}{\beta}}(t)$, we have $(yw+\varepsilon)^p \cdot yw^{-\beta} \geq \varepsilon^p yw^{-\beta} \geq \varepsilon^\kappa$ and hence $\mathcal{P}v \geq 0$ since $\theta > 1$. If conversely $w(x) < \varepsilon^{\frac{p-\kappa}{\beta}} y^{\frac{1}{\beta}}(t)$, then

$$\frac{(\theta-1)yw}{\varepsilon^\kappa} > \frac{(\theta-1)y \cdot \varepsilon^{\frac{p-\kappa}{\beta}} y^{\frac{1}{\beta}}}{\varepsilon^\kappa} \geq (\theta-1)\varepsilon^{\frac{p-(\beta+1)\kappa}{\beta}} c_1^{\frac{\beta+1}{\beta}} \geq (\theta-1)c_1^{\frac{\beta+1}{\beta}} \geq 1$$

due to our choice of c_1. Therefore, the comparison pronciple yields $\tilde{u}_{\eta\varepsilon} \leq v$ in $\Omega \times (0,\infty)$, which in view of (1.5) yields (2.13). ////

2.1   Absence of finite-time gradient blow-up

Using the above boundary estimates, upon another comparison argument we obtain an (exponential) upper bound for the growth of the norm in $L^\infty(\Omega)$ of any maximal solution of (0.1). This will be a consequence of the following lemma.

Lemma 2.4 *Let $p \geq 2$ and $q \in [1, p-1]$. Then there exist $\theta > 1$ and $C > 0$ with the property that for any $T > 0$ one can pick $\varepsilon_0(T) \in (0,1)$ such that for any $\eta \in (0,1)$ and each $\varepsilon \in (0, \varepsilon_0(T))$, the solution of (1.2) fulfills*

$$\|\tilde{u}_{\eta\varepsilon x}(\cdot, t)\|_{L^\infty(\Omega)} \leq C\, e^{\theta t} \qquad \text{for all } t \in (0, T). \tag{2.15}$$

PROOF.   Let us first assume that $u_0 \in C^2(\bar{\Omega})$. Then standard parabolic regularity theory ([13]) ensures that $\tilde{u}_{\eta\varepsilon}$ belongs to $C_{loc}^{1+\gamma, \frac{1+\gamma}{2}}(\bar{\Omega} \times [0,\infty)) \cap C^\infty(\bar{\Omega} \times (0,\infty))$ for some $\gamma \in (0,1)$, which in particular implies that $v := \tilde{u}_{\eta\varepsilon x}$ lies in $C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\Omega \times (0,\infty))$ for each fixed η and ε. Differentiating (1.2) with respect to x, we see that thus v is a classical solution of

$$v_t = (\tilde{u}_{\eta\varepsilon} + \varepsilon)^p v_{xx} + p(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1} v v_x + q(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^{q-1} v \qquad \text{in } \Omega \times (0,\infty)$$

with initial data $v_0 := v(\cdot, 0)$ fulfilling $\|v_0\|_{L^\infty(\Omega)} \leq c_1 := \sup_{\eta \in (0,1)} \|\tilde{u}_{0\eta x}\|_{L^\infty(\Omega)}$. Now in view of (1.8) we can pick $c_2 > 0$ independent of η and ε such that $\tilde{u}_{\eta\varepsilon} \leq c_2$ in $\Omega \times (0,\infty)$, and fix any $\theta > 1$ fulfilling $\theta \geq q \cdot (c_2+1)^{q-1}$. Then Lemma 2.2 and Lemma 2.3 imply that there exists $c_3 \geq c_1$ with the property that given $T > 0$ one can find $\varepsilon_0(T) \in (0,1)$ such that whenever $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_0(T))$, we have $|v(x,t)| \leq c_3 e^{\theta t}$ for all $t \in (0,T)$ and each $x \in \partial\Omega$. Since $c_3 \geq c_1$, this entails that the spatially homogeneous function $V(x,t) := c_3 e^{\theta t}$, $(x,t) \in \bar{\Omega} \times [0,\infty)$, satisfies $V \geq |v|$ on the parabolic boundary of $\Omega \times (0,T)$ for any such η and ε. Since

$$V_t - (\tilde{u}_{\eta\varepsilon} + \varepsilon)^p V_{xx} \pm p(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1} V V_x - q(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{q-1}V = \left\{ \theta - q(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^{q-1} \right\} \cdot c_3 e^{\theta t}$$
$$\geq \left\{ \theta - q(c_2+1)^{q-1} \right\} \cdot c_3 e^{\theta t} \geq 0$$

in $\Omega \times (0,\infty)$, twice applying the comparison principle we conclude that $|v| \leq V$ in $\Omega \times (0,T)$, which yields (2.15) in the case $u_0 \in C^2(\bar{\Omega})$.

If merely $u_0 \in W^{1,\infty}(\Omega)$, we can choose a sequence $(u_0^{(j)})_{j \in \mathbb{N}} \subset C^2(\bar{\Omega})$ of nonnegative functions vanishing on $\partial\Omega$

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such that $u_0^{(j)} \stackrel{\star}{\rightharpoonup} u_0$ in $W^{1,\infty}(\Omega)$ and apply the above result to the corresponding solutions $\tilde{u}_{\eta\varepsilon}^{(j)}$. Since $\tilde{u}_{\eta\varepsilon}^{(j)} \to \tilde{u}_{\eta\varepsilon}$ in $C^{2,1}_{loc}(\Omega \times (0,\infty))$ by a continuous dependence argument applied to the non-degenerate problem (1.2), (2.15) easily follows from the fact that the above constant c_3 can be chosen independently of j. ////

On letting $\varepsilon \searrow 0$ and then $\eta \searrow 0$ in (2.15), we can now without further comment state our main result concerning the impossibility of finite-time gradient blow-up of any maximal solution of (0.1).

Theorem 2.5 *Let $p \geq 2$ and $q \in [1, p-1]$, and suppose that u_0 satisfies (0.2). Then the maximal solution u of (0.1) satisfies $u(\cdot, t) \in W^{1,\infty}(\Omega)$ for all $t \geq 0$, and there exist $\theta > 1$ and $C > 0$ such that*

$$\|u_x(\cdot, t)\|_{L^\infty(\Omega)} \leq C\, e^{\theta t} \qquad \text{for all } t \geq 0.$$

3   Gradient blow-up in infinite time

We proceed to identify some norms involving u_x with respect to which our solutions of (0.1) are unbounded as $t \to \infty$. To this end, we first make sure that trajectories approach steady states of (0.1) in an appropriate sense. We observe that the natural energy $E(\varphi) := \frac{1}{2}\int_\Omega \varphi_x^2 + \frac{1}{p-q-1}\int_\Omega \varphi^{-(p-q-1)}$ associated with (0.1) may be unbounded throughout the evolution. In fact, this is true whenever $q \leq p-2$, or if u_0 is such that $\{u_0 = 0\}$ has nonempty interior, for instance, where we note that in the latter case the set $\{u(\cdot, t)\}$ will have nonempty interior as well ([6], [23]). Accordingly, our approach has to utilize more subtle arguments, relying on Lemma 1.6 on the one hand, and comparison from below with certain time monotone solutions on the other.

Lemma 3.1 *Let $p \geq 2$ and $q \in [1, p-1]$, and suppose that there exist $w \in C^0(\bar{\Omega})$ and a sequence of times $0 < t_j \to \infty$ such that the maximal solution u of (0.1) satisfies*

$$u(\cdot, t_j) \to w \quad \text{in } C^0(\bar{\Omega}) \qquad \text{as } j \to \infty. \tag{3.1}$$

Then $\{w > 0\}$ is nonvoid, and for each connected component $G = (a, b)$ of $\{w > 0\}$ we have the identity

$$w \equiv w_{a,b,p-q} \qquad \text{in } G, \tag{3.2}$$

where $w_{a,b,p-q}$ denotes the solution of (1.4) in G corresponding to $\beta = p - q$.

PROOF.   First, by using appropriate stationary subsolutions of (1.2) as given by Lemma 1.3 it can easily be checked that $\{w > 0\}$ contains the positivity set $\{u_0 > 0\}$ of u_0 and hence is not empty. The proof of (3.2) will be carried out in four steps.

<u>Step 1.</u>   Let us first make sure that for all subintervals $G' = (a', b') \subset\subset G$ there exist $j_0 \in \mathbb{N}, \varepsilon_0 \in (0, 1)$ and $c_{G'} > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the classical solution v_ε of

$$\begin{cases} v_{\varepsilon t} = (v_\varepsilon + \varepsilon)^p v_{\varepsilon xx} + (v_\varepsilon + \varepsilon^\kappa)^q & \text{in } G' \times (t_{j_0}, \infty), \\ v_\varepsilon|_{\partial G'} = 0, \\ v_\varepsilon|_{t=t_{j_0}} = v_0, \end{cases} \tag{3.3}$$

with

$$v_0(x) := c_{G'} \cdot \sin \frac{\pi(x - a')}{b' - a'}, \qquad x \in \bar{G}', \tag{3.4}$$

satisfies $v_{\varepsilon t} \geq 0$ in $G' \times (t_{j_0}, \infty)$ as well as

$$\tilde{u}_{\eta\varepsilon} \geq v_\varepsilon \qquad \text{in } G' \times (t_{j_0}, \infty) \tag{3.5}$$

whenever $\eta \in (0, 1)$.

In fact, given any such G', from (3.1) we obtain that since $\inf_{x \in G'} w(x) > 0$ by our assumptions on G and G', we must have $\inf_{x \in G'} u(x, t_{j_0}) > 0$ for some sufficiently large $j_0 \in \mathbb{N}$. Using that $(\tilde{u}_{\eta\varepsilon}(\cdot, t_{j_0}))_{\eta \in (0,1), \varepsilon \in (0,\varepsilon_0)}$ is relatively compact in $C^0(\bar{G}')$ for some $\varepsilon_0 \in (0, 1)$ by Lemma 2.4, we can thus find $c_2 > 0$ such that

$$\tilde{u}_{\eta\varepsilon}(x, t_{j_0}) \geq c_2 \qquad \text{for all } x \in G' \text{ and each } \eta \in (0, 1) \text{ and } \varepsilon \in (0, \varepsilon_0).$$

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We now apply Lemma 1.3 to see that there exists $c_{G'} \in (0, c_2)$ such that v_0 as defined by (3.4) fulfills $(v_0 + \varepsilon)^p v_{0xx} + (v_0 + \varepsilon^\kappa)^q \geq 0$ in G'. By a standard reasoning (cf. [17, Chapter 52], for instance), this guarantees that the solution v_ε of (3.3) indeed satisfies $v_{\varepsilon t} \geq 0$ in $G' \times (t_{j_0}, \infty)$, whereas (3.5) is a consequence of the comparison pronciple and the fact that $\tilde{u}_{\eta\varepsilon}(\cdot, t_{j_0}) \geq v_0$ in G'.

<u>Step 2.</u> We next assert that for any open subinterval $G' \subset\subset G$ we can find $j_1 \in \mathbb{N}$ and $\varepsilon_1 \in (0, 1)$ such that

$$(\tilde{u}_{\eta\varepsilon})_{\eta \in (0,1), \varepsilon \in (0,\varepsilon_1)} \quad \text{is relatively compact in } C^{2,1}_{loc}(\bar{G}' \times [t_{j_1}, \infty)). \tag{3.6}$$

Indeed, choosing another open interval $G'' = (a'', b'') \subset \mathbb{R}$ such that $\bar{G}' \subset G'' \subset \bar{G}'' \subset G$, from Step 1 we particularly infer the existence of $j_1 \in \mathbb{N}, \varepsilon_1 \in (0, 1)$ and $c_3 > 0$ fulfilling

$$\tilde{u}_{\eta\varepsilon}(x, t) \geq c_3 \cdot \sin \frac{\pi(x - a'')}{b'' - a''} \qquad \text{for all } (x, t) \in G'' \times (t_{j_1}, \infty), \eta \in (0, 1) \text{ and } \varepsilon \in (0, \varepsilon_1).$$

Taking into account the upper bound provided by (1.8), after diminishing ε_1 if necessary we obtain $c_4 > 0$ and $c_5 > 0$ such that

$$c_4 \leq \tilde{u}_{\eta\varepsilon} \leq c_5 \qquad \text{in } G' \times (t_{j_1}, \infty)$$

for all $\eta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_1)$. According to parabolic Schauder estimates ([13]) and the Arzelá-Ascoli theorem, this entails (3.6).

<u>Step 3.</u> We proceed to show that
$$w \geq w_{a,b,p-q} \qquad \text{in } G. \tag{3.7}$$
In view of Lemma 1.1, it is sufficient for this to prove that for all $G' = (a', b') \subset\subset G$ we have
$$w \geq w' := w_{a',b',p-q} \qquad \text{in G'}. \tag{3.8}$$

For this purpose, we fix any such G' and take $j_0 \in \mathbb{N}, \varepsilon_0 \in (0, 1)$ and $v_\varepsilon, \varepsilon \in (0, \varepsilon_0)$, as provided by Step 1. The since $v_0 \leq c_6 \cdot w'$ in G' for some $c_6 \geq 1$ by Lemma 1.1, we conclude using Lemma 1.2, Lemma 1.1 and the comparison principle that there exists $c_7 > 0$ such that

$$v_0(x) \leq v_\varepsilon(x, t) \leq c_6 w'(x) \leq c_7 (\text{dist}(x, \partial G'))^{\frac{2}{p+1-q}} \qquad \text{for all } (x, t) \in G' \times (t_{j_0}, \infty) \tag{3.9}$$

whenever $\varepsilon \in (0, \varepsilon_0)$. Therefore a standard limit procedure (cf. [20] for details) shows that as $\varepsilon \searrow 0$, v_ε converges to the unique positive classical solution v of

$$\begin{cases} v_t = v^p v_{xx} + v^q & \text{in } G' \times (t_{j_0}, \infty), \\ v|_{\partial G'} = 0, \\ v|_{t=t_{j_0}} = v_0, \end{cases} \tag{3.10}$$

in $C^0_{loc}(\bar{G}' \times [t_{j_0}, \infty)) \cap C^{2,1}_{loc}(G' \times (t_{j_0}, \infty))$. Clearly, $v_t \geq 0$ in $G' \times (t_{j_0}, \infty)$, and hence in view of (3.9) and parabolic Schauder theory, $W(x) := \lim_{t\to\infty} v(x, t)$ defines a function $W \in C^0(\bar{G}') \cap C^2(G')$ which clearly must be a positive steady state of (3.10). According to the uniqueness statement in Lemma 1.1, W thus must coincide with w'. This entails (3.8), because taking $\varepsilon \searrow 0$ and then $\eta \searrow 0$ in (3.5) ensures that $u \geq v$ in $G' \times (t_{j_0}, \infty)$, so that in particular $w = \lim_{j\to\infty} u(\cdot, t_j) \geq \lim_{j\to\infty} v(\cdot, t_j) = W = w'$.

<u>Step 4.</u> We complete the proof of (3.2) by showing that
$$w \leq w_{a,b,p-q} \qquad \text{in G.} \tag{3.11}$$

To this end, we again fix an open interval $G' \subset\subset G$ and recall Lemma 1.6 which asserts that

$$-(\tilde{u}_{\eta\varepsilon} + \varepsilon)^p \tilde{u}_{\eta\varepsilon xx} \leq (\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^q + z_\varepsilon(t) \cdot (\tilde{u}_{\eta\varepsilon} + \varepsilon) \qquad \text{in } G'$$

for all $\eta \in (0, 1), \varepsilon \in (0, 1)$ and $t > 0$, with z_ε as defined by (1.10). Using the compactness property (3.6) and the fact that $\lim_{\varepsilon \searrow 0} z_\varepsilon(t) = \frac{1}{pt} \to 0$ as $t \to \infty$, we may let $\varepsilon \searrow 0$, then $\eta \searrow 0$ and finally $t = t_j \to \infty$ here to obtain that $w \in C^2(\bar{G}')$ and $-w^p w_{xx} \leq w^q$ in G'. Since $G' \subset\subset G$ was arbitrary and $w > 0$ in G, this means that $-w_{xx} - w^{q-p} \leq 0$ in G, which in view of the fact that $w|_{\partial G} = 0$ implies (3.11) due to the elliptic comparison principle stated in Lemma 1.1. ////

As a first consequence we state that u cannot remain bounded in $W^{1,\infty}(\Omega)$.

Lemma 3.2 *Let $p \geq 2$ and $q \in [1, p-1]$. Then for the maximal solution u of (0.1) we have*

$$\|u_x(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \qquad as\ t \to \infty. \tag{3.12}$$

PROOF.   If (3.12) was false, we could pick a sequence of numbers $t_j \to \infty$ and a function $w \in W^{1,\infty}(\Omega)$ such that $u(\cdot, t_j) \xrightarrow{\star} w$ in $W^{1,\infty}(\Omega)$ as $j \to \infty$. In accordance to Lemma 3.1, there exists a subinterval $G = (a,b)$ of Ω such that w coincides with the solution $w_{a,b,p-q}$ of (1.4) in G. However, Lemma 1.1 entails that since $p - q \geq 1$, the function $w_{a,b,p-q}$ does not belong to $W^{1,\infty}(G)$. This contradiction shows that actually (3.12) must hold. ////

In the case $q < p-1$ we can go even further and assert blow-up of certain weaker norms of u_x.

Lemma 3.3 *Let $p > 2$ and $q \in [1, p-1)$, and assume that $\alpha \geq 0$ and $m > 1$ are such that*

$$m \geq \frac{p+1-q+2\alpha}{p-q-1}. \tag{3.13}$$

Then the maximal solution u of (0.1) satisfies

$$\int_\Omega u^\alpha(x,t)|u_x(x,t)|^m dx \to \infty \qquad as\ t \to \infty. \tag{3.14}$$

PROOF.   Assuming on the contrary that (3.14) be false, since $W^{1,m}(\Omega)$ is reflexive and compactly embedded into $C^0(\bar{\Omega})$, we could find a sequence of times $t_j \to \infty$ along which

$$u(\cdot, t_j) \to w \quad \text{in } C^0(\bar{\Omega}) \qquad \text{and} \qquad u^{\frac{m+\alpha}{m}}(\cdot, t_j) \rightharpoonup w^{\frac{m+\alpha}{m}} \quad \text{in } W^{1,m}(\Omega) \tag{3.15}$$

would hold for some nonnegative $w \in C^0(\bar{\Omega})$ vanishing on $\partial\Omega$. By Lemma 3.1 we can fix an interval $G = (a,b) \subset \Omega$ such that $w \equiv w_{a,b,p-q}$ in G, where $w_{a,b,p-q}$ denotes the solution of (1.4) corresponding to $\beta = p - q$. Invoking Lemma 1.1 we can thus find $\delta > 0$ and $c_1 > 0$ such that

$$w(x) \geq c_1(x-a)^{\frac{1}{p+1-q}} \quad \text{and} \quad w_x(x) \geq c_1(x-a)^{\frac{2}{p+1-q}-1} \qquad \text{for all } x \in (a, a+\delta).$$

Therefore,

$$\begin{aligned}
\left\| (w^{\frac{m+\alpha}{m}})_x \right\|_{L^m(\Omega)}^m &\geq \left(\frac{m+\alpha}{m}\right)^m \cdot \int_a^{a+\delta} w^\alpha(x) w_x^m(x) dx \\
&\geq c_1^{m+\alpha} \cdot \left(\frac{m+\alpha}{m}\right)^m \cdot \int_a^{a+\delta} (x-a)^{\frac{2\alpha}{p+1-q}+(\frac{2}{p+1-q}-1)\cdot m} dx.
\end{aligned}$$

According to our assumptions on α and m, this means that $w^{\frac{m+\alpha}{m}}$ cannot be an element of $W^{1,m}(\Omega)$, which contradicts (3.15) and thereby completes the proof. ////

4   Algebraic upper bounds for the blow-up rate

4.1   Integral bounds for solutions with arbitrary initial data

Combining Theorem 2.5 with the results from Lemma 3.2 and Lemma 3.3, we obtain that whenever $1 \leq q \leq p-1$, the maximal solution of (0.1) undergoes a gradient blow-up which occurs in infinite time and at a rate no faster than exponential. We proceed to derive some upper bounds on u_x which indicate that if $q > 1$ then this rate in fact is at most algebraic. We first consider estimates for u_x in $L^m(\Omega)$ for *finite* m, possibly involving powers of u as weight functions.

Lemma 4.1 *Assume that $p \geq 2$ and $q \in [1, p-1]$, and let $\alpha \geq 0$ and $m > 2$ be such that $\alpha \geq (\frac{p}{2} - q)m$. Then there exist $C > 0$ and $\varepsilon_0 \in (0,1)$ such that for all $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_0)$ the solution of (1.2) satisfies*

$$\left(\int_\Omega \tilde{u}_{\eta\varepsilon}^\alpha |\tilde{u}_{\eta\varepsilon x}|^m \right)^{\frac{2}{m}} \leq C \left(\frac{e^{B\varepsilon^{q-1}t} - 1}{e^{B\varepsilon^{q-1}} - 1} \right)^{\frac{2(m-1)\alpha}{mp}} + C \int_1^t \left(\frac{e^{B\varepsilon^{q-1}t} - 1}{e^{B\varepsilon^{q-1}s} - 1} \right)^{\frac{2(m-1)\alpha}{mp}} ds \tag{4.1}$$

for all $t \geq 1$, where $B > 0$ is as defined in Lemma 1.6.

PROOF.   Writing u instead of $\tilde{u}_{\eta\varepsilon}$ for convenience, we know from parabolic regularity theory that u is smooth in $\bar{\Omega} \times (0, \infty)$ and satisfies $u_t = 0$ on $\partial\Omega \times (0, \infty)$. Hence, integrating by parts over Ω we compute

$$\frac{d}{dt} \int_\Omega (u + \varepsilon)^\alpha |u_x|^m = -m(m-1) \int_\Omega (u + \varepsilon)^{p+\alpha} |u_x|^{m-2} u_{xx}^2 - m(m-1) \int_\Omega (u + \varepsilon)^\alpha (u + \varepsilon^\kappa)^q |u_x|^{m-2} u_{xx}$$

$$- (m-1)\alpha \int_\Omega (u + \varepsilon)^{p+\alpha-1} |u_x|^m u_{xx} - (m-1)\alpha \int_\Omega (u + \varepsilon)^{\alpha-1} (u + \varepsilon^\kappa)^q |u_x|^m$$

$$=: I_1 + I_2 + I_3 + I_4 \qquad \text{for all } t > 0. \tag{4.2}$$

Here, Lemma 1.6 says that with z_ε as given by (1.10) we have the one-sided estimate

$$-u_{xx} \leq (u + \varepsilon)^{1-p} z_\varepsilon(t) + (u + \varepsilon)^{-p}(u + \varepsilon^\kappa)^q \qquad \text{in } \Omega \times (0, \infty),$$

so that since $\alpha \geq 0$ and $m > 2$,

$$I_3 \leq (m-1)\alpha \cdot z_\varepsilon(t) \cdot \int_\Omega (u + \varepsilon)^\alpha |u_x|^m + (m-1)\alpha \int_\Omega (u + \varepsilon)^{\alpha-1}(u + \varepsilon^\kappa)^q |u_x|^m. \tag{4.3}$$

Moreover, applying Young's inequality and the Hölder inequality we find

$$|I_2| \leq m(m-1) \int_\Omega (u + \varepsilon)^{p+\alpha} |u_x|^{m-2} u_{xx}^2 + \frac{m(m-1)}{4} \int_\Omega (u + \varepsilon)^{-p+\alpha}(u + \varepsilon^\kappa)^{2q} |u_x|^{m-2}$$

$$\leq m(m-1) \int_\Omega (u + \varepsilon)^{p+\alpha} |u_x|^{m-2} u_{xx}^2$$

$$+ \frac{m(m-1)}{4} \left(\int_\Omega (u + \varepsilon)^\alpha |u_x|^m \right)^{\frac{m-2}{m}} \cdot \left(\int_\Omega (u + \varepsilon)^{-\frac{mp}{2} + \alpha} \cdot (u + \varepsilon^\kappa)^{mq} \right)^{\frac{2}{m}} \tag{4.4}$$

for all $t > 0$. Since Lemma 1.4 ensures that $u \leq c_1$ in $\Omega \times (0, \infty)$ with some $c_1 > 0$ independent of η and ε, and since $\kappa > 1$, our restriction on α allows us to estimate

$$\frac{m(m-1)}{4} \left(\int_\Omega (u + \varepsilon)^{-\frac{mp}{2} + \alpha} \cdot (u + \varepsilon^\kappa)^{mq} \right)^{\frac{2}{m}} \leq \frac{m(m-1)}{4} \left(\int_\Omega (u + \varepsilon)^{-\frac{mp}{2} + \alpha + mq} \right)^{\frac{2}{m}}$$

$$\leq c_2 := \frac{m(m-1)}{4} \left(|\Omega| \cdot (c_1 + 1)^{-\frac{mp}{2} + \alpha + mq} \right)^{\frac{2}{m}}$$

for all $t > 0$. Inserted into (4.4), combined with (4.3) and (4.2) this implies that the function y defined by $y(t) := \int_\Omega (u + \varepsilon)^\alpha |u_x|^m$, $t \geq 1$, satisfies

$$y' \leq c_2 y^{\frac{m-2}{m}} + (m-1)\alpha z_\varepsilon(t) \cdot y \qquad \text{for all } t > 0.$$

A straightforward integration of this Bernoulli-type ODI leads to the estimate

$$y^{\frac{2}{m}}(t) \leq y^{\frac{2}{m}}(1) \cdot e^{\frac{2(m-1)\alpha}{m} \cdot Z_\varepsilon(t)} + \frac{2c_2}{m} \cdot \int_1^t e^{\frac{2(m-1)\alpha}{m} \cdot (Z_\varepsilon(t) - Z_\varepsilon(s))} ds \qquad \text{for all } t \geq 1 \tag{4.5}$$

with $Z_\varepsilon(t) := \int_1^t z_\varepsilon(s) ds$, $t \geq 1$. Recalling the definition (1.10) of z_ε, we can explicitly compute Z_ε to obtain

$$Z_\varepsilon(t) = \frac{B\varepsilon^{q-1}}{p} \cdot \int_1^t \frac{ds}{1 - e^{-B\varepsilon^{q-1}t}} = \frac{1}{p} \cdot \ln \frac{e^{B\varepsilon^{q-1}t} - 1}{e^{B\varepsilon^{q-1}} - 1}.$$

Thus, from (4.5) we can can easily derive (4.1) thanks to the fact that $\sup_{\eta \in (0,1), \varepsilon \in (0, \varepsilon_0)} \|\tilde{u}_{\eta\varepsilon x}(\cdot, 1)\|_{L^\infty(\Omega)}$ is finite for some $\varepsilon_0 \in (0,1)$ by Lemma 2.4. ////

After passing to the degenerate limit, for $q > 1$ this yields the following.

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Corollary 4.2 *Let $p > 2$ and $q \in (1, p-1]$, and suppose that $\alpha \geq 0$ and $m > 2$ are such that $\alpha \geq (\frac{p}{2} - q)m$. Then there exists $C > 0$ such that for each $t \geq 2$, the maximal solution u of (0.1) fulfills*

$$\left(\int_\Omega u^\alpha |u_x|^m \right)^{\frac{1}{m}} \leq \begin{cases} C \cdot t^{\frac{1}{2}} & \text{if } \alpha < \frac{mp}{2(m-1)}, \\ C \cdot (t \ln t)^{\frac{1}{2}} & \text{if } \alpha = \frac{mp}{2(m-1)}, \\ C \cdot t^{\frac{(m-1)\alpha}{mp}} & \text{if } \alpha > \frac{mp}{2(m-1)}. \end{cases} \qquad (4.6)$$

PROOF. We fix $t \geq 2$ and let $\varepsilon \searrow 0$ in (4.1), which in view of the fact that $q > 1$ is equivalent to letting $\delta := B\varepsilon^{q-1}$ tend to zero. To justify the limit process on the right of (4.1), we observe that for each $s \in [1, t]$ we have

$$\frac{e^{\delta t} - 1}{e^{\delta s} - 1} \to \frac{t}{s} \qquad \text{as } \delta \to 0 \qquad (4.7)$$

by l'Hospital's rule. Since t is fixed, there exist positive constants c_1 and c_2 such that $e^{\delta t} - 1 \leq c_1 \delta t$ and $e^{\delta s} - 1 \geq c_2 \delta$ for all $s \in [1, t]$, which shows that $1 \leq \frac{e^{\delta t} - 1}{e^{\delta s} - 1} \leq \frac{c_1 t}{c_2}$ for all such s. Therefore the dominated convergence theorem may be applied along with (4.7) and Fatou's lemma to assert that after taking $\varepsilon \searrow 0$ and then $\eta \searrow 0$, (4.1) gives

$$\left(\int_\Omega u^\alpha |u_x|^m \right)^{\frac{2}{m}} \leq C \cdot \left(t^{\frac{2(m-1)\alpha}{mp}} + \int_1^t \left(\frac{t}{s} \right)^{\frac{2(m-1)\alpha}{mp}} ds \right) \qquad \text{for all } t \geq 2$$

with $C > 0$ taken from Lemma 4.1. This easily yields (4.6). ////

Choosing the smallest possible α and hence the largest possible weight in (4.6), recalling Lemma 3.3 we can summarize as follows.

Corollary 4.3 *Let $p > 2$ and $q \in (1, p-1)$. Then as $1 \leq t \to \infty$, for the proper solution u of (0.1) we have*

$$\left.\begin{array}{rll} Ct^{\frac{1}{2}} & \geq & \|u_x(\cdot, t)\|_{L^m(\Omega)} \to \infty \quad \text{for all } m > \max\{2, \frac{p+1-q}{p-q-1}\} \qquad \text{if } \frac{p}{2} \leq q < p-1, \\ Ct^{\frac{(p-2q)(m-1)}{2p}} & \geq & \|(u^{\frac{p+2-2q}{2}})_x\|_{L^m(\Omega)} \to \infty \quad \text{for all } m > \max\{2, \frac{p+1-q}{q-1}, \frac{2(p-q)}{p-2q}\} \qquad \text{if } 1 < q < \frac{p}{2}, \end{array}\right\} \qquad (4.8)$$

with some $C > 0$ depending on m only.

PROOF. If $q \geq \frac{p}{2}$, we choose $\alpha := 0$ in Corollary 4.2, whence the first inequality in (4.8) follows from (4.6). On the other hand, upon this choice of α, Lemma 3.3 says that $\int_\Omega |u_x(\cdot, t)|^m \to \infty$ as $t \to \infty$ whenever $m \geq \frac{p+1-q}{p-q-1}$. The proof in the case $q < \frac{p}{2}$ is similar. ////

4.2 An estimate in $W^{1,\infty}(\Omega)$ in the case $\inf_{x \in \Omega} \frac{u_0(x)}{\text{dist}(x, \partial\Omega)} > 0$

According to the possible loss of regularity due to the degeneracy in (0.1), pointwise estimates for the derivative u_x cannot be derived in a trivial way from estimates for u_x on the lateral boundary. In fact, in [25] it was shown that even some positive classical solutions exist which have the peculiar property that $u_x \equiv 0$ on $\partial\Omega \times (0, \infty)$. Therefore, in order to turn the boundary estimate in Lemma 2.2 into its natural counterpart concerning $\|u_x(\cdot, t)\|_{L^\infty(\Omega)}$, we shall need an additional assumption on the initial data. Requiring u_0 to be bounded from below by a positive multiple of $\text{dist}(\cdot, \partial\Omega)$, this will provide sufficient control the degeneracy near points where u is small.

Lemma 4.4 *Let $p > 2$ and $q \in (1, p-1]$, and suppose that there exists $c > 0$ such that*

$$u_0(x) \geq c\, \text{dist}(x, \partial\Omega) \qquad \text{for all } x \in \Omega. \qquad (4.9)$$

Then for any $\nu > 0$ there exists $C(\nu) > 0$ with the property that for all $T > 0$ there is $\varepsilon_0(T) \in (0, 1)$ such that for each $\eta \in (0, 1)$ and $\varepsilon \in (0, \varepsilon_0(T))$ the solution $\tilde{u}_{\eta\varepsilon}$ of (1.2) satisfies

$$\|\tilde{u}_{\eta\varepsilon x}(\cdot, t)\|_{L^\infty(\Omega)} \leq C(\nu) \cdot (t+1)^{\frac{p-q-1}{2(q-1)} + \nu} \qquad \text{for all } t \in (0, T). \qquad (4.10)$$

PROOF.   We detail the proof for $q < p-1$ only; upon slight modifications, the borderline case $q = p-1$ can be treated in quite the same manner.

We first observe that due to Lemma 2.4 there exist $c_1 > 0$ and $\varepsilon_1 \in (0,1)$ such that whenever $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_1)$,

$$\|\tilde{u}_{\eta \varepsilon x}(\cdot, t)\|_{L^\infty(\Omega)} \leq c_1 \qquad \text{for all } t \in [0,1], \tag{4.11}$$

whence it is sufficient to prove (4.10) for $1 < t < T < \infty$. To this end, we fix $\nu > 0$ and apply Lemma 2.2 to obtain $c_2 \geq 1$ such that for all $T > 1$ we can find $\varepsilon_2(T) \in (0, \varepsilon_1)$ such that

$$\tilde{u}_{\eta \varepsilon}(x, t) \leq c_2 t^{\frac{p-q-1}{2(q-1)} + \frac{\nu}{2}} \cdot x \qquad \text{for all } (x,t) \in \Omega \times (1, T), \tag{4.12}$$

whence in particular

$$\tilde{u}_{\eta \varepsilon x}(0, t) \leq c_2 t^{\frac{p-q-1}{2(q-1)} + \frac{\nu}{2}} \qquad \text{for all } t \in (1, T) \tag{4.13}$$

whenever $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_2(T))$. Moreover, in view of our assumption (4.9), (1.8) provides $c_3 \in (0,1)$, $c_4 > 0$ and $\varepsilon_3 \in (0, \varepsilon_2(T))$ such that for $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_3)$,

$$c_3 \operatorname{dist}(x, \partial\Omega) \leq \tilde{u}_{\eta \varepsilon}(x, t) \leq c_4 \qquad \text{for all } (x,t) \in \Omega \times (0, \infty). \tag{4.14}$$

Therefore interior parabolic regularity theory implies the existence of $c_5 > 0$ fulfilling

$$\left| \tilde{u}_{\eta \varepsilon x}\left(\frac{L}{2}, t \right) \right| \leq c_5 \qquad \text{for all } t \geq 1 \tag{4.15}$$

for $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_3)$.

We now fix $\gamma \in (0,1)$ so small that $\gamma < \frac{p-2}{2}$,

$$\left(\frac{p-q-1}{2(q-1)} + \frac{\nu}{2} \right) \cdot (\gamma + 1) \leq \frac{p-q-1}{2(q-1)} + \nu \qquad \text{and} \qquad \left(\frac{p-q-1}{2(q-1)} + \frac{\nu}{2} \right) \cdot 2\gamma \leq 2\nu, \tag{4.16}$$

and then a constant $y_0 > 0$ satisfying

$$y_0 \geq c_1 c_2^\gamma, \tag{4.17}$$

$$y_0 \geq 4 c_2^{\gamma+1}, \tag{4.18}$$

$$y_0 \geq c_5 \cdot \left(\frac{c_4 + 1}{\frac{L}{2}} \right)^\gamma \tag{4.19}$$

as well as

$$y_0 \geq \left(\frac{4(q+\gamma)c_2^{2\gamma}}{p\gamma} \right)^{\frac{1}{2}} \cdot \left(\frac{2(q-1)(q+\gamma)}{p-q-1} \right)^{\frac{p-q-1}{2(q-1)}} \qquad \text{and} \tag{4.20}$$

$$y_0 \geq \left(\frac{(q+\gamma)c_2^{2\gamma}}{\gamma(p-\gamma-1)} \right)^{\frac{1}{2}} \cdot \left(\frac{2(q-1)(q+\gamma)}{p-q-1} \right)^{\frac{p-q-1}{2(q-1)}}. \tag{4.21}$$

Following [10], for $T > 1$, $\eta \in (0,1)$ and $\varepsilon \in (0, \varepsilon_3)$ we introduce the auxiliary function

$$J(x, t) := \tilde{u}_{\eta \varepsilon x}(x, t) - y(t) \psi(x) f(\tilde{u}_{\eta \varepsilon}(x, t)), \qquad (x, t) \in \left[0, \frac{L}{2} \right] \times [1, T]$$

with

$$y(t) := y_0 t^{\frac{p-q-1}{2(q-1)} + \nu}, \qquad t \geq 1,$$

$$\psi(x) := (x + \varepsilon)^\gamma, \qquad x \in \left[0, \frac{L}{2} \right], \qquad \text{and}$$

$$f(u) := (u - \varepsilon)^{-\gamma}, \qquad u \geq 0.$$

Then at $t = 1$,

$$
\begin{aligned}
J(x, 1) &= \tilde{u}_{\eta \varepsilon x}(x, 1) - y_0 (x + \varepsilon)^\gamma (\tilde{u}_{\eta \varepsilon}(x, 1) + \varepsilon)^{-\gamma} \\
&\leq c_1 - y_0 c_2^{-\gamma} \\
&\leq 0 \qquad \text{for all } x \in \left(0, \frac{L}{2} \right) \tag{4.22}
\end{aligned}
$$

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due to (4.11), (4.12), (4.17) and the fact that $c_2 \geq 1$. On the left lateral boundary $x = 0$,

$$
\begin{aligned}
J(0,t) &= \tilde{u}_{\eta\varepsilon x}(0,t) - y(t) \\
&\leq c_2 t^{\frac{p-q-1}{2(q-1)} + \frac{\nu}{2}} - y_0 t^{\frac{p-q-1}{2(q-1)} + \nu} \\
&\leq 0 \qquad \text{for all } t \in (1, T)
\end{aligned}
\tag{4.23}
$$

because of (4.13), (4.18) and, again, our assumption $c_2 \geq 1$. For $x = \frac{L}{2}$ we have

$$
\begin{aligned}
J\left(\frac{L}{2}, t\right) &= \tilde{u}_{\eta\varepsilon x}\left(\frac{L}{2}, t\right) - y(t) \cdot \left(\frac{L}{2} + \varepsilon\right)^\gamma \cdot \left(\tilde{u}_{\eta\varepsilon}\left(\frac{L}{2}, t\right) + \varepsilon\right)^{-\gamma} \\
&\leq c_5 - y_0 \cdot \left(\frac{L}{2}\right)^\gamma \cdot (c_4 + 1)^{-\gamma} \\
&\leq 0 \qquad \text{for all } t \in (1, T)
\end{aligned}
\tag{4.24}
$$

in view of (4.15), (4.14) and (4.19).

Next, using the identities

$$
\begin{aligned}
J_x &= \tilde{u}_{\eta\varepsilon xx} - y\psi_x f - y\psi f' \tilde{u}_{\eta\varepsilon x} \qquad \text{and} \\
J_{xx} &= \tilde{u}_{\eta\varepsilon xxx} - y\psi_{xx} f - 2y\psi_x f' \tilde{u}_{\eta\varepsilon x} - y\psi f'' \tilde{u}_{\eta\varepsilon x}^2 - y\psi f' \tilde{u}_{\eta\varepsilon xx},
\end{aligned}
$$

we compute

$$
\begin{aligned}
J_t &= (\tilde{u}_{\eta\varepsilon} + \varepsilon)^p J_{xx} + p(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1} \cdot (J + y\psi f) \cdot J_x \\
&\quad + \Big\{ 2y\psi_x (\tilde{u}_{\eta\varepsilon} + \varepsilon)^p f' + y\psi(\tilde{u}_{\eta\varepsilon} + \varepsilon)^p f'' J + 2y\psi(\tilde{u}_{\eta\varepsilon} + \varepsilon)^p f + q(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^{q-1} \\
&\qquad + p(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1}(y\psi_x f + 2y^2\psi^2 ff' + y\psi f' J) \Big\} \cdot J \\
&\quad - y'\psi f + y\psi_{xx}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^p f + 2y^2\psi\psi_x(\tilde{u}_{\eta\varepsilon} + \varepsilon)^p ff' + py^2\psi\psi_x(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1} f^2 \\
&\quad + y^3\psi^3(\tilde{u}_{\eta\varepsilon} + \varepsilon)^p f^2 f'' + py^3\psi^3(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-1} f^2 f' \\
&\quad + qy\psi(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^{q-1} f - y\psi(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^q f'
\end{aligned}
$$

in $(0, \frac{L}{2}) \times (1, T)$. Thus,

$$
J_t = \mathcal{L}J + I(x, t) \qquad \text{in } \left(0, \frac{L}{2}\right) \times (1, T),
\tag{4.25}
$$

where $\mathcal{L}$ is a linear uniformly parabolic operator with smooth coefficients and

$$
\begin{aligned}
\frac{1}{y\psi} \cdot I &= -\frac{y'}{y}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{-\gamma} - \gamma(1 - \gamma)(x + \varepsilon)^{-2}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-\gamma} \\
&\quad + \gamma(p - 2\gamma)y(x + \varepsilon)^{\gamma - 1}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-2\gamma-1} - \gamma(p - \gamma - 1)y^2(x + \varepsilon)^{2\gamma}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-3\gamma-2} \\
&\quad + q(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^{q-1}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{-\gamma} + \gamma(\tilde{u}_{\eta\varepsilon} + \varepsilon^\kappa)^q(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{-\gamma-1} \\
&\leq -\frac{y'}{y}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{-\gamma} + p\gamma y(x + \varepsilon)^{\gamma - 1}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-2\gamma-1} - \frac{p\gamma}{2}y^2(x + \varepsilon)^{2\gamma}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{p-3\gamma-2} \\
&\quad + (q + \gamma)(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{q-\gamma-1} \\
&=: -I_1 + I_2 - I_3 + I_4.
\end{aligned}
\tag{4.26}
$$

Here we have used that $\gamma < 1$, that $\gamma < \frac{p-2}{2}$, and that $\varepsilon^\kappa \leq \varepsilon$ since $\kappa > 1$.

Now (4.12), (4.16) and (4.18) imply

$$
\frac{I_2}{\frac{1}{2}I_3} = \frac{4}{y} \cdot \left(\frac{\tilde{u}_{\eta\varepsilon} + \varepsilon}{x + \varepsilon}\right)^{\gamma+1} \leq \frac{4}{y_0 t^{\frac{p-q-1}{2(q-1)} + \nu}} \cdot c_2^{\gamma+1} \cdot t^{\left(\frac{p-q-1}{2(q-1)} + \frac{\nu}{2}\right) \cdot (\gamma+1)} \leq 1 \quad \text{in } \left(0, \frac{L}{2}\right) \times (1, T).
\tag{4.27}
$$

Moreover, thanks to (4.16) we know that

$$
\frac{I_4}{\frac{1}{2}I_3} = \frac{4(q + \gamma)}{p\gamma y^2}\left(\frac{\tilde{u}_{\eta\varepsilon} + \varepsilon}{x + \varepsilon}\right)^{2\gamma}(\tilde{u}_{\eta\varepsilon} + \varepsilon)^{-(p-q-1)}
$$

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$$\leq \quad \frac{4(q+\gamma)}{p\gamma y_0^2 \cdot t^{\frac{p-q-1}{q-1}+2\nu}} \cdot c_2^{2\gamma} \cdot t^{(\frac{p-q-1}{2(q-1)}+\frac{\nu}{2})\cdot 2\gamma}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-(p-q-1)}$$

$$\leq \quad \frac{4(q+\gamma)c_2^{2\gamma}}{p\gamma y_0^2} \cdot t^{-\frac{p-q-1}{q-1}} \cdot (\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-(p-q-1)}$$

$$\leq \quad 1 \tag{4.28}$$

holds at any point $(x,t) \in (0,\frac{L}{2}) \times (1,T)$ where

$$\tilde{u}_{\eta\varepsilon} + \varepsilon \geq \Big(\frac{4(q+\gamma)c_2^{2\gamma}}{p\gamma y_0^2}\Big)^{\frac{1}{p-q-1}} \cdot t^{-\frac{1}{q-1}}. \tag{4.29}$$

If the latter inequality is violated, however, then since $y' \geq \frac{p-q-1}{2(q-1)} \cdot \frac{y}{t}$ we obtain

$$\frac{I_4}{I_1} = \frac{(q+\gamma)(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{q-1}}{\frac{y'}{y}}$$

$$< \frac{(q+\gamma)\cdot\Big(\frac{4(q+\gamma)c_2^{2\gamma}}{p\gamma y_0^2}\Big)^{\frac{q-1}{p-q-1}}\cdot t^{-1}}{\frac{p-q-1}{2(q-1)}\cdot t^{-1}}$$

$$= \frac{2(q-1)(q+\gamma)}{p-q-1} \cdot \Big(\frac{4(q+\gamma)c_2^{2\gamma}}{p\gamma y_0^2}\Big)^{\frac{q-1}{p-q-1}}$$

$$\leq \quad 1 \tag{4.30}$$

due to (4.20). Combining (4.22)-(4.30), we conclude from the maximum principle that $J \leq 0$ in $(0,\frac{L}{2}) \times (1,T)$ and hence

$$\tilde{u}_{\eta\varepsilon x}(x,t) \quad \leq \quad y_0 t^{\frac{p-q-1}{2(q-1)}+\nu} \cdot \Big(\frac{x+\varepsilon}{\tilde{u}_{\eta\varepsilon}+\varepsilon}\Big)^{\gamma}$$

$$\leq \quad y_0 c_3^{-\gamma} t^{\frac{p-q-1}{2(q-1)}+\nu} \qquad \text{for all } (x,t) \in \Big(0,\frac{L}{2}\Big) \times (1,T) \tag{4.31}$$

for all $\eta \in (0,1)$ and $\varepsilon \in (0,\varepsilon_3)$ because of (4.14).

In order to estimate $\tilde{u}_{\eta\varepsilon x}$ from below, we proceed quite similarly, so that we may confine ourselves with an outline of the proof: We now define

$$\tilde{J}(x,t) := \tilde{u}_{\eta\varepsilon x}(x,t) + y(t)\psi(x)f(\tilde{u}_{\eta\varepsilon}(x,t)), \qquad (x,t) \in \Big[0,\frac{L}{2}\Big] \times [1,T],$$

with y,ψ and f as above. Then

$$\tilde{J}(x,1) \geq -c_1 + c_2^{-\gamma}y_0 \geq 0 \qquad \text{for all } x \in \Big(0,\frac{L}{2}\Big) \tag{4.32}$$

by (4.17). Moreover,

$$\tilde{J}(0,t) \geq \tilde{u}_{\eta\varepsilon x}(0,t) \geq 0 \qquad \text{for all } t \in (1,T), \tag{4.33}$$

since $\tilde{u}_{\eta\varepsilon} \geq 0$ and $\tilde{u}_{\eta\varepsilon}|_{\partial\Omega} = 0$, while (4.19) asserts that

$$\tilde{J}\Big(\frac{L}{2},t\Big) \geq -c_5 + y_0 \cdot \Big(\frac{L}{2}\Big)^{\gamma} \cdot (c_4+1)^{-\gamma} \geq 0 \qquad \text{for all } t \in (1,T). \tag{4.34}$$

Furthermore,

$$\tilde{J}_t = \widetilde{\mathcal{L}}\tilde{J} + \tilde{I}(x,t) \qquad \text{in } \Big(0,\frac{L}{2}\Big) \times (1,T) \tag{4.35}$$

with a smooth linear uniformly parabolic operator $\widetilde{\mathcal{L}}$ and

$$
\begin{aligned}
\frac{1}{y\psi} \cdot \tilde{I} \;=\;\; & \frac{y'}{y}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-\gamma} + \gamma(1-\gamma)(x+\varepsilon)^{-2}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{p-\gamma} \\
& +\gamma(p-2\gamma)y(x+\varepsilon)^{\gamma-1}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{p-2\gamma-1} + \gamma(p-\gamma-1)y^2(x+\varepsilon)^{2\gamma}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{p-3\gamma-2} \\
& -q(\tilde{u}_{\eta\varepsilon}+\varepsilon^{\kappa})^{q-1}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-\gamma} - \gamma(\tilde{u}_{\eta\varepsilon}+\varepsilon^{\kappa})^{q}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-\gamma-1} \\
\geq\;\; & \frac{y'}{y}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-\gamma} + \gamma(p-\gamma-1)y^2(x+\varepsilon)^{2\gamma}(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{p-3\gamma-2} - (q+\gamma)(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{q-\gamma-1} \\
=:\;\; & \tilde{I}_1 + \tilde{I}_2 - \tilde{I}_3. \tag{4.36}
\end{aligned}
$$

Here, if $(x,t) \in (0,\frac{L}{2}) \times (1,T)$ is such that

$$
\tilde{u}_{\eta\varepsilon}+\varepsilon \geq \left(\frac{(q+\gamma)c_2^{2\gamma}}{\gamma(p-\gamma-1)y_0^2}\right)^{\frac{1}{p-q-1}} t^{-\frac{1}{q-1}}, \tag{4.37}
$$

then

$$
\begin{aligned}
\frac{\tilde{I}_3}{\tilde{I}_2} \;=\;\; & \frac{q+\gamma}{\gamma(p-\gamma-1)}\left(\frac{\tilde{u}_{\eta\varepsilon}+\varepsilon}{x+\varepsilon}\right)^{2\gamma} \cdot (\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-(p-q-1)} \\
\leq\;\; & \frac{q+\gamma}{\gamma(p-\gamma-1)y_0^2 \cdot t^{\frac{p-q-1}{q-1}+2\nu}} \cdot c_2^{2\gamma} \cdot t^{(\frac{p-q-1}{2(q-1)}+\frac{\nu}{2})\cdot 2\gamma} \cdot (\tilde{u}_{\eta\varepsilon}+\varepsilon)^{-(p-q-1)} \\
\leq\;\; & 1.
\end{aligned}
$$

If (4.37) is false, however, then

$$
\begin{aligned}
\frac{\tilde{I}_3}{\tilde{I}_1} \;=\;\; & \frac{(q+\gamma)(\tilde{u}_{\eta\varepsilon}+\varepsilon)^{q-1}}{\frac{y'}{y}} \\
<\;\; & \frac{(q+\gamma)\cdot\left(\frac{(q+\gamma)c_2^{2\gamma}}{\gamma(p-\gamma-1)y_0^2}\right)^{\frac{q-1}{p-q-1}}\cdot t^{-1}}{\frac{p-q-1}{2(q-1)}\cdot t^{-1}} \\
\leq\;\; & 1
\end{aligned}
$$

according to (4.21). Therefore (4.32)-(4.36) along with the maximum principle and (4.14) ensure that

$$
\tilde{u}_{\eta\varepsilon x}(x,t) \geq -y_0 t^{\frac{p-q-1}{2(q-1)}+\nu} \cdot \left(\frac{x+\varepsilon}{\tilde{u}_{\eta\varepsilon}+\varepsilon}\right)^{\gamma} \geq -y_0 c_3^{-\gamma} \cdot t^{\frac{p-q-1}{2(q-1)}+\nu}
$$

is valid for all $(x,t) \in (0,\frac{L}{2}) \times (1,T)$, $\eta \in (0,1)$ and $\varepsilon \in (0,\varepsilon_3)$. Combined with (4.31) and (4.11) this establishes (4.10). ////

Without further difficulty we can pass to the proof of our final result which essentially sharpens the outcome of Lemma 3.2 under the assumption (4.9).

Theorem 4.5 *Let $p > 2$ and $q \in (1, p-1]$, and assume that u_0 satisfies (0.2) and*

$$
u_0(x) \geq c\,\mathrm{dist}(x,\partial\Omega) \qquad \text{for all } x \in \Omega
$$

with some $c > 0$. Then for all $\nu > 0$ there exists $C(\nu) > 0$ such that the maximal solution u of (0.1) satisfies

$$
\|u_x(\cdot,t)\|_{L^\infty(\Omega)} \leq C(\nu) \cdot (t+1)^{\frac{p-q-1}{2(q-1)}+\nu} \qquad \text{for all } t > 0.
$$

PROOF.   The claim immediately follows on letting $\varepsilon \to 0$ and then $\eta \to 0$ in (4.10). ////

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