Stabilization in a two-dimensional chemotaxis-Navier-Stokes system

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Abstract

This paper deals with an initial-boundary value problem for the system

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0,\end{cases}$$

which has been proposed as a model for the spatio-temporal evolution of populations of swimming aerobic bacteria.

It is known that in bounded convex domains $\Omega\subset\mathbb{R}^2$ and under appropriate assumptions on the parameter functions χ, f and ϕ, for each $\kappa\in\mathbb{R}$ and all sufficiently smooth initial data this problem possesses a unique global-in-time classical solution. The present work asserts that this solution stabilizes to the spatially uniform equilibrium $(\overline{n_0},0,0)$, where $\overline{n_0}:=\frac{1}{|\Omega|}\int_\Omega n(x,0)dx$, in the sense that as $t\to\infty$,

$$n(\cdot,t)\to\overline{n_0},\qquad c(\cdot,t)\to 0\qquad\text{and}\qquad u(\cdot,t)\to 0$$

hold with respect to the norm in $L^\infty(\Omega)$.

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1 Introduction

This work deals with a mathematical model for the interaction of bacterial populations with a surrounding fluid in which their nutrient is dissolved. Indeed, experimental findings reveal the occurrence of rather complex spatio-temporal behavior in colonies of the aerobic species *Bacillus subtilis* when suspended into sessile drops of water: As reported in [6] and [29], establishing such a simple setting may result in a formation of plume-like aggregates of cells as well as a spontaneous emergence of large-scale fluid motion and convection patterns. In [29], the authors propose to decribe such processes by the system of evolution equations

$$\begin{cases} n_t + u\cdot\nabla n = \Delta n - \nabla\cdot(n\chi(c)\nabla c), & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c = \Delta c - nf(c), & x\in\Omega,\ t>0,\\ u_t + \kappa(u\cdot\nabla)u = \Delta u + \nabla P + n\nabla\phi, & x\in\Omega,\ t>0,\\ \nabla\cdot u = 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

for the unknown (n,c,u,P), where χ, f and ϕ are given parameter functions and $\kappa\in\mathbb{R}$. This model is based on the hypothesis that besides the bacteria, with population density $n=n(x,t)$, and the incompressible fluid, as represented by the velocity field $u=u(x,t)$ and the associated pressure $P=P(x,t)$, the only further component relevant for such phenomena is the oxygen with concentration denoted by $c=c(x,t)$. Moreover, it is assumed that bacterial motion is governed by random diffusion and transport through the fluid, and by chemotactic migration toward increasing gradients of the attractive oxygen. In turn, the quantity n affects both the evolution of the chemical via consumption, and of the fluid motion via buoyant forces.

The challenge of describing qualitative behavior in (1.1). Clearly, evaluating the efficiency of this model in the context of the above observations amounts to estimating its ability to generate nontrivial dynamics of solutions. With regard to this, some numerical evidence in [29] indeed reports the emergence of patterns on intermediate time scales. To the best of our knowledge, however, no analytical result is available yet which rigorously describes the qualitative behavior of such solutions.

This may reflect the circumstance that mathematically, (1.1) couples two mechanisms which are far from being fully understood even when considered separately. On the one hand, unless $\kappa=0$, as a subsystem the incompressible Navier-Stokes equations are included, which are still lacking a complete existence and regularity theory despite substantial and elaborate research since Leray's pioneering work (cf. e.g. [30] for a survey, and also [24], [28]). On the other hand, the first two equations in (1.1) form a variant of the Keller-Segel chemotaxis system from mathematical biology. As for the latter, it is known that the cross-diffusive term $-\nabla\cdot(n\chi(c)\nabla c)$ as its most characteristic ingredient may destabilize homogeneity and even enforce blow-up of solutions; however, rigorous results on this topic are still very few, and all of these are restricted to rather special settings such as the standard Keller-Segel system ([17])

$$\begin{cases} n_t = \Delta n - \nabla\cdot(n\nabla c), & x\in\Omega,\ t>0,\\ c_t = \Delta c - c + n, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

or simplications thereof (cf. [14], [34], [16], [21], [22] and also [15] and [23] for broader discussions, as well as e.g. [18] for dynamical effects of chemotactic cross-diffusion on a related system with transport through a given fluid). Even less is known for the chemotaxis system with consumption of the chemoattractant such as contained in (1.1),

$$\begin{cases} n_t = \Delta n - \nabla \cdot (n\chi(c)\nabla c), & x \in \Omega,\ t > 0, \\ c_t = \Delta c - nf(c), & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

for which it seems that only some results on global classical solvability in the spatially two-dimensional case, and on global existence of weak but eventually smooth solutions in the three-dimensional case are available in certain special cases ([25]).

Accordingly, analytical results on (1.1) up to now seem to concentrate on the basic issues related to questions of well-posedness: The work [20] provides some statements on local existence of certain weak solutions in the spatially two- or three-dimensional setting under various boundary conditions and the assumptions that $\chi \equiv const.$ and that f be nondecreasing with $f(0) = 0$. A more subtle approach making use of quasi-energy functionals associated with (1.1) is performed in [8]. In the case $\Omega = \mathbb{R}^2$ when the nonlinear convective term in (1.1) is removed by setting $\kappa = 0$, weak solutions are constructed there which exist globally in time, provided that either $c(\cdot, 0)$ or $\nabla\phi$ are sufficiently small, that χ and f satisfy some structural assumptions slightly more restrictive than (1.7)-(1.10) below, and that the initial data decay sufficiently fast at spatial infinity. For the same two-dimensional Cauchy problem, this smallness assumption on the initial data could be removed in [19] so as to ensure global weak solvability even in the case of Navier-Stokes fluid evolution when $\kappa \neq 0$. In [33], global classical solutions and their uniqueness could be established in bounded convex two-dimensional domains under the boundary conditions (1.4) below and milder assumptions on χ and f than those in [8], unlike the latter paper thereby allowing for the choices $\chi \equiv const.$ and $f(c) = c$. Recently, for nonnegative and noncecreasing χ and f it was shown in [3] by using a slightly modified energy-like functional that unique global classical solutions also exist under the alternative requirements that $\Omega = \mathbb{R}^2$ and $\|\chi - \mu f\|_{L^\infty((0,\infty))}$ be small for some $\mu \geq 0$.
In the spatially three-dimensional case, the existence theory seems much less complete: As far as we know, the only results on global existence of solutions to (1.1) with large initial data can be found in [33] and [3]: The former paper addresses the Stokes case $\kappa = 0$ and asserts global existence of weak solutions in bounded convex domains $\Omega \subset \mathbb{R}^3$ under the assumptions (1.7)-(1.10) below; in the latter work, global weak solutions for the corresponding Cauchy problem in $\Omega = \mathbb{R}^3$ are constructed in the special situation when χ precisely coincides with a fixed multiple of f.
Let us mention that recently, some existence results have also been derived for the variant of (1.1) obtained on replacing linear cell diffusion by porous medium-type nonlinear diffusion ([7], [5], [26], [27]).

Main results. In contrast to the growing literature concerned with global solvability, very few seems to be known about the qualitative behavior of solutions to (1.1). In fact, with regard to this we are aware of one result only which addresses the issue of stability of the constant steady states $(n_\infty, 0, 0)$, $n_\infty \geq 0$, of (1.1) in the case when $\Omega = \mathbb{R}^3$. Namely, it was shown in [8] that each of these equalibria attracts all solutions emanating from initial data which are sufficiently small perturbations thereof with respect to the topology of $(W^{3,2}(\mathbb{R}^3))^3$ when f is nondecreasing with $f(0) = 0$ and $\phi \in W^{3,\infty}(\mathbb{R}^3)$ has bounded first moment.

Of course, this leaves open the possibility of nontrivial large-time asymptotics of solutions emanating from large initial data possibly far from equilibrium, but apparently the literature provides no results in this direction yet.

The goal of the present work is to give a complete description of the large time behavior in (1.1) in the situation when $\Omega \subset \mathbb{R}^2$ is a bounded convex domain with smooth boundary. Indeed, we shall see that then *all* solutions approach a spatially homogeneous equilibrium in the large time limit. In order to state our main results more precisely, let us specify the precise mathematical framework that will be considered below. We shall close the PDE system (1.1) by imposing no-flux boundary conditions for both n and c and no-slip boundary conditions for u,

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{for } x \in \partial\Omega \text{ and } t > 0, \tag{1.4}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.5}$$

With A denoting the realization of the Stokes operator in the solenoidal subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0\}$ of $L^2(\Omega)$ ([24]), we assume that here we have

$$\begin{cases} n_0 \in C^0(\bar\Omega), \quad n_0 > 0 \quad \text{in } \bar\Omega, \\ c_0 \in W^{1,\vartheta}(\Omega) \text{ for some } \vartheta > 2, \quad c_0 > 0 \quad \text{in } \bar\Omega, \\ u_0 \in D(A^\alpha) \text{ for some } \alpha \in (\frac{1}{2}, 1). \end{cases} \tag{1.6}$$

Moreover, adopting the hypotheses from [33] on the parameter functions in (1.1) we shall suppose throughout that

$$\begin{cases} \chi \in C^2([0,\infty)), \quad \chi > 0 \quad \text{in } [0,\infty), \\ f \in C^2([0,\infty)), \quad f(0) = 0, \quad f > 0 \quad \text{in } (0,\infty), \\ \phi \in C^2(\bar\Omega), \end{cases} \tag{1.7}$$

and that

$$\Big(\frac{f}{\chi}\Big)' > 0 \qquad \text{on } [0,\infty), \tag{1.8}$$

$$\Big(\frac{f}{\chi}\Big)'' \le 0 \qquad \text{on } [0,\infty) \qquad \text{and} \tag{1.9}$$

$$(\chi \cdot f)' \ge 0 \qquad \text{on } [0,\infty). \tag{1.10}$$

Within this framework, it is known ([33]) that a globally defined classical solution (n, c, u, P) exists which is unique, up to addition of constants in the pressure variable P, and which satisfies $n > 0$ and $c > 0$ in $\bar\Omega \times [0,\infty)$. Our main result describes the large time behavior of this solution as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and let $\kappa \in \mathbb{R}$. Assume that n_0, c_0 and u_0 satisfy (1.6), and that χ, f and ϕ fulfill (1.7)-(1.10). Then the global classical solution of (1.1), (1.4), (1.5) satisfies*

$$\begin{aligned} &\|n(\cdot,t) - \overline{n_0}\|_{L^\infty(\Omega)} \to 0, \\ &\|c(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad\qquad \textit{and} \\ &\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \end{aligned} \tag{1.11}$$

as $t \to \infty$, where $\overline{n_0} = \frac{1}{|\Omega|}\int_\Omega n_0$.

In particular, this indicates that structure generating dynamics in the spatially two-dimensional version of (1.1), (1.4), (1.5), if at all, occur on intermediate time scales rather than in the sense of a stable large-time pattern formation process. Correspondingly, collecting any type of analytical evidence for the ability of (1.1) to describe complex dynamical phenomena, as experimentally detected in [6] and [29], forms a natural but challenging next step. We have to leave open here the questions whether a three-dimensionality of the physical setting can enforce more colorful large time asymptotics in (1.1), and in how far the solution behavior may be influenced by boundary effects not considered here. For instance, the framework underlying the numerical simulations in [29] accounts for a certain oxygen influx through the boundary of the fluid drop.

Outline of our approach. Before going into details, let us briefly outline the main steps of our analysis: As a starting point we shall recall from [33] an energy-type inequality which will imply

$$\frac{d}{dt}\Big\{\int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{\chi(c)|\nabla c|^2}{f(c)}\Big\} + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{C}\int_\Omega \frac{|\nabla c|^4}{c^3} \le C\int_\Omega |u|^4 \qquad \text{for all } t>0,$$

with some $C > 0$ (Lemma 2.4). Since here the right-hand side does not vanish, this does by no means directly entail stabilization of any of the involved quantities. At least, from this we shall derive through a first series of estimates that the boundedness properties

$$\int_t^{t+1}\int_\Omega n^2 \le C, \qquad \int_t^{t+1}\int_\Omega |\nabla c|^4 \le C \quad \text{and} \quad \int_\Omega |\nabla c(\cdot,t)|^2 \le C$$

are valid for all $t > 1$ with some $C > 0$ (Lemma 3.6).
We shall next use this in properly exploiting the decay information implicitly contained in the easily gained inequalities $\int_0^\infty \int_\Omega |\nabla c|^2 < \infty$ and $\int_0^\infty \int_\Omega nf(c) < \infty$ (Lemma 4.1) in order to show that c decays with respect to the norm in $L^\infty(\Omega)$ (Corollary 4.4).
This will then enable us to obtain, inter alia, that

$$\int_2^\infty \int_\Omega |\nabla n|^2 < \infty,$$

upon studying the time evolution of $t \mapsto \int_\Omega \frac{n^p(\cdot,t)}{\delta - c(\cdot,t)}$ with arbitrary $p > 2$ and some conveniently small $\delta > 0$ (Lemma 5.1).
Thanks to the smoothing effects of the Stokes and heat semigroups in (1.1), this will first tell us that $u(\cdot,t) \to 0$ in $L^\infty(\Omega)$ (Lemma 6.3), and then enable us to finally assert the claimed stabilization property of n (Lemma 8.2).

Throughout the rest of the paper we shall assume without further comment that (1.6)-(1.10) hold, and let (n, c, u) denote the corresponding solution of (1.1), (1.4), (1.5).

2 Preliminaries. An energy inequality

To begin with, let us collect some basic solution properties which essentially have already been used in [33], and partly also in [8].
The first two statements immediately result from an integration of the first PDE in (1.1), and from an application of the maximum principle to the second.

Lemma 2.1 *We have*

$$\int_\Omega n(x,t)dx = \int_\Omega n_0 \qquad \text{for all } t>0 \tag{2.1}$$

and

$$t \mapsto \|c(\cdot,t)\|_{L^\infty(\Omega)} \quad \text{is nonincreasing.} \tag{2.2}$$

In particular,

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \le \|c_0\|_{L^\infty(\Omega)} \qquad \text{for all } t>0. \tag{2.3}$$

We next recall from [33, Lemma 3.4] an energy inequality associated with the subsystem generated by the first two equations in (1.1), and which will play an essential role in our subsequent analysis. Actually this is the only place where the convexity of Ω is explicitly used (cf. also [8, Section 3] for a related approach).

Lemma 2.2 *There exists $C>0$ such that*

$$\frac{d}{dt}\Big\{ \int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{g(c)} \Big\} + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{4}\int_\Omega g(c)|D^2\rho(c)|^2 \le C\int_\Omega |u|^4 \qquad \text{for all } t>0, \tag{2.4}$$

where we have set

$$g(s) := \frac{f(s)}{\chi(s)} \qquad \text{and} \qquad \rho(s) := \int_1^s \frac{d\sigma}{g(\sigma)} \qquad \text{for } s>0. \tag{2.5}$$

In order to take full advantage of the last term on the left of (2.4), we shall utilize the following general integral estimate which in the case of a power-type function h falls among a class of inequalities frequently used in the analysis of higher-order thin film equations (see [2], [4], for instance). A proof of Lemma 2.3 can be found in [33, Lemma 3.3].

Lemma 2.3 *Let $h\in C^1((0,\infty))$ be positive and nondecreasing, and let $\Theta(s) := \int_1^s \frac{d\sigma}{h(\sigma)}$ for $s>0$. Then for all positive $\varphi\in C^2(\bar\Omega)$ fulfilling $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, the inequality*

$$\int_\Omega \frac{h'(\varphi)}{h^3(\varphi)}|\nabla\varphi|^4 \le (2+\sqrt{2})^2 \int_\Omega \frac{h(\varphi)}{h'(\varphi)}|D^2\Theta(\varphi)|^2 \tag{2.6}$$

holds.

The latter enables us to modify the energy inequality (2.4) so as to contain an integral involving $|\nabla c|^4$ in its dissipated part.

Lemma 2.4 *There exists $C>0$ such that*

$$\frac{d}{dt}\Big\{ \int_\Omega n\ln n + \frac{1}{2}\int_\Omega \frac{|\nabla c|^2}{g(c)} \Big\} + \int_\Omega \frac{|\nabla n|^2}{n} + \frac{1}{C}\int_\Omega \frac{|\nabla c|^4}{c^3} \le C\int_\Omega |u|^4 \qquad \text{for all } t>0, \tag{2.7}$$

where g is as defined in (2.5).

Proof. In view of (2.4), we only need to find $C_1 > 0$ such that

$$\int_\Omega \frac{|\nabla c|^4}{c^3} \le C_1 \int_\Omega g(c)|D^2\rho(c)|^2 \qquad \text{for all } t > 0, \tag{2.8}$$

where ρ is as in (2.5). To see this, applying Lemma 2.3 to $h := g$ we obtain

$$\int_\Omega \frac{g'(c)}{g^3(c)}|\nabla c|^4 \le (2+\sqrt{2})^2 \int_\Omega \frac{g(c)}{g'(c)}|D^2\rho(c)|^2 \qquad \text{for all } t > 0. \tag{2.9}$$

Now according to (2.3), writing $C_2 := \|c_0\|_{L^\infty(\Omega)}$ we have $0 \le c \le C_2$ in $\Omega \times (0,\infty)$. Thus, thanks to (1.8) and the positivity of χ on $[0,\infty)$ we can find positive constants C_3 and C_4 fulfilling $C_3 \le g'(c) \le C_4$ in $\Omega \times (0,\infty)$. Therefore,

$$\frac{g(c)}{g'(c)} \le \frac{g(c)}{C_3} \qquad \text{in } \Omega \times (0,\infty),$$

while on the other hand the additional hypothesis $f(0) = 0$ entails $g(c) \le C_4 c$ and hence

$$\frac{g'(c)}{g^3(c)} \ge \frac{C_3}{C_4^3 c^3} \qquad \text{in } \Omega \times (0,\infty).$$

Therefore (2.8) is a consequence of (2.9). □

3 Time-independent integral estimates for n and c

Now a natural next step consists of turning the above energy inequality (2.7) into useful time-independent bounds, and our main outcome in this direction will be the inequalities (3.17)-(3.19) below. Here an apparently challenging issue is to estimate $\int_\Omega |u|^4$ appropriately in terms of expressions involving the dissipation rate in (2.7). This will be achieved in Lemma 3.3 by means of an interpolation involving the members $\int_\Omega |\nabla u|^2$ and $\int_\Omega |u|^2$ of the standard energy inequality associated with the Navier-Stokes equations. The following lemma provides a formulation of the latter which is convenient in the present setting in that it adequately accounts for the external force induced by n.

Lemma 3.1 *Let $q > 1$. Then one can find $\lambda > 0$ and $C > 0$ with the property that*

$$\begin{aligned} \|u(\cdot,t)\|^2_{L^2(\Omega)} \quad &+ \quad \frac{1}{2}\int_0^t e^{-\lambda(t-s)}\|\nabla u(\cdot,s)\|^2_{L^2(\Omega)}ds \\ &\le \quad \|u_0\|^2_{L^2(\Omega)} + \int_0^t e^{-\lambda(t-s)}\|n(\cdot,s)\|^2_{L^q(\Omega)}ds \qquad \textit{for all } t > 0. \end{aligned} \tag{3.1}$$

Proof. We multiply the third equation in (1.1) by u and integrate by parts over $x \in \Omega$. Since it is well-known that $\nabla \cdot u \equiv 0$ implies $\int_\Omega [(u\cdot\nabla)u]\cdot u = 0$, we thereby obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega n\nabla\phi \cdot u \qquad \text{for all } t > 0. \tag{3.2}$$

Here we use the Hölder inequality along with the assumed boundedness of ϕ to estimate

$$\int_\Omega n\nabla\phi \cdot u \le \|\nabla\phi\|_{L^\infty(\Omega)} \cdot \|n\|_{L^q(\Omega)} \cdot \|u\|_{L^{q'}(\Omega)} \tag{3.3}$$

with $q' := \frac{q}{q-1}$. Now since the spatial dimension is two, for any such q the space $W_0^{1,2}(\Omega)$ is continuously embedded into $L^{q'}(\Omega)$, and hence by means of the Poincaré inequality in $W_0^{1,2}(\Omega)$ we can find $C_1 > 0$ and $C_2 > 0$ such that

$$\|\varphi\|_{L^{q'}(\Omega)} \le C_1\|\varphi\|_{W_0^{1,2}(\Omega)} \quad \text{and} \quad \|\varphi\|_{W_0^{1,2}(\Omega)} \le C_2\|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega). \tag{3.4}$$

Thus, by Young's inequality and (3.3),

$$\int_\Omega n\nabla\phi \cdot u \le \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + C_3\|n\|^2_{L^q(\Omega)}$$

holds with $C_3 := \frac{C_1^2 C_2^2 \|\nabla\phi\|^2_{L^\infty(\Omega)}}{2}$, so that (3.2) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le C_3\|n\|^2_{L^q(\Omega)} \qquad \text{for all } t > 0.$$

In order to create an appropriate absorptive term, we once more use (3.4) to find that

$$\frac{1}{4}\int_\Omega |\nabla u|^2 \ge \frac{1}{4C_2^2}\|u\|^2_{W_0^{1,2}(\Omega)} \ge \frac{1}{4C_2^2}\int_\Omega |u|^2,$$

and therefore we see that

$$\frac{d}{dt}\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le -\lambda\int_\Omega |u|^2 + 2C_3\|n\|^2_{L^q(\Omega)} \qquad \text{for all } t > 0$$

with $\lambda := \frac{1}{2C_2^2}$. Integrating the ODI

$$y'(t) \le -\lambda y(t) + h_1(t) - h_2(t), \qquad t > 0,$$

thus obtained for $y(t) := \int_\Omega |u(x,t)|^2 dx$, $h_1(t) := 2C_3\|n(\cdot,t)\|_{L^q(\Omega)}$ and $h_2(t) := \frac{1}{2}\int_\Omega |\nabla u(x,t)|^2 dx$, we derive the inequality

$$y(t) \le e^{-\lambda t}y(0) + \int_0^t e^{-\lambda(t-s)}[h_1(s) - h_2(s)]ds \qquad \text{for all } t > 0,$$

which readily implies (3.1). □

The integral appearing on the right of (3.1) can be related to an expression involving $\nabla n^{\frac{1}{2}}$ by interpolation using the mass identity (2.1):

Lemma 3.2 *Let $q \in (1,2)$. Then there exists $C > 0$ such that with $\lambda > 0$ as provided by Lemma 3.1 we have*

$$\int_0^t e^{-\lambda(t-s)}\|n(\cdot,s)\|^2_{L^q(\Omega)}ds \le C \cdot \left\{1 + \left(\int_0^t e^{-\lambda(t-s)}\|\nabla n^{\frac{1}{2}}(\cdot,s)\|^2_{L^2(\Omega)}ds\right)^{\frac{2(q-1)}{q}}\right\} \qquad \textit{for all } t > 0. \tag{3.5}$$

PROOF. We first apply the Gagliardo-Nirenberg inequality to find $C_1 > 0$ such that

$$\|n(\cdot,s)\|_{L^q(\Omega)}^2 = \|n^{\frac{1}{2}}(\cdot,s)\|_{L^{2q}(\Omega)}^4 \le C_1 \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{4(q-1)}{q}} \cdot \|n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{4}{q}} + C_1 \|n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^4$$

for all $s > 0$. Since $\|n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 = \int_\Omega n(\cdot,s) = \int_\Omega n_0$ for all $s > 0$ by (2.1), we thus find $C_2 > 0$ fulfilling

$$\|n(\cdot,s)\|_{L^2(\Omega)} \le C_2 \Big\{ 1 + \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{4(q-1)}{q}} \Big\} \qquad \text{for all } s > 0,$$

and hence the integral in question can be estimated according to

$$\int_0^t e^{-\lambda(t-s)} \|n(\cdot,s)\|_{L^q(\Omega)}^2 ds \;\le\; C_2 \int_0^t e^{-\lambda(t-s)} ds + C_2 \int_0^t e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{4(q-1)}{q}} ds \tag{3.6}$$

for all $t > 0$, where clearly

$$\int_0^t e^{-\lambda(t-s)} ds \le \frac{1}{\lambda} \qquad \text{for all } t > 0. \tag{3.7}$$

As for the rightmost term in (3.6), we invoke the Hölder inequality with exponents $\frac{q}{2(q-1)} > 1$ and $\frac{q}{2-q}$ to see that

$$\begin{aligned}
\int_0^t e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^{\frac{4(q-1)}{q}} ds &= \int_0^t \Big(e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 \Big)^{\frac{2(q-1)}{q}} \cdot e^{-\frac{2-q}{q}\lambda(t-s)} ds \\
&\le \Big(\int_0^t e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 \Big)^{\frac{2(q-1)}{q}} \cdot \Big(\int_0^t e^{-\lambda(t-s)} ds \Big)^{\frac{2-q}{q}} \\
&\le \Big(\int_0^t e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 \Big)^{\frac{2(q-1)}{q}} \cdot \Big(\frac{1}{\lambda} \Big)^{\frac{2-q}{q}} \qquad \text{for all } t > 0,
\end{aligned}$$

again because of (3.7). Therefore, (3.6) implies (3.5). □

Combining the above two lemmata along with another interpolation, we can now find an estimate for an integral involving $\int_\Omega |u|^4$ in a similar flavor as the inequality achieved in Lemma 3.2.

Lemma 3.3 *Let $q \in (1,2)$. Then there exists $C > 0$ such that with $\lambda > 0$ taken from Lemma 3.1 we have*

$$\int_0^t e^{-\lambda(t-s)} \|u(\cdot,s)\|_{L^4(\Omega)}^4 ds \le C \Bigg\{ 1 + \sup_{t' \in (0,t)} \Big(\int_0^{t'} e^{-\lambda(t'-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 ds \Big)^{\frac{4(q-1)}{q}} \Bigg\} \qquad \textit{for all } t > 0. \tag{3.8}$$

PROOF. According to the Gagliardo-Nirenberg inequality and the Poincaré inequality we can find $C_1 > 0$ such that

$$\|u(\cdot,s)\|_{L^4(\Omega)}^4 \le C_1 \|\nabla u(\cdot,s)\|_{L^2(\Omega)}^2 \|u(\cdot,s)\|_{L^2(\Omega)}^2 \qquad \text{for all } s > 0,$$

and hence we obtain

$$\int_0^t e^{-\lambda(t-s)}\|u(\cdot,s)\|_{L^4(\Omega)}^4 ds \le C_1\cdot\Big(\sup_{t'\in(0,t)}\|u(\cdot,t')\|_{L^2(\Omega)}^2\Big)\cdot\int_0^t e^{-\lambda(t-s)}\|\nabla u(\cdot,s)\|_{L^2(\Omega)}^2 ds \qquad \text{for all } t>0. \tag{3.9}$$

Here by means of Lemma 3.1 we can estimate both

$$\begin{aligned}\int_0^t e^{-\lambda(t-s)}\|\nabla u(\cdot,s)\|_{L^2(\Omega)}^2 ds &\le C_2\Big\{1+\int_0^t e^{-\lambda(t-s)}\|n(\cdot,s)\|_{L^q(\Omega)}^2 ds\Big\}\\ &\le C_2\Big\{1+\sup_{t'\in(0,t)}\int_0^{t'} e^{-\lambda(t'-s)}\|n(\cdot,s)\|_{L^q(\Omega)}^2 ds\Big\} \qquad \text{for all } t>0\end{aligned}$$

and

$$\sup_{t'\in(0,t)}\|u(\cdot,t')\|_{L^2(\Omega)}^2 \le C_2\Big\{1+\sup_{t'\in(0,t)}\int_0^{t'} e^{-\lambda(t'-s)}\|n(\cdot,s)\|_{L^q(\Omega)}^2 ds\Big\} \qquad \text{for all } t>0$$

with some $C_2>0$. Thus, (3.9) shows that there exists $C_3>0$ satisfying

$$\int_0^t e^{-\lambda(t-s)}\|u(\cdot,s)\|_{L^4(\Omega)}^4 ds \le C_3\Big\{1+\sup_{t'\in(0,t)}\Big(\int_0^{t'} e^{-\lambda(t'-s)}\|n(\cdot,s)\|_{L^q(\Omega)}^2 ds\Big)^2\Big\} \qquad \text{for all } t>0,$$

and hence (3.8) immediately results upon an application of Lemma 3.2. $\Box$

Now the integrals on the right-hand sides of (3.5) and (3.8) appear in a natural way in a correspondingly integrated version of the energy inequality (2.7) as follows.

Lemma 3.4 *Let $\lambda>0$ be as in Lemma 3.1. Then there exists $C>0$ such that*

$$\begin{aligned}\int_\Omega|\nabla c(x,t)|^2dx &+ \int_0^t e^{-\lambda(t-s)}\|\nabla n^{\frac12}(\cdot,s)\|_{L^2(\Omega)}^2 ds+\int_0^t e^{-\lambda(t-s)}\|\nabla c(\cdot,s)\|_{L^4(\Omega)}^4 ds\\ &\le C\cdot\Big\{1+\int_0^t e^{-\lambda(t-s)}\|u(\cdot,s)\|_{L^4(\Omega)}^4 ds\Big\} \qquad \textit{for all } t>0.\end{aligned} \tag{3.10}$$

Proof. We first claim that with $g=\frac{f}{\chi}$ as already introduced in (2.5), the function z defined by

$$z(t):=\int_\Omega n\ln n+\frac12\int_\Omega\frac{|\nabla c|^2}{g(c)}, \qquad t\ge0,$$

satisfies

$$z'(t)\le-\lambda z(t)+h_1(t)-h_2(t) \qquad \text{for all } t>0, \tag{3.11}$$

where

$$h_1(t):=C_1+C_1\|u(\cdot,t)\|_{L^4(\Omega)}^4, \qquad t>0, \tag{3.12}$$

and

$$h_2(t):=C_2\|\nabla n^{\frac12}(\cdot,t)\|_{L^2(\Omega)}^2+C_2\|\nabla c(\cdot,t)\|_{L^4(\Omega)}^4, \qquad t>0, \tag{3.13}$$

with some suitably large $C_1 > 0$ and appropriately small $C_2 > 0$.
To this end, we start from Lemma 2.4 which says that there exist $C_3 > 0$ and $C_4 > 0$ such that

$$z'(t) \le -C_3 \int_\Omega \frac{|\nabla n|^2}{n} - C_3 \int_\Omega \frac{|\nabla c|^4}{c^3} + C_4 \int_\Omega |u|^4 \qquad \text{for all } t > 0. \tag{3.14}$$

In order to link the second term on the right to z, we once again make use of the boundedness of c and our assumption (1.8) to find $C_5 > 0$ fulfilling $g(c) \ge C_5 c$ in $\Omega \times (0, \infty)$. Accordingly, thanks to Young's inequality we can find $C_6 > 0$ such that

$$\frac{1}{2} \int_\Omega \frac{|\nabla c|^2}{g(c)} \le \frac{1}{2C_5} \int_\Omega \frac{|\nabla c|^2}{c} = \frac{1}{2C_5} \int_\Omega \frac{|\nabla c|^2}{c^{\frac{3}{2}}} \cdot c^{\frac{1}{2}} \le \frac{C_3}{2\lambda} \int_\Omega \frac{|\nabla c|^4}{c^3} + C_6 \int_\Omega c,$$

so that again using the boundedness of c we obtain

$$\frac{C_3}{2} \int_\Omega \frac{|\nabla c|^4}{c^3} \ge \frac{\lambda}{2} \int_\Omega \frac{|\nabla c|^2}{g(c)} - C_7 \qquad \text{for all } t > 0 \tag{3.15}$$

with $C_7 := C_6 \lambda \|c_0\|_{L^\infty(\Omega)} |\Omega|$. To achieve a similar lower bound for the first term on the right of (3.14), we fix any $p \in (1, 2)$ and then observe that

$$\xi \ln \xi \le \frac{1}{p(p-1)} \xi^p \qquad \text{for all } \xi > 0,$$

which is an immediate consequence of the fact that $\xi \mapsto \xi \ln \xi - \frac{1}{p(p-1)} \xi^p$ is concave on $(1, \infty)$. Accordingly, an application of the Gagliardo-Nirenberg inequality yields $C_8 > 0$ such that

$$\int_\Omega n \ln n \le \frac{1}{p(p-1)} \int_\Omega n^p = \frac{1}{p(p-1)} \|n^{\frac{1}{2}}\|_{L^{2p}(\Omega)}^{2p} \le C_8 \|\nabla n^{\frac{1}{2}}\|_{L^2(\Omega)}^{2(p-1)} \cdot \|n^{\frac{1}{2}}\|_{L^2(\Omega)}^2 + C_8 \|n^{\frac{1}{2}}\|_{L^2(\Omega)}^{2p}$$

for all $t > 0$, and hence by (2.1) we have

$$\int_\Omega n \ln n \le C_9 \|\nabla n^{\frac{1}{2}}\|_{L^2(\Omega)}^{2(p-1)} + C_9 \qquad \text{for all } t > 0$$

with some $C_9 > 0$. Since $p < 2$ implies that $2(p-1) < 2$, we may thus again use Young's inequality to find $C_{10} > 0$ fulfilling

$$\int_\Omega n \ln n \le \frac{2C_3}{\lambda} \|\nabla n^{\frac{1}{2}}\|_{L^2(\Omega)}^2 + C_{10} \qquad \text{for all } t > 0,$$

which means that

$$\frac{C_3}{2} \int_\Omega \frac{|\nabla n|^2}{n} = 2C_3 \|\nabla n^{\frac{1}{2}}\|_{L^2(\Omega)}^2 \ge \lambda \int_\Omega n \ln n - C_{10} \lambda \qquad \text{for all } t > 0. \tag{3.16}$$

Combining (3.14)-(3.16), we infer that z satisfies the ODI

$$z'(t) \le -\lambda z(t) + C_7 + C_{10}\lambda + C_4 \int_\Omega |u|^4 - \frac{C_3}{2} \int_\Omega \frac{|\nabla n|^2}{n} - \frac{C_3}{2} \int_\Omega \frac{|\nabla c|^4}{c^3} \qquad \text{for all } t > 0.$$

Since finally $\int_\Omega \frac{|\nabla c|^4}{c^3} \ge \frac{1}{\|c_0\|^3_{L^\infty(\Omega)}} \int_\Omega |\nabla c|^4$ by (2.3), this entails (3.11) upon evident choices of C_1 and C_2.

We next integrate (3.11) in time to obtain

$$z(t) + \int_0^t e^{-\lambda(t-s)} h_2(s)ds \le e^{-\lambda t} z(0) + \int_0^t e^{-\lambda(t-s)} h_1(s)ds \qquad \text{for all } t > 0.$$

Since evidently

$$e^{-\lambda t} z(0) \le C_{11} := \int_\Omega |n_0 \ln n_0| + \frac{1}{2} \int_\Omega \frac{|\nabla c_0|^2}{g(c_0)} \qquad \text{for all } t > 0,$$

and since $\xi \ln \xi \ge -\frac{1}{e}$ for all $\xi > 0$ implies that

$$z(t) \ge -\frac{|\Omega|}{e} + \frac{1}{2} \int_\Omega \frac{|\nabla c|^2}{g(c)} \qquad \text{for all } t > 0,$$

in light of (3.13) and (3.12) this yields the inequality

$$\begin{aligned} \frac{1}{2} \int_\Omega \frac{|\nabla c|^2}{g(c)} \quad &+ \quad C_2 \int_0^t e^{-\lambda(t-s)} \Big(\|\nabla n^{\frac{1}{2}}(\cdot,s)\|^2_{L^2(\Omega)} + \|\nabla c(\cdot,s)\|^4_{L^4(\Omega)} \Big) ds \\ &\le \quad C_{11} + \frac{|\Omega|}{e} + C_1 \int_0^t e^{-\lambda(t-s)} ds + C_1 \int_0^t e^{-\lambda(t-s)} \|u(\cdot,s)\|^4_{L^4(\Omega)} ds \qquad \text{for all } t > 0. \end{aligned}$$

Since the boundedness of c clearly entails that of $g(c)$ in $\Omega \times (0,\infty)$, (3.10) follows from this upon observing that $C_1 \int_0^t e^{-\lambda(t-s)} ds \le \frac{C_1}{\lambda}$ for all $t > 0$. $\Box$

We finally use that for the exponent in (3.8) we can achieve $\frac{4(q-1)}{q} < 1$ on choosing $q > 1$ appropriately. We thereby obtain from the above two lemmata the following.

Lemma 3.5 *There exists $C > 0$ such that*

$$\int_0^t e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|^2_{L^2(\Omega)} ds \le C \qquad \textit{for all } t > 0.$$

Proof. Let us abbreviate

$$K(t) := \int_0^t e^{-\lambda(t-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|^2_{L^2(\Omega)} ds, \qquad t > 0,$$

and fix any $q > 1$ such that $q < \frac{4}{3}$. Then Lemma 3.4 in conjunction with Lemma 3.3 provides $C_1 > 0$ such that

$$K(t) \le C_1 \cdot \Big(1 + \sup_{t' \in (0,t)} K^{\frac{4(q-1)}{q}}(t') \Big) \qquad \text{for all } t > 0.$$

For given $T > 0$, this entails that writing $K_T := \sup_{t \in (0,T)} K(t)$ we have

$$K(t) \le C_1 \cdot \Big(1 + K_T^{\frac{4(q-1)}{q}} \Big) \qquad \text{for all } t \in (0,T)$$

and hence

$$K_T \le C_1 \cdot \Big(1 + K_T^{\frac{4(q-1)}{q}}\Big).$$

Since $\frac{4(q-1)}{q} < 1$ thanks to our assumption $q < \frac{4}{3}$, we can pick $C_2 > 0$ such that $C_1 \xi^{\frac{4(q-1)}{q}} < \frac{\xi}{2} + C_2$ for all $\xi > 0$ and thus infer that

$$\frac{1}{2} K_T \le C_1 + C_2.$$

As both C_1 and C_2 are independent of T, this implies $K(t) \le 2(C_1 + C_2)$ for all $t > 0$ and thereby completes the proof. $\Box$

The latter observation has immediate but useful consequences which form the main results of this section.

Lemma 3.6 *For some $C > 0$ we have*

$$\int_t^{t+1} \int_\Omega n^2 \le C \qquad \textit{for all } t > 0 \tag{3.17}$$

and

$$\int_t^{t+1} \int_\Omega |\nabla c|^4 \le C \qquad \textit{for all } t > 0 \tag{3.18}$$

as well as

$$\int_\Omega |\nabla c(x,t)|^2 dx \le C \qquad \textit{for all } t > 0. \tag{3.19}$$

PROOF. From Lemma 3.5 we obtain $C_1 > 0$ such that

$$\int_0^{t+1} e^{-\lambda(t+1-s)} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|^2_{L^2(\Omega)} ds \le C_1 \qquad \text{for all } t > 0, \tag{3.20}$$

so that successively applying Lemma 3.3 and then Lemma 3.4 we see that (3.19) holds, and that

$$\int_0^{t+1} e^{-\lambda(t+1-s)} \|u(\cdot,s)\|^4_{L^4(\Omega)} ds \le C_2 \qquad \text{for all } t > 0$$

and

$$\int_0^{t+1} e^{-\lambda(t+1-s)} \|\nabla c(\cdot,s)\|^4_{L^4(\Omega)} ds \le C_2 \qquad \text{for all } t > 0 \tag{3.21}$$

with certain positive constants C_2 and C_3. In particular, (3.21) implies (3.18), because

$$\int_t^{t+1} \|\nabla c(\cdot,s)\|^4_{L^4(\Omega)} ds \le e^\lambda \cdot \int_t^{t+1} e^{-\lambda(t+1-s)} \|\nabla c(\cdot,s)\|^4_{L^4(\Omega)} ds \le C_3 e^\lambda \qquad \text{for all } t > 0,$$

and similarly from (3.20) it follows that

$$\int_t^{t+1} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|^2_{L^2(\Omega)} ds \le C_1 e^\lambda \qquad \text{for all } t > 0. \tag{3.22}$$

In order to derive (3.17) from this, we only need to interpolate using the Gagliardo-Nirenberg inequality and recall (2.1) to find $C_4 > 0$ and $C_5 > 0$ such that

$$
\begin{aligned}
\int_t^{t+1} \int_\Omega n^2 &= \int_t^{t+1} \|n^{\frac{1}{2}}(\cdot,s)\|_{L^4(\Omega)}^4 ds \\
&\le C_4 \int_t^{t+1} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 \|n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 ds + C_4 \int_t^{t+1} \|n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^4 ds \\
&\le C_5 \int_t^{t+1} \|\nabla n^{\frac{1}{2}}(\cdot,s)\|_{L^2(\Omega)}^2 ds + C_5 \qquad \text{for all } t > 0,
\end{aligned}
$$

which due to (3.22) indeed yields (3.17). □

4 Uniform decay of c

Having dealt with issues of boundedness so far, we next turn our attention to the derivation of properties indicating decay of solutions. We start with two simple observations which provide some, yet rather weak, information on the decay of ∇c and the product $nf(c)$ in the large time limit. Our goal in this section will be to improve this first piece of knowledge, using the regularity properties collected above, so as to ensure that actually c converges to zero with respect to the norm in $L^\infty(\Omega)$ as $t \to \infty$.

Lemma 4.1 *We have*

$$\int_0^\infty \int_\Omega nf(c) < \infty \tag{4.1}$$

and

$$\int_0^\infty \int_\Omega |\nabla c|^2 < \infty. \tag{4.2}$$

Proof. We integrate the second equation in (1.1) over Ω to obtain

$$\frac{d}{dt} \int_\Omega c = -\int_\Omega nf(c) - \int_\Omega u \cdot \nabla c = -\int_\Omega nf(c) \qquad \text{for all } t > 0, \tag{4.3}$$

where we have used that $\frac{\partial c}{\partial \nu} = 0$ on $\partial\Omega$ and that $\int_\Omega u \cdot \nabla c = -\int_\Omega c \nabla \cdot u = 0$, because $u = 0$ on $\partial\Omega$ and $\nabla \cdot u \equiv 0$. Integrating (4.3) in time readily yields

$$\int_0^t \int_\Omega nf(c) \le \int_\Omega c_0 \qquad \text{for all } t > 0$$

and thus proves (4.1). To see (4.2), we multiply the second equation in (1.1) by c to see that

$$\frac{1}{2} \frac{d}{dt} \int_\Omega c^2 = -\int_\Omega |\nabla c|^2 - \int_\Omega ncf(c) - \int_\Omega uc \cdot \nabla c \qquad \text{for all } t > 0.$$

Again, $\nabla \cdot u \equiv 0$ and $u|_{\partial\Omega} = 0$ imply that $\int_\Omega uc \cdot \nabla c = \frac{1}{2} \int_\Omega u \cdot \nabla c^2 = 0$, and hence a time integration shows that

$$\int_0^t \int_\Omega |\nabla c|^2 \le \frac{1}{2} \int_\Omega c_0^2 \qquad \text{for all } t > 0,$$

for n, c and $f(c)$ are all nonnegative. $\square$

Based on the estimate (3.17) in Lemma 3.6, we can turn the above information into the following decay property of the quantity c itself.

Lemma 4.2 *There exists $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ such that $t_k \to \infty$ and*

$$\int_{t_k}^{t_k+1}\int_\Omega c \to 0 \qquad \textit{as } k\to\infty. \tag{4.4}$$

PROOF. We begin by recalling (4.1) which entails that

$$\int_j^{j+1}\int_\Omega nf(c) \to 0 \qquad \text{as } j\to\infty. \tag{4.5}$$

Here, writing $\overline{f}(t) := \frac{1}{|\Omega|}\int_\Omega f(c(x,t))dx$ for $t>0$ we can decompose

$$\begin{aligned}\int_j^{j+1}\int_\Omega nf(c) &= \int_j^{j+1}\int_\Omega n(x,t)\Big(f(c(x,t))-\overline{f}(t)\Big)dxdt + \int_j^{j+1}\int_\Omega n(x,t)\overline{f}(t)dxdt \\ &=: I_1(j)+I_2(j), \qquad j\in\mathbb{N},\end{aligned} \tag{4.6}$$

where the Cauchy-Schwarz inequality allows us to estimate

$$I_1(j) \le \Big(\int_j^{j+1}\int_\Omega n^2\Big)^{\frac12}\cdot\Big(\int_j^{j+1}\int_\Omega \Big|f(c(x,t))-\overline{f}(t)\Big|^2\Big)^{\frac12} \qquad \text{for all } j\in\mathbb{N}. \tag{4.7}$$

Invoking the Poincaré inequality in the form

$$\int_\Omega \Big|\varphi(x)-\frac{1}{|\Omega|}\int_\Omega \varphi(y)dy\Big|^2 dx \le C_1\int_\Omega |\nabla\varphi|^2 \qquad \text{for all } \varphi\in W^{1,2}(\Omega),$$

valid for some $C_1>0$, thanks to (2.3) we find that

$$\begin{aligned}\int_j^{j+1}\int_\Omega \Big|f(c(x,t))-\overline{f}(t)\Big|^2 dxdt &\le C_1\int_j^{j+1}\int_\Omega \Big|\nabla f(c(x,t))\Big|^2 dxdt \\ &\le C_1\|f'\|^2_{L^\infty((0,C_2))}\int_j^{j+1}\int_\Omega |\nabla c|^2 \qquad \text{for all } j\in\mathbb{N}\end{aligned} \tag{4.8}$$

with $C_2 := \|c_0\|_{L^\infty(\Omega)}$. Since now (4.2) ensures that

$$\int_j^{j+1}\int_\Omega |\nabla c|^2 \to 0 \qquad \text{as } j\to\infty,$$

whereas

$$\sup_{j\in\mathbb{N}}\int_j^{j+1}\int_\Omega n^2 < \infty$$

by Lemma 3.6, (4.7) and (4.8) show that $I_1(j) \to 0$ as $j \to \infty$.
Combined with (4.6) and (4.5), this implies that also $I_2(j) \to 0$ as $j \to \infty$. However, since $\overline{f}(t)$ is constant in space, in view of (2.1) we have

$$I_2(j) = \overline{n_0}|\Omega| \cdot \int_j^{j+1} \overline{f}(t)dt = \overline{n_0} \int_j^{j+1} \int_\Omega f(c(x,t))dxdt \qquad \text{for all } j \in \mathbb{N}$$

with $\overline{n_0} = \frac{1}{|\Omega|} \int_\Omega n_0 > 0$. Accordingly,

$$\int_j^{j+1} \int_\Omega f(c(x,t))dxdt \to 0 \qquad \text{as } j \to \infty,$$

which means that defining $c_j(x,s) := c(x,j+s)$, $(x,s) \in \Omega \times (0,1)$, $j \in \mathbb{N}$, we have $f \circ c_j \to 0$ in $L^1(\Omega \times (0,1))$ as $j \to \infty$. We may thus extract a subsequence $(j_k)_{k\in\mathbb{N}} \subset \mathbb{N}$ such that $j_k \to \infty$ and $f \circ c_{j_k} \to 0$ a.e. in $\Omega \times (0,1)$ as $k \to \infty$. Since f is positive on $(0,\infty)$, this necessarily requires that $c_{j_k} \to 0$ a.e. in $\Omega \times (0,1)$ as $k \to \infty$. As on the other hand $(c_{j_k})_{k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega \times (0,1))$ by (2.3), the dominated convergence theorem ensures that $c_{j_k} \to 0$ in $L^1(\Omega \times (0,1))$ as $k \to \infty$. Restated in the original variables, this precisely means that (4.4) holds for $t_k := j_k$. $\square$

Another application of Lemma 3.6, this time focussing on the inequality (3.18), allows us to improve the latter.

Lemma 4.3 *There exists $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ such that $t_k \to \infty$ and*

$$\int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^\infty(\Omega)}dt \to 0 \qquad \textit{as } k \to \infty. \tag{4.9}$$

Proof. We let $(t_k)_{k\in\mathbb{N}} \subset (0,\infty)$ be as given by Lemma 4.2. Then from Lemma 3.6 we know that there exists $C_1 > 0$ such that

$$\int_{t_k}^{t_k+1} \int_\Omega |\nabla c|^4 \le C_1 \qquad \text{for all } k \in \mathbb{N}. \tag{4.10}$$

Moreover, employing the Gagliardo-Nirenberg inequality let us pick $C_2 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)} \le C_2 \|\nabla\varphi\|_{L^4(\Omega)}^{\frac{4}{5}} \|\varphi\|_{L^1(\Omega)}^{\frac{1}{5}} + C_2\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,4}(\Omega). \tag{4.11}$$

Then given $\varepsilon > 0$ we fix $\delta > 0$ small satisfying $C_1^{\frac{1}{4}}\delta < \varepsilon$ and apply Young's inequality to (4.11) to achieve

$$\|\varphi\|_{L^\infty(\Omega)} \le \delta \|\nabla\varphi\|_{L^4(\Omega)} + C_3\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi \in W^{1,4}(\Omega)$$

with some $C_3 > 0$ which of course depends on δ. We apply this Ehrling-type inequality to $c(\cdot,t)$ for $t \in (t_k, t_k+1)$, integrate in time and use the Hölder inequality and (4.10) to obtain

$$\begin{aligned}
\int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^\infty(\Omega)}dt &\le \delta \int_{t_k}^{t_k+1} |\nabla c(\cdot,t)\|_{L^4(\Omega)}dt + C_3 \int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^1(\Omega)}dt \\
&\le \delta\Big(\int_{t_k}^{t_k+1} \|\nabla c(\cdot,t)\|_{L^4(\Omega)}^4 dt\Big)^{\frac{1}{4}} + C_3 \int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^1(\Omega)}dt \\
&\le \delta C_1^{\frac{1}{4}} + C_3 \int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^1(\Omega)}dt \qquad \text{for all } k \in \mathbb{N}.
\end{aligned}$$

According to our choice of δ, in light of Lemma 4.2 this shows that

$$\limsup_{k\to\infty} \int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^\infty(\Omega)} dt < \varepsilon + \limsup_{k\to\infty} \int_{t_k}^{t_k+1} \|c(\cdot,t)\|_{L^1(\Omega)} dt = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the proof thereby becomes complete. $\Box$

Now the main result of this section is an immediate consequence of Lemma 4.3 and the fact that the spatial L^∞ norm of c is nonincreasing with time.

Corollary 4.4 *We have*

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{4.12}$$

Proof. As a consequence of Lemma 4.3, we must have $\liminf_{t\to\infty} \|c(\cdot,t)\|_{L^\infty(\Omega)} = 0$. Combined with the monotonicity of $t \mapsto \|c(\cdot,t)\|_{L^\infty(\Omega)}$ as asserted by Lemma 2.1, this implies (4.12). $\Box$

5 L^p bounds and a weak stabilization result for n

According to the above result, we now know that $c(x,t)$ may be assumed arbitrarily small on choosing t suitably large. In particular, for arbitrary $p > 1$ and $\delta > 0$ the functional

$$\int_\Omega \frac{n^p(x,t)}{\delta - c(x,t)} dx$$

is well-defined and positive for sufficiently large t. Pursuing the time evolution thereof will yield the following result which, via (5.2), includes a first indication that $n(\cdot,t)$ will become spatially homogeneous as $t \to \infty$.

Lemma 5.1 *Let $p \geq 2$. Then there exists $C > 0$ such that*

$$\int_\Omega n^p(x,t) dx \leq C \qquad \textit{for all } t > 1, \tag{5.1}$$

and moreover we have

$$\int_1^\infty \int_\Omega (n+1)^{p-2} |\nabla n|^2 < \infty. \tag{5.2}$$

Proof. We clearly may assume that $p > 2$. Then $\frac{4p}{p-1} < 8$, so that it is possible to fix $\delta \in (0, \|c_0\|_{L^\infty(\Omega)})$ small enough satisfying

$$pC_1\delta < 2 \tag{5.3}$$

and

$$p(p-1)C_1^2\delta^2 + \frac{4p}{p-1} < 8, \tag{5.4}$$

where $C_1 := \|\chi\|_{L^\infty((0,C_2))}$ with $C_2 := \|c_0\|_{L^\infty(\Omega)}$.
Now thanks to Corollary 4.4, we know that there exists some large $t_0 > 1$ with the property that

$$c \leq \frac{\delta}{2} \qquad \text{in } \Omega \times (t_0, \infty). \tag{5.5}$$

Then $\frac{(n+1)^p}{\delta-c}$ is well-defined and smooth in $\bar\Omega\times[t_0,\infty)$, and hence for $t>t_0$ we may use (1.1) to compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega \frac{(n+1)^p}{\delta-c} &= p\int_\Omega \frac{(n+1)^{p-1}}{\delta-c}\cdot\Big\{\Delta n-\nabla\cdot(n\chi(c)\nabla c)-u\cdot\nabla n\Big\}\\ &\quad+\int_\Omega \frac{(n+1)^p}{(\delta-c)^2}\cdot\Big\{\Delta c-nf(c)-u\cdot\nabla c\Big\}\qquad\text{for all } t>t_0.\end{aligned}\tag{5.6}$$

Here several integrations by parts yield

$$p\int_\Omega \frac{(n+1)^{p-1}}{\delta-c}\Delta n=-p(p-1)\int_\Omega \frac{(n+1)^{p-2}}{\delta-c}|\nabla n|^2-p\int_\Omega\frac{(n+1)^{p-1}}{(\delta-c)^2}\nabla n\cdot\nabla c$$

and

$$-p\int_\Omega \frac{(n+1)^{p-1}}{\delta-c}\nabla\cdot(n\chi(c)\nabla c)=p(p-1)\int_\Omega\frac{n(n+1)^{p-2}}{\delta-c}\chi(c)\nabla n\cdot\nabla c+p\int_\Omega\frac{n(n+1)^{p-1}}{(\delta-c)^2}\chi(c)|\nabla c|^2$$

as well as

$$\int_\Omega\frac{(n+1)^p}{(\delta-c)^2}\Delta c=-p\int_\Omega\frac{(n+1)^{p-1}}{(\delta-c)^2}\nabla n\cdot\nabla c-2\int_\Omega\frac{(n+1)^p}{(\delta-c)^3}|\nabla c|^2$$

for $t>t_0$. By incompressibility, the integrals involving u cancel each other: Indeed, since $\nabla\cdot u\equiv 0$ and $u|_{\partial\Omega}=0$, for all $t>t_0$ we have

$$p\int_\Omega\frac{(n+1)^{p-1}}{\delta-c}u\cdot\nabla n=\int_\Omega\frac{1}{\delta-c}u\cdot\nabla(n+1)^p=-\int_\Omega(n+1)^p u\cdot\nabla\frac{1}{\delta-c}=-\int_\Omega\frac{(n+1)^p}{(\delta-c)^2}u\cdot\nabla c.$$

As moreover $f\ge 0$, from (5.6) we all in all obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega\frac{(n+1)^p}{\delta-c} &\le -p(p-1)\int_\Omega\frac{(n+1)^{p-2}}{\delta-c}|\nabla n|^2\\ &\quad-\int_\Omega(n+1)^p\cdot\Big\{\frac{2}{(\delta-c)^3}-\frac{p\chi(c)}{(\delta-c)^2}\cdot\frac{n}{n+1}\Big\}\cdot|\nabla c|^2\\ &\quad+\int_\Omega(n+1)^{p-1}\cdot\Big\{\frac{p(p-1)\chi(c)}{\delta-c}\cdot\frac{n}{n+1}-\frac{2p}{(\delta-c)^2}\Big\}\cdot(\nabla n\cdot\nabla c)\end{aligned}\tag{5.7}$$

for all $t>t_0$. According to (5.3), the summand containing $|\nabla c|^2$ is nonpositive, because

$$\frac{\frac{p\chi(c)}{(\delta-c)^2}\cdot\frac{n}{n+1}}{\frac{2}{(\delta-c)^3}}=\frac{p\chi(c)\cdot(\delta-c)}{2}\cdot\frac{n}{n+1}\le\frac{pC_1\delta}{2}<1\qquad\text{in }\Omega\times(t_0,\infty).$$

We therefore may invoke Young's inequality to estimate

$$\begin{aligned}&\int_\Omega(n+1)^{p-1}\cdot\Big\{\frac{p(p-1)\chi(c)}{\delta-c}\cdot\frac{n}{n+1}-\frac{2p}{(\delta-c)^2}\Big\}\cdot(\nabla n\cdot\nabla c)\\ &\qquad\le\int_\Omega(n+1)^p\cdot\Big\{\frac{2}{(\delta-c)^3}-\frac{p\chi(c)}{(\delta-c)^2}\cdot\frac{n}{n+1}\Big\}\cdot|\nabla c|^2\\ &\qquad\quad+\int_\Omega(n+1)^{p-2}h(n,c)|\nabla n|^2\qquad\text{for all } t>t_0\end{aligned}\tag{5.8}$$

with

$$h(\eta,\xi) := \frac{\Big\{\frac{p(p-1)\chi(\xi)}{\delta-\xi}\cdot\frac{\eta}{\eta+1} - \frac{2p}{(\delta-\xi)^2}\Big\}^2}{4\cdot\Big\{\frac{2}{(\delta-\xi)^3} - \frac{p\chi(\xi)}{(\delta-\xi)^2}\cdot\frac{\eta}{\eta+1}\Big\}} \qquad \text{for } \eta \ge 0 \text{ and } \xi \in [0,\delta).$$

By straightforward rearrangements, we obtain

$$\begin{aligned}\frac{h(\eta,\xi)}{\frac{p(p-1)}{\delta-\xi}} &= \frac{p(p-1)\chi^2(\xi)\cdot(\delta-\xi)^2\cdot\frac{\eta^2}{(\eta+1)^2} - 4p\chi(\xi)\cdot(\delta-\xi)\cdot\frac{\eta}{\eta+1} + \frac{4p}{p-1}}{8-4p\chi(\xi)\cdot(\delta-\xi)\cdot\frac{\eta}{\eta+1}} \\ &=: \frac{h_1(\eta,\xi)}{h_2(\eta,\xi)} \qquad \text{for all } \eta\ge 0 \text{ and } \xi\in[0,\delta),\end{aligned} \tag{5.9}$$

where since $\delta < C_2$ we can use that $\chi \le C_1$ on $[0,\delta)$ to estimate

$$\begin{aligned}h_1(\eta,\xi)-h_2(\eta,\xi) &= p(p-1)\chi^2(\xi)\cdot(\delta-\xi)^2\cdot\frac{\eta^2}{(\eta+1)^2} + \frac{4p}{p-1} - 8 \\ &\le p(p-1)C_1^2\delta^2 + \frac{4p}{p-1} - 8 \qquad \text{for all } \eta\ge 0 \text{ and } \xi\in[0,\delta).\end{aligned}$$

Therefore, our restriction (5.4) on δ, asserting that $C_3 := 8 - p(p-1)C_1^2\delta^2 - \frac{4p}{p-1}$ is positive, along with the observation that

$$h_2(n,c) \ge C_4 := 8 - 4pC_1\delta > 0 \qquad \text{in } \Omega\times(t_0,\infty)$$

by (5.3), shows that $\frac{h_1(n,c)}{h_2(n,c)} \le 1 - C_5$ in $\Omega\times(t_0,\infty)$ wih $C_5 := \frac{C_3}{C_4} > 0$. Together with (5.7)-(5.9), this implies that

$$\frac{d}{dt}\int_\Omega \frac{(n+1)^p}{\delta-c} \le -p(p-1)C_5\cdot\int_\Omega \frac{(n+1)^{p-2}}{\delta-c}|\nabla n|^2 \qquad \text{for all } t>t_0$$

and hence, upon a time integration over (t_0,t), that

$$\int_\Omega \frac{(n+1)^p(x,t)}{\delta-c(x,t)}dx + p(p-1)C_5\int_{t_0}^t\int_\Omega \frac{(n+1)^{p-2}}{\delta-c}|\nabla n|^2 \le C_6 := \int_\Omega \frac{(n+1)^p(x,t_0)}{\delta-c(x,t_0)}dx \qquad \text{for all } t>t_0.$$

This clearly entails that

$$\int_\Omega n^p(x,t)dx \le \delta C_6 \quad \text{and} \quad \int_{t_0}^t\int_\Omega (n+1)^{p-2}|\nabla n|^2 \le \frac{\delta C_6}{p(p-1)C_5} \qquad \text{for all } t>t_0$$

and thereby proves both (5.1) and (5.2), because n is bounded and smooth in $\bar\Omega\times[1,t_0]$. $\square$

6 Decay of u

We are now prepared to prove the claimed asymptotic behavior of u, which will be accomplished in Lemma 6.3. As a preparatory step, we apply the decay information (5.2) to the energy identity for u to derive decay of $u(\cdot,t)$ with respect to the norm in $L^2(\Omega)$ in the first instance.

Lemma 6.1 *We have*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \to 0 \qquad \text{as } t\to\infty \tag{6.1}$$

and

$$\int_1^\infty \int_\Omega |\nabla u|^2 < \infty. \tag{6.2}$$

Proof. We once more test the third equation in (1.1) by u to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega n\nabla\phi\cdot u \qquad \text{for all } t>0, \tag{6.3}$$

but now unlike in Lemma 3.1 in the latter term we first integrate by parts before estimating, because our goal is to use the decay property of ∇n contained in (5.2). More precisely, since $\nabla\cdot u\equiv 0$ and $u|_{\partial\Omega}=0$ we have

$$\int_\Omega n\nabla\phi\cdot u = -\int_\Omega \phi u\cdot\nabla n \qquad \text{for all } t>0,$$

and taking $C_1>0$ from the Poincaré inequality ensuring

$$\int_\Omega |u|^2 \le C_1\int_\Omega |\nabla u|^2 \qquad \text{for all } t>0, \tag{6.4}$$

we use Young's inequalty to estimate

$$-\int_\Omega \phi u\cdot\nabla n \le \frac{1}{2C_1}\int_\Omega |u|^2 + C_2\int_\Omega |\nabla n|^2 \qquad \text{for all } t>0$$

with $C_2 := \frac{C_1\|\phi\|^2_{L^\infty(\Omega)}}{2}$.
Then using (6.4) we infer from (6.3) that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 \le C_2\int_\Omega |\nabla n|^2 \qquad \text{for all } t>0, \tag{6.5}$$

which upon integration over $(1,t)$, $t>1$, implies that

$$\frac{1}{2}\int_1^t\int_\Omega |\nabla u|^2 \le \frac{1}{2}\int_\Omega |u(x,1)|^2dx + C_2\int_1^\infty\int_\Omega |\nabla n|^2 \qquad \text{for all } t>1.$$

Since an application of Lemma 5.1 to $p:=2$ gives

$$\int_1^\infty\int_\Omega |\nabla n|^2 < \infty, \tag{6.6}$$

this proves (6.2).
To see (6.1), we further estimate the dissipative term in (6.5), again using (6.4), to see that $y(t) := \int_\Omega |u(x,t)|^2 dx, t \geq 0$, satisfies

$$y'(t) \leq -C_3 y(t) + h(t) \qquad \text{for all } t > 0$$

with $C_3 := \frac{1}{C_1}$ and $h(t) := 2C_2 \int_\Omega |\nabla n(x,t)|^2 dx, t > 0$. Integrating this yields

$$y(t) \leq e^{-C_3(t-1)} y(1) + \int_1^t e^{-C_3(t-s)} h(s) ds \qquad \text{for all } t > 1.$$

Since here for $t > 2$ we can split

$$\begin{aligned} \int_1^t e^{-C_3(t-s)} h(s) ds &= \int_1^{\frac{t}{2}} e^{-C_3(t-s)} h(s) ds + \int_{\frac{t}{2}}^t e^{-C_3(t-s)} h(s) ds \\ &\leq e^{-\frac{C_3 t}{2}} \cdot \int_1^\infty h(s) ds + \int_{\frac{t}{2}}^\infty h(s) ds, \end{aligned}$$

the fact that $\int_1^\infty h(s) ds < \infty$ asserted by (6.6) guarantees that $y(t) \to 0$ as $t \to \infty$, as desired. $\Box$

A standard bootstrap procedure, again based on Lemma 5.1, asserts that since the spatial dimension is two, u is actually more regular than guaranteed by Lemma 6.1.

Lemma 6.2 *There exists $C > 0$ such that*

$$\int_\Omega |\nabla u(x,t)|^2 dx \leq C \qquad \text{for all } t > 1. \tag{6.7}$$

Proof. We let A denote the Stokes operator in $L^2_\sigma(\Omega)$ with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$. Then it is well-known that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $D(A)$ ([12]). Therefore the Gagliardo-Nirenberg inequality yields $C_1 > 0$ and $C_2 > 0$ such that

$$\|\varphi\|_{L^4(\Omega)} \leq C_1 \|A\varphi\|_{L^2(\Omega)}^{\frac{1}{4}} \|\varphi\|_{L^2(\Omega)}^{\frac{3}{4}} \qquad \text{for all } \varphi \in D(A) \tag{6.8}$$

and

$$\|\nabla\varphi\|_{L^4(\Omega)} \leq C_2 \|A\varphi\|_{L^2(\Omega)}^{\frac{3}{4}} \|\varphi\|_{L^2(\Omega)}^{\frac{1}{4}} \qquad \text{for all } \varphi \in D(A). \tag{6.9}$$

We then pick $\delta > 0$ small enough fulfilling

$$C_1 C_2 |\kappa| \delta < \frac{1}{2} \tag{6.10}$$

and finally choose $t_0 > 0$ large enough such that

$$\|u(\cdot,t)\|_{L^2(\Omega)} < \delta \qquad \text{for all } t > t_0, \tag{6.11}$$

which is possible due to Lemma 6.1.
We now apply the Helmholtz projection $\mathcal{P}$ in $L^2_\sigma(\Omega)$ to the third equation in (1.1), multiply the resulting identity by Au and integrate over Ω to find

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}}u|^2 + \int_\Omega |Au|^2 = \kappa \int_\Omega (\mathcal{P}u \cdot \nabla) \cdot Au + \int_\Omega (\mathcal{P}n\nabla\phi) \cdot Au \qquad \text{for all } t > 0, \tag{6.12}$$

where we have made use of the identity $\int_\Omega \varphi\cdot A\varphi = \int_\Omega |A^{\frac{1}{2}}\varphi|^2 = \int_\Omega |\nabla\varphi|^2$ for $\varphi\in D(A)$. In order to estimate the convective term in (6.12) following a standard argument, we use the Hölder inequality as well as (6.8), (6.9), (6.11) and (6.10) to estimate

$$\begin{aligned}
\kappa\int_\Omega (\mathcal{P}u\cdot\nabla)\cdot Au &\le |\kappa|\cdot\|u\|_{L^4(\Omega)}\cdot\|\nabla u\|_{L^4(\Omega)}\cdot\|Au\|_{L^2(\Omega)}\\
&\le |\kappa|\cdot C_1\|Au\|_{L^2(\Omega)}^{\frac{1}{4}}\|u\|_{L^2(\Omega)}^{\frac{3}{4}}\cdot C_2\|Au\|_{L^2(\Omega)}^{\frac{3}{4}}\|u\|_{L^2(\Omega)}^{\frac{1}{4}}\cdot\|Au\|_{L^2(\Omega)}\\
&\le |\kappa|C_1C_2\|Au\|_{L^2(\Omega)}^2\|u\|_{L^2(\Omega)}\\
&\le \frac{1}{2}\|Au\|_{L^2(\Omega)}^2 \qquad \text{for all } t>0.
\end{aligned}$$

As to the last term in (6.12), we invoke Young's inequality to estimate

$$\int_\Omega (\mathcal{P}n\nabla\phi)\cdot Au \le \frac{1}{2}\int_\Omega |Au|^2 + C_3\int_\Omega n^2$$

with $C_3 := \frac{\|\nabla\phi\|_{L^\infty(\Omega)}^2}{2}$, whence (6.12) altogether yields

$$\frac{d}{dt}\int_\Omega |\nabla u|^2 \le 2C_3\int_\Omega n^2 \qquad \text{for all } t>t_0. \tag{6.13}$$

Now since Lemma 6.1 in particular implies that for some $C_4>0$ we have $\int_k^{k+1}\int_\Omega |\nabla u|^2 \le C_4$ for all $k\in\mathbb{N}$, given any such k we can pick $t_k\in(k,k+1)$ such that $\int_\Omega |\nabla u(x,t_k)|^2dx \le C_4$. As furthermore $\int_\Omega n^2(x,t)dx \le C_5$ for all $t>1$ with some $C_5>0$ by Lemma 5.1, we may integrate (6.13) with respect to t to obtain

$$\begin{aligned}
\int_\Omega |\nabla u(x,t)|^2dx &\le \int_\Omega |\nabla u(\cdot,t_k)|^2dx + 2C_3\int_{t_k}^t\int_\Omega n^2\\
&\le C_6 := C_4 + 4C_3C_5 \qquad \text{for all } t\in(t_k,t_k+2),
\end{aligned}$$

provided that $k>t_0$. Since $(t_k,t_k+2)\supset[k+1,k+2]$, this entails that

$$\sup_{t\in[k+1,k+2]}\int_\Omega |\nabla u(x,t)|^2dx \le C_6 \qquad \text{for all } k>t_0$$

and hence clearly proves (6.7). □

We can now use the regularity information of u and n gained above in order to establish uniform decay of u by means of a variation-of-constants representation of u.

Lemma 6.3 *The solution component u is bounded in $\Omega\times(0,\infty)$ and satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t\to\infty. \tag{6.14}$$

PROOF. We claim that for all $r\in[2,\infty)$ we have

$$\sup_{t>2}\|u(\cdot,t)\|_{W^{1,r}(\Omega)} < \infty. \tag{6.15}$$

Since $W^{1,r}(\Omega)$ is compactly embedded in $L^\infty(\Omega)$ for any such r, in combination with the decay property (6.1) and a straightforward interpolation argument this will prove (6.14), while since clearly u is bounded in $\Omega \times (0,2)$, (6.15) will also imply global pointwise boundedness of u.
To verify (6.15), we first pick $\alpha \in (0,1)$ such that $\alpha > 1 - \frac{1}{r}$. Then $\alpha - \frac{1}{2} + \frac{1}{r} > \frac{1}{2}$, so that we can fix $p \in (1,2)$ close enough to 2 such that still

$$\alpha - \frac{1}{2} + \frac{1}{r} > \frac{1}{p}. \tag{6.16}$$

We now consider the Stokes operator A in $L^p_\sigma(\Omega) := \{\varphi \in L^p(\Omega) \mid \nabla \cdot \varphi = 0 \text{ in } \mathcal{D}'(\Omega)\}$, with domain $D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega)$. Then the choices in (6.16) ensure that the domain $D(A^\alpha)$ of the fractional power A^α satisfies $D(A^\alpha) \hookrightarrow W^{1,r}(\Omega)$ ([11, p.201], [13, p.77]), so that there exists $C_1 > 0$ such that

$$\|\varphi\|_{W^{1,r}(\Omega)} \le C_1 \|A^\alpha \varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in D(A^\alpha). \tag{6.17}$$

We next rewrite the third equation in (1.1) in the form

$$u_t = \Delta u + \nabla P + h(x,t), \qquad x \in \Omega,\ t > 0, \tag{6.18}$$

where $h := h_1 + h_2$ with $h_1 := -\kappa(u \cdot \nabla)u$ and $h_2 := n\nabla\phi$. By Lemma 5.1, for some $C_2 > 0$ we have

$$\|h_2(\cdot,t)\|_{L^p(\Omega)} \le C_2 \qquad \text{for all } t > 1, \tag{6.19}$$

whereas using the Hölder inequality with exponents $\frac{2}{p} > 1$ and $\frac{2}{2-p}$ we obtain from Lemma 6.2 that

$$\begin{aligned} \|h_1(\cdot,t)\|_{L^p(\Omega)} &\le |\kappa| \cdot \|u(\cdot,t)\|_{L^{\frac{2p}{2-p}}(\Omega)} \cdot \|\nabla u(\cdot,t)\|_{L^2(\Omega)} \\ &\le C_3 \qquad \text{for all } t > 1 \end{aligned} \tag{6.20}$$

with some $C_3 > 0$, because $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2p}{2-p}}(\Omega)$.
Now the variation-of-constants formula associated with (6.18) represents u according to

$$u(\cdot,t) = e^{-(t-k)A}u(\cdot,k) + \int_k^t e^{-(t-s)A}\mathcal{P}h(\cdot,s)ds, \qquad t > k,$$

where e^{-tA} and $\mathcal{P}$ denote the semigroup generated by A and the Helmholtz projection in $L^p(\Omega)$, respectively, and $k \ge 1$ is an arbitrary integer. Here we apply A^α to both sides and recall the well-known smoothing estimate

$$\|A^\alpha e^{-tA}\varphi\|_{L^p(\Omega)} \le C_4 t^{-\alpha}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in L^p_\sigma(\Omega),$$

valid for all $t > 0$ and some $C_4 > 0$ ([11], [9]). Since $\mathcal{P}$ is a bounded operator from $L^p(\Omega)$ to $L^p_\sigma(\Omega)$, we thereupon obtain

$$\|A^\alpha u(\cdot,t)\|_{L^p(\Omega)} \le C_4(t-k)^{-\alpha}\|u(\cdot,k)\|_{L^p(\Omega)} + C_4\int_k^t (t-s)^{-\alpha}\|h(\cdot,s)\|_{L^p(\Omega)}ds \qquad \text{for all } t > k.$$

As clearly $C_5 := \sup_{t>0} \|u(\cdot,t)\|_{L^p(\Omega)}$ is finite due to Lemma 6.1 and the fact that $p < 2$, by (6.19) and (6.20) we have

$$\begin{aligned} \|A^\alpha u(\cdot,t)\|_{L^p(\Omega)} &\le C_4 C_5 (t-k)^{-\alpha} + C_4 (C_2 + C_3) \int_k^t (t-s)^{-\alpha} ds \\ &\le C_4 C_5 + C_4 (C_2 + C_3) \cdot \frac{2^{1-\alpha}}{1-\alpha} \qquad \text{for all } t \in [k+1, k+2]. \end{aligned}$$

Since $k \ge 1$ was arbitrary, in view of (6.17) this establishes (6.15) and hence completes the proof. $\Box$

As an appendix to this section, we finally exploit the boundedness statement contained in the above lemma in order to derive a regularity property for c which goes beyond those in Lemma 2.1 and Lemma 4.1.

Lemma 6.4 *For all $r \in (2,\infty)$ there exists $C > 0$ such that*

$$\int_\Omega |\nabla c(x,t)|^r dx \le C \qquad \textit{for all } t > 2. \tag{6.21}$$

PROOF. We write the second PDE in (1.1) in the form

$$c_t = \Delta c - c + h(x,t), \qquad x \in \Omega,\ t > 0, \tag{6.22}$$

with $h := h_1 + h_2 + h_3$, where $h_1 := c, h_2 := -nf(c)$ and $h_3 := -u \cdot \nabla c$. Then Lemma 5.1 and (2.3) guarantee that

$$\|h_1(\cdot,t)\|_{L^2(\Omega)} \le C_1 \quad \text{for all } t > 0 \qquad \text{and} \qquad \|h_2(\cdot,t)\|_{L^2(\Omega)} \le C_2 \quad \text{for all } t > 1, \tag{6.23}$$

while Lemma 6.3 together with Lemma 3.6 assert that

$$\|h_3(\cdot,t)\|_{L^2(\Omega)} \le C_3 \qquad \text{for all } t > 1 \tag{6.24}$$

with positive constants C_1, C_2 and C_3.
We now let B denote the realization of $-\Delta + 1$ in $L^2(\Omega)$ subject to homogeneous Neumann boundary conditions, and then obtain from (6.22) that for each integer $k \ge 1$ we have

$$c(\cdot,t) = e^{-tB} c(\cdot,k) + \int_k^t e^{-(t-s)B} h(\cdot,s) ds, \qquad t > k. \tag{6.25}$$

Next, given $r \in (2,\infty)$ we pick some $\beta \in (\frac{1}{2}, 1)$ fulfilling $\beta > 1 - \frac{1}{r}$, which guarantees that $D(B^\beta) \hookrightarrow W^{1,r}(\Omega)$ ([13], [9]), and hence there exists $C_4 > 0$ such that

$$\|\nabla\varphi\|_{L^r(\Omega)} \le C_4 \|B^\beta \varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in D(B^\beta).$$

Since moreover $\|B^\beta e^{-tB}\varphi\|_{L^2(\Omega)} \le C_5 t^{-\beta} \|\varphi\|_{L^2(\Omega)}$ for all $\varphi \in L^2(\Omega)$ with some $C_5 > 0$, applying B^β to both sides of (6.25) and using (6.23) and (6.24) we obtain

$$\begin{aligned} \|\nabla c(\cdot,t)\|_{L^r(\Omega)} &\le C_4 \|B^\beta c(\cdot,t)\|_{L^2(\Omega)} \\ &\le C_4 C_5 (t-k)^{-\beta} \|c(\cdot,k)\|_{L^2(\Omega)} + C_4 C_5 \int_k^t (t-s)^{-\beta} (C_1 + C_2 + C_3) ds \qquad \text{for all } t > k. \end{aligned}$$

In view of (2.3) and the fact that $\beta < 1$, this shows that for some $C_6 > 0$ we have $\|\nabla c(\cdot,t)\|_{L^r(\Omega)} \le C_6$ for any $t \in [k+1, k+2]$ and each $k \ge 1$. $\Box$

7 Boundedness of n

According to Lemma 6.3 and Lemma 6.4, the nonlinearities in the first equation in (1.1) are bounded in $L^\infty((2,\infty);(W^{1,p}(\Omega))^\star)$ for any finite p. In a straightforward manner this can be turned into the following.

Lemma 7.1 *There exists $C > 0$ such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{7.1}$$

Moreover,

$$(n(\cdot,t))_{t>3} \textit{ is relatively compact in } C^0(\bar\Omega). \tag{7.2}$$

Proof. By the first equation in (1.1) and the fact that $\nabla\cdot u \equiv 0$, we have

$$n_t = \Delta n - n + h_1(x,t) - \nabla\cdot(h_2(x,t)+h_3(x,t)), \qquad x\in\Omega,\ t>0, \tag{7.3}$$

with $h_1 := n, h_2 := n\chi(c)\nabla c$ and $h_3 := un$. Fixing any $p > 2$, from Lemma 5.1 we obtain $C_1 > 0$ such that

$$\|h_1(\cdot,t)\|_{L^p(\Omega)} \le C_1 \qquad \text{for all } t>2, \tag{7.4}$$

and Lemma 5.1, (2.3) and Lemma 6.4 yield $C_2 > 0$ satisfying

$$\|h_2(\cdot,t)\|_{L^p(\Omega)} \le C_2 \qquad \text{for all } t>2, \tag{7.5}$$

while Lemma 6.3 along with Lemma 5.1 provides $C_3 > 0$ fulfilling

$$\|h_3(\cdot,t)\|_{L^p(\Omega)} \le C_3 \qquad \text{for all } t>2. \tag{7.6}$$

Now

$$n(\cdot,t) = e^{-tB}n(\cdot,k) + \int_k^t e^{-(t-s)B}h_1(\cdot,s)ds - \int_k^t e^{-(t-s)B}\nabla\cdot(h_2(\cdot,s)+h_3(\cdot,s))ds \qquad \text{for all } t>k, \tag{7.7}$$

where $k\in\{2,3,4,...\}$ and, similar to the proof of Lemma 6.4, B represents the sectorial extension of $-\Delta+1$ in $L^p(\Omega)$ under homogeneous Neumann data. Then since $p > 2$, we may first choose $\beta\in(0,\frac12)$ fulfilling $\beta > \frac1p$ and then $\theta > 0$ small satisfying $\theta < 2\beta - \frac2p$, so that $D(B^\beta) \hookrightarrow C^\theta(\bar\Omega)$ ([13], [9]) and hence

$$\|\varphi\|_{C^\theta(\bar\Omega)} \le C_4\|B^\beta\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi\in D(B^\beta) \tag{7.8}$$

with some $C_4 > 0$. Since furthermore there exist $C_5 > 0$ and $C_6 > 0$ such that for all $\varphi\in L^p(\Omega)$ we have

$$\|B^\beta e^{-tB}\varphi\|_{L^p(\Omega)} \le C_5 t^{-\beta}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0 \tag{7.9}$$

and

$$\|B^\beta e^{-tB}\nabla\cdot\varphi\|_{L^p(\Omega)} \le C_6 t^{-\frac12-\beta}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } t>0 \tag{7.10}$$

(cf. [32, Lemma 1.3]), from (7.7) and (7.4)-(7.6) we infer that for any such k,

$$\begin{aligned} \|B^\beta n(\cdot,t)\|_{L^p(\Omega)} &\le C_5(t-k)^{-\beta}\|n(\cdot,k)\|_{L^p(\Omega)} + C_5\int_k^t (t-s)^{-\beta}\cdot C_1 ds \\ &\quad + C_6\int_k^t (t-s)^{-\frac{1}{2}-\beta}\cdot (C_2+C_3)ds \qquad \text{for all } t>k. \end{aligned}$$

Thanks to the fact that $\frac{1}{2}+\beta<1$, in view of Lemma 5.1 and (7.8) this entails that with some $C_7>0$ we have

$$\|n(\cdot,t)\|_{C^\theta(\bar\Omega)} \le C_7 \qquad \text{for all } t\ge 3.$$

By means of the Arzelà-Ascoli theorem we thereby infer that (7.2) holds, whereupon (7.1) results from this and the observation that n clearly is bounded in $\Omega\times(0,3)$. □

8 Convergence of n

We finally need to make sure that $n(\cdot,t)$ stabilizes toward the constant $\overline{n_0}$ as $t\to\infty$. Here Lemma 5.1 provides a first step by implying that $\int_1^\infty\int_\Omega |\nabla n|^2$ is finite, and that hence $\nabla n(\cdot,t_k)\to 0$ in $L^2(\Omega)$ along a suitable sequence of numbers $t_k\to\infty$. In order to conclude convergence along the entire net $t\to\infty$, let us derive a certain, though rather weak, decay property of n_t.

Lemma 8.1 *We have*

$$\int_1^\infty \|n_t(\cdot,t)\|^2_{(W^{1,2}(\Omega))^\star} dt < \infty. \tag{8.1}$$

PROOF. We fix $t>1$ and let $\varphi\in W^{1,2}(\Omega)$ be given. Then multiplying the first equation in (1.1) by φ and integrating by parts over Ω we obtain

$$\begin{aligned} \int_\Omega n_t(x,t)\varphi(x)dx &= \int_\Omega \Big(-\nabla n + n\chi(c)\nabla c\Big)\cdot\nabla\varphi - \int_\Omega (u\cdot\nabla n)\varphi \\ &\le \Big(\|\nabla n\|_{L^2(\Omega)} + \|n\chi(c)\nabla c\|_{L^2(\Omega)}\Big)\cdot\|\nabla\varphi\|_{L^2(\Omega)} + \|u\cdot\nabla n\|_{L^2(\Omega)}\cdot\|\varphi\|_{L^2(\Omega)} \end{aligned}$$

by the Cauchy-Schwarz inequality. Since φ was arbitrary, this implies that

$$\|n_t(\cdot,t)\|_{(W^{1,2}(\Omega))^\star} \le \|\nabla n\|_{L^2(\Omega)} + \|n\chi(c)\nabla c\|_{L^2(\Omega)} + \|u\cdot\nabla n\|_{L^2(\Omega)} \qquad \text{for all } t>1$$

and hence

$$\begin{aligned} \int_1^T \|n_t(\cdot,t)\|^2_{(W^{1,2}(\Omega))^\star}dt &\le C_1\cdot\Bigg\{\int_1^T \|\nabla n(\cdot,t)\|^2_{L^2(\Omega)}dt \\ &\quad + \|n\|^2_{L^\infty(\Omega\times(1,\infty))}\cdot\|\chi(c)\|^2_{L^\infty(\Omega\times(1,\infty))}\cdot\int_1^T\|\nabla c(\cdot,t)\|^2_{L^2(\Omega)}dt \\ &\quad + \|u\|^2_{L^\infty(\Omega\times(1,\infty))}\cdot\int_1^T\|\nabla n(\cdot,t)\|^2_{L^2(\Omega)}dt\Bigg\} \qquad \text{for all } T>1 \end{aligned}$$

with some $C_1 > 0$. Since n, c and u are bounded in $\Omega \times (1, \infty)$ by Lemma 7.1, (2.3) and Lemma 6.3, the convergence of the integrals $\int_1^\infty \int_\Omega |\nabla n|^2$ and $\int_1^\infty \int_\Omega |\nabla c|^2$ asserted by Lemma 5.1 and Lemma 4.1 yields the claim. $\Box$

Based on the above, a variant of a standard argument ([1]) now allows us to conclude that n stabilizes in the claimed sense.

Lemma 8.2 *We have*

$$\|n(\cdot, t) - \overline{n_0}\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty, \tag{8.2}$$

where $\overline{n_0} = \frac{1}{|\Omega|} \int_\Omega n_0$.

PROOF. Since $(n(\cdot, t))_{t>3}$ is relatively compact in $C^0(\bar{\Omega})$ by Lemma 7.1, according to a standard reasoning in order to prove (8.2) we only need to make sure that $\overline{n_0}$ is the sole element of the corresponding ω-limit set on n; that is, we need to show that whenever $(t_k)_{k\in\mathbb{N}} \subset (3, \infty)$ and $n_\infty \in C^0(\bar{\Omega})$ are such that $t_k \to \infty$ and $n(\cdot, t_k) \to n_\infty$ in $C^0(\bar{\Omega})$ as $k \to \infty$, we necessarily have $n_\infty \equiv \overline{n_0}$.
To see this, given any such $(t_k)_{k\in\mathbb{N}}$ and n_∞, we introduce

$$n_k(x, s) := n(x, t_k + s), \qquad x \in \Omega,\ s \in (0, 1),\ k \in \mathbb{N}.$$

Then letting $C_1 > 0$ denote a Poincaré constant satisfying

$$\Big\|\varphi - \frac{1}{|\Omega|} \int_\Omega \varphi\Big\|_{L^2(\Omega)}^2 \le C_1 \|\nabla\varphi\|_{L^2(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

recalling Lemma 5.1 and (2.1) we see that

$$\begin{aligned} \|n_k - \overline{n_0}\|_{L^2(\Omega\times(0,1))}^2 &= \int_{t_k}^{t_k+1} \int_\Omega |n(x,t) - \overline{n_0}|^2 dxdt \\ &\le C_1 \int_{t_k}^{t_k+1} \int_\Omega |\nabla n(x,t)|^2 dxdt \\ &\to 0 \qquad \text{as } k \to \infty. \end{aligned} \tag{8.3}$$

On the other hand, writing $\tilde{n}_\infty(x, s) := n_\infty(x)$ for $(x, s) \in \Omega \times (0, 1)$, we can estimate

$$\begin{aligned} \|n_k - \tilde{n}_\infty\|_{L^2((0,1);(W^{1,2}(\Omega))^\star)}^2 &= \int_{t_k}^{t_k+1} \|n(\cdot,t) - n_\infty\|_{(W^{1,2}(\Omega))^\star}^2 dt \\ &\le 2\int_{t_k}^{t_k+1} \|n(\cdot,t) - n(\cdot,t_k)\|_{(W^{1,2}(\Omega))^\star}^2 dt \\ &\quad +2\int_{t_k}^{t_k+1} \|n(\cdot,t_k) - n_\infty\|_{(W^{1,2}(\Omega))^\star}^2 dt \\ &=: I_1(k) + I_2(k) \qquad \text{for all } k \in \mathbb{N}, \end{aligned}$$

where clearly

$$I_2(k) = 2\|n(\cdot,t_k) - n_\infty\|_{(W^{1,2}(\Omega))^\star}^2 \to 0 \qquad \text{as } k \to \infty$$

due to our hypothesis on $n(\cdot,t_k)$ and the fact that $C^0(\bar{\Omega}) \hookrightarrow (W^{1,2}(\Omega))^\star$. As for $I_1(k)$, we invoke Lemma 8.1 and apply the Cauchy-Schwarz inequality to infer that

$$\begin{aligned} I_1(k) &= 2\int_{t_k}^{t_k+1} \bigg\| \int_{t_k}^t n_t(\cdot,s)ds \bigg\|^2_{(W^{1,2}(\Omega))^\star} dt \\ &\le 2\int_{t_k}^{t_k+1} \bigg(\int_{t_k}^t \|n_t(\cdot,s)\|^2_{(W^{1,2}(\Omega))^\star} ds \bigg) \cdot (t-t_k)dt \\ &\le 2\int_{t_k}^{\infty} \|n_t(\cdot,s)\|^2_{(W^{1,2}(\Omega))^\star} ds \\ &\to 0 \qquad \text{as } k\to\infty, \end{aligned}$$

because $t_k \to \infty$ as $k \to \infty$ and $\int_1^\infty \|n_t(\cdot,t)\|^2_{(W^{1,2}(\Omega))^\star} dt$ is finite according to Lemma 8.1. Consequently, $\|n_k - \tilde{n}_\infty\|_{L^2((0,1);(W^{1,2}(\Omega))^\star)} \to 0$ as $k\to\infty$, which combined with (8.3) implies that $\tilde{n}_\infty \equiv \overline{n_0}$ in $\Omega\times(0,1)$ and hence indeed $n_\infty \equiv \overline{n_0}$ throughout Ω. □

9 Proof of Theorem 1.1

PROOF of Theorem 1.1. We only need to collect Lemma 8.2, Corollary 4.4 and Lemma 6.3. □

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