Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system

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Abstract

We study the Neumann initial-boundary value problem for the fully parabolic Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega, \ t > 0, \end{cases}$$

where Ω is a ball in $\mathbb{R}^n$ with $n \geq 3$.
It is proved that for any prescribed $m > 0$ there exist radially symmetric positive initial data $(u_0, v_0) \in C^0(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ with $\int_\Omega u_0 = m$ such that the corresponding solution blows up in finite time. Moreover, by providing an essentially explicit blow-up criterion it is shown that within the space of all radial functions, the set of such blow-up enforcing initial data indeed is large in an appropriate sense; in particular, this set is dense with respect to the topology of $L^p(\Omega) \times W^{1,2}(\Omega)$ for any $p \in (1, \frac{2n}{n+2})$.

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1 Introduction

We consider the parabolic initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$, with given nonnegative initial data u_0 and v_0.
In a celebrated work by Keller and Segel ([21]), this system was proposed as a macroscopic model for chemotactic cell migration, that is, for the motion of cells which, besides diffusing randomly, partly orient their movement towards increasing concentrations of a chemical signal substance. In this context, $u = u(x,t)$ then denotes the cell density, whereas $v = v(x,t)$ represents the concentration of the chemical.

In prototypical processes such as the collective behavior of the slime mold *Dictyostelium Discoideum*, the signal is produced by the cells themselves, and the possibly most striking consequence thereof appears to be the ability of cell populations to spontaneously form aggregates in small spatial regions after a finite time. Correspondingly, verifying the validity of any mathematical model for such processes is closely linked to investigating its capability to adequately describe such phenomena of self-organization. A commonly accepted mathematical concept for this consists of identifying the emergence of aggregation with the collapse of the corresponding solution into a singularity with respect to the norm in $L^\infty(\Omega)$ ([17], [30]).

The challenge of proving blow-up. Accordingly, since the work of Keller and Segel extensive mathematical efforts have been undertaken to detect unbounded solutions in (1.1) or, more generally, to determine conditions on the initial data and on certain parameters which either guarantee or rule out the existence of such blow-up solutions in (1.1) and related models.
It turned out that the mathematical difficulties linked to the subtle task of finding unbounded solutions can significantly be reduced upon replacing (1.1) with associated parabolic-elliptic variants, the second equation of which being either

$$0 = \Delta v - v + u, \qquad x \in \Omega,\ t > 0, \tag{1.2}$$

or

$$0 = \Delta v - \overline{m} + u, \qquad x \in \Omega,\ t > 0. \tag{1.3}$$

Here $\overline{m} := \frac{m}{|\Omega|}$, where $m := \int_\Omega u_0$ denotes the total mass of cells which remains constant in time in the sense that $\int_\Omega u(x,t)dx \equiv m$ for $t > 0$. According to the applicability of much a larger repertoire of mathematical tools, the knowledge for the resulting simplified systems has been rather complete in respect of the occurrence of blow-up for quite a while. In fact, it was shown in [20], [26] and [28] that the corresponding initial-boundary value problems in the spatially two-dimensional setting indeed possess some solutions which blow up in finite time provided that the mass m is large enough and concentrated around some point to a suitable extent, whereas if m is small then solutions remain bounded; the precise threshold values for the mass could be identified as 8π in the radially symmetric

setting and 4π in the general case (cf. [26], [28] and [1], for instance, and also [17] for a survey). In the three-dimensional framework, finite-time blow-up in the systems related to (1.2) and (1.3) may occur for arbitrarily small values of m, meaning that no mass threshold for aggregation exists in that case ([27], [13]).
Actually, the understanding of these simplified systems is even elaborate: For instance, for both the two- and three-dimensional cases the studies in [14] and [13] provide examples of unbounded solutions the asymptotic behavior of which can be described rather precisely near their blow-up time.

In contrast to this, the knowledge on the full parabolic-parabolic system (1.1) appears to be much less comprehensive except for the case $n = 1$ where blow-up is entirely ruled out ([31]). For example, a counterpart of the above two-dimensional mass threshold phenomenon could rigorously be proved to exist only in a weakened sense. Namely, it is known that all solutions of (1.1) with mass satisfying $m < 4\pi$ remain bounded for all times ([12], [29]), while for any $\varepsilon > 0$ there exist some unbounded solutions mith mass $m < 4\pi + \varepsilon$ ([18]). In general it is not known, however, whether the blow-up time of the latter solutions is finite or infinite; it is thus conceivable that these solutions are global in time and become unbounded only in the large time limit. Only in the radial symmetric setting certain particular solutions of (1.1) satisfying $m > 8\pi$ were constructed which blow up in finite time and, moreover, their asymptotic behavior near the blow-up time was described in [15]. However, this does not clarify whether or not this phenomenon is exceptional; indeed, nothing is known about the size or the structure of the set of initial data enforcing finite-time blow-up.
Corresponding confinements in the analogy with the parabolic-elliptic systems appear also in the higher-dimensional setting: It has been shown in [34] that when $n \geq 3$, (1.1) possesses unbounded solutions with arbitrarily small total mass of cells, but again it has been left open there whether or not the associated blow-up time is finite. To the best of our knowledge, not even a single example of a solution of (1.1) which undergoes finite-time blow-up is known in the case $n \geq 3$.

As a more general observation, let us note that due to their diffusive and hence essentially dissipative structure, systems of type (1.1) are amenable to various powerful techniques of regularity theory. This becomes manifest in the literature not only on the original model (1.1), but also on quite a large variety of related models involving, for instance, different mechanisms of diffusion and chemotactic cross-diffusion in the equation for the cell density. Namely, there is a fairly rich literature addressing boundedness issues in such systems of both parabolic-parabolic and parabolic-elliptic type (see e.g. [32], [22], [23], [24], [9], [4], [19], [33] and also the survey [16]). In some special cases even more subtle analytical results on bounded solutions are available, such as e.g. on attractors ([36]) or on two-dimensional forward self-similar solutions with supercritical mass (cf. [2] and the references therein). As opposed to this, the few results addressing blow-up also in the setting of nonlinear diffusion concentrate on parabolic-elliptic simplifications (see [5], [6], [3], [8], [10], for instance) or on unboundedness in possibly infinite time ([19], [35]), the apparently only exception being a recent result on finite-time blow-up in a quasilinear one-dimensional Keller-Segel system with sufficiently weak nonlinear diffusion of cells and sufficiently fast diffusion of chemoattractant ([7]).

Main results. In view of the underlying biological background, we find it worthwhile to firstly investigate whether cell aggregation in the mathematically extreme flavor of finite-time blow-up at all occurs in (1.1) also in the three-dimensional setting, and secondly to make sure whether this is a rare phenomenon which can only be expected under very special assumptions on the initial framework, or whether the set of initial data enforcing a finite-time collapse is rich in an appropriate sense. Accord-

ingly, the purpose of the present paper consists of deriving sufficient conditions on (u_0, v_0) which lead to finite-time blow-up of solutions to (1.1) in the more general case $n \geq 3$.
In order to formulate our main results in this direction, and thereby demonstrate that moreover these criteria will be essentially explicit, let us recall that any solution of (1.1) satisfies the energy inequality

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t), v(\cdot,t)) \leq -\mathcal{D}(u(\cdot,t), v(\cdot,t)) \qquad \text{for all } t \in (0, T_{max}(u_0, v_0)) \tag{1.4}$$

where $T_{max}(u_0, v_0) \in (0, \infty]$ denotes the maximum existence time of (u, v) and where, for arbitrary smooth positive functions u and v, the energy is defined by

$$\mathcal{F}(u, v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u \tag{1.5}$$

and the dissipation rate is given by

$$\mathcal{D}(u, v) := \int_\Omega v_t^2 + \int_\Omega u \cdot \Big|\frac{\nabla u}{u} - \nabla v\Big|^2 \tag{1.6}$$

(see [29], [34] and also Lemma 2.1 below).

This energy inequality plays an essential role in deriving boundedness of solutions in subcritical cases (see [29], for instance). But also the detection of unbounded (possibly global) solutions in [18] and [34] crucially relies on the use of (1.4) through an indirect argument: Indeed, if $\mathcal{F}(u_0, v_0)$ is a sufficiently large negative number then (u, v) cannot be both global and bounded, because due to (1.4) any such solution must approach a low-energy equilibrium which ruled out by a corresponding a-priori estimate for energies of steady states. This type of reasoning, well-established in scalar parabolic problems ([25]) and also applicable in larger classes of quasilinear Keller-Segel models ([18], [19], [35]), evidently cannot give any information on the actual blow-up time, and hence cannot rule out the possibility that $T_{max}(u_0, v_0) = \infty$. However, it will turn out in this work that a more subtle analysis of (1.4) can be used to derive a basically explicit sufficient condition on the initial data which ensure that finite-time blow-up occurs. More precisely, the first of our main results reads as follows.

Theorem 1.1 *Let $\Omega = B_R \subset \mathbb{R}^n$ with some $n \geq 3$ and $R > 0$, and let $m > 0$ and $A > 0$. Then there exist $T(m, A) > 0$ and $K(m, A) > 0$ with the property that given any (u_0, v_0) from the set*

$$\begin{aligned}\mathcal{B}(m, A) \quad := \quad & \Big\{(u_0, v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega) \ \Big| \ u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } \bar\Omega \\ & \qquad \text{with } \textstyle\int_\Omega u_0 = m,\ \|v_0\|_{W^{1,2}(\Omega)} \leq A \text{ and } \mathcal{F}(u_0, v_0) \leq -K(m, A)\Big\},\end{aligned} \tag{1.7}$$

for the corresponding solution (u, v) of (1.1) we have $T_{max}(u_0, v_0) \leq T(m, A) < \infty$; that is, (u, v) blows up before or at time $T(m, A)$.

Secondly, we shall address the question in how far the above set of low-energy initial data can be considered large. In fact, this set turns out to be even dense in the space of positive functions in $C^0(\bar\Omega) \times W^{1,\infty}(\Omega)$ when equipped with an appropriate topology:

Theorem 1.2 *Let Ω be as in Theorem 1.1, and suppose that $p \in (1, \frac{2n}{n+2})$. Then for each $m > 0$ and $A > 0$, the set $\mathcal{B}(m, A)$ defined in (1.7) is dense in the space of all radially symmetric positive functions in $C^0(\bar\Omega) \times W^{1,\infty}(\Omega)$ with respect to the topology in $L^p(\Omega) \times W^{1,2}(\Omega)$.*
In particular, for any positive radial $(u_0, v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega)$ and any $\varepsilon > 0$ one can find some radial positive $(u_{0\varepsilon}, v_{0\varepsilon}) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega)$ such that

$$\|u_{0\varepsilon} - u_0\|_{L^p(\Omega)} + \|v_{0\varepsilon} - v_0\|_{W^{1,2}(\Omega)} < \varepsilon,$$

but such that the solution $(u_\varepsilon, v_\varepsilon)$ of (1.1) with initial data $(u_\varepsilon, v_\varepsilon)|_{t=0} = (u_{0\varepsilon}, v_{0\varepsilon})$ blows up in finite time.

Let us underline that to the best of our knowledge, this is the first result asserting the occurrence of finite-time blow-up in the Keller-Segel system (1.1) in space dimension $n \geq 3$. But Theorem 1.1 and Theorem 1.2 evidently go much further: For instance, they especially say that each of the constant steady states $(u, v) \equiv (c, c)$, $c > 0$, is highly unstable in that any of its neighborhoods in the above topology contains initial data which evolve into a singularity in finite time.

Plan of the paper. Our technical approach is based on the idea to estimate the dissipated quantity in (1.4) from below in order to turn (1.4) into an inequality of the form

$$\frac{d}{dt}\Big(-\mathcal{F}(u(\cdot,t), v(\cdot,t))\Big) \geq \Big(c \cdot (-\mathcal{F}(u(\cdot,t), v(\cdot,t))) - 1\Big)_+^\lambda \qquad \text{for all } t \in (0, T_{max}(u_0, v_0))$$

with some $\lambda > 1$ and $c > 0$. We shall thus be concerned with deriving an upper bound for $\mathcal{F}(u, v)$ in terms of a sublinear power of $\mathcal{D}(u, v)$, and the main step towards this will be provided by the estimate

$$\int_\Omega uv \leq C \cdot \left(\Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{2\theta} + \Big\|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big\|_{L^2(\Omega)} + 1\right)$$

with some $\theta \in (0, 1)$ and $C > 0$, to be given in Lemma 4.1. Here, u and v will be allowed to be rather arbitrary smooth positive radial functions satisfying appropriate mass constraints and an additional pointwise upper estimate for v that can be shown to be fulfilled by the component $v(\cdot, t)$ for any $t \in (0, T_{max}(u_0, v_0))$ of any solution of (1.1) in question (Corollary 3.3).
The application to the parabolic problem is then straightforward (see Section 5), whereas Theorem 1.2 will be proved using an explicit construction of appropriate initial data in Section 6.

2 Preliminaries

To begin with, let us collect some basic statements on local well-posedness and elementary properties of solutions to (1.1).

Lemma 2.1 *Let $(u_0, v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega)$ be radially symmetric and positive in $\bar\Omega$, and fix $q \in (n, \infty)$. Then there exist $T_{max}(u_0, v_0) \in (0, \infty]$ and a uniquely determined couple (u, v) of radially symmetric functions, satisfying the inclusions*

$$u \in C^0([0, T_{max}(u_0, v_0)); C^0(\bar\Omega)) \cap C^{2,1}(\bar\Omega \times (0, T_{max}(u_0, v_0))) \qquad \textit{and}$$
$$v \in C^0([0, T_{max}(u_0, v_0)); W^{1,q}(\Omega)) \cap C^{2,1}(\bar\Omega \times (0, T_{max}(u_0, v_0))),$$

which solves (1.1) classically in $\Omega \times (0, T_{max}(u_0, v_0))$ and has the extensibility property

$$\textit{either } T_{max}(u_0, v_0) = \infty, \textit{ or } \|u(\cdot, t)\|_{L^\infty(\Omega)} \to \infty \qquad \textit{as } t \nearrow T_{max}(u_0, v_0).$$

In addition, this solution fulfils

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 \qquad \textit{for all } t \in (0, T_{max}(u_0, v_0)) \tag{2.1}$$

and

$$\int_\Omega v(x,t)dx \le \max\Big\{ \int_\Omega u_0, \int_\Omega v_0 \Big\} \qquad \textit{for all } t \in (0, T_{max}(u_0, v_0)), \tag{2.2}$$

and moreover the energy inequality (1.4) holds.

Proof. The statements concerning existence, uniqueness, regularity and extensibility are well-known, and thus for details covering the present and more general frameworks we may refer the reader to [19], [1] and [37], for instance.
The identity (2.1) immediately follows from integration of the first equation in (1.1) in space, whereupon integrating the second one yields

$$\frac{d}{dt} \int_\Omega v(x,t)dx = -\int_\Omega v(x,t)dx + \int_\Omega u_0 \qquad \text{for all } t \in (0, T_{max}(u_0, v_0)).$$

Combined with a straightformward ODE comparison, this proves (2.2). $\Box$

The following consequences of the Gagliardo-Nirenberg inequality and Young's inequality are immediate. Since they will be used in several places in the sequel, let us briefly state them separately and refer to [11] for the underlying interpolation estimates.

Lemma 2.2 *There exists $C > 0$ such that*

$$\|\varphi\|_{L^2(\Omega)} \le C \|\nabla\varphi\|_{L^2(\Omega)}^{\frac{n}{n+2}} \|\varphi\|_{L^1(\Omega)}^{\frac{2}{n+2}} + C\|\varphi\|_{L^1(\Omega)} \qquad \textit{for all } \varphi \in W^{1,2}(\Omega). \tag{2.3}$$

Moreover, for each $\varepsilon > 0$ one can find $C(\varepsilon) > 0$ with the property that

$$\|\varphi\|_{L^2(\Omega)}^2 \le \varepsilon \|\nabla\varphi\|_{L^2(\Omega)}^2 + C(\varepsilon)\|\varphi\|_{L^1(\Omega)}^2 \qquad \textit{for all } \varphi \in W^{1,2}(\Omega). \tag{2.4}$$

3 A pointwise upper bound for solutions of (1.1)

Let us first adapt a basically well-known regularity property of the second solution component v which is a straightforward consequence of standard parabolic regularity arguments and thereby it does in fact not require any symmetry assumption on the initial data.

Lemma 3.1 *Let $p \in (1, \frac{n}{n-1})$. Then there exists $C(p) > 0$ such that for any choice of positive functions $u_0 \in C^0(\bar\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$, the solution of (1.1) satisfies*

$$\|\nabla v(\cdot, t)\|_{L^p(\Omega)} \le C(p) \cdot \Big(\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)} \Big) \qquad \textit{for all } t \in (0, T_{max}(u_0, v_0)). \tag{3.1}$$

Proof. It s well-known (cf. [34], for instance) that the Neumann heat semigroup $(e^{t\Delta})_{t\ge0}$ in Ω has the property

$$\|\nabla e^{t\Delta}\varphi\|_{L^p(\Omega)} \le c_1 t^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{p})}\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in L^1(\Omega) \tag{3.2}$$

with some $c_1>0$. Moreover, since it can easily be checked that $\frac{d}{dt}\|\nabla e^{t\Delta}\varphi\|^2_{L^2(\Omega)}\le 0$, using the Hölder inequality along with the fact that $p<2$ we can find $c_2>0$ such that

$$\|\nabla e^{t\Delta}\varphi\|_{L^p(\Omega)} \le c_2\|\nabla e^{t\Delta}\varphi\|_{L^2(\Omega)} \le c_2\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi\in W^{1,2}(\Omega). \tag{3.3}$$

Applying (3.2) and (3.3) to the variation-of-constants representation of v,

$$v(\cdot,t)=e^{t(\Delta-1)}v_0+\int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds, \qquad t\in(0,T_{max}(u_0,v_0)),$$

we obtain

$$\begin{aligned}\|\nabla v(\cdot,t)\|_{L^p(\Omega)} &\le c_2\|\nabla v_0\|_{L^2(\Omega)}+c_1\int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{p})}\cdot e^{-(t-s)}\|u(\cdot,s)\|_{L^1(\Omega)}ds\\ &= c_2\|\nabla v_0\|_{L^2(\Omega)}+c_1\|u_0\|_{L^1(\Omega)}\cdot\int_0^t \sigma^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{p})}e^{-\sigma}d\sigma\end{aligned}$$

for all $t\in(0,T_{max}(u_0,v_0))$,

because $\|u(\cdot,t)\|_{L^1(\Omega)}=\|u_0\|_{L^1(\Omega)}$ for all $t\in(0,T_{max}(u_0,v_0))$ by (2.1). Now since our restriction $p<\frac{n}{n-1}$ ensures that $\frac{1}{2}+\frac{n}{2}(1-\frac{1}{p})<1$, this implies

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le c_2\|\nabla v_0\|_{L^2(\Omega)}+c_1\|u_0\|_{L^1(\Omega)}\cdot\Gamma\Big(\frac{1}{2}-\frac{n}{2}\Big(1-\frac{1}{p}\Big)\Big) \qquad \text{for all } t\in(0,T_{max}(u_0,v_0))$$

and thereby proves (3.1). □

As a consequence of Lemma 3.1 and (2.2), in the case when (u,v) is radially symmetric we obtain a pointwise upper bound for v which is valid up to the blow-up time and hence gives a first, though rather rough, information on what might finally be called the spatial blow-up profile of (u,v).

Lemma 3.2 *Let $p\in(1,\frac{n}{n-1})$. Then there exists $C(p)>0$ such that whenever $u_0\in C^0(\bar\Omega)$ and $v_0\in W^{1,\infty}(\Omega)$ are positive and radially symmetric, the solution of (1.1) satisfies*

$$v(r,t)\le C(p)\cdot\Big(\|u_0\|_{L^1(\Omega)}+\|v_0\|_{L^1(\Omega)}+\|\nabla v_0\|_{L^2(\Omega)}\Big)\cdot r^{-\frac{n-p}{p}} \qquad \textit{for all } (r,t)\in(0,R)\times(0,T_{max}(u_0,v_0)). \tag{3.4}$$

Proof. Abbreviating $M:=\max\Big\{\int_\Omega u_0,\int_\Omega v_0\Big\}$, from (2.2) we know that $\int_\Omega v(\cdot,t)\le M$ for all $t\in(0,T_{max}(u_0,v_0))$. Thus, for each $t\in(0,T_{max}(u_0,v_0))$ we can pick some $r_0(t)\in(\frac{R}{2},R)$ such that

$$v(r_0(t),t)\le\frac{M}{|B_R\setminus B_{\frac{R}{2}}|},$$

for supposing the opposite would lead to the absurd conclusion

$$\int_\Omega v(\cdot,t) \ge \int_{B_R\setminus B_{\frac{R}{2}}} v(\cdot,t) > \int_{B_R\setminus B_{\frac{R}{2}}} \frac{M}{|B_R \setminus B_{\frac{R}{2}}|} = M.$$

Therefore, using the Hölder inequality and Lemma 3.1, we can find $c_1(p) > 0$ such that

$$\begin{aligned} v(r,t) - v(r_0(t),t) &= \int_{r_0(t)}^r v_r(\rho,t)d\rho \\ &\le \left| \int_{r_0(t)}^r \rho^{n-1}|v_r(\rho,t)|^p d\rho \right|^{\frac{1}{p}} \cdot \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} d\rho \right|^{\frac{p-1}{p}} \\ &\le c_1(p)\cdot \left(\|u_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)}\right) \cdot \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} d\rho \right|^{\frac{p-1}{p}} \end{aligned} \tag{3.5}$$

for all $(r,t) \in (0,R)\times(0,T_{max}(u_0,v_0))$. Now since $p < \frac{n}{n-1} < n$, for small r we can estimate

$$\begin{aligned} \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} d\rho \right|^{\frac{p-1}{p}} &= \left(\frac{r^{-\frac{n-p}{p-1}} - (r_0(t))^{-\frac{n-p}{p-1}}}{\frac{n-p}{p-1}} \right)^{\frac{p-1}{p}} \\ &\le \left(\frac{p-1}{n-p}\right)^{\frac{p-1}{p}} \cdot r^{-\frac{n-p}{p}} \qquad \text{for all } r \in (0, r_0(t)], \end{aligned} \tag{3.6}$$

whereas for large r we similarly find that

$$\begin{aligned} \left| \int_{r_0(t)}^r \rho^{-\frac{n-1}{p-1}} d\rho \right|^{\frac{p-1}{p}} &\le \left(\frac{p-1}{n-p}\right)^{\frac{p-1}{p}} \cdot (r_0(t))^{-\frac{n-p}{p}} \le \left(\frac{p-1}{n-p}\right)^{\frac{p-1}{p}} \cdot \left(\frac{R}{2}\right)^{-\frac{n-p}{p}} \\ &\le \left(\frac{p-1}{n-p}\right)^{\frac{p-1}{p}} \cdot 2^{\frac{n-p}{p}} r^{-\frac{n-p}{p}} \qquad \text{for all } r \in (r_0(t), R), \end{aligned} \tag{3.7}$$

because $r_0(t) > \frac{R}{2}$. As finally

$$v(r_0(t),t) \le \frac{M}{|B_R\setminus B_{\frac{R}{2}}|} \le \frac{M}{|B_R\setminus B_{\frac{R}{2}}|} \cdot R^{\frac{n-p}{p}} r^{-\frac{n-p}{p}} \qquad \text{for all } r\in(0,R),$$

(3.5)-(3.7) imply (3.4). □

Adjusting p in Lemma 3.2 appropriately, we can achieve an estimate showing that the singularity of v can essentially not be stronger than that of the fundamental solution of the Laplacian.

Corollary 3.3 *Let $\kappa > n-2$. Then one can find $C(\kappa) > 0$ such that for all radially symmetric and positive functions $u_0 \in C^0(\bar\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$, the corresponding solution of (1.1) satisfies*

$$v(r,t) \le C(\kappa)\cdot\Big(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)}\Big)\cdot r^{-\kappa} \qquad \textit{for all } (r,t)\in(0,R)\times(0,T_{max}(u_0,v_0)). \tag{3.8}$$

Proof. Since $\kappa > n-2$, we have $\frac{n}{\kappa+1} < \frac{n}{n-1}$, so that it is possible to fix $p > 1$ such that $\frac{n}{\kappa+1} \le p < \frac{n}{n-1}$. An application of Lemma 3.2 then easily yields (3.8), because $p \ge \frac{n}{\kappa+1}$ implies $\frac{n-p}{p} \le \kappa$. □

4 An estimate for $\int_\Omega uv$ in terms of the dissipation rate

Guided by our knowledge on the solutions of (1.1) gained above, in asserting a lower estimate of the desired form

$$\frac{\mathcal{F}(u,v)}{\mathcal{D}^\theta(u,v)+1} \ge -C$$

with some $\theta \in (0,1)$ and $C > 0$, we shall concentrate henceforth on positive radial functions satisfying the mass constraints

$$\int_\Omega u = m \qquad \text{and} \qquad \int_\Omega v \le M \tag{4.1}$$

and the additional pointwise restriction

$$v(x) \le B|x|^{-\kappa} \qquad \text{for all } x \in \Omega, \tag{4.2}$$

where $m > 0, M > 0, B > 0$ and $\kappa > n-2$ are given fixed parameters.

More precisely, our goal will be to derive an inequality of the form

$$\frac{\mathcal{F}(u,v)}{\mathcal{D}^\theta(u,v)+1} \ge -C(m,M,B,\kappa) \qquad \text{for all } (u,v) \in \mathcal{S}(m,M,B,\kappa) \tag{4.3}$$

with some $\theta \in (0,1)$ and $C(m,M,B,\kappa) > 0$, where

$$\mathcal{S}(m,M,B,\kappa) := \Big\{(u,v) \in C^1(\bar\Omega) \times C^2(\bar\Omega) \ \Big| \ u \text{ and } v \text{ are positive and radially symmetric with } \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \text{ and such that (4.1) and (4.2) hold} \Big\} \tag{4.4}$$

and $\mathcal{F}$ and $\mathcal{D}$ are as defined in (1.5) and (1.6), respectively.
In view of the latter, establishing (4.3) essentially amounts to showing that the integral $\int_\Omega uv$ is bounded from above by a sublinear power of the sum of the norms in $L^2(\Omega)$ of the functions f and g which for convenience in subsequent notation are introduced by abbreviating

$$f := -\Delta v + v - u \tag{4.5}$$

and

$$g := \Big(\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big) \cdot \frac{x}{|x|} \qquad (\text{for } x \ne 0) \tag{4.6}$$

for $(u,v) \in \mathcal{S}(m,M,B,\kappa)$; since (u,v) is radial, these definitions actually reduces to the identities $f = -r^{1-n}(v^{n-1}v_r)_r + v - u$ and $g = \frac{u_r}{\sqrt{u}} - \sqrt{u}v_r$.
The essential step toward (4.3) will be contained in the following main result of this section.

Lemma 4.1 *There exists $C(m,M,B,\kappa) > 0$ such that for all $(u,v) \in \mathcal{S}(m,M,B,\kappa)$ we have*

$$\int_\Omega uv \le C(m,M,B,\kappa) \cdot \Big(\Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{2\theta} + \Big\|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big\|_{L^2(\Omega)} + 1\Big) \tag{4.7}$$

with $\theta \in (\frac{1}{2}, 1)$ given by (4.32).

The proof of Lemma 4.1 will be accomplished through a series of auxiliary statements. The first of these shows that proving (4.7) actually amounts to estimating $\int_\Omega |\nabla v|^2$.

Lemma 4.2 *There exists $C(M)>0$ such that for all $(u,v)\in\mathcal{S}(m,M,B,\kappa)$ we have*

$$\int_\Omega uv \le 2\int_\Omega |\nabla v|^2 + C(M)\cdot\Big(\Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1\Big). \tag{4.8}$$

PROOF. We use the notation in (4.5) and multiply the latter by v to obtain upon integrating by parts over Ω that

$$\int_\Omega uv = \int_\Omega |\nabla v|^2 + \int_\Omega v^2 - \int_\Omega fv. \tag{4.9}$$

In order to estimate the right-hand side appropriately, we note that by Lemma 2.2 and (4.1) there exists $c_1=c_1(M)>0$ and $c_2=c_2(M)>0$ such that

$$\|v\|_{L^2(\Omega)} \le c_1\Big(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + 1\Big) \tag{4.10}$$

and

$$\int_\Omega v^2 \le \frac{1}{2}\int_\Omega |\nabla v|^2 + c_2. \tag{4.11}$$

Furthermore, combining (4.10) with the Cauchy-Schwarz inequality and Young's inequality applied with exponents $\frac{2n+4}{n}$ and $\frac{2n+4}{n+4}$ provides $c_3=c_3(M)>0$ such that

$$\begin{aligned}
-\int_\Omega fv &\le \|f\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} \\
&\le c_1\cdot(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + 1)\cdot\|f\|_{L^2(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |\nabla v|^2 + c_3\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_1\|f\|_{L^2(\Omega)}.
\end{aligned} \tag{4.12}$$

Since $\frac{2n+4}{n+4}>1$, again by Young's inequality we can find $c_4=c_4(M)$ fulfilling

$$c_1\|f\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_4,$$

whereupon it becomes clear that (4.9), (4.11) and (4.12) imply (4.8). □

Accordingly, our next goal is to bound $\int_\Omega |\nabla v|^2$ appropriately. This will be done by splitting this expression into an integral over a small inner ball B_{r_0} and a corresponding outer annulus, the precise value of r_0 remaining at our disposal until it will be fixed in Lemma 4.5 below. Let us first concentrate on the outer region.

Lemma 4.3 *Let $r_0\in(0,R)$ and $\varepsilon\in(0,1)$. Then one can find a constant $C(\varepsilon,m,M,B,\kappa)>0$ such that for all $(u,v)\in\mathcal{S}(m,M,B,\kappa)$ the estimate*

$$\int_{\Omega\setminus B_{r_0}} |\nabla v|^2 \le \varepsilon\int_\Omega uv + \varepsilon\int_\Omega |\nabla v|^2 + C(\varepsilon,m,M,B,\kappa)\cdot\Big\{r_0^{-\frac{2n+4}{n}\kappa} + \Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}\Big\} \tag{4.13}$$

holds.

PROOF. We fix an arbitrary $\alpha \in (0,1)$. Then observing that $v \ge 0$, we may multiply the identity (4.5) defining f by v^α and integrate by parts over Ω to obtain

$$\alpha \int_\Omega v^{\alpha-1}|\nabla v|^2 + \int_\Omega v^{\alpha+1} = \int_\Omega uv^\alpha + \int_\Omega fv^\alpha. \tag{4.14}$$

Here we apply (4.2) and use the fact that $\alpha \in (0,1)$ to estimate

$$\alpha \int_\Omega v^{\alpha-1}|\nabla v|^2 \ge \alpha B^{\alpha-1} r_0^{(1-\alpha)\kappa} \cdot \int_{\Omega\setminus B_{r_0}} |\nabla v|^2$$

and thus infer from (4.14) upon dropping a nonnegative term that

$$\int_{\Omega\setminus B_{r_0}} |\nabla v|^2 \quad\le\quad \frac{B^{1-\alpha}}{\alpha} r_0^{-(1-\alpha)\kappa} \int_\Omega uv^\alpha \;+\; \frac{B^{1-\alpha}}{\alpha} r_0^{-(1-\alpha)\kappa} \int_\Omega fv^\alpha. \tag{4.15}$$

Now according to Young's inequality, to each $\eta > 0$ there corresponds some $c_1(\eta, B) > 0$ such that

$$\frac{B^{1-\alpha}}{\alpha} r_0^{-(1-\alpha)\kappa} v^\alpha(r) \le \eta v(r) + c_1(\eta, B) r_0^{-\kappa} \qquad \text{for all } r \in (0,R), \tag{4.16}$$

which applied to $\eta := \varepsilon$ yields

$$\begin{aligned} \frac{B^{1-\alpha}}{\alpha} r_0^{-(1-\alpha)\kappa} \int_\Omega uv^\alpha \quad &\le\quad \varepsilon \int_\Omega uv + c_1(\varepsilon, B) r_0^{-\kappa} \int_\Omega u \\ &=\quad \varepsilon \int_\Omega uv + c_1(\varepsilon, B) m r_0^{-\kappa} \\ &\le\quad \varepsilon \int_\Omega uv + c_1(\varepsilon, B) m R^{\frac{n+4}{n}\kappa} r_0^{-\frac{2n+4}{n}\kappa} \end{aligned} \tag{4.17}$$

in view of the nonnegativity of u and (4.1).
Moreover, an application of (4.16) to $\eta := 1$ shows that

$$\frac{B^{1-\alpha}}{\alpha} r_0^{-(1-\alpha)\kappa} \int_\Omega fv^\alpha \le \int_\Omega |f|v + c_1(1, B) r_0^{-\kappa} \int_\Omega |f|, \tag{4.18}$$

where by the Cauchy-Schwarz inequality we have

$$\int_\Omega |f|v \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \qquad \text{and} \qquad \int_\Omega |f| \le \sqrt{|\Omega|}\|f\|_{L^2(\Omega)}.$$

In order to further estimate the first expression, we invoke Lemma 2.2 which in conjunction with (4.1) provides $c_2(M) > 0$ such that

$$\|v\|_{L^2(\Omega)} \le c_2(M) \cdot \Big(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + 1 \Big) \le c_2(M) \cdot \Big(\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + R^\kappa r_0^{-\kappa} \Big),$$

whence (4.18) becomes

$$\frac{B^{1-\alpha}}{\alpha} r_0^{-(1-\alpha)\kappa} \int_\Omega fv^\alpha \le c_3(M, B, \kappa) \cdot \Big(\|f\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} + r_0^{-\kappa} \|f\|_{L^2(\Omega)} \Big)$$

with some $c_3(M,B,\kappa)>0$. Here by means of Young's inequality, we can find $c_4(\varepsilon,M,B,\kappa)>0$ and $c_5(M,B,\kappa)>0$ such that

$$c_3(M,B,\kappa)\|f\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}^{\frac{n}{n+2}} \le \varepsilon\|\nabla v\|_{L^2(\Omega)}^2 + c_4(\varepsilon,M,B,\kappa)\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}$$

and

$$c_3(M,B,\kappa)r_0^{-\kappa}\|f\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_5(M,B,\kappa)r_0^{-\frac{2n+4}{n}\kappa}.$$

Therefore, (4.18) all in all becomes

$$\frac{B^{1-\alpha}}{\alpha}r_0^{-(1-\alpha)\kappa}\int_\Omega fv^\alpha \le \varepsilon\int_\Omega|\nabla v|^2 + \Big(c_4(\varepsilon,M,B,\kappa)+1\Big)\cdot\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + c_5(M,B,\kappa)r_0^{-\frac{2n+4}{n}\kappa},$$

which combined with (4.17) and (4.15) yields (4.13). □

We next estimate ∇v in the corresponding interior part, where we emphasize the importance of the factor r_0 in the term $r_0\cdot\|\Delta v-v+u\|_{L^2(\Omega)}^2$ on the right-hand side of (4.19). Indeed, r_0 will eventually be chosen in dependence of $\|\Delta v-v+u\|_{L^2(\Omega)}$ in such a way that the above product essentially becomes a suitable *sub*quadratic power of $\|\Delta v-v+u\|_{L^2(\Omega)}$ (see Lemma 4.5). Let us also mention that as (4.30) will show, our assumption $n\ge 3$ is crucially needed here.

Lemma 4.4 *There exists $C(m)>0$ such that for any $r_0\in(0,R)$ and all $(u,v)\in\mathcal{S}(m,M,B,\kappa)$ we have*

$$\int_{B_{r_0}}|\nabla v|^2 \le C(m)\cdot\left\{r_0\cdot\Big\|\Delta v-v+u\Big\|_{L^2(\Omega)}^2 + \Big\|\frac{\nabla u}{\sqrt{u}}-\sqrt{u}\nabla v\Big\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}^2+1\right\}. \tag{4.19}$$

Proof. Abbreviating as in (4.5), we rewrite $-\Delta v+v=u+f$ in polar coordinates to see that

$$(r^{n-1}v_r)_r = -r^{n-1}u - r^{n-1}f + r^{n-1}v, \qquad r\in(0,R),$$

which we multiply by $r^{n-1}v_r$ to obtain

$$\frac{1}{2}\Big((r^{n-1}v_r)^2\Big)_r = -r^{2n-2}uv_r - r^{2n-2}fv_r + r^{2n-2}vv_r, \qquad r\in(0,R). \tag{4.20}$$

In the first term on the right, referring to the notation in (4.6)
we substitute $v_r=\frac{u_r}{u}-\frac{g}{\sqrt{u}}$, so that

$$-r^{2n-2}uv_r = -r^{2n-2}u_r + r^{2n-2}\sqrt{u}g, \qquad r\in(0,R), \tag{4.21}$$

whereas for the last term in (4.20) we clearly have

$$r^{2n-2}vv_r = \frac{1}{2}r^{2n-2}(v^2)_r, \qquad r\in(0,R). \tag{4.22}$$

As for the expression involving f, we pick any $\delta\in(0,\frac{2n-2}{R}]$ and apply Young's inequality to obtain

$$-r^{2n-2}fv_r \le \frac{\delta}{2}(r^{n-1}v_r)^2 + \frac{1}{2\delta}r^{2n-2}f^2, \qquad r\in(0,R). \tag{4.23}$$

In light of (4.21)-(4.23), (4.20) shows that $y(r) := (r^{n-1}v_r(r))^2$, $r \in [0,R]$, satisfies

$$y_r \le -2r^{2n-2}u_r + 2r^{2n-2}\sqrt{u}g + \delta y + \frac{1}{\delta}r^{2n-2}f^2 + r^{2n-2}(v^2)_r \qquad \text{for all } r \in (0,R).$$

Since $y(0) = 0$ thanks to the smoothness of v, an integration of this ODI yields

$$\begin{aligned} r^{2n-2}v_r^2(r) = y(r) \quad &\le \quad -2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}u_r(\rho)d\rho + 2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}\sqrt{u(\rho)}g(\rho)d\rho \\ &\quad +\frac{1}{\delta}\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}f^2(\rho)d\rho + \int_0^r e^{\delta(r-\rho)}\rho^{2n-2}(v^2)_r(\rho)d\rho \end{aligned} \tag{4.24}$$

for all $r \in (0,R)$.
Here an integration by parts gives

$$\begin{aligned} -2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}u_r(\rho)d\rho \quad &= \quad 4(n-1)\int_0^r e^{\delta(r-\rho)}\rho^{2n-3}u(\rho)d\rho \\ &\qquad -2\delta\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}u(\rho)d\rho - 2r^{2n-2}u(r) \\ &\le \quad 4(n-1)\int_0^r e^{\delta(r-\rho)}\rho^{2n-3}u(\rho)d\rho \\ &\le \quad 4(n-1)e^{\delta R}\int_0^r \rho^{2n-3}u(\rho)d\rho \qquad \text{for all } r \in (0,R), \end{aligned} \tag{4.25}$$

because u is nonnegative.
Next, the Cauchy-Schwarz inequality shows that

$$2\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}\sqrt{u(\rho)}g(\rho)d\rho \le 2\Big(\int_0^R \rho^{n-1}u(\rho)d\rho\Big)^{\frac{1}{2}} \cdot \Big(\int_0^r e^{2\delta(r-\rho)}\cdot\rho^{3n-3}g^2(\rho)d\rho\Big)^{\frac{1}{2}} \tag{4.26}$$

for all $r \in (0,R)$, where

$$\int_0^R \rho^{n-1}u(\rho)d\rho = \frac{m}{\omega_n} \tag{4.27}$$

and

$$\begin{aligned} \int_0^r e^{2\delta(r-\rho)}\cdot\rho^{3n-3}g^2(\rho)d\rho \quad &\le \quad e^{2\delta R}\cdot r^{2n-2}\int_0^R \rho^{n-1}g^2(\rho)d\rho \\ &= \quad e^{2\delta R}\cdot r^{2n-2}\cdot\frac{\|g\|^2_{L^2(\Omega)}}{\omega_n} \qquad \text{for all } r \in (0,R) \end{aligned} \tag{4.28}$$

with ω_n denoting the $(n-1)$-dimensional measure of ∂B_1.
By a similar idea using pointwise estimates, the second last term in (4.24) can be controlled according to

$$\begin{aligned} \frac{1}{\delta}\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}f^2(\rho)d\rho \quad &\le \quad \frac{e^{\delta R}}{\delta}\cdot r^{n-1}\cdot\int_0^R \rho^{n-1}f^2(\rho)d\rho \\ &= \quad \frac{e^{\delta R}}{\delta}\cdot r^{n-1}\cdot\frac{\|f\|^2_{L^2(\Omega)}}{\omega_n} \qquad \text{for all } r \in (0,R). \end{aligned} \tag{4.29}$$

Finally, upon another integration by parts we find that

$$\begin{aligned}\int_0^r e^{\delta(r-\rho)}\rho^{2n-2}(v^2)_r(\rho)d\rho &= r^{2n-2}v^2(r)\\ &\quad -\int_0^r e^{\delta(r-\rho)}\cdot\big[(2n-2)\rho^{2n-3}-\delta\rho^{2n-2}\big]\cdot v^2(\rho)d\rho\\ &\le r^{2n-2}v^2(r) \qquad \text{for all } r\in(0,R),\end{aligned}$$

because $(2n-2)\rho^{2n-3}\ge\delta\rho^{2n-2}$ for all $\rho\in(0,R)$ due to our restriction $\delta\le\frac{2n-2}{R}$. Combined with (4.24)-(4.29), this shows that

$$r^{2n-2}v_r^2(r)\le c_1(m)\int_0^r\rho^{2n-3}u(\rho)d\rho+c_1(m)r^{n-1}\|g\|_{L^2(\Omega)}+c_1(m)r^{n-1}\|f\|^2_{L^2(\Omega)}+r^{2n-2}v^2(r)$$

for all $r\in(0,R)$ with some $c_1(m)>0$. On division by r^{n-1} and integration over $r\in(0,r_0)$ we therefore obtain

$$\begin{aligned}\int_0^{r_0}r^{n-1}v_r^2(r)dr &\le c_1(m)\int_0^{r_0}\frac{1}{r^{n-1}}\cdot\int_0^r\rho^{2n-3}u(\rho)d\rho dr+c_1(m)r_0\|g\|_{L^2(\Omega)}\\ &\quad +c_1(m)r_0\|f\|^2_{L^2(\Omega)}+\int_0^{r_0}r^{n-1}v^2(r)dr\\ &\le c_1(m)\int_0^{r_0}\frac{1}{r^{n-1}}\cdot\int_0^r\rho^{2n-3}u(\rho)d\rho dr+c_1(m)R\|g\|_{L^2(\Omega)}\\ &\quad +c_1(m)r_0\|f\|^2_{L^2(\Omega)}+\frac{c_1(m)}{\omega_n}\|v\|^2_{L^2(\Omega)}.\end{aligned}$$

Here the Fubini theorem applies to show that

$$\begin{aligned}\int_0^{r_0}\frac{1}{r^{n-1}}\cdot\int_0^r\rho^{2n-3}u(\rho)d\rho dr &= \int_0^{r_0}\left(\int_\rho^{r_0}\frac{dr}{r^{n-1}}\right)\cdot\rho^{2n-3}u(\rho)d\rho\\ &= \frac{1}{n-2}\int_0^{r_0}(\rho^{2-n}-r_0^{2-n})\cdot\rho^{2n-3}u(\rho)d\rho\\ &\le \frac{1}{n-2}\int_0^{r_0}\rho^{n-1}u(\rho)d\rho\\ &\le \frac{m}{(n-2)\omega_n},\end{aligned}\tag{4.30}$$

regardless of the size of $r_0\in(0,R)$. In view of (4.5) and (4.6), this completes the proof. $\square$

A combination of the above two lemmata now yields an estimate for $\int_\Omega|\nabla v|^2$ that is adequate for our purpose.

Lemma 4.5 *For all $\varepsilon>0$ there exists $C(\varepsilon,m,M,B,\kappa)>0$ such that each $(u,v)\in\mathcal{S}(m,M,B,\kappa)$ satisfies*

$$\int_\Omega|\nabla v|^2\le\varepsilon\int_\Omega uv+C(\varepsilon,m,M,B,\kappa)\cdot\left(\Big\|\Delta v-v+u\Big\|^{2\theta}_{L^2(\Omega)}+\Big\|\frac{\nabla u}{\sqrt{u}}-\sqrt{u}\nabla v\Big\|_{L^2(\Omega)}+1\right),\tag{4.31}$$

where

$$\theta := \frac{1}{1+\frac{n}{(2n+4)\kappa}} \in \Big(\frac{1}{2},1\Big). \tag{4.32}$$

Proof. Let us set $\beta := \frac{(2n+4)\kappa}{n}$, so that $\theta = \frac{\beta}{\beta+1}$. Then given $\varepsilon \in (0,1)$, with notation as in (4.5) and (4.6) we apply Lemma 4.3 to $r_0 := \min\{\frac{R}{2}, \|f\|_{L^2(\Omega)}^{-\frac{2}{\beta+1}}\} \in (0,R)$ and thus obtain $c_1 = c_1(\varepsilon,m,M,B,\kappa) > 0$ such that

$$\int_{\Omega\setminus B_{r_0}} |\nabla v|^2 \le \frac{\varepsilon}{4}\int_\Omega uv + \frac{1}{2}\int_\Omega |\nabla v|^2 + c_1\cdot\Big(r_0^{-\beta} + \|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}\Big). \tag{4.33}$$

With this value of r_0 being fixed henceforth, Lemma 4.4 provides $c_2 = c_2(m) > 0$ such that

$$\int_{B_{r_0}} |\nabla v|^2 \le c_2\cdot\Big(r_0\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}^2 + 1\Big),$$

whence altogether we infer that

$$\int_\Omega |\nabla v|^2 \le \frac{\varepsilon}{2}\int_\Omega uv + 2c_1 r_0^{-\beta} + 2c_1\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 2c_2 r_0\|f\|_{L^2(\Omega)}^2 + 2c_2(\|g\|_{L^2(\Omega)}+1) + 2c_2\|v\|_{L^2(\Omega)}^2.$$

Here Lemma 2.2 and (4.1) say that for some $c_3 = c_3(M)$ we have

$$2c_2\|v\|_{L^2(\Omega)}^2 \le \frac{1}{2}\int_\Omega |\nabla v|^2 + c_3,$$

so that

$$\int_\Omega |\nabla v|^2 \le \varepsilon\int_\Omega uv + 4c_2(\|g\|_{L^2(\Omega)}+1) + 2c_3 + I, \tag{4.34}$$

where we abbreviate

$$I := 4c_1 r_0^{-\beta} + 4c_1\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 4c_2 r_0\|f\|_{L^2(\Omega)}^2.$$

Now in the case $\|f\|_{L^2(\Omega)} \le (\frac{2}{R})^{\frac{\beta+1}{2}}$ we have $r_0 = \frac{R}{2}$ and hence it follows that

$$I \le 4c_1\cdot\Big(\frac{2}{R}\Big)^\beta + 4c_1\cdot\Big(\frac{2}{R}\Big)^{\frac{\beta+1}{2}\cdot\frac{2n+4}{n+4}} + 4c_2\cdot\frac{R}{2}\cdot\Big(\frac{2}{R}\Big)^{\beta+1},$$

which clearly entails (4.31).
If, conversely, $\|f\|_{L^2(\Omega)} > (\frac{2}{R})^{\frac{\beta+1}{2}}$ and thus $r_0 = \|f\|_{L^2(\Omega)}^{-\frac{2}{\beta+1}}$ then

$$\begin{aligned} I &\le 4c_1\|f\|_{L^2(\Omega)}^{\frac{2\beta}{\beta+1}} + 4c_1\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 4c_2\|f\|_{L^2(\Omega)}^{2-\frac{2}{\beta+1}} \\ &= 4(c_1+c_2)\|f\|_{L^2(\Omega)}^{\frac{2\beta}{\beta+1}} + 4c_1\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}}. \end{aligned}$$

Since $\kappa > n-2$ implies

$$\frac{\beta}{\frac{n+2}{2}} = \frac{2}{n+2} \cdot \frac{(2n+4)\kappa}{n} > \frac{4(n-2)}{n} \geq \frac{4}{3} > 1$$

due to the fact that $n \geq 3$, it can easily be checked that $2\theta \equiv \frac{2\beta}{\beta+1} > \frac{2n+4}{n+4}$. This verifies the inclusion in (4.32), and furthermore Young's inequality yields $c_4 = c_4(\varepsilon, m, M, B, \kappa) > 0$ such that

$$I \leq 4(c_1 + c_2 + 1)\|f\|_{L^2(\Omega)}^{\frac{2\beta}{\beta+1}} + c_4,$$

so that (4.34) shows that (4.31) is also valid when $\|f\|_{L^2(\Omega)} > (\frac{2}{R})^{\frac{\beta+1}{2}}$. $\square$

Thereupon, the main result of this section actually reduces to a corollary.

Proof of Lemma 4.1. We only need to start from Lemma 4.2 and then apply Lemma 4.5 to $\varepsilon := \frac{1}{4}$ to obtain $c_1 = c_1(M) > 0$ and $c_2 = c_2(m, M, B, \kappa) > 0$ such that with f and g as in (4.5) and (4.6) we have

$$\int_\Omega uv \leq \frac{1}{2}\int_\Omega uv + c_1 \cdot \Big(\|f\|_{L^2(\Omega)}^{\frac{2n+4}{n+4}} + 1\Big) + c_2 \cdot \Big(\|f\|_{L^2(\Omega)}^{2\theta} + \|g\|_{L^2(\Omega)} + 1\Big).$$

Since $\frac{2n+4}{n+4} < 2\theta$ by (4.32), using Young's inequality we immediately arrive at (4.7). $\square$

5 Blow-up. Proof of Theorem 1.1

We now plan to apply the above estimates to solutions of the dynamical problem (1.1). Still referring to the definition of $\mathcal{S}(m, M, B, \kappa)$ as introduced in the beginning of Section 4, we first turn the outcome of Lemma 4.1 into a statement of type (4.3), still for arbitrary functions in $\mathcal{S}(m, M, B, \kappa)$.

Theorem 5.1 *There exists $C(m, M, B, \kappa) > 0$ such that for all $(u, v) \in \mathcal{S}(m, M, B, \kappa)$ we have*

$$\mathcal{F}(u, v) \geq -C(m, M, B, \kappa) \cdot \Big(\mathcal{D}^\theta(u, v) + 1\Big) \tag{5.1}$$

with $\theta \in (\frac{1}{2}, 1)$ given by (4.32).

Proof. Since $\theta > \frac{1}{2}$, we may apply Young's inequality to (4.7) to find $c_1 = c_1(m, M, B, \kappa) > 0$ such that

$$\int_\Omega uv \leq c_1\Bigg(\Big(\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2\Big)^\theta + 1\Bigg)$$

with f and g as given by (4.5) and (4.6). Therefore, using that $\xi \ln \xi \geq -\frac{1}{e}$ for all $\xi > 0$ we obtain the inequality

$$\begin{aligned} \mathcal{F}(u, v) &= \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u \ln u \\ &\geq -\int_\Omega uv - \frac{|\Omega|}{e} \\ &\geq -c_2 \cdot \Big((\|f\|_{L^2(\Omega)}^2 + \|g\|_{L^2(\Omega)}^2)^\theta + 1\Big) \end{aligned}$$

with $c_2 \equiv c_2(m,M,B,\kappa) := c_1 + \frac{|\Omega|}{e}$. Since by definition of f and g we have $\mathcal{D}(u,v) = \|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)}$, this already establishes (5.1). $\Box$

Now given a solution (u,v) of (1.1), the fact that in (5.1) we have $\theta < 1$ will enable us to derive an ODI for $t \mapsto -\mathcal{F}(u(\cdot,t),v(\cdot,t))$ with superlinearly growing nonlinearity. For initial data (u_0,v_0) with large negative energy $\mathcal{F}(u_0,v_0)$, this means that (u,v) cannot exist globally.

Lemma 5.2 *Let $m > 0, A > 0$ and $\kappa > n-2$. Then there exist $K = K(m,A,\kappa) > 0$ and $C = C(m,A,\kappa) > 0$ such that for each (u_0,v_0) from the set*

$$\widetilde{\mathcal{B}}(m,A,\kappa) \;:=\; \left\{ (u_0,v_0) \in C^0(\bar\Omega) \times W^{1,\infty}(\Omega) \;\middle|\; u_0 \text{ and } v_0 \text{ are radially symmetric and positive in } \bar\Omega \right.$$
$$\left. \text{with } \int_\Omega u_0 = m,\ \|v_0\|_{W^{1,2}(\Omega)} \le A \text{ and } \mathcal{F}(u_0,v_0) \le -K \right\}, \qquad (5.2)$$

the corresponding solution (u,v) of (1.1) has the property

$$\mathcal{F}(u(\cdot,t),v(\cdot,t)) \le \frac{\mathcal{F}(u_0,v_0)}{(1-Ct)^{\frac{\theta}{1-\theta}}} \qquad \text{for all } t \in (0,T_{max}(u_0,v_0)), \qquad (5.3)$$

where $\theta \in (\frac{1}{2},1)$ is as given by (4.32).
In particular, for any such solution we have $T_{max}(u_0,v_0) < \infty$, that is, (u,v) blows up in finite time.

Proof. Let us fix $c_1 > 0$ such that

$$\|\varphi\|_{L^1(\Omega)} \le c_1 \|\varphi\|_{W^{1,2}(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \qquad (5.4)$$

According to Corollary 3.3, we can pick $c_2 = c_2(\kappa) > 0$ such that whenever $u_0 \in C^0(\bar\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$ are radial and positive, the corresponding solution (u,v) of (1.1) satisfies

$$v(r,t) \le c_2 \cdot \Big(\|u_0\|_{L^1(\Omega)} + \|v_0\|_{L^1(\Omega)} + \|\nabla v_0\|_{L^2(\Omega)} \Big) \cdot r^{-\kappa} \qquad \text{for all } (r,t) \in (0,R) \times (0,T_{max}(u_0,v_0)). \qquad (5.5)$$

Next, writing $B := c_2(m + c_1 A + A)$ and $M := \max\{m, c_1 A\}$ we invoke Theorem 5.1 to obtain $c_3 = c_3(m,M,B,\kappa) > 0$ such that

$$\mathcal{F}(\tilde u,\tilde v) \ge -c_3 \cdot \Big(\mathcal{D}^\theta(\tilde u,\tilde v) + 1 \Big) \qquad \text{for all } (\tilde u,\tilde v) \in \mathcal{S}(m,M,B,\kappa). \qquad (5.6)$$

We will see that then (5.3) holds for all $(u_0,v_0) \in \widetilde{B}(m,A,\kappa)$ if we define

$$K(m,A,\kappa) := 2c_3 \qquad (5.7)$$

and

$$C(m,A,\kappa) := \frac{1-\theta}{2c_3\theta}. \qquad (5.8)$$

Indeed, given $(u_0,v_0) \in \widetilde{B}(m,A,\kappa)$ we know from (5.5) and (5.4) that the solution (u,v) of (1.1) emanating from (u_0,v_0) is smooth and radially symmetric with $u > 0$ and $v > 0$ in $\bar\Omega \times [0,T_{max}(u_0,v_0))$ and

$$v(r,t) \le c_2 \cdot (m + c_1 A + A) \cdot r^{-\kappa} = B r^{-\kappa} \qquad \text{for all } (r,t) \in (0,R) \times (0,T_{max}(u_0,v_0)).$$

Since moreover $\int_\Omega u(\cdot,t) \equiv \int_\Omega u_0 = m$ and $v(\cdot,t) \le \max\{\int_\Omega u_0, \int_\Omega v_0\} \le \max\{m, c_1 A\} = M$ for all $t \in (0, T_{max}(u_0,v_0))$ by (2.1) and (2.2), it follows that $(u(\cdot,t), v(\cdot,t)) \in \mathcal{S}(m,M,B,\kappa)$ for all $t \in (0, T_{max}(u_0,v_0))$ and hence (5.6) may be applied to $(\tilde{u},\tilde{v}) := (u(\cdot,t), v(\cdot,t))$ for any such t.
In order to derive (5.3) from this and the energy inequality (1.4), let us make sure that

$$y(t) := -\mathcal{F}(u(\cdot,t), v(\cdot,t)), \qquad t \in [0, T_{max}(u_0,v_0)),$$

defines a positive function $y \in C^0([0,T_{max}(u_0,v_0))) \cap C^1((0,T_{max}(u_0,v_0)))$ which satisfies

$$y'(t) \ge c_4 y^{\frac{1}{\theta}}(t) \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)) \tag{5.9}$$

with $c_4 = c_4(m,M,B,\kappa) := (2c_3)^{-\frac{1}{\theta}}$.
In fact, the claimed regularity properties of y immediately result from those of (u,v), whereas (1.4) ensures that y is nondecreasing and thus

$$y(t) \ge y(0) \ge K(m,A,\kappa) = 2c_3 > 0 \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)). \tag{5.10}$$

Therefore we may invert so as to obtain from (5.6) and (5.10) that

$$\mathcal{D}^\theta(u(\cdot,t), v(\cdot,t)) \ge \frac{y(t)}{c_3} - 1 \ge \frac{y(t)}{c_3} - \frac{y(t)}{2c_3} = \frac{y(t)}{2c_3} \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)).$$

In light of (1.4), y thus satisfies

$$y'(t) \ge \mathcal{D}(u(\cdot,t), v(\cdot,t)) \ge \Big(\frac{y(t)}{2c_3}\Big)^{\frac{1}{\theta}} \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)),$$

which precisely yields (5.9).
Now by straightforward integration, we see that

$$y(t) \ge y(0) \cdot \Big\{1 - \frac{1-\theta}{\theta} c_4 y^{\frac{1-\theta}{\theta}}(0) \cdot t\Big\}^{-\frac{\theta}{1-\theta}} \qquad \text{for all } t \in (0, T_{max}(u_0,v_0)),$$

which implies (5.3) upon the observation that (5.10) entails

$$\frac{1-\theta}{\theta} c_4 \cdot y^{\frac{1-\theta}{\theta}}(0) \ge \frac{1-\theta}{\theta} c_4 \cdot (2c_3)^{\frac{1-\theta}{\theta}} = \frac{1-\theta}{\theta} \cdot \frac{1}{2c_3} = C(m,A,\kappa).$$

The proof is complete. $\square$

In a last step we can remove the auxiliary parameter κ appearing in the latter result so as to obtain the blow-up criterion stated in Theorem 1.1.

Proof of Theorem 1.1. We only need to fix an arbitrary $\kappa > n-2$ and apply Lemma 5.2 to find that the conclusion holds if we let $K(m,A) := K(m,A,\kappa)$ and $T(m,A) := \frac{1}{C(m,A,\kappa)}$ with $K(m,A,\kappa)$ and $C(m,A,\kappa)$ as given by Lemma 5.2. $\square$

6 A density property of $\mathcal{B}(m, A)$. Proof of Theorem 1.2

The proof of Theorem 1.2 will be an immediate consequence of the following lemma which roughly says that arbitrary initial data can be approximated by low-energy initial data in the claimed manner.

Lemma 6.1 *Let $m > 0$ and $u \in C^0(\bar{\Omega})$ and $v \in W^{1,\infty}(\Omega)$ be radially symmetric and positive in $\bar{\Omega}$ with $\int_\Omega u = m$. Then for each $p \in (1, \frac{2n}{n+2})$ there exist sequences $(u_k)_{k\in\mathbb{N}} \subset C^0(\bar{\Omega})$ and $(v_k)_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ of radially symmetric positive functions satisfying $\int_\Omega u_k = m$ for all $k \in \mathbb{N}$ and*

$$u_k \to u \quad \text{in } L^p(\Omega) \quad \text{and} \quad v_k \to v \quad \text{in } W^{1,2}(\Omega) \qquad \text{as } k \to \infty, \tag{6.1}$$

but such that with $\mathcal{F}$ as defined in (1.5) we have

$$\mathcal{F}(u_k, v_k) \to -\infty \qquad \text{as } k \to \infty. \tag{6.2}$$

Proof. We fix an arbitrary sequence $(r_k)_{k\in\mathbb{N}} \subset (0, R)$ such that $r_k \to 0$ as $k \to \infty$, and let

$$\varphi(\xi) := \int_0^1 \rho^{n-1}(\rho^2 + \xi)^{-\frac{n}{2}} d\rho, \qquad \xi > 0.$$

Then by monotone convergence we have $\varphi(\xi) \nearrow \infty$ as $\xi \searrow 0$, so that for each $k \in \mathbb{N}$ it is possible to fix $\eta_k \in (0, R^2)$ appropriately small such that

$$r_k^n \cdot \varphi\Big(\frac{\eta_k}{r_k^2}\Big) \geq k. \tag{6.3}$$

Now given $p \in (1, \frac{2n}{n+2})$, we can fix $\alpha > 0$ fulfilling $n - \frac{n}{p} < \alpha < \frac{n-2}{2}$ and thereupon introduce positive radial functions $\tilde{u}_k, u_k$ and v_k on $\bar{\Omega}$ by defining

$$\tilde{u}_k(r) := \begin{cases} a_k \cdot (r^2 + \eta_k)^{-\frac{n-\alpha}{2}}, & r \in [0, r_k], \\ u(r), & r \in (r_k, R], \end{cases}$$

and

$$u_k := \frac{m\tilde{u}_k}{\|\tilde{u}_k\|_{L^1(\Omega)}} \tag{6.4}$$

as well as

$$v_k(r) := \begin{cases} b_k \cdot (r^2 + \eta_k)^{-\frac{\alpha}{2}}, & r \in [0, r_k], \\ v(r), & r \in (r_k, R], \end{cases}$$

with

$$a_k := (r_k^2 + \eta_k)^{\frac{n-\alpha}{2}} \cdot u(r_k) \qquad \text{and} \qquad b_k := (r_k^2 + \eta_k)^{\frac{\alpha}{2}} \cdot v(r_k)$$

for $k \in \mathbb{N}$. Then clearly $v_k \in W^{1,\infty}(\Omega)$, whereas $\tilde{u}_k$ and u_k belong to $C^0(\bar{\Omega})$ with $\int_\Omega u_k = m$. Moreover, once again writing $\omega_n := |\partial B_1|$ we have

$$\begin{aligned}
\|\tilde{u}_k\|^p_{L^p(B_{r_k})} &= \omega_n \cdot \int_0^{r_k} r^{n-1} \cdot a_k^p (r^2+\eta_k)^{-\frac{(n-\alpha)p}{2}} dr \\
&= \omega_n \cdot u^p(r_k) \cdot \int_0^{r_k} r^{n-1} \cdot \Big(\frac{r_k^2+\eta_k}{r^2+\eta_k}\Big)^{\frac{(n-\alpha)p}{2}} dr \\
&\le \omega_n \cdot u^p(r_k) \cdot \int_0^{r_k} r^{n-1-(n-\alpha)p} dr \\
&= \frac{\omega_n}{n-(n-\alpha)p} \cdot u^p(r_k) r_k^{n-(n-\alpha)p} \qquad \text{for all } k \in \mathbb{N},
\end{aligned} \tag{6.5}$$

because $\alpha > n - \frac{n}{p}$ implies $(n-\alpha)p < n$. Since $r_k \to 0$ and u is bounded, we thus infer from

$$\|\tilde{u}_k - u\|_{L^p(\Omega)} = \|\tilde{u}_k - u\|_{L^p(B_{r_k})} \le \|\tilde{u}_k\|_{L^p(B_{r_k})} + \|u\|_{L^\infty(\Omega)} |B_{r_k}|^{\frac{1}{p}} \tag{6.6}$$

that $\tilde{u}_k \to u$ in $L^p(\Omega)$ as $k \to \infty$. In particular, this entails that $\tilde{u}_k \to u$ also in $L^1(\Omega)$ and accordingly

$$\|\tilde{u}_k\|_{L^1(\Omega)} \to m \qquad \text{as } k \to \infty, \tag{6.7}$$

so that combining (6.4) and (6.6) yields

$$u_k \to u \qquad \text{in } L^p(\Omega) \qquad \text{as } k \to \infty.$$

Next, computing $v_{kr} = -\alpha b_k r (r^2+\eta_k)^{-\frac{\alpha+2}{2}}$ for $r \in (0, r_k)$ and observing that $b_k \le c_1 := (2R^2)^{\frac{\alpha}{2}} \|v\|_{L^\infty(\Omega)}$ for all $k \in \mathbb{N}$, we find that

$$\begin{aligned}
\|\nabla v_k\|^2_{L^2(B_{r_k})} &= \omega_n \cdot \alpha^2 b_k^2 \cdot \int_0^{r_k} r^{n+1} (r^2+\eta_k)^{-\alpha-2} dr \\
&\le \omega_n \cdot \alpha^2 c_1^2 \cdot \int_0^{r_k} r^{n-2\alpha-3} dr \\
&= \frac{\omega_n \alpha^2 c_1^2}{n-2\alpha-2} \cdot r_k^{n-2\alpha-2} \qquad \text{for all } k \in \mathbb{N}
\end{aligned}$$

and, similarly,

$$\|v_k\|^2_{L^2(B_{r_k})} = \omega_n \cdot b_k^2 \cdot \int_0^{r_k} r^{n-1} (r^2+\eta_k)^{-\alpha} dr \le \frac{\omega_n c_1^2}{n-2\alpha} \cdot r_k^{n-2\alpha} \qquad \text{for all } k \in \mathbb{N},$$

for our restriction $\alpha < \frac{n-2}{2}$ ensures that both $n-2\alpha-2$ and $n-2\alpha$ are positive. Again since $r_k \to 0$, in view of the estimate

$$\|v_k - v\|_{W^{1,2}(\Omega)} = \|v_k - v\|_{W^{1,2}(B_{r_k})} \le \|v_k\|_{W^{1,2}(B_{r_k})} + \|v\|_{W^{1,\infty}(\Omega)} (2|B_{r_k}|)^{\frac{1}{2}}$$

and the boundedness of v in $W^{1,\infty}(\Omega)$ this shows that that

$$v_k \to v \quad \text{in } W^{1,2}(\Omega) \qquad \text{as } k \to \infty$$

and thus completes the proof of (6.1).
To verify (6.2), we first note that since $\sup_{\xi>0}\frac{\xi\ln\xi}{\xi^p}$ is finite thanks to our assumption $p>1$, from the boundedness of $(u_k)_{k\in\mathbb{N}}$ in $L^p(\Omega)$ and of $(v_k)_{k\in\mathbb{N}}$ in $W^{1,2}(\Omega)$ asserted by (6.1) we obtain $c_2>0$ such that

$$\frac{1}{2}\int_\Omega|\nabla v_k|^2+\frac{1}{2}\int_\Omega v_k^2+\int_\Omega u_k\ln u_k\le c_2 \qquad \text{for all } k\in\mathbb{N}. \tag{6.8}$$

On the other hand, using that

$$a_kb_k=u(r_k)v(r_k)\cdot(r_k^2+\eta_k)^{\frac{n}{2}}\ge u(r_k)v(r_k)\cdot r_k^n \qquad \text{for all } k\in\mathbb{N}$$

and recalling the definition of φ we have

$$\begin{aligned}\int_\Omega u_kv_k &\ge \int_{B_{r_k}}u_kv_k\\ &= \omega_n\cdot\frac{m}{\|\tilde u_k\|_{L^1(\Omega)}}\cdot a_kb_k\cdot\int_0^{r_k}r^{n-1}(r^2+\eta_k)^{-\frac{n}{2}}dr\\ &= \omega_n\cdot\frac{m}{\|\tilde u_k\|_{L^1(\Omega)}}\cdot a_kb_k\cdot\int_0^1\rho^{n-1}\cdot\Big(\rho^2+\frac{\eta_k}{r_k^2}\Big)^{-\frac{n}{2}}d\rho\\ &\ge \omega_n\cdot\frac{m}{\|\tilde u_k\|_{L^1(\Omega)}}\cdot u(r_k)v(r_k)r_k^n\cdot\varphi\Big(\frac{\eta_k}{r_k^2}\Big) \qquad \text{for all } k\in\mathbb{N}.\end{aligned}$$

Hence, (6.3) warrants that

$$\int_\Omega u_kv_k\ge\omega_n\cdot\frac{m}{\|\tilde u_k\|_{L^1(\Omega)}}\cdot u(r_k)v(r_k)\cdot k \qquad \text{for all } k\in\mathbb{N},$$

so that from (6.7) we infer that

$$\liminf_{k\to\infty}\frac{1}{k}\cdot\int_\Omega u_kv_k\ge\omega_n\cdot u(0)v(0)>0 \qquad \text{as } k\to\infty,$$

because u and v are continuous. Thus, by positivity of u and v we conclude that $\int_\Omega u_kv_k\to\infty$ as $k\to\infty$, and in conjunction with (6.8) this proves (6.2). □

PROOF of Theorem 1.2. In view of Theorem 1.1, the claim directly results from Lemma 6.1. □

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