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A Singular Differential Equation Stemming from an Optimal Control Problem in Financial Economics

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Abstract We consider the ordinary differential equation

$$x^2u'' = axu' + bu - c\big(u' - 1\big)^2, \quad x \in (0, x_0),$$

with $a \in \mathbb{R}$, $b \in \mathbb{R}$, $c > 0$ and the singular initial condition $u(0) = 0$, which in financial economics describes optimal disposal of an asset in a market with liquidity effects. It is shown in the paper that if $a + b < 0$ then no continuous solutions exist, whereas if $a + b > 0$ then there are infinitely many continuous solutions with indistinguishable asymptotics near 0. Moreover, it is proved that in the latter case there is precisely one solution u corresponding to the choice $x_0 = \infty$ which is such that $0 \le u(x) \le x$ for all $x > 0$, and that this solution is strictly increasing and concave.

Keywords Singular · ODE · Initial value problem · Supersolution · Subsolution · Nonuniqueness

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1 Introduction

The paper is concerned with solutions of the problem

$$x^2u'' = axu' + bu - c\big(u' - 1\big)^2, \quad x > 0, \tag{1.1}$$

$$u(0) = 0, \tag{1.2}$$

where a and b are real numbers and $c > 0$. By a solution of (1.1) in $[0, x_0]$ we mean a function $u \in C^0([0, x_0]) \cap C^2((0, x_0))$ which satisfies (1.1) for $x > 0$.

Equation (1.1) arises in the study of a specific stochastic optimization similar to the classical LQ problem. The equation is singular at $x = 0$ which in itself is not particularly noteworthy, since stochastic LQ problems with geometric Brownian state variable invariably give rise to nonlinear singular ODEs/PDEs of the type seen in (1.1) and in (1.3) below, see for example [9]. Our problem derives its rich structure from the fact that the initial condition (1.2), too, refers to the singular point $x = 0$. This, as we demonstrate below, poses certain technical obstacles in establishing existence and, more importantly, gives rise to infinitely many solutions with indistinguishable asymptotics near zero (Corollary 4.4).

As was already highlighted, the ODE (1.1), (1.2) is not artificial, rather it stems from a well-defined optimization problem in financial economics. Specifically, it is obtained from the PDE

$$\begin{cases} \frac{1}{2}y^2\sigma^2 w_{yy} + \lambda y w_y + r^* z w_z - \rho w + \frac{(y-w_z)^2}{4\eta} = 0, \\ w(y,0) = 0, \end{cases} \tag{1.3}$$

using the scaling

$$w(y,z) = \frac{y^2}{\eta}u(x), \quad x = \eta\frac{z}{y}.$$

The PDE (1.3) in turn represents the Hamilton-Jacobi-Bellman equation for the optimal value function w of the following dynamic optimization problem:

$$w\big(y(0), z(0)\big) = \max E\left(\int_0^{T(z=0)} e^{-\rho s} f\big(y(s), z(s)\big)\big(y(s) - \eta f\big(y(s), z(s)\big)\big)ds\right), \tag{1.4}$$

subject to

$$dy(t) = \lambda y(t)dt + \sigma y(t)dB(t), \tag{1.5}$$

$$dz(t) = \big(r^* z(t) - f\big(y(t), z(t)\big)\big)dt, \tag{1.6}$$

over the controls $f : \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}$, $T(z = 0)$ being the first arrival time at $z = 0$ and $B(t)$ the standard Wiener process.

Optimization (1.4) with dynamics given by (1.5), (1.6) models optimal liquidation of a large quantity of an asset whose market price is adversely affected by its ongoing sale. In this context $z(0)$ represents the quantity of the asset yet to be sold, $y(0)$ is the prevailing price and w captures the expected revenue of an optimal sale of quantity z conditional on the current price being y. For more details we refer the reader to [3].

The problem of existence is the first principal subject of this paper. It is shown that for $a+b>0$ the problem (1.1), (1.2) has a continuum of local solutions and at least one global solution bounded between 0 and x (Sect. 2), while for $a+b<0$ no solutions exist (Sect. 3).

If one admits the possibility that there are multiple solutions to (1.1), (1.2), one immediately has to deal with the additional challenge of identifying "the" right solution relevant to the associated optimization problem. The economic nature of the optimization (1.4)–(1.6) strongly suggests that the relevant solution of (1.1), (1.2) should be increasing (larger amount of asset means larger revenue) but concave (decreasing returns to scale, since larger volume of sales has greater adverse effect on the sale price of the asset). However, there is no indication in the form of (1.1) that a solution with these properties should exist in the first place. In Sect. 4 we thus analyze monotonicity and convexity properties of a solution bounded between 0 and x (Proposition 4.5), the upper bound corresponding to an immediate sale of the entire stock of the asset without any adverse price effect.

In Sect. 5 we address the question of global uniqueness. We show that there is exactly one solution on $\mathbb{R}_+$ which remains bounded between 0 and x, and this solution is necessarily increasing and concave (Proposition 5.1). Finally, in Sect. 6 we examine finer aspects of local non-uniqueness.

In the current paper we focus on the intricacies of the initial value problem (1.1), (1.2). The implication of the results for the underlying optimal control problem is a delicate issue left to further research.

A paper similar in spirit to ours is [4]. It studies a specific second order equation with a singularity at 0, arising in the theory of general relativity. As in our case, the existence of infinitely many solutions is established by the method of upper and lower solutions and then the properties of the set of solutions are studied.

As far as the local existence is concerned, Liang in [7] carried out a systematic study of second order singular initial value problems of the form

$$u'' = \frac{1}{x} F\big(x, u, u'\big), \tag{1.7}$$

where F is a continuous function and the initial conditions satisfy $F(0, u(0), u'(0)) = 0$. The key quantity in this study is $\gamma := \frac{\partial}{\partial u'} F(0, u(0), u'(0))$. It is shown that for $\gamma < 0$ local uniqueness holds, while for $\gamma > 0$ solutions become unique only after the asymptotics of u' have been fixed to the order x^γ near $x = 0$. The case $\gamma = 0$ is not treated. Each solution has an asymptotic expansion in powers of x and x^γ (provided γ is not an integer), and asymptotic expansion of $u^{(n)}$ is obtained by differentiating n-times the asymptotic expansion for u.

In contrast, we study a specific singular IVP from a wider class

$$u'' = \frac{1}{x^\alpha} F\big(x, u, u'\big), \tag{1.8}$$

with $\alpha = 2$, $u(0) = 0$, $u'(0) = 1$ and $F(x, u, u') := axu' + bu - c(u' - 1)^2$. Like in [7] our ODE arises from a self-similar solution of a PDE. However, we deal with a borderline case where $\frac{\partial}{\partial u'} F(0, u(0), u'(0)) = 0$. As a result, standard blow-up techniques are not productive and we have to resort to the method of sub-supersolutions.

Finally, we remark that it is not uncommon for HJB equations associated with stochastic optimization to exhibit multiple solutions. The meaningful solution then has to be selected by employing additional criteria. In the case of linear-quadratic problems the relevant solution is identified as the maximal/minimal one. In other cases the optimal solution can be singled out as the unique viscosity solution of the HJB equation, cf. [1]. In our case these criteria do not seem to be helpful. Rather, the significant solution is uniquely determined by its global monotonicity and concavity properties.

2 Existence for $a+b\geq 0$

In essence, existence will be proved similarly as in [4]. That is, ordered pairs of sub- and a supersolutions of (1.1) will be found, and an application of a standard existence result for second order boundary-value problems will provide solutions lying in between, cf. [5]. As in [4], due to the singularity in the ODE (1.1), an approximation procedure will be involved in the proof. However, compared to [4], the presence of u' in the equation will require additional arguments. We isolate technical arguments in the following propositions.

Proposition 2.1 *Let $0<x_1<x_2$ and suppose that there exist $\underline{u},\overline{u}\in C^2([x_1,x_2])$ such that*

$$\underline{u}\leq\overline{u}\quad \text{in } [x_1,x_2]; \tag{2.1}$$

$$\mathcal{E}\underline{u}<0\quad \text{in } [x_1,x_2]; \tag{2.2}$$

$$\mathcal{E}\overline{u}>0\quad \text{in } [x_1,x_2]; \tag{2.3}$$

the operator $\mathcal{E}$ being defined according to

$$\mathcal{E}u:=-x^2u''+axu'+bu-c\big(u'-1\big)^2 \tag{2.4}$$

for functions u which belong to $C^2([x_1,x_2])$. Then for each $u_1\in[\underline{u}(x_1),\overline{u}(x_1)]$, $u_2\in[\underline{u}(x_2),\overline{u}(x_2)]$ there exists a solution $u\in C^2([x_1,x_2])$ to (1.1) in $[x_1,x_2]$ satisfying $\underline{u}\leq u\leq\overline{u}$, $u(x_1)=u_1$, $u(x_2)=u_2$.

Proof Rewrite (1.1) as

$$u''=f\big(x,u,u'\big)$$

with

$$f\big(x,u,u'\big)=x^{-1}au'+x^{-2}bu-x^{-2}c\big(u'-1\big)^2.$$

For $x_1\leq x\leq x_2$ and $\underline{u}(x)\leq u\leq\overline{u}(x)$, f satisfies the Bernstein condition [2]

$$\big|f\big(x,u,u'\big)\big|\leq A+Bu'^2$$

for suitable $A,B>0$. Therefore, the result follows from Nagumo [8], Satz 2, cf. also [5], Theorem II-1.3 for a more recent reference. □

Remark Recall that regularity of a differential equation is inherited by its solutions (cf. [6], Chap. V, Corollary 4.1). In particular, since the expression for u'' is C^∞ in x, u, u' for $x > 0$, any solution u of (1.1) in $[x_1, x_2]$ with $0 < x_1$ is in $C^\infty([x_1, x_2])$.

Proposition 2.2

(i) *Let $x_0 \in (0, \infty)$. Suppose that there exist $\underline{u}, \overline{u} \in C^0[0, x_0] \cap C^2(0, x_0)$ satisfying*

$$\underline{u}(0) = \overline{u}(0) = 0 \tag{2.5}$$

in addition to (2.1)–(2.3) with $x_1 = 0, x_2 = x_0$. Then, for each $u_0 \in [\underline{u}(x_0), \overline{u}(x_0)]$ there exists a solution of (1.1), (1.2) in $[0, x_0]$ such that $\underline{u} \le u \le \overline{u}$ in $(0, x_0)$ and $u(x_0) = u_0$.

(ii) *Let $\underline{u}, \overline{u}$ satisfy (2.1)–(2.3) for $x_1 = 0$ and $x_2 = \infty$ as well as (2.5). Then, there exists a solution of (1.1), (1.2) in $[0, \infty)$ such that $\underline{u} \le u \le \overline{u}$.*

Proof (i) By Proposition 2.1, for each $\varepsilon \in (0, x_0)$ and each $u_0 \in [\underline{u}(x_0), \overline{u}(x_0)]$ there exists a solution $u_\varepsilon \in C^2([\varepsilon, x_0])$ of

$$\begin{cases} \mathcal{E} u_\varepsilon = 0 \quad \text{in } [\varepsilon, x_0], \\ u_\varepsilon(\varepsilon) = \underline{u}(\varepsilon), \qquad u_\varepsilon(x_0) = u_0, \end{cases} \tag{2.6}$$

which satisfies

$$\underline{u}(x) \le u_\varepsilon(x) \le \overline{u}(x) \quad \text{for all } x \in (\varepsilon, x_0). \tag{2.7}$$

Let now $\varepsilon_n \searrow 0$ for $n \to \infty$. For fixed n, the functions u_{ε_k} with $k \ge n$ are uniformly bounded on $[\varepsilon_n, x_0]$. By [2] (cf. also [5], I.4.3 page 45), the same holds for their derivatives u'_{ε_k}. Therefore, on $[\varepsilon_n, x_0]$, u_{ε_k} are equicontinuous and, moreover, from (1.1) it follows that u''_{ε_k}, $k \ge n$ are uniformly bounded on $[\varepsilon_n, x_0]$. Thus u'_{ε_k} are equicontinuous and, in turn, because od (1.1), u''_{ε_k} are equicontinuous as well on $[\varepsilon_n, x_0]$. Therefore, one can pick a subsequence $u_{\varepsilon_{k_j}}$ which converges C^2 uniformly to a C^2 function u^n satisfying (1.1) on $[\varepsilon_n, x_0]$ together with $u^n(x_0) = u_0$ and $\underline{u}(x) \le u^n(x) \le \overline{u}(x)$. By standard diagonal selection we can pick a subsequence from the sequence $u_{\varepsilon_{k_j}}$ which converges pointwise in $[0, x_0]$ and uniformly in $[\varepsilon, x_0]$ for each $0 < \varepsilon \le x_0$ to a function $u \in C^0[0, x_0] \cap C^2(0, x_0]$ and satisfying the requirements of item (i) of the proposition.
(ii) By (i), for each ε we have a solution of (1.1), (1.2) such that $u_\varepsilon(0) = 0$ and $\underline{u} \le u \le \overline{u}$ in $[0, 1/\varepsilon]$. Applying for $\varepsilon > 0$ the same extraction idea as in (i) we obtain the claimed solution in $[0, \infty)$. □

Proposition 2.3

(i) *For $a + b > 0$ and any $x_0 > 0$ there is a continuum of solutions to (1.1), (1.2) on $[0, x_0]$ such that $0 \le u \le x$.*

(ii) *For $a + b \ge 0$ there is at least one solution of (1.1), (1.2) on $[0, \infty)$ such that $0 \le u \le x$.*

Proof For $a+b>0$ it is readily checked that $\underline{u}(x)\equiv 0$ is a subsolution and $\overline{u}(x)=x$ is a supersolution in $[0,\infty)$. The claim thus follows from Proposition 2.2. For $a+b=0$, $u(x)=x$ is a global solution. □

The problem (1.1), (1.2) can for $a+b>0$ be formally solved by a power series. We let

$$k_0 := 1, \qquad k_1 := -\frac{2}{3}\sqrt{\frac{a+b}{c}}, \tag{2.8}$$

and inductively define

$$f_n(x) := \sum_{i=0}^{n} k_i x^{1+i/2}, \tag{2.9}$$

where

$$k_{n+1} := \lim_{x\to 0_+} \frac{2\mathcal{E} f_n}{3ck_1(n+3)x^{(n+2)/2}} \tag{2.10}$$

for $n\geq 1$.

Lemma 2.4 *Let $a+b>0$. Then the coefficients $\{k_i\}_{i=0}^n$ are well-defined and $\mathcal{E} f_n = O(x^{(n+2)/2})$ as $x\searrow 0$ for all $n\in\mathbb{N}$.*

Proof The statement clearly holds for $n=1$. Arguing by induction, we suppose that it is valid for some $n\geq 1$. Then

$$\begin{aligned}\mathcal{E} f_{n+1} &= \mathcal{E} f_n - 2c\big(f_n'-1\big)\big(1+(n+1)/2\big)k_{n+1}x^{(n+1)/2} + O\big(x^{(n+3)/2}\big)\\ &= \mathcal{E} f_n - \frac{3}{2}ck_1(n+3)k_{n+1}x^{(n+2)/2} + O\big(x^{(n+3)/2}\big) \quad \text{as } x\searrow 0. \end{aligned}\tag{2.11}$$

Since $\mathcal{E} f_n$ is a polynomial in powers of $\sqrt{x}$ and $\mathcal{E} f_n = O(x^{(n+2)/2})$ it follows that k_{n+1} is well defined and that $\mathcal{E} f_n - (3/2)ck_1(n+3)k_{n+1}x^{(n+2)/2} = O(x^{(n+3)/2})$. In view of (2.11) this implies that $\mathcal{E} f_{n+1} = O(x^{(n+3)/2})$ and thus completes the proof. □

Easy calculations show that the coefficients $\{k_n\}_{n\geq 2}$ satisfy the recursion

$$\begin{aligned} k_{2i} = \frac{1}{6(i+1)k_1}\Bigg[&2\frac{k_{2i-1}}{c}\left(a+b+\left(i-\frac{1}{2}\right)a-\left(i^2-\frac{1}{4}\right)\right)\\ &-\sum_{j=1}^{i-1}(3+j)(2+2i-j)k_{j+1}k_{2i-j}\Bigg],\end{aligned}\tag{2.12}$$

$$\begin{aligned} k_{2i+1} = \frac{1}{3(2i+3)k_1}\Bigg[&2\frac{k_{2i}}{c}\big(a+b+ia-i(1+i)\big)-\frac{1}{2}(3+i)^2k_{i+1}^2\\ &-\sum_{j=1}^{i-1}(3+j)(3+2i-j)k_{j+1}k_{2i-j+1}\Bigg].\end{aligned}\tag{2.13}$$

From here it is readily seen that the radius of convergence of the power series (2.9) is nil when $a < \frac{3}{2}$ and $b \in (-a, \frac{3}{4} - \frac{3}{2}a]$, firstly by showing inductively $k_i > 0$ for $i \geq 2$ and subsequently neglecting all quadratic terms in k_i in (2.12), (2.13) and proving the easy estimate $k_{n+1}/k_n \geq -2(n-1)/(3k_1 c)$ for sufficiently large n. Hence the power series f_n does not define a solution directly via $\lim_{n\to\infty} f_n(x)$ outside $x = 0$. We conjecture this remains to be the case for arbitrary parameter values as long as $a + b > 0$.

We will show later (Corollary 4.4) that every local solution of (0.1), (0.2) with the property $u(x) \leq x$ satisfies

$$u^{(k)}(x) = f_n^{(k)}(x) + o\big(x^{(n+3)/2-k}\big),$$

for $k \in \{0, 1\}$ and $n = 1$. Whether this is true for $n > 1$ or $k > 1$ remains an open question.

3 Nonexistence for $a + b < 0$

In this second part we shall deduce Proposition 3.2 below which will exclude the existence of any continuous solution to (1.1) for any $x_0 > 0$ under the assumption $a + b < 0$ which is complementary to the hypothesis of Proposition 2.3.

To this end we first prove that any supposedly existing continuous solution must satisfy $u'(x) \to 1$ as $x \to 0$. This property can formally easily be guessed upon tracing the possible solution behavior near $x = 0$.

Lemma 3.1 *Suppose that for some $x_0 > 0$, the function $u \in C^0([0, x_0]) \cap C^2((0, x_0))$ is a solution of (1.1), (1.2). Then*

$$\lim_{x \searrow 0} u'(x) = 1. \tag{3.1}$$

Proof Letting $v := u' - 1$ we can rewrite (1.1) as

$$u' = v + 1,$$
$$x^2 v' = ax(v+1) + bu - cv^2.$$

Let $X(t) := -t^{-1}$ for $t < 0$. Then $X'(t) = t^{-2}$ and $X(t) \searrow 0$ as $t \to -\infty$. We next introduce $U(t) := u(X(t))$ and $V(t) := v(X(t))$ for $t < 0$. Then the pair (U, V) solves the following system of differential equations

$$U' = t^{-2}(V + 1),$$
$$V' = -at^{-1}(V + 1) + bU - cV^2.$$

By assumption, we have $U(t) \to 0$ as $t \to -\infty$ and thus

$$V'(t) = p(t) + q(t)V(t) - cV^2(t) \tag{3.2}$$

with

$$p(t) \to 0 \quad \text{and} \quad q(t) \to 0 \quad \text{as } t \to -\infty. \tag{3.3}$$

We wish to show that if $V(t)$ is defined for all $t \le -x_0^{-1}$ then $V(t) \to 0$ as $t \to -\infty$. The proof proceeds in several steps.

(i) Given $\varepsilon > 0$ there is $T < -x_0^{-1}$ such that $|p(t)| < \varepsilon^2 c/3$ and $|q(t)| < \varepsilon c/3$ for all $t \le T$, by virtue of (3.3).

(ii) Consider $t_0 \le T$. We claim that if $|V(t)| \ge \varepsilon$ for all $t \le t_0$ then

$$V(t) \ge \frac{1}{V(t_0)^{-1} + \frac{c}{3}(t - t_0)} \quad \text{for all } t \le t_0, \tag{3.4}$$

while defined. To this end note that (3.2) and (i) yield

$$V'(t) \le -\frac{c}{3} V^2(t) \quad \text{if } t \le T \text{ and } |V(t)| \ge \varepsilon. \tag{3.5}$$

By the comparison theorem for ordinary differential equations we conclude that

$$V(t) \ge Y(t) \quad \text{for } t \le t_0,$$

where Y solves the differential equation $Y' = -\frac{c}{3} Y^2$ with $Y(t_0) = V(t_0)$. On solving for Y we obtain (3.4).

(iii) Now we prove that there exists $t_1 \le T$ such that $V(t_1) > -\epsilon$. Suppose to the contrary that $V(t) \le -\epsilon$ for all $t \le T$. Then, (3.4) gives $V(t) \ge -\varepsilon/2$ for $t < t_0 - \frac{6}{c\varepsilon}$, yielding the desired contradiction.

Next we show that $V(t) > -\varepsilon$ *for all* $t \le t_1$. Arguing by contradiction, suppose this is not the case. Then there is t_2 such that $-\infty < t_2 = \sup\{t \le t_1 : V(t) \le -\varepsilon\} < t_1$. By continuity we have $V(t_2) = -\varepsilon$. From (3.5) we obtain $V'(t_2) < 0$ which is in conflict with $V(t_2) = -\varepsilon$ and $V(t) > -\varepsilon$ for $t \in (t_2, t_1)$.

(iv) Finally, we show that $V(t) \le \epsilon$ for all $t \le T$. If not, there is $t_3 \le T$ such that $V(t_3) > \epsilon$ and we have $t_4 := \sup\{t \le t_3 : V(t) \le \epsilon\} < t_3$. The same argument as in part (iii) shows that $t_4 = -\infty$ and therefore $V(t) > \varepsilon$ for all $t \le t_3$. From (3.4) we now obtain $V(t) \to \infty$ for $t \searrow t_3 - \frac{3}{c} V(t_3)^{-1}$. Therefore, $V(t)$ is not defined for some $t \le -x_0^{-1}$ which is inconsistent with differentiability of U in $(-\infty, 0)$.

Since ϵ was arbitrary this completes the proof of the lemma. □

It is now possible to rule out local existence of a continuous solution of (1.1), (1.2) under the condition that $a + b$ be strictly negative.

Proposition 3.2 *Suppose that* $a + b < 0$. *Then for each* $x_0 > 0$, *the problem* (1.1), (1.2) *does not possess any solution* u *in* $[0, x_0]$.

Proof Suppose that such a solution exists for some $x_0 > 0$. Then from Lemma 3.1 we know that u actually belongs $C^1([0, x_0])$ with $u'(0) = 1$, and hence the functions

φ_1 and φ_2 defined by

$$\varphi_1(x) := u'(x) - 1, \quad x \in (0, x_0), \quad \text{and} \quad \varphi_2(x) := \frac{u(x) - x}{x}, \quad x \in (0, x_0),$$

satisfy $\varphi_1(x) \to 0$ and $\varphi_2(x) \to 0$ as $x \to 0$. Since $a + b < 0$, we can thus find $\overline{x} \in (0, x_0)$ such that

$$a + b + a\varphi_1(x) + b\varphi_2(x) \le \frac{a+b}{2} \quad \text{for all } x \in (0, \overline{x}).$$

Therefore, (1.1) shows that

$$\begin{aligned}
x^2 u''(x) &= axu'(x) + bu(x) - c\big(u'(x) - 1\big)^2 \\
&\le axu'(x) + bu(x) \\
&= ax\big(1 + \varphi_1(x)\big) + bx\big(1 + \varphi_2(x)\big) \\
&= \big(a + b + a\varphi_1(x) + b\varphi_2(x)\big) \cdot x \\
&\le -\delta x \quad \text{for all } x \in (0, \overline{x})
\end{aligned}$$

holds with $\delta := -\frac{a+b}{2} > 0$. By integration we find that

$$u'(\overline{x}) - u'(x) \le -\delta \ln \frac{\overline{x}}{x} \quad \text{for all } x \in (0, \overline{x}).$$

This implies that $u'(x) \to +\infty$ as $x \to 0$ and thereby contradicts Lemma 3.1. □

4 Monotonicity and Concavity Properties of Solutions

In this section we assume $a + b > 0$ and we study monotonicity and convexity properties of solutions to (1.1), whose existence was established in Sect. 2.

The following lemma is the key to establishing monotonicity, concavity, and ultimately also uniqueness in a certain restricted class of solutions.

Lemma 4.1 *Consider a nonconstant function $y \in C^0([0, \infty)) \cap C^2((0, \infty))$ satisfying*

$$x^2 y''(x) = f(x)y'(x) + g\big(x, y(x)\big), \tag{4.1}$$

for some continuous functions f and g. Suppose there is a constant $y^* \in [-\infty, \infty]$ such that for all $x > 0$ one has $g(x, y) > 0$ for $y > y^*$ and $g(x, y) < 0$ for $y < y^*$. Then there is at most one $x_0 \in (0, \infty)$ such that $y'(x_0) = 0$. If such x_0 exists then one, and only one, of the following two alternatives is possible: Either

- *$y'(x) < 0$ for $x < x_0$, $y'(x) > 0$ for $x > x_0$, and $y(x) > y(x_0) > y^*$ for all $x \neq x_0$,*

or

- *$y'(x) > 0$ for $x < x_0$, $y'(x) < 0$ for $x > x_0$, and $y(x) < y(x_0) < y^*$ for all $x \neq x_0$.*

Proof We first note that because of continuity of g we have $g(x, y^*) = 0$ for all $x > 0$ whenever y^* is finite. By an ODE uniqueness argument, $y(x_0) = y^*$ and $y'(x_0) = 0$ implies $y(x) \equiv y^*$. Therefore, if $y(x)$ is not constant and $y'(x_0) = 0$ then $y(x_0) \neq y^*$.

Now suppose that $y(x_0) > y^*$. Then from (4.1) it follows that $y''(x_0) > 0$, hence $y'(x) < 0$ for $x < x_0$ sufficiently close to x_0. Arguing by contradiction, let us suppose that there exists $0 < x_1 < x_0$ such that $y'(x_1) \geq 0$. Then there is $x_2 \in [x_1, x_0)$ such that

$$y'(x_2) = 0, \qquad y'(x) < 0 \quad \text{for } x_2 < x < x_0, \tag{4.2}$$

which implies $y''(x_2) < 0$ and also $y(x) > y(x_0) > y^*$ for $x_2 \leq x < x_0$. On the other hand, (4.1) together with $y(x_2) > y^*$ and $y'(x_2) = 0$ entails that $y''(x_2) > 0$, yielding the desired contradiction. Therefore, $y'(x) < 0$ for all $x \in (0, x_0)$. The proof of $y'(x) > 0$ for $x > x_0$ follows the same lines. Finally, the case $y(x) < y^*$ can be reduced to the case $y(x) > y^*$ by the transformation $y \mapsto -y$, $y^* \mapsto -y^*$. □

We now apply this to derive some monotonicity properties of solutions. Here in order to abbreviate notation, we call a function $\phi : [0, \infty) \to \mathbb{R}$ *eventually monotonic* if it is monotonic on $[x_0, \infty)$ for some $x_0 \geq 0$.

Lemma 4.2 *Let u be a nonconstant solution of* (1.1) *on* $(0, \infty)$.

(i) *If $b \neq 0$ and u is bounded and eventually monotonic, then $u(x)$ converges to the unique stationary solution $\hat{u} = c/b$ as $x \to \infty$. If $b = 0$ and u is eventually monotonic, then u is unbounded.*

(ii) *If $b > 0$ and $u \geq 0$, then one of the following alternatives occurs: Either*

- *$u'(x) < 0$ for all $x > 0$, $u(x) > c/b$ for all $x > 0$ and $u(x) \to c/b$ as $x \to \infty$,*

or

- *$u'(x) > 0$ for all $x > 0$, and either $u(x) < c/b$ for all $x > 0$ and $u(x) \to c/b$ as $x \to \infty$, or u is unbounded,*

or finally

- *there exists a unique $x_0 > 0$ such that $u'(x_0) = 0$, and we have $u''(x_0) > 0$, $u(x) > u(x_0) > c/b$ for all $x \neq x_0$, $u'(x) < 0$ for $x < x_0$, $u'(x) > 0$ for $x > x_0$, and u is unbounded.*

(iii) *If $b \leq 0$ and $u \geq 0$ then $u'(x) > 0$ for all $x > 0$ and u is unbounded.*

Proof

(i) The substitution $x(t) = e^t$, $\tilde{u}(t) = u(x(t))$ transforms (1.1) into

$$\tilde{u}'' = (a+1)\tilde{u}' + b\tilde{u} - c\big(e^{-t}\tilde{u}' - 1\big)^2. \tag{4.3}$$

Being bounded and eventually monotonic, $\tilde{u}$ has a limit l as $t \to \infty$ and consequently $\lim_{t\to\infty} \tilde{u}'(t) = 0$. From (4.3) it now follows that $\lim_{t\to\infty} \tilde{u}''(t) = bl - c$. If $bl - c \neq 0$ then $\lim_{t\to\infty} \tilde{u}''(t) \neq 0$ which is inconsistent with the convergence of $\tilde{u}'$. This proves $bl - c = 0$. For $b = 0$ this is a contradiction with $c > 0$, for $b \neq 0$ it yields $l = c/b$.

(ii) If $u'(x) < 0$ for all x then u is bounded and, by (i), tends to c/b as $x \to \infty$ which is possible only if $u(x) > c/b$ for all x. If $u'(x) > 0$ for all x then thanks to monotonicity, $u(x)$ approaches a limit in $[0, \infty]$ as $x \to \infty$. If this limit is finite it has to equal c/b by virtue of (i), and in the remaining case u is unbounded.

Suppose now there is $x_0 > 0$ such that $u'(x_0) = 0$. Lemma 4.1 applied to (1.1) with $y \equiv u \geq 0$, $g(x, y) := by - c$ and $y^* := c/b$ yields two alternatives, the first of which is stated in part (ii). The second alternative is not possible since it implies $0 \leq u \leq c/b$ but at the same time $u'(x) < 0$ for $x > x_0$ which means that $u \not\to c/b$ as $x \to \infty$. A bounded solution not converging to c/b contradicts part (i).

(iii) If u' is not positive everywhere then Lemma 4.1 applied to (1.1) with $y \equiv u \geq 0$, $g(x, y) := by - c$ and $y^* := \infty$ implies that there is x_0 such that $u(x) < u(x_0)$ and $u'(x) < 0$ for $x > x_0$. Therefore u is bounded and eventually monotonic. This contradicts (i) when $b = 0$. For $b < 0$, (i) dictates that u should converge to c/b as $x \to \infty$, which contradicts $u \geq 0$ since $c/b < 0$. □

Proposition 4.3 *Suppose that $a + b > 0$ and that $u(x) \leq x$ is a solution of* (1.1), (1.2) *on $(0, x_0)$ with some $x_0 > 0$. Then there exists $x_1 \in (0, x_0)$ such that $u'(x) > 0$, $u''(x) < 0$ and $u'''(x) > 0$ for all $x \in (0, x_1)$. Furthermore, in this case we have*

$$\lim_{x \to 0} \frac{u'(x) - 1}{\sqrt{x}} = -\sqrt{\frac{a+b}{c}}. \tag{4.4}$$

Proof We recall that by Lemma 3.1 $u'(0) = 1$ and that by the remark following Proposition 2.1, $u(x)$ is C^∞ for $x > 0$. As an immediate consequence we must have $u'(x) > 0$ for all sufficiently small $x > 0$. On differentiating (1.1) we obtain

$$x^2 u''' + 2xu'' = axu'' + (a + b)u' - 2c\big(u' - 1\big)u'' \quad \text{on } (0, x_0). \tag{4.5}$$

Lemma 4.1 applied to (4.5) with $y \equiv u'$, $g(x, y) := (a + b)y$, $y^* := 0$ implies that u'' has a constant non-zero sign near $x = 0$. This, together with $u(x) \leq x$ and $u'(0) = 1$, yields that necessarily $u''(x) < 0$ for all sufficiently small $x > 0$.

We now differentiate (4.5) once more to obtain

$$x^2 u'''' = \big((a - 4)x - 2c\big(u' - 1\big)\big)u''' + (2a + b - 2)u'' - 2c\big(u''\big)^2 \quad \text{on } (0, x_0). \tag{4.6}$$

Lemma 4.1 applied to (4.6) with $y \equiv u'' \leq 0$, $g(x, y) := (2a + b - 2)y - 2cy^2$ and $y^* := (2a + b - 2)/(2c)$ implies that $u'''(x)$ has a constant non-zero sign near $x = 0$. Arguing by contradiction, we suppose that $u''' < 0$ near $x = 0$. Since $u'' < 0$, this implies that $L := \lim_{x \searrow 0} u''(x)$ exists and is finite. This however contradicts (1.1), since on integrating we find $x^2 u''(x) = Lx^2 + o(x^2)$, $xu'(x) = x + Lx^2 + o(x^2)$ and $u(x) = x + Lx^2/2 + o(x^2)$ as $x \to 0$, and on substituting these expressions into (1.1) one concludes that it cannot hold near $x = 0$. We have thus proved $u''' > 0$ near zero.

Next, dividing (1.1) by x we obtain

$$xu''(x) = au'(x) + b\,\frac{u(x)}{x} + c\,\frac{(u'(x) - 1)^2}{x} \quad \text{for all } x \in (0, x_0). \tag{4.7}$$

Since u'' is increasing and negative, by (3.1) we find that

$$u'(x) - 1 = \int_0^x u''(\xi)d\xi \le xu''(x) \le 0,$$

and, consequently,

$$xu''(x) \to 0 \quad \text{as } x \to 0. \tag{4.8}$$

Substituting this into (4.7) we obtain

$$\lim_{x\to 0} c\frac{(u'(x)-1)^2}{x} = a + b.$$

Since $u'(x) - 1 \le 0$, this is equivalent to (4.4). □

Corollary 4.4 *There is a continuum of local solutions of* (1.1), (1.2), *with the property* $0 \le u(x) \le x$ *and they all satisfy*

$$u(x) = x - \frac{2}{3}\sqrt{\frac{a+b}{c}}x^{3/2} + o\big(x^{3/2}\big) \tag{4.9}$$

$$u'(x) = 1 - \sqrt{\frac{a+b}{c}}x^{1/2} + o\big(x^{1/2}\big). \tag{4.10}$$

Proof Multiplicity of solutions was proved in Proposition 2.3. Expansion (4.10) follows from (4.4), and (4.9) follows by integration of (4.10). □

Proposition 4.5 *Let* u *be a solution of* (1.1), (1.2) *with* $x_0 = \infty$ *such that* $0 \le u(x) \le x$ *for all* $x > 0$. *Then, in addition to* (4.9) *and* (4.10), *we have that* $u'(x) > 0$, $u''(x) < 0$ *and* $u'''(x) > 0$ *for all* $x > 0$. *Moreover,*

$$\lim_{x\to\infty} u'(x) = 0. \tag{4.11}$$

Proof The conclusion $u'(x) > 0$ for all $x > 0$ is a trivial consequence of Lemma 4.2(iii) for $b \le 0$. In the case $b > 0$, Lemma 4.2(ii) implies that if u is not increasing everywhere then there is $x_0 > 0$ such that $u'(x) < 0$ on $(0, x_0)$ and this contradicts the facts that $u(0) = 0$ and $u \ge 0$.

Next, Lemma 4.1 applied to (4.5) with $y \equiv u' \ge 0$, $g(x, y) := (a + b)y \ge 0$ and $y^* := 0$ shows that if $u'' < 0$ does not hold over $(0, \infty)$ then $u''(x) > 0$ for all sufficiently large $x > 0$. We show that the latter alternative is impossible. To this end, we let $v := u'$ and $\tilde{v}(t) := v(x(t))$ with $x(t) = e^t$. Arguing by contradiction, since v is eventually increasing and $u(x) \le x$, we must have $v \le 1$, which implies that v and $\tilde{v}$ converge as $x \to \infty$ and $t \to \infty$, respectively. This in turn implies $v'(x(t)) \to 0$ and $\tilde{v}'(t) \to 0$ as $t \to \infty$. Since from (4.5) we see that

$$\tilde{v}''(t) = (a-1)\tilde{v}'(t) + (a+b)\tilde{v}(t) - 2c\big(\tilde{v}(t) - 1\big)v'\big(x(t)\big), \tag{4.12}$$

we therefore obtain $\tilde{v}''(t) - (a+b)\tilde{v}(t) \to 0$ as $t \to \infty$. Since $\tilde{v} > 0$ is increasing and convergent, so is $\tilde{v}''$. This is inconsistent with $\tilde{v}'(t) \to 0$ as $t \to \infty$. We have thus proved $u''(x) < 0$ for all $x > 0$.

Now since v is decreasing and bounded below by 0, it converges as $t \to \infty$. Therefore, both $v'(x(t))$ and $\tilde{v}'(t)$ converge to 0 as $t \to \infty$. From (4.12) we thus obtain $\tilde{v}''(t) - (a+b)\tilde{v}(t) \to 0$ as $t \to \infty$. Should the limit of $\tilde{v}$ for $t \to \infty$ not be zero, the same would hold for $\tilde{v}''$. This, however, is in conflict with the convergence of $\tilde{v}'$ for $t \to \infty$. This proves statement (4.11).

Finally, in order to verify that $u''' > 0$ throughout $(0, \infty)$, we note that from Lemma 4.3 we know that u''' is positive near zero. Arguing by contradiction, we assume that u'' is not increasing everywhere. Lemma 4.1 applied to (4.6) implies that in such case u'' must be eventually decreasing and therefore $\lim_{x\to\infty} u''(x) \neq 0$ which contradicts statement (4.11). □

The following inequality is related to the theory of speculative attacks, in which an agency artificially supports a (low) fixed price of an asset, using a limited amount of reserves. The inequality indicates that in order for the speculator to make expected profit or at least for her to break even, the price of the asset must always jump upwards after the speculative attack has exhausted the entire supply of the asset at the subsidized price.

Corollary 4.6 *Let u be a solution of* (1.1), (1.2) *with $a+b>0$ such that $0 \le u(x) \le x$ for all $x > 0$. Then*

$$1 + u'(x) > 2\frac{u(x)}{x} \quad \textit{for all } x > 0. \tag{4.13}$$

Proof Since $u'''(x) > 0$ for $x > 0$ by Proposition 4.5, the function u' is strictly convex on $[0, \infty)$. Therefore

$$\frac{u'(x) - u'(0)}{x} < u''(x) \quad \text{for all } x > 0.$$

On multiplying both sides by x, utilizing $u'(0) = 1$ and integrating we obtain

$$u(x) - x < xu'(x) - u(x) \quad \text{for all } x > 0,$$

which yields the desired inequality. □

5 Uniqueness of the Global Solution Bounded Between 0 and x

Proposition 5.1 *There is one, and only one, solution u of* (1.1), (1.2) *in* $[0, \infty)$ *which has the additional property that $0 \le u(x) \le x$ for all $x > 0$. This solution necessarily satisfies $u > 0$, $u' > 0$, $u'' < 0$, and $u''' > 0$ on* $(0, \infty)$.

The proposition stems from the following result:

Lemma 5.2 *Let $u \neq v$ be two solutions of* (1.1) *in* $[0, \infty)$ *which satisfy* $u(0) = v(0)$, $u'(0) = v'(0)$ *and* $u'' \le 0$ *on* $(0, \infty)$. *Then* $w := v - u$ *satisfies either* $w'' > 0$ *on* $(0, \infty)$ *or* $w'' < 0$ *throughout* $(0, \infty)$.

Proof The function w solves

$$x^2 w'' = axw' + bw - 2c\big(u' - 1\big)w' - cw'^2 \quad \text{on } (0, \infty),$$

which on differentiation yields

$$x^2 w''' = (a-2)xw'' + (a+b)w' - 2cu''w' - 2c\big(u' - 1\big)w'' - 2cw'w'' \quad \text{on } (0, \infty). \tag{5.1}$$

Lemma 4.1 applied to (5.1) with $y \equiv w'$, $g(x, y) := (a + b - 2cu''(x))y$ and $y^* := 0$ shows that w'' can have at most one root. Now the existence of such a root $x_0 > 0$ would imply either $w' > 0$, $w''(x) < 0$ for $0 < x < x_0$ and $w''(x) > 0$ for $x > x_0$; or $w' < 0$, $w''(x) > 0$ for $0 < x < x_0$ and $w''(x) < 0$ for $x > x_0$. This however contradicts $w'(0) = 0$. Thus one must have either $w''(x) > 0$ or $w''(x) < 0$ for all $x > 0$. □

Proof of Proposition 5.1 Suppose there are two solutions $u \neq v$ bounded between 0 and x and let $w := v - u$. Proposition 4.5 yields $\lim_{x\to\infty} u'(x) = \lim_{x\to\infty} v'(x) = 0$ and therefore

$$\lim_{x\to\infty} w'(x) = 0. \tag{5.2}$$

On the other hand, Proposition 4.5 also gives $u'(0) = v'(0) = 1$, implying $w'(0) = 0$. We can thus employ Lemma 5.2 to obtain that either $w''(x) > 0$ or $w''(x) < 0$ for all $x > 0$. In view of $w'(0) = 0$ both alternatives contradict (5.2). The claimed further properties of the unique solution bounded between 0 and x follow from Proposition 4.5. □

6 Finer Aspects of Non-uniqueness

By Corollary 4.4 there is a continuum of local solutions sharing the same asymptotics up to degree 3/2 at 0. The solutions are parametrized by their values at any $x > 0$. In this section we establish existence of disjoint continua of local solutions of (1.1), (1.2) distinguished by refinement of their asymptotic at 0.

We first show that the higher order terms specification of (4.10) can be somewhat sharpened. Denote

$$k = \sqrt{\frac{a+b}{c}}.$$

Lemma 6.1 *Let* u, x_0, x_1 *be as in Corollary* 4.4. *Then there exists* $l > 0$ *such that*

$$\left|\frac{u'(x) - 1}{\sqrt{x}} + k\right| \le l\sqrt{x} \quad \textit{for all } x \in (0, x_1). \tag{6.1}$$

If u is the unique solution provided by Proposition 5.1 *then* (6.1) *holds for all* $0 \le x < \infty$ *for some* $l > 0$.

Proof Since $\frac{u'(x)-1}{\sqrt{x}} \to -k$ as $x \searrow 0$ by Proposition 4.3, we can find $x_1 > 0$, $c_1 > 0$ such that

$$\left|u'(x) - 1\right| \le c_1\sqrt{x} \quad \text{for } 0 < x < x_1. \tag{6.2}$$

Therefore trivially

$$\left|au'(x) - a\right| \le c_1|a|\sqrt{x} \quad \text{for all } 0 < x < x_1 \tag{6.3}$$

and also

$$\begin{aligned} \left|\frac{bu(x)}{x} - b\right| &= \frac{|b|}{x} \cdot \left|\int_0^x \big(u'(y) - 1\big)dy\right| \le \frac{c_1|b|}{x} \cdot \int_0^x \sqrt{y}dy \\ &= \frac{2c_1|b|}{3}\sqrt{x} \quad \text{for all } 0 < x < x_1. \end{aligned} \tag{6.4}$$

Taking x_1 sufficiently small, using Proposition 4.3 we obtain that $u'' \le 0$ and $u''' \ge 0$ and hence $|xu''(x)| \le |u'(x) - 1|$ for all $0 < x < x_1$, from (6.2) we also infer that

$$\left|xu''(x)\right| \le c_1\sqrt{x} \quad \text{for all } 0 < x < x_1.$$

Accordingly, from (1.1) we obtain that $\varphi(x) := \frac{u'(x)-1}{\sqrt{x}}$, $0 < x < x_1$, satisfies

$$\begin{aligned} \left|c\varphi^2(x) - ck^2\right| &= \left|\big(au'(x) - a\big) + \left(\frac{bu(x)}{x} - b\right) - xu''(x)\right| \\ &\le c_2\sqrt{x} \quad \text{for all } 0 < x < x_1 \end{aligned} \tag{6.5}$$

with $c_2 := c_1|a| + \frac{2c_1|b|}{3} + c_1$. Therefore $\varphi^2(x) \le k^2 + \frac{c_2}{c}\sqrt{x}$, so that

$$-\varphi(x) = \left|\varphi(x)\right| \le k \cdot \sqrt{1 + \frac{c_2\sqrt{x}}{ck^2}} \le k \cdot \left(1 + \frac{c_2\sqrt{x}}{2ck^2}\right) \quad \text{for all } 0 < x < x_1, \tag{6.6}$$

where we have used that $\sqrt{1+\xi} \le 1 + \frac{\xi}{2}$ for all $\xi > 0$. Likewise, (6.5) entails that $\varphi^2(x) \ge k^2 - \frac{c_2}{c}\sqrt{x}$ for all $x > 0$. Thus, since $\sqrt{1-\xi} \ge 1 - \frac{\xi}{\sqrt{2}}$ for all $\xi \in (0, \frac{1}{2})$, we see that

$$-\varphi(x) \ge k \cdot \sqrt{1 - \frac{c_2\sqrt{x}}{ck^2}} \ge k - \frac{c_2\sqrt{x}}{\sqrt{2}ck} \quad \text{for all } x \in (0, x_2), \tag{6.7}$$

where $x_2 := \min\{x_1, (\frac{ck^2}{2c_2})^2\}$.

If u is the global solution of Proposition 5.1, then (6.1) is obvious in $[x_1, \infty)$ because $u'(x) \to 0$ as $x \to \infty$. This, together with (6.6) and (6.7) concludes the proof of the lemma. □

The core of our approach will be formed by the usage of on ordered pair of sub- and supersolutions which deviate from our original solution by an exponentially small term near $x = 0$. As a preparation, let us compute the action of the operator $\mathcal{E}$, as defined in (2.4), on such functions.

Lemma 6.2 *Suppose that* u *is a solution of* (1.1), (1.2) *in* $(0, x_0)$ *for some* $x_0 > 0$. *Then for all* $\varepsilon > 0$, $\alpha \in \mathbb{R}$ *and* $\beta \in \mathbb{R}$, *the function* v *defined by*

$$v(x) := u(x) - \varepsilon x^\alpha e^{-\frac{\beta}{\sqrt{x}}}, \quad x \in (0, x_0), \tag{6.8}$$

satisfies

$$\begin{aligned} \mathcal{E}v = \varepsilon e^{-\frac{\beta}{\sqrt{x}}} \cdot \Bigg\{ & \frac{\beta^2}{4} x^{\alpha-1} + \left[\left(\alpha - \frac{3}{2}\right)\frac{\beta}{2} + \frac{\alpha\beta}{2} - \frac{a\beta}{2}\right] \cdot x^{\alpha-\frac{1}{2}} \\ & + \big[\alpha(\alpha-1) - a\alpha - b\big] \cdot x^\alpha + 2c \cdot \frac{u'-1}{\sqrt{x}} \cdot \left[\frac{\beta}{2}x^{\alpha-1} + \alpha x^{\alpha-\frac{1}{2}}\right] \\ & - \varepsilon c \cdot \left[\frac{\beta}{2} x^{\alpha-\frac{3}{2}} + \alpha x^{\alpha-1}\right]^2 \cdot e^{-\frac{\beta}{\sqrt{x}}} \Bigg\} \end{aligned} \tag{6.9}$$

for all $x \in (0, x_0)$.

Proof We write

$$w(x) := \varepsilon x^\alpha e^{-\frac{\beta}{\sqrt{x}}}, \quad x \in (0, x_0),$$

so that $v = u - w$ and consequently

$$\begin{aligned} \mathcal{E}v = & -x^2 u'' + axu' + bu \\ & + x^2 w'' - axw' - bw \\ & + c\big(u' - w' - 1\big)^2 \quad \text{in } (0, x_0). \end{aligned}$$

Since $(u' - w' - 1)^2 = (u'-1)^2 - 2(u'-1)w' + w'^2$, using that $\mathcal{E}u \equiv 0$ we see that

$$\mathcal{E}v = x^2 w'' - axw' - bw + 2c\big(u'-1\big)w' - cw'^2 \quad \text{in } (0, x_0). \tag{6.10}$$

We now compute

$$w'(x) = \varepsilon \frac{\beta}{2} x^{\alpha-\frac{3}{2}} e^{-\frac{\beta}{\sqrt{x}}} + \varepsilon \alpha x^{\alpha-1} e^{-\frac{\beta}{\sqrt{x}}}$$

and

$$\begin{aligned} w''(x) = \varepsilon \frac{\beta^2}{4} x^{\alpha-3} e^{-\frac{\beta}{\sqrt{x}}} + \varepsilon \left(\alpha - \frac{3}{2}\right)\frac{\beta}{2} x^{\alpha-\frac{5}{2}} e^{-\frac{\beta}{\sqrt{x}}} + \varepsilon \frac{\alpha\beta}{2} x^{\alpha-\frac{5}{2}} e^{-\frac{\beta}{\sqrt{x}}} \\ + \varepsilon\alpha(\alpha-1) x^{\alpha-2} e^{-\frac{\beta}{\sqrt{x}}} \end{aligned}$$

and hence obtain from (6.10) that

$$\mathcal{E}v = \varepsilon e^{-\frac{\beta}{\sqrt{x}}} \cdot \left\{ \frac{\beta^2}{4} x^{\alpha-1} + \left(\alpha - \frac{3}{2}\right)\frac{\beta}{2} x^{\alpha-\frac{1}{2}} + \frac{\alpha\beta}{2} x^{\alpha-\frac{1}{2}} + \alpha(\alpha-1)x^{\alpha} \right.$$
$$- \frac{a\beta}{2} x^{\alpha-\frac{1}{2}} - a\alpha x^{\alpha} - bx^{\alpha} + 2c \cdot \frac{u'-1}{\sqrt{x}} \cdot \sqrt{x} \cdot \left[\frac{\beta}{2} x^{\alpha-\frac{3}{2}} + \alpha x^{\alpha-1}\right]$$
$$\left. - \varepsilon c \left[\frac{\beta}{2} x^{\alpha-\frac{3}{2}} + \alpha x^{\alpha-1}\right]^2 e^{-\frac{\beta}{\sqrt{x}}} \right\} \quad \text{in } (0, x_0).$$

On straightforward rearrangements, this yields (6.9). □

We can now identify an appropriate family of subsolutions for (1.1).

Lemma 6.3 *Suppose that for some* $x_0 > 0$, u *is a solution of* (1.1), (1.2) *in* $(0, x_0)$ *satisfying* (6.1) *with some* $l > 0$. *Then for any real number* $\alpha < \frac{3}{2} + a - 2cl$ *there exists* $x_3 = x_3(\alpha) \in (0, x_0)$ *such that for each* $\varepsilon > 0$, *the function* $\underline{v}$ *defined by*

$$\underline{v}(x) := u(x) - \varepsilon x^{\alpha} e^{-\frac{4ck}{\sqrt{x}}}, \quad x \in (0, x_3), \tag{6.11}$$

satisfies

$$\mathcal{E}\underline{v} < 0 \quad \textit{in } (0, x_3). \tag{6.12}$$

Proof Given $\alpha < \frac{3}{2} + a - 2cl$, thanks to the positivity of c and k we can find $x_1 \in (0, x_0)$ such that both

$$\frac{2ck}{\sqrt{x_1}} + \alpha \ge 0 \tag{6.13}$$

and

$$\frac{2ck \cdot (\frac{3}{2} + a - 2cl - \alpha)}{\sqrt{x_1}} > \alpha(\alpha-1) - a\alpha - b + 2cl\alpha \tag{6.14}$$

hold. Then for $\varepsilon > 0$ we define $\underline{v}$ as in (6.11) and thus obtain from Lemma 6.2, applied to $\beta := 4ck$, that (6.9) holds for $v = \underline{v}$ and all $x \in (0, x_1)$. In order to utilize (6.1) appropriately, we note that in view of (6.13) we have

$$\frac{\beta}{2} x^{\alpha-1} + \alpha x^{\alpha-\frac{1}{2}} = \left(\frac{2ck}{\sqrt{x}} + \alpha\right) \cdot x^{\alpha-\frac{1}{2}} \ge 0 \quad \text{for all } x \in (0, x_1).$$

We may thus multiply this by the inequality

$$\frac{u'(x) - 1}{\sqrt{x}} \le -k + l\sqrt{x} \quad \text{for all } x \in (0, x_1),$$

as resulting from (6.1), to infer from (6.9) on dropping a nonpositive term that

$$\mathcal{E}\underline{v} \le \varepsilon e^{-\frac{\beta}{\sqrt{x}}} \cdot \left\{ \frac{\beta^2}{4} x^{\alpha-1} + \left[\left(\alpha - \frac{3}{2}\right)\frac{\beta}{2} + \frac{\alpha\beta}{2} - \frac{a\beta}{2} \right] \cdot x^{\alpha-\frac{1}{2}} \right.$$
$$\left. + \left[\alpha(\alpha-1) - a\alpha - b\right] \cdot x^{\alpha} + 2c(-k + l\sqrt{x}) \cdot \left[\frac{\beta}{2} x^{\alpha-1} + \alpha x^{\alpha-\frac{1}{2}} \right] \right\}$$
$$= \varepsilon e^{-\frac{\beta}{\sqrt{x}}} \cdot \left\{ \left[\frac{\beta^2}{4} - ck\beta \right] \cdot x^{\alpha-1} \right.$$
$$+ \left[\left(\alpha - \frac{3}{2}\right)\frac{\beta}{2} + \frac{\alpha\beta}{2} - \frac{a\beta}{2} - 2ck\alpha + cl\beta \right] \cdot x^{\alpha-\frac{1}{2}}$$
$$\left. + \left[\alpha(\alpha-1) - a\alpha - b + 2cl\alpha\right] \cdot x^{\alpha} \right\} \quad \text{in } (0, x_1).$$

Here the leading term disappears because $\beta = 4ck$, whereas

$$\left(\alpha - \frac{3}{2}\right)\frac{\beta}{2} + \frac{\alpha\beta}{2} - \frac{a\beta}{2} - 2ck\alpha + cl\beta = 2ck \cdot \left[\alpha - \frac{3}{2} + \alpha - a - \alpha + 2cl \right]$$
$$= 2ck \cdot \left[\alpha - \frac{3}{2} - a + 2cl \right] < 0.$$

Hence, using (6.14) we infer that

$$\mathcal{E}\underline{v} \le \varepsilon e^{-\frac{\beta}{\sqrt{x}}} \cdot x^{\alpha} \cdot \left\{ -\frac{2ck \cdot (\frac{3}{2} + a - 2cl - \alpha)}{\sqrt{x}} + \alpha(\alpha-1) - a\alpha - b + 2cl\alpha \right\}$$
$$< 0 \quad \text{in } (0, x_1),$$

as desired. □

Our construction of supersolutions to (1.1) is similar.

Lemma 6.4 *Let u be a solution of (1.1), (1.2) in $(0, x_0)$ for some $x_0 > 0$, which satisfies (6.1) with some $l > 0$. Then for each $\varepsilon_0 > 0$ and $\alpha > \frac{3}{2} + a + 2cl$ one can pick $x_4 = x_4(\alpha) \in (0, x_0)$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the function $\overline{v}$ defined by*

$$\overline{v}(x) := u(x) - \varepsilon x^{\alpha} e^{-\frac{4ck}{\sqrt{x}}}, \quad x \in (0, x_4), \tag{6.15}$$

satisfies

$$\mathcal{E}\overline{v} > 0 \quad \text{in } (0, x_4). \tag{6.16}$$

Proof Relying on our assumption on α, let us first pick $x_4 \in (0, x_0)$ such that besides again

$$\frac{2ck}{\sqrt{x_3}} + \alpha \ge 0, \tag{6.17}$$

the inequality

$$\frac{2ck(\alpha-\frac{3}{2}-a-2cl)}{\sqrt{x_4}} > \left|\alpha(\alpha-1)-a\alpha-b-2cl\alpha\right| + c\cdot\sup_{x\in(0,x_4)}\left\{x^{-\alpha}\cdot\left[2ckx^{\alpha-\frac{3}{2}}+\alpha x^{\alpha-1}\right]^2\cdot e^{-\frac{4ck}{\sqrt{x}}}\right\} \quad (6.18)$$

holds. Then writing $\beta := 4ck$, we again have $\frac{\beta}{2}x^{\alpha-1}+\alpha x^{\alpha-\frac{1}{2}} \ge 0$ in $(0,x_4)$. Hence, using that (6.1) implies that

$$\frac{u'(x)-1}{\sqrt{x}} \ge -k-l\sqrt{x} \quad \text{for all } x\in(0,x_0),$$

from (6.9) we obtain

$$\begin{aligned}\mathcal{E}\overline{v} \ge \varepsilon e^{-\frac{4ck}{\sqrt{\beta}}}\cdot\Bigg\{&\frac{\beta^2}{4}x^{\alpha-1}+\left[\left(\alpha-\frac{3}{2}\right)\frac{\beta}{2}+\frac{\alpha\beta}{2}-\frac{a\beta}{2}\right]\cdot x^{\alpha-\frac{1}{2}}\\ &+\left[\alpha(\alpha-1)-a\alpha-b\right]\cdot x^\alpha+2c\cdot(-k-l\sqrt{x})\cdot\left[\frac{\beta}{2}x^{\alpha-1}+\alpha x^{\alpha-\frac{1}{2}}\right]\\ &-\varepsilon_0 c\cdot\left[\frac{\beta}{2}x^{\alpha-\frac{3}{2}}+\alpha x^{\alpha-1}\right]^2\cdot e^{-\frac{\beta}{\sqrt{x}}}\Bigg\} \quad \text{in } (0,x_4).\end{aligned}$$

Since $\beta = 4ck$, this reduces to

$$\begin{aligned}\mathcal{E}\overline{v} \ge \varepsilon e^{-\frac{4ck}{\sqrt{x}}}\cdot x^\alpha\cdot\Bigg\{&\frac{2ck(\alpha-\frac{3}{2}-a-2cl)}{\sqrt{x}}+\alpha(\alpha-1)-a\alpha-b-2cl\alpha\\ &-\varepsilon_0 c x^{-\alpha}\cdot\left[2ckx^{\alpha-\frac{3}{2}}+\alpha x^{\alpha-1}\right]^2\cdot e^{-\frac{4ck}{\sqrt{x}}}\Bigg\} \quad \text{in } (0,x_4),\end{aligned}$$

and hence (6.18) asserts (6.16). □

By means of Proposition 2.2, we can now infer the existence of infinitely many classes of continua of local solutions to (1.1), (1.2). Here we shall strongly rely on the fact that the numbers α in Lemmas 6.3 and 6.4 can be chosen in such a way that the functions $\underline{v}$ and $\overline{v}$ defined in (6.11) and (6.15) are ordered appropriately.

Proposition 6.5 *Suppose that for some* $x_0>0$, *u is a solution of* (1.1), (1.2) *in* $(0,x_0)$ *satisfying* (6.1) *with some* $l>0$. *Then for all* $\underline{\alpha}\in(-\infty,\frac{3}{2}+a-2cl)$ *and* $\overline{\alpha}\in(\frac{3}{2}+a+2cl,\infty)$ *there exists* $\widehat{x}_0\in(0,x_0)$ *such that given* $\varepsilon\in(0,1)$ *and any* $\widehat{u}_0$ *satisfying*

$$\widehat{u}_0\in\left(u(x)-\varepsilon x^{\underline{\alpha}}e^{-\frac{4ck}{\sqrt{x}}},\ u(x)-\varepsilon x^{\overline{\alpha}}e^{-\frac{4ck}{\sqrt{x}}}\right) \quad \textit{at } x=\widehat{x}_0,$$

(1.1), (1.2) *possesses a solution* $\widehat{u}$ *on* $(0,\widehat{x}_0)$ *fulfilling*

$$u(x)-\varepsilon x^{\underline{\alpha}}e^{-\frac{4ck}{\sqrt{x}}}\le\widehat{u}(x)\le u(x)-\varepsilon x^{\overline{\alpha}}e^{-\frac{4ck}{\sqrt{x}}} \quad \textit{for all } x\in(0,\widehat{x}_0) \quad (6.19)$$

as well as $\widehat{u}(\widehat{x}_0)=\widehat{u}_0$.

Proof We invoke Lemmas 6.3 and 6.4 with $\varepsilon_0 := 1$ to obtain $x_3 \in (0, x_0)$ and $x_4 \in (0, x_0)$ such that for any $\varepsilon \in (0, 1)$, the functions $\underline{v}$ and $\overline{v}$ defined by $\underline{v}(x) := u(x) - \varepsilon x^{\underline{\alpha}} e^{-\frac{4ck}{\sqrt{x}}}$, $x \in (0, x_3)$, and $\overline{v}(x) := u(x) - \varepsilon x^{\overline{\alpha}} e^{-\frac{4ck}{\sqrt{x}}}$, $x \in (0, x_4)$, have the properties $\mathcal{E}\underline{v} < 0$ on $(0, x_3)$ and $\mathcal{E}\overline{v} > 0$ on $(0, x_4)$. Writing $\widehat{x}_0 := \min\{1, x_3, x_4\}$, we moreover see using $\underline{\alpha} < \overline{\alpha}$ that $\underline{v} < \overline{v}$ throughout $(0, \widehat{x}_0)$. Therefore the claim results upon an application of Proposition 2.2(i). □

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References

1. Barles, G.: Convergence of numerical schemes for degenerate parabolic equations arising in finance theory. In: Rogers, L.C.G., Talay, D. (eds.) Numerical Methods in Finance, pp. 1–21 (1997)
2. Bernstein, C.: Sur certaines équations différentielles ordinaires du second ordre. C. R. Math. Acad. Sci. **138**, 950–951 (1904)
3. Černý, A.: Currency crises: introduction of spot speculators. J. Financ. Econom. **4**, 75–89 (1999)
4. Cid, J.Á., López-Pouso, O., López-Pouso, R.: Existence of infinitely many solutions for second-order singular initial value problems with an application to nonlinear massive gravity. Nonlinear Anal., Real World Appl. **12**, 2596–2606 (2011)
5. de Coster, C., Habets, P.: Two-point boundary value problems: upper and lower solutions. In: Mathematics in Science and Engineering, vol. 205. Elsevier, Amsterdam (2006)
6. Hartman, P.: Ordinary Differential Equations. Wiley, New York (1964)
7. Liang, J.: A singular initial value problem and self-similar solutions of a nonlinear dissipative wave equation. J. Differ. Equ. **246**, 819–844 (2009)
8. Nagumo, M.: Über die Differentialgleichung $y'' = f(x, y, y')$. Proc. Phys. Math. Soc. Jpn. **19**, 861–866 (1937)
9. Wirl, F.: Energy prices and carbon taxes under uncertainty about global warming. Environ. Resour. Econ. **36**, 313–340 (2007)