How far can chemotactic cross-diffusion enforce exceeding carrying capacities?

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Abstract

We consider nonnegative solutions of the Neumann initial-boundary value problem for the chemotaxis-growth system

$$\begin{cases} u_t = \varepsilon u_{xx} - (uv_x)_x + ru - \mu u^2, & x \in \Omega,\ t > 0, \\ 0 = v_{xx} - v + u, & x \in \Omega,\ t > 0, \end{cases} \qquad (\star)$$

in $\Omega := (0, L) \subset \mathbb{R}$ with $L > 0, \varepsilon > 0, r \geq 0$ and $\mu > 0$, along with the corresponding limit problem formally obtained upon taking $\varepsilon \searrow 0$.

For the latter hyperbolic-elliptic problem, we establish results on local existence and uniqueness within an appropriate generalized solution concept. In this context we shall moreover derive an extensibility criterion involving the norm of $u(\cdot, t)$ in $L^\infty(\Omega)$. This will enable us to conclude that in this case $\varepsilon = 0$,

- if $\mu \geq 1$, then all solutions emanating from sufficiently regular initial data are global in time, whereas
- if $\mu < 1$, then some solutions blow up in finite time.

The latter will reveal that the original parabolic-elliptic problem $(\star)$, though known to possess no such exploding solutions, exhibits the following property of dynamical structure generation: Given any $\mu \in (0, 1)$ one can find smooth bounded initial data with the property that for each prescribed number $M > 0$ the solution of $(\star)$ will attain values above M at some time, provided that ε is sufficiently small. In particular, this means that the associated carrying capacity given by $\frac{r}{\mu}$ can be exceeded during evolution to an arbitrary extent.

We finally present some numerical simulations which illustrate this type of solution behavior, and which moreover inter alia indicate that achieving large population densities is a transient dynamical phenomenon occuring at intermediate time scales only.

Key words: chemotaxis, logistic source, blow-up, hyperbolic-elliptic system
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1 Introduction

We consider the parabolic-elliptic evolution problem

$$\begin{cases} u_t = \varepsilon u_{xx} - (uv_x)_x + ru - \mu u^2, & x \in \Omega,\ t > 0, \\ 0 = v_{xx} - v + u, & x \in \Omega,\ t > 0, \\ u_x(x,t) = v_x(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in the bounded interval $\Omega = (0, L) \subset \mathbb{R}$ with parameters $L > 0, \varepsilon > 0, r \geq 0$ and $\mu > 0$, and a given nonnegative function u_0.

Being a particular variant of the celebrated Keller-Segel system ([19]), (1.1) is used in mathematical biology to describe the collective behavior of cell populations, with density denoted by $u(x,t)$, in which the individual cells not only proliferate according to a logistic law and diffuse randomly, but also partially direct their movement toward increasing concentrations $v(x,t)$ of a chemical signal produced by themselves. This mechanism, also referred to as chemotaxis, is known to play an important role in many biological situations, ranging from the paradigmatic process of slime mold formation in *Dictyostelium Discoideum* ([19]) over pattern formation in colonies of *Salmonella typhimurium* ([45]), to invasion of tumor cells into healthy tissue ([5]), and also to self-organization during embryonic development ([33]). Accordingly, Keller-Segel-type systems form a natural core of numerous more complex PDE systems arising in the macroscopic modeling of such processes (cf. also the survey [16] for a broader overview).

Challenges originating from chemotactic cross-diffusion. As compared to other second-order evolution systems of dissipative type, already the simple problem (1.1) is far from being fully understood; the destabilizing potential of the cross-diffusive term in (1.1) has only partially been described by rigorous analysis so far. After all, some results reveal the striking phenomenon of spontaneous formation of singularities in some related Keller-Segel systems, thus reflecting the formation of cell aggregates in a mathematically rather extreme flavor: It is known, for instance, that in higher-dimensional analogues of (1.1) when there is no cell kinetics, that is, when $r = \mu = 0$, blow-up of solutions may occur in the sense that the spatial norm of u in $L^\infty(\Omega)$ becomes unbounded either in finite or in infinite time (see [18], [25], [2] for parabolic-elliptic and [15], [23] and [43] for fully parabolic cases). When nonlinear variants of diffusion and cross-diffusion are considered, according to refined modeling approaches e.g. accounting for volume-filling effects ([31]), further results detect such unbounded solutions inter alia even in some spatially one-dimensional situations (see [8], [10] and [44] for parabolic-elliptic and [7], [9] and [41] for parabolic-parabolic systems).

In many applications, blow-up phenomena do not appropriately reflect the respective experimentally observable behavior; even in cases where cell aggregation occurs it is not completely clear whether adequate models should enforce the emergence of locally infinite cell densities, or rather yield stabilization toward nonconstant equilibria (cf. the discussions in [17] and in [16, Sect. 2.1], for instance). Accordingly, considerable efforts are undertaken in order to develop models in which explosions are precluded. Such variants focus e.g. on weakening of cross-diffusion due to saturation effects ([31], [30], [46]), on enhancing diffusion in densely populated regions ([11]), on inhibition of signal production at large cell densities due to quorum-sensing ([21]), or also on additional cross-diffusive mechanisms in the evolution of the chemical ([4]). Cell proliferation terms of logistic type, such as contained in (1.1) when $r > 0$ and $\mu > 0$, form the possibly simplest among such blow-up preventing model elements: Indeed, cell division and death usually become relevant when sufficiently large time scales are involved, and accordingly such cell kinetics are included in many corresponding models, e.g. for tumor invasion ([5], [36], [22]).

In fact, the presence of such logistic terms, and in particular of the quadratic absorption term $-\mu u^2$, is sufficient to suppress any blow-up in many relevant situations: In the multi-dimensional analogue of (1.1) in bounded domains $\Omega \subset \mathbb{R}^n, n \ge 1$, for instance, it is known that if either $n \le 2$ and $\mu > 0$ is arbitrary, or alternatively $n \ge 3$ and $\mu > \frac{n-2}{n}$, then for all suitably regular inital data u_0, global classical solutions exist whih remain uniformly bounded for all times. This is explicitly contained in [37] for the prototypical choice $\varepsilon = 1$, but can easily be seen to extend to actually any choice of $\varepsilon > 0$. For corresponding results in fully parabolic versions thereof, we refer to [29] and [27] in the case $n = 2$ and to [39] for $n \ge 3$. Even weaker death effects, represented by subquadratic absorption terms of the form $-\mu u^\alpha$ with $\alpha \in (1, 2)$, can ensure the global existence of solutions at least in a certain generalized sense; for instance, such solutions can be constructed whenever $\alpha > 2 - \frac{1}{n}$ ([42], cf. also [26]).

Main results: Exceeding carrying capacities. The latter boundedness results are all in good accordance with the biological concept of *carrying capacity*. In fact, in the associated ODE $u_t = ru - \mu u^2$ all positive solutions approach this carrying capacity $u_c := \frac{r}{\mu}$ in the large time limit, and moreover they are uniformly bounded according to

$$u \le \max\Big\{\overline{u}_0, u_c\Big\} \tag{1.2}$$

for all times, where $\overline{u}_0$ denotes an upper bound for the initial data. By maximum principle arguments, the same can be derived when diffusion is involved such as in $u_t = \varepsilon\Delta u + ru - \mu u^2$, $\varepsilon > 0$, complemented with homogeneous Neumann boundary conditions in bounded domains $\Omega \subset \mathbb{R}^n, n \ge 1$, meaning that also in this case the carrying capacity cannot be substantially exceeded during evolution.

The purpose of the present paper is to investigate in how far chemotactic cross-diffusion can affect this property. It is known that spatially inhomogeneous steady states of (1.1) in its n-dimensional counterpart exist for generic choices of the parameters ensuring dominance of cross-diffusion, e.g. for small positive values of $r = \mu = \varepsilon$ not lying in an exceptional countable set ([20], [37]). Of course, it is evident that at any of these nonconstant equilibria, the cell density must lie above the carrying capacity near its maximum. Due to a lack of knowledge on the global dynamics in (1.1), however, this does not clarify whether such excesses are artificial in the sense that they must be present already initially to their full extent, possibly still limited according to (1.2).
The first of our main results asserts that going beyond carrying capacities actually is a genuinely dynamical feature of (1.1) when $\mu < 1$ and diffusion is sufficiently weak. In fact, for such μ chemotactic cross-diffusion can enforce the spontaneous emergence of arbitrarily large population densities if ε is suitably small, even in the simple one-dimensional model (1.1):

Theorem 1.1 *Let $r \ge 0$ and $\mu \in (0, 1)$. Then for all $p > \frac{1}{1-\mu}$ there exists $C(p) > 0$ satisfying the following: Whenever $q > 1$ and $u_0 \in W^{1,q}(\Omega)$ is nonnegative and such that*

$$\|u_0\|_{L^p(\Omega)} > C(p) \cdot \max\left\{\frac{1}{|\Omega|}\int_\Omega u_0\,,\ \frac{r}{\mu}\right\}, \tag{1.3}$$

there exists $T > 0$ such that to each $M > 0$ there corresponds some $\varepsilon_0(M) > 0$ with the property that for any $\varepsilon \in (0, \varepsilon_0(M))$ one can find $t_\varepsilon \in (0,T)$ and $x_\varepsilon \in \Omega$ such that the solution (u, v) of (1.1) satisfies

$$u(x_\varepsilon, t_\varepsilon) > M. \tag{1.4}$$

Let us emphasize that here the hypothesis (1.3) does neither involve ε nor M. Accordingly, the above statement says that the first solution component u can exceed both the carrying capacity u_c and its initial upper bound $\|u_0\|_{L^\infty(\Omega)}$ to an arbitrary extent. Theorem 1.1 may therefore be understood as a first step toward a more comprehensive qualitative understanding of the evolution in chemotaxis-growth systems, going beyond the basic knowledge of boundedness of solutions ([37]) and the existence of global attractors ([12], [27], [1]), complementing results on the occurrence of wave-like solution behavior as in the standard Fisher-KPP equation with diffusion ([24]), and rigorously capturing at least part of the rich variety of impressive dynamical properties which numerical experiments indicate to exist ([32]).

Blow-up in a hyperbolic-elliptic limit problem. In the course of our analysis we shall consider (1.1) along with the associated hyperbolic-elliptic problem

$$\begin{cases} u_t = -(uv_x)_x + ru - \mu u^2, & x \in \Omega,\ t > 0, \\ 0 = v_{xx} - v + u, & x \in \Omega,\ t > 0, \\ v_x(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

formally obtained in the limit $\varepsilon \searrow 0$. This system, by the nature of its naive derivation, clearly is entirely different from classical hyperbolic chemotaxis models resulting upon stochastic consideration of microscopic run-and-tumble processes. Whereas problems of the latter type have been widely studied ([34], [3]), the only result we are aware of which addresses a Keller-Segel system in the limit of vanishing cell diffusion is contained in [35] where, inter alia, for the two-dimensional variant of (1.5) with $r = \mu = 0$ a generalized global solvability statement is derived in the framework of measure-valued solutions (cf. also [13]). Beyond this, however, systems of type (1.5), especially when accounting for dampening effects stemming from cell kinetics, seem to lack any rigorous analysis in the literature.

Accordingly, we first need to make sure that (1.5) is locally well-posed in the following sense.

Theorem 1.2 *Let $r \geq 0$ and $\mu > 0$, and suppose that for some $q > 1$, $u_0 \in W^{1,q}(\Omega)$ is nonnegative. Then there exist $T_{max} \in (0, \infty]$ and a uniquely determined pair (u, v) of functions*

$$u \in C^0(\bar\Omega \times [0, T_{max})) \cap L^\infty_{loc}([0, T_{max}); W^{1,q}(\Omega)) \qquad \textit{and}$$
$$v \in C^{2,0}(\bar\Omega \times [0, T_{max})),$$

which form a strong $W^{1,q}$-solution of (1.5) in $\Omega \times (0, T_{max})$ in the sense of Definition 4.1 below, and which are such that

$$\textit{either } T_{max} = \infty, \quad \textit{ or } \quad \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.6}$$

By means of the extensibility criterion (1.6) and appropriate a priori estimates, we shall see that when the absorptive effect of the cell kinetic term in (1.5) is adequately large in the sense that $\mu \geq 1$, then all the above solutions are in fact global in time.

Proposition 1.3 *Let $r \geq 0$ and $\mu \geq 1$. Then for each nonnegative u_0 belonging to $W^{1,q}(\Omega)$ for some $q > 1$, the problem (1.5) possesses a unique global strong $W^{1,q}$-solution (u, v). Furthermore, if either $\mu > 1$ or $(r, \mu) = (0, 1)$, then both u and v are bounded in $\Omega \times (0, \infty)$.*

However, if $\mu < 1$ then finite-time blow-up occurs in (1.5) for all suitably large initial data.

Theorem 1.4 *Let $r \geq 0$ and $\mu \in (0, 1)$. Then for all $p > \frac{1}{1-\mu}$ one can find $C(p) > 0$ with the following property: Whenever $q > 1$ and $u_0 \in W^{1,q}(\Omega)$ is nonnegative and such that*

$$\|u_0\|_{L^p(\Omega)} > C(p) \cdot \max\left\{\frac{1}{|\Omega|}\int_\Omega u_0\,,\,\frac{r}{\mu}\right\}, \tag{1.7}$$

the strong $W^{1,q}$-solution (u, v) of (1.5) blows up in finite time; that is, in Theorem 1.2 we have $T_{max} < \infty$ and

$$\limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.8}$$

Organization of the paper. After a short preliminary section collecting basic properties of solutions to the second equation in (1.5) and (1.1), in Section 3 we shall derive a series of estimates for solutions of (1.1). These will on the one hand enable us to rediscover global solvability of (1.1) as a by-product, and on the other hand form the starting point for our analysis of (1.5). In particular, the extensibility criterion (1.6) will be prepared by a key gradient estimate provided by Corollary 3.6, the derivation of which essentially makes use of the spatially one-dimensional framework. In Section 4 we then introduce the concept of strong $W^{1,q}$-solutions of (1.5) and first prove a corresponding uniqueness statement for solutions within this class (Lemma 4.2). Combined with appropriate compactness arguments, in Section 4.4 this will allow for the important conclusion that actually along the entire net $\varepsilon \searrow 0$, solutions of (1.1) approach such a solution locally in time (Lemma 4.5). Together with an extensibility argument, this will complete the proof of Theorem 1.2, from which by a simple comparison argument in Section 4.5 we will firstly obtain the global existence result for $\mu \geq 1$ in Proposition 1.3. Secondly, in Section 4.6 a suitable testing procedure will enable us to track the time evolution of the functional $\int_\Omega u^p(\cdot, t)$ for such generalized solutions of (1.5), and to derive an integral version of a Ricatti-like ODI for the latter. A corresponding Grønwall-type argument will thereupon lead to the blow-up assertion in Theorem 1.4, from which our main result, Theorem 1.1, can finally be deduced by a continuous dependence argument in Section 5.

In Section 6 we finally illustrate and supplement our results by presenting some numerical simulations. These will inter alia indicate that the above phenomenon of achieving large population densities is transient in the sense that it occurs at intermediate time scales only.

2 Preliminaries: Some estimates for v

For later reference, let us briefly collect some elementary properties of the solution $v \in C^2(\bar{\Omega})$ of the elliptic problem

$$\begin{cases} -v_{xx} + v = u, & x \in \Omega, \\ v_x = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

for a given function $u \in C^0(\bar{\Omega})$. As we intend to apply our results also to possibly sign-changing inhomogeneities in our uniqueness proof in Lemma 4.2, we do not require u to be nonnegative here.

Lemma 2.1 *Let $u \in C^0(\bar{\Omega})$. Then the solution v of (2.1) satisfies*

$$\inf_{x\in\Omega} u(x) \le v \le \sup_{x\in\Omega} u(x) \qquad in\ \Omega \tag{2.2}$$

and

$$\|v\|_{L^p(\Omega)} \le \|u\|_{L^p(\Omega)} \qquad for\ all\ p \in [1,\infty] \tag{2.3}$$

as well as

$$\|v_x\|_{L^\infty(\Omega)} \le 2\|u\|_{L^1(\Omega)}. \tag{2.4}$$

If moreover u is nonnegative, then so is v and

$$\|v_x\|_{L^\infty(\Omega)} \le \|u\|_{L^1(\Omega)}. \tag{2.5}$$

Proof. Both inequalities in (2.2) are direct consequences of the elliptic maximum principle. To verify (2.3), we first consider the case $p \in (1,\infty)$, in which we test (2.1) by $v(v^2+\eta)^{\frac{p}{2}-1}$ for $\eta > 0$ to find that

$$\int_\Omega \Big((p-1)v^2+\eta\Big)\cdot(v^2+\eta)^{\frac{p}{2}-2}v_x^2 + \int_\Omega v^2(v^2+\eta)^{\frac{p}{2}-1} = \int_\Omega uv(v^2+\eta)^{\frac{p}{2}-1} \le \int_\Omega |u|\cdot|v|\cdot(v^2+\eta)^{\frac{p}{2}-1}$$

for all $\eta > 0$. Here we may drop the first nonnegative term on the left and invoke the monotone convergence theorem in taking $\eta \searrow 0$ to obtain

$$\int_\Omega |v|^p \le \int_\Omega |u|\cdot|v|^{p-1} \le \Big(\int_\Omega |u|^p\Big)^{\frac{1}{p}} \cdot \Big(\int_\Omega |v|^p\Big)^{\frac{p-1}{p}}$$

by the Hölder inequality. This proves (2.3) for all $p \in (1,\infty)$, whereupon the cases $p = 1$ and $p = \infty$ can easily be covered by limit procedures. Since

$$v_x(x) = \int_0^x v_{xx}(y)dy = \int_0^x v(y)dy - \int_0^x u(y)dy \qquad \text{for all } x \in \Omega \tag{2.6}$$

due to the fact that $v_x(0) = 0$, (2.4) immediately results from (2.3). Finally, if $u \ge 0$ then also $v \ge 0$ by (2.2), and therefore (2.6) readily implies (2.5). $\Box$

In our blow-up proof in Lemma 4.8 we shall also need the following variant of (2.3).

Lemma 2.2 *Let $p > 0$. Then for all $\eta > 0$ there exists $C(\eta,p) > 0$ such that whenever $u \in C^0(\bar{\Omega})$ is nonnegative, the solution v of (2.1) satisfies*

$$\int_\Omega v^{p+1} \le \eta \int_\Omega u^{p+1} + C(\eta,p)\Big(\int_\Omega u\Big)^{p+1}. \tag{2.7}$$

Proof. We test (2.1) by v^p and apply Young's inequality to obtain

$$p\int_\Omega v^{p-1}v_x^2 + \int_\Omega v^{p+1} = \int_\Omega uv^p \le \frac{1}{p+1}\int_\Omega u^{p+1} + \frac{p}{p+1}\int_\Omega v^{p+1},$$

so that in particular

$$\frac{4p}{p+1}\int_\Omega \Big(v^{\frac{p+1}{2}}\Big)_x^2 \le \int_\Omega u^{p+1}. \tag{2.8}$$

Now since the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$ is compact, Ehrling's lemma provides $c_1 = c_1(\eta, p) > 0$ such that

$$\|\psi\|^2_{L^2(\Omega)} \le \frac{4p\eta}{p+1}\|\psi_x\|^2_{L^2(\Omega)} + c_1\|\psi\|^2_{L^{\frac{2}{p+1}}(\Omega)} \qquad \text{for all } \psi \in W^{1,2}(\Omega).$$

Since $\int_\Omega v \le \int_\Omega u$ by Lemma 2.1, applying this to $\psi := v^{\frac{p+1}{2}}$ we thus obtain using (2.8) that

$$\int_\Omega v^{p+1} \le \frac{4p\eta}{p+1}\int_\Omega \Big(v^{\frac{p+1}{2}}\Big)^2_x + c_1\Big(\int_\Omega u\Big)^{p+1} \le \eta \int_\Omega u^{p+1} + c_1\Big(\int_\Omega u\Big)^{p+1},$$

as claimed. $\square$

3 Estimates for the parabolic-elliptic problem

3.1 Basic estimates and global existence

Let us start our analysis with a standard estimate.

Lemma 3.1 *Let $r \ge 0, \mu > 0$ and $u_0 \in C^0(\bar\Omega)$ be nonnegative, and suppose that (u, v) is a classical solution of (1.1) in $\Omega \times (0, T)$ for some $T > 0$ and $\varepsilon > 0$. Then for each $p \ge 1$ we have*

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\varepsilon\int_\Omega u^{p-2}u_x^2 \le pr\int_\Omega u^p - (1-p+\mu p)\int_\Omega u^{p+1} \qquad \textit{for all } t \in (0,T). \tag{3.1}$$

Proof. We multiply the first equation in (1.1) by u^{p-1} and integrate by parts to see that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + (p-1)\varepsilon\int_\Omega u^{p-2}u_x^2 = (p-1)\int_\Omega u^{p-1}u_xv_x + r\int_\Omega u^p - \mu\int_\Omega u^{p+1} \qquad \text{for all } t \in (0,T).$$

Since one more integration by parts along with the identity $v_{xx} = v - u$ shows that

$$(p-1)\int_\Omega u^{p-1}u_xv_x = -\frac{p-1}{p}\int_\Omega u^p(v-u) \le \frac{p-1}{p}\int_\Omega u^{p+1},$$

this readily implies (3.1). $\square$

A comparison argument immediately yields the following.

Corollary 3.2 *Let $r \ge 0, \mu > 0$ and $u_0 \in C^0(\bar\Omega)$ be nonnegative, and suppose that (u, v) is a classical solution of (1.1) in $\Omega \times (0, T)$ for some $T > 0$ and $\varepsilon > 0$. Then for all $p \ge 1$ with $p < \frac{1}{(1-\mu)_+}$, we have*

$$\int_\Omega u^p(\cdot, t) \le C \qquad \textit{for all } t \in (0,T) \tag{3.2}$$

and

$$\int_t^{t+1}\int_\Omega u^{p+1} \le \frac{pr+1}{1-\mu+\mu p}C \qquad \textit{for all } t \in (0, T-1) \tag{3.3}$$

as well as

$$(p-1)\varepsilon\int_t^{t+1}\int_\Omega u^{p-2}u_x^2 \le \frac{pr+1}{p}C \qquad \textit{for all } t\in(0,T-1) \tag{3.4}$$

with

$$C := \max\Big\{\int_\Omega u_0^p\,,\,\Big(\frac{pr}{1-p+\mu p}\Big)^p\cdot|\Omega|\Big\}.$$

PROOF. On the right of (3.1), we use the Hölder inequality to estimate

$$\int_\Omega u^{p+1} \ge |\Omega|^{-\frac{1}{p}}\Big(\int_\Omega u^p\Big)^{\frac{p+1}{p}},$$

whence Lemma 3.1 shows that

$$\frac{d}{dt}\int_\Omega u^p \le rp\int_\Omega u^p - (1-p+\mu p)|\Omega|^{-\frac{1}{p}}\Big(\int_\Omega u^p\Big)^{\frac{p+1}{p}} \qquad \text{for all } t\in(0,T).$$

An ODE comparison therefore yields (3.2), whereupon (3.3) and (3.4) easily result from an integration of (3.1). □

As a particular consequence of the latter we obtain that all the problems (1.1) admit global classical solutions for all nonnegative $u_0\in C^0(\bar\Omega)$. Results of this type are essentially well-known and far from surprising; indeed, in the one-dimensional setting solutions of (1.1) and also its parabolic counterpart remain bounded even in the case $r=\mu=0$ without the dampening logistic influence (see [6], [47] and [37], for instance). For completeness, however, we include a basically self-contained proof here which, unlike the literature we are aware of, precisely covers the precise the particular system considered here.

Lemma 3.3 *Let $r\ge 0, \mu>0$ and $u_0\in C^0(\bar\Omega)$ be nonnegative. Then for all $\varepsilon>0$, the problem (1.1) possesses a uniquely determined global classical solution $(u_\varepsilon,v_\varepsilon)$ for which both u_ε and v_ε are bounded in $\Omega\times(0,\infty)$.*

PROOF. By straightforward adaptation of standard arguments (see e.g. [10] and [39]) it is possible to prove the existence of $T_{max,\varepsilon}\in(0,\infty]$ and a unique solution $(u_\varepsilon,v_\varepsilon)$ of (1.1) in $\Omega\times(0,T_{max,\varepsilon})$ such that

$$\text{either } T_{max,\varepsilon}=\infty, \text{ or } \limsup_{t\nearrow T_{max,\varepsilon}}\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}=\infty. \tag{3.5}$$

In order to show that actually the former alternative must occur, we first apply Corollary 3.2 to find $c_1>0$ such that

$$\int_\Omega u_\varepsilon(\cdot,t)\le c_1 \qquad \text{for all } t\in(0,T_{max,\varepsilon}). \tag{3.6}$$

We next use Young's inequality to derive the pointwise estimate

$$(r+2)u_\varepsilon - \mu u_\varepsilon^2 \le c_2 := \frac{(r+2)^2}{4\mu} \qquad \text{in } \Omega\times(0,T_{max,\varepsilon}),$$

so that

$$u_{\varepsilon t}\le \varepsilon u_{\varepsilon xx} - (u_\varepsilon v_{\varepsilon x})_x + c_2 \qquad \text{in } \Omega\times(0,T_{max,\varepsilon}). \tag{3.7}$$

To derive an appropriate boundedness property from this and (3.6), let us fix an arbitrary $\alpha \in (0, \frac{1}{2})$ and then choose $q > 2$ such that $2\alpha - \frac{1}{q} > 0$ and

$$\frac{1}{q} > 2\alpha - \frac{1}{2}. \tag{3.8}$$

Then if A_ε denotes the realization of the sectorial operator $-\varepsilon(\cdot)_{xx} + 1$ under homogeneous Neumann boundary conditions in $L^q(\Omega)$, the domain of its fractional power A_ε^α satisfies $D(A_\varepsilon^\alpha) \hookrightarrow L^\infty(\Omega)$ ([14]), whence there exists $c_3(\varepsilon) > 0$ fulfilling

$$\|\psi\|_{L^\infty(\Omega)} \le c_3(\varepsilon)\|A_\varepsilon^\alpha \psi\|_{L^q(\Omega)} \qquad \text{for all } \psi \in D(A_\varepsilon^\alpha). \tag{3.9}$$

Moreover, we recall ([40]) that with some $c_4(\varepsilon) > 0$ and $c_5(\varepsilon) > 0$, the corresponding semigroup $(e^{-\tau A_\varepsilon})_{\tau \ge 0}$ satisfies

$$\|e^{-\tau A_\varepsilon}\psi_x\|_{L^q(\Omega)} \le c_4(\varepsilon)\tau^{-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})}\|\psi\|_{L^2(\Omega)} \qquad \text{for all } \tau > 0 \text{ and } \psi \in C^1(\bar\Omega) \text{ with } \psi_x = 0 \text{ on } \partial\Omega \tag{3.10}$$

and

$$\|A_\varepsilon^\alpha e^{-\tau A_\varepsilon}\psi\|_{L^q(\Omega)} \le c_5(\varepsilon)\tau^{-\alpha}\|\psi\|_{L^q(\Omega)} \qquad \text{for all } \tau > 0 \text{ and } \psi \in L^q(\Omega), \tag{3.11}$$

and that by the maximum principle,

$$\|e^{-\tau A_\varepsilon}\psi\|_{L^\infty(\Omega)} \le \|\psi\|_{L^\infty(\Omega)} \qquad \text{for all } \tau > 0 \text{ and } \psi \in C^0(\bar\Omega). \tag{3.12}$$

Now according to (3.7) and another comparison argument, we have

$$\begin{aligned}
\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{-t(A_\varepsilon+1)}u_0\|_{L^\infty(\Omega)} + \left\| \int_0^t e^{-(t-s)(A_\varepsilon+1)} \Big(u_\varepsilon(\cdot,s)v_{\varepsilon x}(\cdot,s)\Big)_x ds \right\|_{L^\infty(\Omega)} \\
&\quad + \left\| \int_0^t e^{-(t-s)(A_\varepsilon+1)} c_1 ds \right\|_{L^\infty(\Omega)} \\
&\le \|u_0\|_{L^\infty(\Omega)} + c_3(\varepsilon) \int_0^t e^{-(t-s)} \left\| A_\varepsilon^\alpha e^{-(t-s)A_\varepsilon} \Big(u_\varepsilon(\cdot,s)v_{\varepsilon x}(\cdot,s)\Big)_x \right\|_{L^q(\Omega)} ds \\
&\quad + c_5(\varepsilon)c_1 \int_0^t e^{-(t-s)} ds \\
&\le \|u_0\|_{L^\infty(\Omega)} + c_3(\varepsilon)c_4(\varepsilon)c_5(\varepsilon) \int_0^t e^{-(t-s)} \Big(\frac{t-s}{2}\Big)^{-\alpha-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})} \|u_\varepsilon(\cdot,s)v_{\varepsilon x}(\cdot,s)\|_{L^2(\Omega)} ds \\
&\quad + c_5(\varepsilon)c_1 \qquad \text{for all } t \in (0, T_{max,\varepsilon}).
\end{aligned} \tag{3.13}$$

Since by (3.6), Lemma 2.1 and the Hölder inequality we have

$$\begin{aligned}
\|u_\varepsilon(\cdot,s)v_{\varepsilon x}(\cdot,s)\|_{L^2(\Omega)} &\le \|u_\varepsilon(\cdot,s)\|_{L^2(\Omega)} \cdot \|v_{\varepsilon x}(\cdot,s)\|_{L^\infty(\Omega)} \\
&\le \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{1}{2}} \cdot \|u_\varepsilon(\cdot,s)\|_{L^1(\Omega)}^{\frac{1}{2}} \cdot \|v_{\varepsilon x}(\cdot,s)\|_{L^\infty(\Omega)} \\
&\le \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{1}{2}} \cdot \|u_\varepsilon(\cdot,s)\|_{L^1(\Omega)}^{\frac{3}{2}} \\
&\le c_1^{\frac{3}{2}} \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{1}{2}} \quad \text{for all } s \in (0, T_{max,\varepsilon}),
\end{aligned}$$

this means that

$$\sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + c_1 c_5(\varepsilon)$$
$$+\Big\{ c_1^{\frac{3}{2}} c_3(\varepsilon) c_4(\varepsilon) c_5(\varepsilon) \cdot \int_0^\infty e^{-\sigma} \big(\tfrac{\sigma}{2}\big)^{-\alpha-\frac{1}{2}-\frac{1}{2}(\frac{1}{2}-\frac{1}{q})} d\sigma \Big\} \cdot \Big\{ \sup_{t\in(0,T)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \Big\}^{\frac{1}{2}}$$

for all $T \in (0, T_{max,\varepsilon})$. As here the integral on the right is finite thanks to (3.8), in light of (3.5) this first implies that $T_{max,\varepsilon} = \infty$ and then yields boundedness of u_ε, and thus by Lemma 2.1 also of v_ε, in $\Omega \times (0,\infty)$. $\square$

3.2 An estimate for the time derivative

For passing to the limit $\varepsilon \searrow 0$ following a standard procedure, we prepare an appropriate estimate for the time derivatives of solutions to (1.1).

Lemma 3.4 *Let $r \ge 0, \mu > 0$ and $u_0 \in C^0(\bar\Omega)$ be nonnegative. Then for all $\varepsilon_0 > 0$ and each $p \in (1,2]$ fulfilling $p < \frac{1}{(1-\mu)_+}$ there exists $C(\varepsilon_0, p) > 0$ such that whenever $\varepsilon \in (0, \varepsilon_0)$, the solution $(u_\varepsilon, v_\varepsilon)$ of (1.1) satisfies*

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|_{(W^{1,\frac{p}{p-1}}(\Omega))^\star}^{\frac{p+1}{2}} dt \le C(\varepsilon_0,p)\cdot(T+1) \qquad \textit{for all } T>0. \tag{3.14}$$

Proof. We multiply the first equation in (1.1) by an arbitrary $\psi \in C^1(\bar\Omega)$ and integrate by parts to obtain

$$\Big| \int_\Omega u_{\varepsilon t}(\cdot,t)\psi \Big| = \Big| -\varepsilon \int_\Omega u_{\varepsilon x}\psi_x + \int_\Omega u_\varepsilon v_{\varepsilon x}\psi_x + r\int_\Omega u_\varepsilon \psi - \mu \int_\Omega u_\varepsilon^2 \psi \Big| \tag{3.15}$$

for all $t > 0$. Here two applications of the Hölder inequality and Corollary 3.2 yield $c_1 > 0$ such that

$$\begin{aligned}
\Big| -\varepsilon \int_\Omega u_{\varepsilon x}\psi_x \Big| &\le \varepsilon \Big(\int_\Omega u_\varepsilon^{p-2} u_{\varepsilon x}^2 \Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega u_\varepsilon^{2-p}\psi_x^2 \Big)^{\frac{1}{2}} \\
&\le \varepsilon \Big(\int_\Omega u_\varepsilon^{p-2} u_{\varepsilon x}^2 \Big)^{\frac{1}{2}} \cdot \Big(\int_\Omega u_\varepsilon^{p} \Big)^{\frac{2-p}{2p}} \cdot \|\psi_x\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\le c_1 \varepsilon \Big(\int_\Omega u_\varepsilon^{p-2} u_{\varepsilon x}^2 \Big)^{\frac{1}{2}} \cdot \|\psi_x\|_{L^{\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t>0,
\end{aligned}$$

and combining Corollary 3.2 with Lemma 2.1 we find $c_2 > 0$ such that

$$\begin{aligned}
\Big| \int_\Omega u_\varepsilon v_{\varepsilon x}\psi_x \Big| &\le \|v_{\varepsilon x}\|_{L^\infty(\Omega)} \cdot \Big(\int_\Omega u_\varepsilon^p \Big)^{\frac{1}{p}} \cdot \|\psi_x\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\le \|u_\varepsilon\|_{L^1(\Omega)} \cdot \Big(\int_\Omega u_\varepsilon^p \Big)^{\frac{1}{p}} \cdot \|\psi_x\|_{L^{\frac{p}{p-1}}(\Omega)} \\
&\le c_2 \|\psi_x\|_{L^{\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t>0.
\end{aligned}$$

Also by Corollary 3.2,

$$\Big| r \int_\Omega u_\varepsilon \psi \Big| \le r \|u_\varepsilon\|_{L^1(\Omega)} \cdot \|\psi\|_{L^\infty(\Omega)} \le c_3 \|\psi\|_{W^{1,\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t>0$$

and

$$\begin{aligned} \Big| -\mu \int_\Omega u_\varepsilon^2 \psi \Big| &\le \mu \Big(\int_\Omega u_\varepsilon^2 \Big) \cdot \|\psi\|_{L^\infty(\Omega)} \\ &\le c_4 \Big(\int_\Omega u_\varepsilon^{p+1} \Big)^{\frac{2}{p+1}} \|\psi\|_{W^{1,\frac{p}{p-1}}(\Omega)} \qquad \text{for all } t>0 \end{aligned}$$

with some $c_3 > 0$ and $c_4 > 0$, because $W^{1,\frac{p}{p-1}}(\Omega) \hookrightarrow L^\infty(\Omega)$. Correspondingly, (3.15) implies that

$$\|u_{\varepsilon t}(\cdot,t)\|_{(W^{1,\frac{p}{p-1}}(\Omega))^\star} \le c_1 \varepsilon \Big(\int_\Omega u_\varepsilon^{p-2} u_{\varepsilon x}^2 \Big)^{\frac12} + c_2 + c_3 + c_4 \Big(\int_\Omega u_\varepsilon^{p+1} \Big)^{\frac{2}{p+1}} \qquad \text{for all } t>0.$$

Since $\frac{p+1}{2} \le \frac32 < 2$, by using Young's inequality we thus find $c_5 > 0$ fulfilling

$$\|u_{\varepsilon t}(\cdot,t)\|^{\frac{p+1}{2}}_{(W^{1,\frac{p}{p-1}}(\Omega))^\star} \le c_5 \Big(\varepsilon \int_\Omega u_\varepsilon^{p-2} u_{\varepsilon x}^2 + \int_\Omega u_\varepsilon^{p+1} + 1 \Big) \qquad \text{for all } t>0.$$

A time integration thereof immediately proves (3.14). $\square$

3.3 A gradient estimate

Now the crucial ingredient for our construction of local-in-time solutions to (1.5) consists of an ODI for the functional $y(t) := \int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}}$, where $\varepsilon > 0$ and $\eta > 0$. In view of the intended blow-up result in Theorem 1.4, it is not surprising that this inequality will contain a production term which can be viewed as essentially superlinear with respect to y.

Lemma 3.5 *Let $r \ge 0$ and $\mu > 0$, and suppose that for some $q > 1$, $u_0 \in W^{1,q}(\Omega)$ is nonnegative. Then for all $\eta > 0$, the solution of (1.1) satisfies*

$$\frac{d}{dt} \int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} \le q\Big(4\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + r\Big) \cdot \int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} + q \Big(\int_\Omega u_\varepsilon(\cdot,t) \Big)^{q+1} \qquad \textit{for all } t>0. \tag{3.16}$$

Proof. For $\eta > 0$ we let $\phi_\eta(s) := (s^2+\eta)^{\frac{q}{2}}$, $s \in \mathbb{R}$, differentiate in (1.1) and integrate by parts to compute

$$\begin{aligned} \frac{d}{dt} \int_\Omega \phi_\eta(u_{\varepsilon x}) &= \int_\Omega \phi_\eta'(u_{\varepsilon x}) u_{\varepsilon xt} \\ &= \int_\Omega \phi_\eta'(u_{\varepsilon x}) \cdot \Big\{ \varepsilon u_{\varepsilon xxx} - u_{\varepsilon xx} v_{\varepsilon x} - 2 u_{\varepsilon x} v_{\varepsilon xx} - u_\varepsilon v_{\varepsilon xxx} + r u_{\varepsilon x} - 2\mu u_\varepsilon u_{\varepsilon x} \Big\} \\ &= -\varepsilon \int_\Omega \phi_\eta''(u_{\varepsilon x}) u_{\varepsilon xx}^2 + \int_\Omega \phi_\eta(u_{\varepsilon x}) v_{\varepsilon xx} - 2 \int_\Omega \phi_\eta'(u_{\varepsilon x}) u_{\varepsilon x} v_{\varepsilon xx} \\ &\quad - \int_\Omega \phi_\eta'(u_{\varepsilon x}) u_\varepsilon v_{\varepsilon xxx} + r \int_\Omega \phi_\eta'(u_{\varepsilon x}) u_{\varepsilon x} - 2\mu \int_\Omega \phi_\eta'(u_{\varepsilon x}) u_\varepsilon u_{\varepsilon x} \qquad \text{for all } t>0. \end{aligned}$$

Here we use that $v_{\varepsilon xx} = v_\varepsilon - u_\varepsilon$ and that for all $\eta > 0$ and $s \in \mathbb{R}$ we have

$$\begin{aligned}
\phi_\eta'(s) &= qs(s^2+\eta)^{\frac{q}{2}-1} \qquad \text{and} \\
\phi_\eta''(s) &= q(s^2+\eta)^{\frac{q}{2}-1} + q(q-2)s^2(s^2+\eta)^{\frac{q}{2}-2} \\
&= q(s^2+\eta)^{\frac{q}{2}-2} \cdot \Big\{(q-1)s^2+\eta\Big\} \ \geq 0,
\end{aligned}$$

to see that

$$\phi_\eta(s) - 2s\phi_\eta'(s) = (s^2+\eta)^{\frac{q}{2}-1} \cdot \Big\{s^2+\eta-2qs^2\Big\} = -(2q-1)s^2(s^2+\eta)^{\frac{q}{2}-1} + \eta(s^2+\eta)^{\frac{q}{2}-1}$$

for all $\eta > 0$ and $s \in \mathbb{R}$, and that hence

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}} \ \leq\ & -(2q-1)\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_{\varepsilon x}^2 v_\varepsilon + (2q-1)\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_{\varepsilon x}^2 u_\varepsilon \\
& +\eta\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} v_\varepsilon - \eta\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon \\
& -q\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x} v_{\varepsilon x} + q\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x}^2 \\
& +qr\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_{\varepsilon x}^2 - 2q\mu\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x}^2 \qquad \text{for all } t > 0.
\end{aligned}$$

Further dropping nonpositive terms on the right, we thus obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}} \ \leq\ & (3q-1)\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x}^2 + \eta\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} v_\varepsilon \\
& -q\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x} v_{\varepsilon x} + qr\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_{\varepsilon x}^2 \quad \text{for all } t > 0. \quad (3.17)
\end{aligned}$$

Here

$$\begin{aligned}
(3q-1)&\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x}^2 + qr\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_{\varepsilon x}^2 \\
&\leq \Big\{(3q-1)\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + qr\Big\} \cdot \int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_{\varepsilon x}^2 \\
&\leq \Big\{(3q-1)\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + qr\Big\} \cdot \int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}} \qquad \text{for all } t > 0, \qquad (3.18)
\end{aligned}$$

and since $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ according to Lemma 2.1, we can estimate

$$\eta\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} v_\varepsilon \leq \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \cdot \int_\Omega \eta(u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} \leq \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \cdot \int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}} \qquad \text{for all } t > 0. \qquad (3.19)$$

Moreover, by Young's inequality and Lemma 2.1,

$$-q\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}-1} u_\varepsilon u_{\varepsilon x} v_{\varepsilon x} \ \leq\ q\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q-1}{2}} u_\varepsilon |v_{\varepsilon x}|$$

$$\begin{aligned}
&\le q\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}}u_\varepsilon + q\int_\Omega u_\varepsilon |v_{\varepsilon x}|^q \\
&\le q\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\cdot\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}} + q\|v_{\varepsilon x}(\cdot,t)\|_{L^q(\Omega)}\cdot\int_\Omega u_\varepsilon \\
&\le q\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}\cdot\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}} + q\Big(\int_\Omega u_\varepsilon\Big)^{q+1} \qquad (3.20)
\end{aligned}$$

for all $t>0$. Inserting (3.18)-(3.20) into (3.17) yields (3.16). $\Box$

A first application of the latter can be obtained by a simple comparison argument.

Corollary 3.6 *Let $r\ge 0$ and $\mu>0$, and suppose that for some $q>1$, $u_0\in W^{1,q}(\Omega)$ is nonnegative. Then the solution of (1.1) satisfies*

$$\int_\Omega |u_{\varepsilon x}(\cdot,t)|^q \le \left\{\int_\Omega |u_{0x}|^q + q\int_0^t \Big(\int_\Omega u_\varepsilon(\cdot,s)\Big)^{q+1}ds\right\}\cdot\exp\left(4q\int_0^t \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds + qrt\right) \qquad (3.21)$$

for all $t>0$.

Proof. From Lemma 3.5, upon an ODE comparison we obtain

$$\begin{aligned}
\int_\Omega (u_{\varepsilon x}(\cdot,t)^2+\eta)^{\frac{q}{2}} &\le \Big(\int_\Omega (u_{0x}^2+\eta)^{\frac{q}{2}}\Big)\cdot\exp\Big(\int_0^t (4q\|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}+qr)ds\Big) \\
&\quad + q\int_0^t \exp\Big(\int_s^t (4q\|u_\varepsilon(\cdot,\sigma)\|_{L^\infty(\Omega)}+qr)d\sigma\Big)\cdot\Big(\int_\Omega u_\varepsilon(\cdot,s)\Big)^{q+1}ds
\end{aligned}$$

for all $t>0$ and each $\eta>0$. Since

$$\begin{aligned}
\int_s^t (4q\|u_\varepsilon(\cdot,\sigma)\|_{L^\infty(\Omega)}+qr)d\sigma &= 4q\int_s^t \|u_\varepsilon(\cdot,\sigma)\|_{L^\infty(\Omega)}d\sigma + qr(t-s) \\
&\le 4q\int_0^t \|u_\varepsilon(\cdot,\sigma)\|_{L^\infty(\Omega)}d\sigma + qrt \qquad \text{for all } t>0 \text{ and any } s\in[0,t],
\end{aligned}$$

this entails that for all $t>0$ and $\eta>0$,

$$\begin{aligned}
\int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} &\le \Big(\int_\Omega (u_{0x}^2+\eta)^{\frac{q}{2}}\Big)\cdot\exp\Big(4q\int_0^t \|u_\varepsilon(\cdot,s)\|_{L^\infty(\Omega)}ds + qrt\Big) \\
&\quad + q\exp\Big(4q\int_0^t \|u_\varepsilon(\cdot,\sigma)\|_{L^\infty(\Omega)}d\sigma + qrt\Big)\cdot\int_0^t \Big(\int_\Omega u_\varepsilon(\cdot,s)\Big)^{q+1}ds.
\end{aligned}$$

By means of Beppo Levi's theorem, from this we immediately derive (3.21). $\Box$

4 The hyperbolic-elliptic problem

4.1 Strong $W^{1,q}$-solutions

The solution concept for (1.5) that we shall pursue throughout the sequel will be the following.

Definition 4.1 *Let $r \geq 0, \mu > 0, q > 1$ and $T \in (0,\infty]$. Then by a* strong $W^{1,q}$-solution *of (1.5) in $\Omega \times (0,T)$ we mean a pair (u,v) of nonnegative functions*

$$u \in L^\infty_{loc}([0,T); W^{1,q}(\Omega)) \cap C^0(\bar{\Omega} \times [0,T)) \tag{4.1}$$

and

$$v \in C^{2,0}(\bar{\Omega} \times [0,T)) \tag{4.2}$$

such that v satisfies the second and third equations in (1.5) in the classical sense, and such that the identity

$$-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = \int_0^T \int_\Omega u v_x \varphi_x + r\int_0^T \int_\Omega u\varphi - \mu \int_0^T \int_\Omega u^2 \varphi \tag{4.3}$$

is valid for all $\varphi \in C_0^\infty(\bar{\Omega} \times [0,T))$.
In the case $T = \infty$ we also call (u,v) a global strong $W^{1,q}$-solution *of (1.5).*

Remark. Under the above hypotheses, let (u,v) be a strong $W^{1,q}$-solution of (1.5) in $\Omega \times (0,T)$.
i) In view of (4.1) and (4.2), it is clear upon a completion argument that then (4.3) actually even holds for any $\varphi \in L^1((0,T); W^{1,1}(\Omega))$ which has compact support in $\bar{\Omega} \times [0,T)$ and is such that $\varphi_t \in L^1(\Omega \times (0,T))$.
ii) Likewise, (4.1) and (4.2) allow for integrating by parts on the right of (4.3) so as to verify the validity of

$$-\int_0^T \int_\Omega u\varphi_t - \int_\Omega u_0 \varphi(\cdot,0) = \int_0^T \int_\Omega \Big\{ - u_x v_x - u v_{xx} + ru - \mu u^2 \Big\} \cdot \varphi \tag{4.4}$$

for any φ as specified in i).

A first elementary property of such solutions is boundedness of their norm in $L^1(\Omega)$.

Lemma 4.1 *Let $r \geq 0$ and $\mu > 0$, and assume that for some $T > 0$ and $q > 1$, u is a strong $W^{1,q}$-solution of (1.5) in $\Omega \times (0,T)$ with a certain nonnegative $u_0 \in W^{1,q}(\Omega)$. Then*

$$\int_\Omega u(x,t)dx \leq \max\Big\{ \int_\Omega u_0, \frac{r|\Omega|}{\mu} \Big\} \qquad \textit{for all } t \in (0,T). \tag{4.5}$$

Proof. In view of a straightforward ODE comparison argument, it is sufficient to show that the function $y \in C^0([0,T))$ defined by $y(t) := \int_\Omega u(x,t)dx, t \in [0,T)$, also belongs to $C^1((0,T))$ and satisfies

$$y'(t) \leq ry(t) - \frac{\mu}{|\Omega|} y^2(t) \qquad \text{for all } t \in (0,T). \tag{4.6}$$

To verify this, given $t_0 \in (0,T)$ and $t_1 \in (t_0,T)$ we let $\chi_\delta \in W^{1,\infty}(\mathbb{R})$ be given by

$$\chi_\delta(t) := \begin{cases} 0 & \text{if } t < t_0 - \delta \text{ or } t > t_1 + \delta, \\ \frac{t-t_0+\delta}{\delta} & \text{if } t \in [t_0-\delta, t_0], \\ 1 & \text{if } t \in (t_0,t_1), \\ \frac{t_1-t+\delta}{\delta} & \text{if } t \in [t_1, t_1+\delta], \end{cases}$$

for $\delta \in (0, \delta_0)$ with $\delta_0 := \min\{t_0, T - t_1\}$. Then according to the remark following Definition 4.1, we may use $\varphi(x,t) := \chi_\delta(t)$, $(x,t) \in \Omega \times (0,T)$, as a test function in (4.3) to infer that

$$\begin{aligned} -\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega u(x,t)dxdt + \frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega u(x,t)dxdt \\ = r\int_0^T\int_\Omega \chi_\delta(t)u(x,t)dxdt - \mu\int_0^T\int_\Omega \chi_\delta(t)u^2(x,t)dxdt \end{aligned} \tag{4.7}$$

for all such δ. Here since u is continuous,

$$-\frac{1}{\delta}\int_{t_0-\delta}^{t_0}\int_\Omega u(x,t)dxdt \to -\int_\Omega u(x,t_0)dx$$

and

$$\frac{1}{\delta}\int_{t_1}^{t_1+\delta}\int_\Omega u(x,t)dxdt \to \int_\Omega u(x,t_1)dx$$

as $\delta \searrow 0$, whence using the dominated convergence theorem in the two integrals on the right of (4.7) we obtain that in the limit $\delta \searrow 0$, (4.7) becomes

$$\int_\Omega u(x,t_1)dx - \int_\Omega u(x,t_0)dx = r\int_{t_0}^{t_1}\int_\Omega u - \mu\int_{t_0}^{t_1}\int_\Omega u^2. \tag{4.8}$$

Upon division by $t_1 - t_0$ and taking $t_1 \searrow t_0$, since u is continuous we thereby see that indeed y is differentiable at any $t_0 \in (0,T)$ with continuous derivative fulfilling

$$y'(t) = ry(t) - \mu\int_\Omega u^2(x,t)dx \qquad \text{for all } t \in (0,T).$$

Since by the Cauchy-Schwarz inequality we have $\int_\Omega u^2 \geq \frac{1}{|\Omega|}(\int_\Omega u)^2$, this implies (4.6) and thereby completes the proof. $\square$

4.2 Uniqueness

Within the above framework, solutions are uniquely determined.

Lemma 4.2 *Let $r \geq 0, \mu > 0$ and $q > 1$, and let $u_0 \in W^{1,q}(\Omega)$ be nonnegative. Then for all $T > 0$, the problem (1.5) possesses at most one strong $W^{1,q}$-solution in $\Omega \times (0,T)$.*

Proof. Given two strong $W^{1,q}$-solutions (u,v) and $(\tilde u, \tilde v)$ of (1.5) in $\Omega \times (0,T)$, we let $w := u - \tilde u$ and $z := v - \tilde v$ in $\Omega \times (0,T)$. We fix $T_0 \in (0,T)$ and $t_0 \in (0,T_0)$ and define a cut-off function $\chi_\delta \in W^{1,\infty}(\mathbb{R})$ by letting

$$\chi_\delta(t) := \begin{cases} 1 & \text{if } t < t_0, \\ \frac{t_0 - t + \delta}{\delta} & \text{if } t \in [t_0, t_0+\delta], \\ 0 & \text{if } t > t_0 + \delta, \end{cases} \tag{4.9}$$

for $\delta \in (0, \frac{T_0-t_0}{2})$. According to the remark following Definition 4.1, we may then use the function defined by

$$\varphi(x,t) := \chi_\delta(t) \cdot \frac{1}{h} \int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} ds, \qquad (x,t) \in \Omega \times (0,T),$$

as a test function in (4.3) for any $\delta \in (0, \frac{T_0-t_0}{2})$, $h \in (0, \frac{T_0-t_0}{2})$ and $\eta > 0$. Upon integrating by parts and then subtracting the respective identities thereby gained for u and $\tilde{u}$, we obtain

$$\begin{aligned}
-\int_0^T \int_\Omega \chi_\delta'(t) w(x,t) \cdot \frac{1}{h} \int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdxdt& \\
-\int_0^T \int_\Omega \chi_\delta(t) w(x,t) \cdot \frac{w(x,t+h)\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1} - w(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}}{h} dxdt& \\
= \int_0^T \int_\Omega \chi_\delta(t) \cdot \Big\{ - w_x v_x - w v_{xx} - \tilde{u}_x z_x - \tilde{u} z_{xx} + rw - \mu(u+\tilde{u})w \Big\} \times& \\
\times \frac{1}{h} \int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdxdt.& \qquad (4.10)
\end{aligned}$$

Here, since

$$[0,T_0] \ni t \mapsto \rho(t) := \int_\Omega w(x,t) \cdot \frac{1}{h} \int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdx$$

is continuous by (4.1), computing $\chi_\delta'(t)$ by (4.9) we see that

$$\begin{aligned}
-\int_0^T \int_\Omega \chi_\delta'(t) w(x,t) \cdot \frac{1}{h} \int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdxdt& \\
= \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \rho(t) dt& \\
\to \rho(t_0) = \int_\Omega w(x,t_0) \cdot \frac{1}{h} \int_{t_0}^{t_0+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdx& \qquad \text{as } \delta \searrow 0.
\end{aligned}$$

Since from (4.1) and (4.2) we moreover know that

$$w, u, \tilde{u}, v_x, v_{xx}, z_x \text{ and } z_{xx} \text{ are continuous in } \bar{\Omega} \times [0,T_0] \text{ and } w_x \text{ and } \tilde{u}_x \text{ belong to } L^\infty((0,T_0); L^q(\Omega)), \qquad (4.11)$$

we may invoke the dominated convergence theorem to infer from (4.10) upon taking $\delta \searrow 0$ that

$$\begin{aligned}
I_1(h,\eta) + I_2(h,\eta) := \int_\Omega w(x,t_0) \cdot \frac{1}{h} \int_{t_0}^{t_0+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdx& \\
-\int_0^{t_0} \int_\Omega w(x,t) \cdot \frac{w(x,t+h)\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1} - w(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}}{h} dxdt& \\
= \int_0^{t_0} \int_\Omega \Big\{ - w_x v_x - w v_{xx} - \tilde{u}_x z_x - \tilde{u} z_{xx} + rw - \mu(u+\tilde{u})w \Big\} \times&
\end{aligned}$$

$$\times\frac{1}{h}\int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} dsdxdt$$
$$=: \quad I_3(h,\eta). \tag{4.12}$$

Here by Young's inequality and a series of straightforward rearrangements,

$$\begin{aligned}
I_2(h,\eta) &= -\frac{1}{h}\int_0^{t_0}\int_\Omega w(x,t)w(x,t+h)\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\quad +\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\geq -\frac{1}{h}\int_0^{t_0}\int_\Omega \Big(w^2(x,t)+\eta\Big)^{\frac{1}{2}}\Big(w^2(x,t+h)+\eta\Big)^{\frac{q-1}{2}}dxdt \\
&\quad +\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\geq -\frac{1}{h}\cdot\Big\{\frac{1}{q}\int_0^{t_0}\int_\Omega \Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}}dxdt+\frac{q-1}{q}\int_0^{t_0}\int_\Omega\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}}dxdt\Big\} \\
&\quad +\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&= -\frac{1}{h}\cdot\Big\{\frac{1}{q}\int_0^{t_0}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt+\frac{\eta}{q}\int_0^{t_0}\int_\Omega\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\qquad\qquad +\frac{q-1}{q}\int_0^{t_0}\int_\Omega w^2(x,t+h)\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\qquad\qquad +\frac{(q-1)\eta}{q}\int_0^{t_0}\int_\Omega\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1}dxdt\Big\} \\
&\quad +\frac{1}{h}\int_0^{t_0}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&= \frac{q-1}{qh}\Big\{\int_0^{t_0}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\qquad\qquad -\int_0^{t_0}\int_\Omega w^2(x,t+h)\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1}dxdt\Big\} \\
&\quad -\frac{\eta}{qh}\int_0^{t_0}\int_\Omega\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt-\frac{(q-1)\eta}{qh}\int_0^{t_0}\int_\Omega\Big(w^2(x,t+h)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&= \frac{q-1}{qh}\int_0^{h}\int_\Omega w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\quad -\frac{q-1}{qh}\int_{t_0}^{t_0+h} w^2(x,t)\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt \\
&\quad -\frac{\eta}{qh}\int_0^{t_0}\int_\Omega\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt-\frac{(q-1)\eta}{qh}\int_h^{t_0+h}\int_\Omega\Big(w^2(x,t)+\eta\Big)^{\frac{q}{2}-1}dxdt
\end{aligned} \tag{4.13}$$

Now since for fixed $s_0 > 0$, all $s \in [-s_0, s_0]$ and any $\eta > 0$ we have

$$\eta(s^2+\eta)^{\frac{q}{2}-1} \le \begin{cases} \eta(s_0^2+\eta)^{\frac{q}{2}-1} & \text{if } q \ge 2, \\ \eta^{\frac{q}{2}} & \text{if } q < 2, \end{cases}$$

the two last summands in (4.13) vanish in the limit $\eta \searrow 0$. Therefore, invoking Beppo Levi's theorem we conclude that

$$\liminf_{\eta \searrow 0} I_2(h,\eta) \ge \frac{q-1}{qh} \int_0^h \int_\Omega |w(x,t)|^q dxdt - \frac{q-1}{qh} \int_{t_0}^{t_0+h} \int_\Omega |w(x,t)|^q dxdt. \tag{4.14}$$

Moreover, estimating $|s(s^2+\eta)^{\frac{q}{2}-1}| \le (s^2+\eta)^{\frac{q}{2}}$ for all $s \in \mathbb{R}$ and $\eta > 0$, from the continuity of w in $\bar\Omega \times [0, T_0]$ we easily deduce that

$$\Omega \ni x \mapsto \frac{1}{h} \int_{t_0}^{t_0+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} ds \stackrel{\star}{\rightharpoonup} \frac{1}{h}\int_{t_0}^{t_0+h} w(x,s)|w(x,s)|^{q-2} ds \qquad \text{in } L^\infty(\Omega),$$

and that similarly

$$\Omega \times (0,t_0) \ni (x,t) \mapsto \frac{1}{h}\int_t^{t+h} w(x,s)\Big(w^2(x,s)+\eta\Big)^{\frac{q}{2}-1} ds \quad \stackrel{\star}{\rightharpoonup} \quad \frac{1}{h}\int_t^{t+h} w(x,s)|w(x,s)|^{q-2} ds \qquad \text{in } L^\infty(\Omega \times (0,t_0))$$

as $\eta \searrow 0$. Along with (4.11), this enables us to take $\eta \searrow 0$ also in $I_1(h,\eta)$ and $I_3(h,\eta)$ to all in all infer from (4.12) and (4.14) that

$$\begin{aligned} &\int_\Omega w(x,t_0) \cdot \frac{1}{h}\int_{t_0}^{t_0+h} w(x,s)|w(x,s)|^{q-2} dsdx \\ &\qquad + \frac{q-1}{qh}\int_0^h \int_\Omega |w(x,t)|^q dxdt - \frac{q-1}{qh}\int_{t_0}^{t_0+h}\int_\Omega |w(x,t)|^q dxdt \\ &\le \int_0^{t_0}\int_\Omega \Big\{ -w_x v_x - w v_{xx} - \tilde u_x z_x - \tilde u z_{xx} + rw - \mu(u+\tilde u)w \Big\} \cdot \frac{1}{h}\int_t^{t+h} w(x,s)|w(x,s)|^{q-2} dsdxdt \end{aligned}$$

for all $h \in (0, \frac{T_0-t_0}{2})$. Finally, again since w is continuous in $\bar\Omega \times [0,T_0]$, and since $w(\cdot,0) = 0$ in Ω, we may let $h \searrow 0$ to obtain

$$\begin{aligned} \frac{1}{q}\int_\Omega |w(x,t_0)|^q dx &\equiv \int_\Omega |w(x,t_0)|^q dx - \frac{q-1}{q}\int_\Omega |w(x,t_0)|^q dx \\ &\le \int_0^{t_0}\int_\Omega \Big\{ -w_x v_x - w v_{xx} - \tilde u_x z_x - \tilde u z_{xx} + rw - \mu(u+\tilde u)w \Big\} \cdot w(x,t)|w(x,t)|^{q-2} dxdt. \end{aligned} \tag{4.15}$$

Here we split the integral on the right-hand side and use that $v_{xx} = v - u$ to compute

$$-\int_0^{t_0}\int_\Omega w_x v_x \cdot w|w|^{q-2} = \frac{1}{q}\int_0^{t_0}\int_\Omega |w|^q (v-u) \le c_1(T_0)\int_0^{t_0}\int_\Omega |w|^q, \tag{4.16}$$

where $c_1(T_0) := \frac{1}{q}\|v\|_{L^\infty(\Omega\times(0,T_0))}$ is finite by (4.2), because $T_0 < T$. Similarly,

$$-\int_0^{t_0}\int_\Omega wv_{xx}\cdot w|w|^{q-2} = \int_0^{t_0}\int_\Omega |w|^q(u-v) \le c_2(T_0)\int_0^{t_0}\int_\Omega |w|^q \tag{4.17}$$

with $c_2(T_0) := \|u\|_{L^\infty(\Omega\times(0,T_0))}$ and

$$\int_0^{t_0}\int_\Omega \Big\{rw - \mu(u+\tilde{u})w\Big\}\cdot w|w|^{q-2} \le r\int_0^{t_0}\int_\Omega |w|^q. \tag{4.18}$$

In order to prepare an appropriate estimation of the respective terms in (4.15) containing z, we first observe that since $-z_{xx} + z = w$ in $\Omega\times(0,T)$ with $z_x|_{\partial\Omega} = 0$, Lemma 2.1 applies to yield

$$\|z(\cdot,t)\|_{L^q(\Omega)} \le \|w(\cdot,t)\|_{L^q(\Omega)} \qquad \text{for all } t\in(0,T) \tag{4.19}$$

and, by the Hölder inequality,

$$\|z_x(\cdot,t)\|_{L^\infty(\Omega)} \le 2\|w(\cdot,t)\|_{L^1(\Omega)} \le 2|\Omega|^{\frac{q-1}{q}}\|w(\cdot,t)\|_{L^q(\Omega)} \qquad \text{for all } t\in(0,T). \tag{4.20}$$

Taking $c_3(T_0) > 0$ such that $\|\tilde{u}_x(\cdot,t)\|_{L^q(\Omega)} \le c_3(T_0)$ for a.e. $t\in(0,T_0)$, once more by the Hölder inequality we hence infer that

$$\begin{aligned} -\int_0^{t_0}\int_\Omega \tilde{u}_x z_x\cdot w|w|^{q-2} &\le \int_0^{t_0}\|\tilde{u}_x(\cdot,t)\|_{L^q(\Omega)}\|z_x(\cdot,t)\|_{L^\infty(\Omega)}\cdot\Big(\int_\Omega |w|^q\Big)^{\frac{q-1}{q}}dt \\ &\le c_3(T_0)\cdot 2|\Omega|^{\frac{q-1}{q}}\int_0^{t_0}\int_\Omega |w|^q, \end{aligned} \tag{4.21}$$

and with $c_4(T_0) := 2\|\tilde{u}\|_{L^\infty(\Omega\times(0,T_0))}$ we likewise obtain from (4.19) that

$$\begin{aligned} -\int_0^{t_0}\int_\Omega \tilde{u}z_{xx}\cdot w|w|^{q-2} &= \int_0^{t_0}\int_\Omega \tilde{u}|w|^q - \int_0^{t_0}\int_\Omega \tilde{u}zw|w|^{q-2} \\ &\le \int_0^{t_0}\|\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)}\cdot\int_\Omega |w|^q \\ &\quad + \int_0^{t_0}\|\tilde{u}(\cdot,t)\|_{L^\infty(\Omega)}\|z(\cdot,t)\|_{L^q(\Omega)}\cdot\Big(\int_\Omega |w|^q\Big)^{\frac{q-1}{q}}dt \\ &\le c_4(T_0)\int_0^{t_0}\int_\Omega |w|^q. \end{aligned} \tag{4.22}$$

We now only need to collect (4.15)-(4.18), (4.21) and (4.22) to find $c_5(T_0) > 0$ fulfilling

$$\int_\Omega |w(x,t_0)|^q dx \le c_5(T_0)\int_0^{t_0}\int_\Omega |w|^q \qquad \text{for all } t_0\in(0,T_0),$$

and thereby conclude by means of Grønwall's lemma that w and hence also z vanish identically in $\Omega\times(0,T_0)$. Since $T_0\in(0,T)$ was arbitrary, this proves the lemma. $\square$

4.3 A general convergence result

In several places below we shall refer to the following general statement on convergence of solutions to (1.1) toward strong $W^{1,q}$-solutions of (1.5). This will serve as an essential ingredient in proving local existence of solutions for any $\mu > 0$ in Lemma 4.5, in establishing global existence for $\mu \ge 1$ in Corollary 4.7, and finally also in verifying Theorem 1.1.

Lemma 4.3 *Let $r \ge 0, \mu > 0$ and $q > 1$, and assume that $u_0 \in W^{1,q}(\Omega)$ is nonnegative. Moreover, suppose that $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,\infty)$, $T > 0$ and $M > 0$ are such that $\varepsilon_j \searrow 0$ as $j \to \infty$ and such that whenever $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, for the solution $(u_\varepsilon, v_\varepsilon)$ of (1.1) we have*

$$u_\varepsilon(x,t) \le M \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T). \tag{4.23}$$

Then there exists a strong $W^{1,q}$-solution (u,v) of (1.5) in $\Omega \times (0,T)$ such that

$$u_\varepsilon \to u \qquad \text{in } C^0(\bar\Omega \times [0,T]), \tag{4.24}$$

$$u_\varepsilon \overset{\star}{\rightharpoonup} u \qquad \text{in } L^\infty((0,T); W^{1,q}(\Omega)) \qquad \text{and} \tag{4.25}$$

$$v_\varepsilon \to v \qquad \text{in } C^{2,0}(\bar\Omega \times [0,T]) \tag{4.26}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

In the proof of Lemma 4.3 we shall make use of a variant of the classical Aubin-Lions lemma ([38]). For convenience, we include a short proof here.

Lemma 4.4 *Let X, Y and Z be Banach spaces such that X is compactly embedded into Y and Y is continuously embedded into Z. Then for each $T > 0$ and any $p \in (1,\infty]$, the space*

$$\mathcal{X} := \Big\{ w \in L^\infty([0,T]; X) \ \Big| \ w_t \in L^p((0,T); Z) \Big\}$$

is compactly embedded into $C^0([0,T]; Y)$.

Proof. Let $(w_k)_{k\in\mathbb{N}} \subset \mathcal{X}$ be bounded. Then since $p > 1$ and X is compactly embedded into Z, $(w_k)_{k\in\mathbb{N}}$ is relatively compact in $C^0([0,T]; Z)$, whence on passing to a subsequence we may assume that $w_k \to w$ in $C^0([0,T]; Z)$. In order to show that actually $(w_k)_{k\in\mathbb{N}}$ forms a Cauchy sequence in $C^0([0,T]; Y)$, we abbreviate $c_1 := \sup_{k\in\mathbb{N}} \|w_k\|_{L^\infty((0,T);X)}$ and let $\delta > 0$ be given. Then since $X \hookrightarrow\hookrightarrow Y \hookrightarrow Z$, Ehrling's lemma yields $c_2 > 0$ such that $\|z\|_Y \le \frac{\delta}{4c_1}\|z\|_X + c_2\|z\|_Z$ for all $z \in X$, so that

$$\begin{aligned} \|w_k(t) - w_l(t)\|_Y &\le \frac{\delta}{4c_1}\Big(\|w_k(t)\|_X + \|w_l(t)\|_X\Big) + c_2\|w_k(t) - w_l(t)\|_Z \\ &\le \frac{\delta}{4c_1} \cdot 2c_1 + c_2\|w_k - w_l\|_{C^0([0,T];Z)} \end{aligned}$$

for all $k, l \in \mathbb{N}$ and each $t \in [0,t]$. Thus, taking $k_0 \in \mathbb{N}$ large enough such that $\|w_k - w_l\|_{C^0([0,T];Z)} < \frac{\delta}{2c_2}$, we infer that $\|w_k - w_l\|_{C^0([0,T];Y)} < \delta$ for all $k, l \ge k_0$. $\Box$

Proof of Lemma 4.3. As a consequence of (4.23), Corollary 3.6 implies that

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is bounded in } L^\infty((0,T); W^{1,q}(\Omega)), \tag{4.27}$$

whereas Lemma 3.4 says that for each $p \in (1,2]$ with $p < \frac{1}{(1-\mu)_+}$,

$$(u_{\varepsilon_j t})_{j\in\mathbb{N}} \quad \text{is bounded in } L^{\frac{p+1}{2}}((0,T);(W^{1,\frac{p}{p-1}}(\Omega))^\star). \tag{4.28}$$

In light of Lemma 4.4, a combination of (4.27) and (4.28) now ensures that

$$(u_{\varepsilon_j})_{j\in\mathbb{N}} \quad \text{is relatively compact in } C^0(\bar\Omega\times[0,T]).$$

According to this and (4.27), given any subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ we can pick a further subsequence $(\varepsilon_{j_i})_{i\in\mathbb{N}}$ thereof such that

$$u_{\varepsilon_{j_i}} \to u \qquad \text{in } C^0(\bar\Omega\times[0,T]) \tag{4.29}$$

and

$$u_{\varepsilon_{j_i}} \stackrel{\star}{\rightharpoonup} u \qquad \text{in } L^\infty((0,T);W^{1,q}(\Omega)), \tag{4.30}$$

and that hence also

$$v_{\varepsilon_{j_i}} \to v \qquad \text{in } C^{2,0}(\bar\Omega\times[0,T]) \tag{4.31}$$

as $i\to\infty$ with some (u,v). In view of a standard compactness argument, in order to prove that (4.24)-(4.26) actually hold along the entire sequence $\varepsilon=\varepsilon_j \searrow 0$, it is sufficient to identify all possible limits of such subsequences. To achieve this, because of the uniqueness statement in Lemma 4.2 we only need to show that (u,v) is a strong $W^{1,q}$-solution of (1.5) in $\Omega\times(0,T)$. To see this, we test (1.1) by an arbitrary $\varphi\in C_0^\infty(\bar\Omega\times[0,T))$ to obtain

$$-\int_0^T\int_\Omega u_\varepsilon\varphi_t - \int_\Omega u_0\varphi(\cdot,0) \;=\; -\varepsilon\int_0^T\int_\Omega u_{\varepsilon x}\varphi_x + \int_0^T\int_\Omega u_\varepsilon v_{\varepsilon x}\varphi_x + r\int_0^T\int_\Omega u_\varepsilon\varphi - \mu\int_0^T\int_\Omega u_\varepsilon^2\varphi$$

for all $\varepsilon>0$. Since (4.29)-(4.31) allow for taking $\varepsilon=\varepsilon_{j_i}\searrow 0$ in each of the integrals here separately, it readily follows that indeed (4.3) holds for (u,v), whence the proof is complete. $\square$

4.4 Local existence. Proof of Theorem 1.2

A first application of Lemma 4.3 asserts local existence of strong $W^{1,q}$-solutions with a quantitative control of the existence time in the following sense.

Lemma 4.5 *Let $r\ge 0, \mu>0$ and $q>1$. Then for all $D>0$ there exists $T(D)>0$ such that whenever $u_0\in W^{1,q}(\Omega)$ is nonnegative with*

$$\|u_0\|_{W^{1,q}(\Omega)} \le D, \tag{4.32}$$

the problem (1.5) possesses a unique strong $W^{1,q}$-solution (u,v) in $\Omega\times(0,T(D))$. This solution can be obtained as the limit of the corresponding solutions $(u_\varepsilon,v_\varepsilon)$ of (1.1) in the sense that

$$u_\varepsilon \to u \qquad \text{in } C^0(\bar\Omega\times[0,T(D)]), \tag{4.33}$$

$$u_\varepsilon \stackrel{\star}{\rightharpoonup} u \qquad \text{in } L^\infty((0,T(D));W^{1,q}(\Omega)) \qquad \text{and} \tag{4.34}$$

$$v_\varepsilon \to v \qquad \text{in } C^{2,0}(\bar\Omega\times[0,T(D)]) \tag{4.35}$$

as $\varepsilon \searrow 0$. Moreover, this solution satisfies

$$\int_\Omega |u_x(\cdot,t)|^q \le \left\{ \int_\Omega |u_{0x}|^q + q\int_0^t \Big(\int_\Omega u(\cdot,s)\Big)^{q+1} ds \right\} \cdot \exp\left(4q\int_0^t \|u(\cdot,s)\|_{L^\infty(\Omega)} ds + qrt\right) \qquad (4.36)$$

for a.e. $t \in (0,T(D))$.

PROOF. Given $D>0$, in order to define $T(D)$ we first fix constants $c_1>0$ and $c_2>0$ such that

$$\|\psi\|_{L^\infty(\Omega)} \le c_1\|\psi_x\|_{L^q(\Omega)} + c_1\|\psi\|_{L^1(\Omega)} \qquad \text{for all } \psi \in W^{1,q}(\Omega) \qquad (4.37)$$

and

$$\|\psi\|_{L^1(\Omega)} \le c_2\|\psi\|_{W^{1,q}(\Omega)} \qquad \text{for all } \psi \in W^{1,q}(\Omega), \qquad (4.38)$$

and let

$$c_3(D) := \max\Big\{c_2 D, \frac{r|\Omega|}{\mu}\Big\}. \qquad (4.39)$$

By continuity and an argument based on local well-posedness, we can then find $T(D)>0$ such that the initial-value problem

$$\begin{cases} y_D'(t) = 4qc_1 y_D^{1+\frac{1}{q}}(t) + q\Big(4c_1c_3(D)+r\Big)\cdot y_D(t) + qc_3^{q+1}(D), & t\in(0,T(D)),\\ y_D(0) = 2D^q+1, \end{cases} \qquad (4.40)$$

possesses a solution y_D satisfying

$$y_D(t) \le 2D^q + 2 \qquad \text{for all } t \in (0,T(D)). \qquad (4.41)$$

To derive the conclusion of the lemma for this choice of $T(D)$, we suppose that $0 \le u_0 \in W^{1,q}(\Omega)$ is such that (4.32) holds, and let $(u_\varepsilon, v_\varepsilon)$ denote the corresponding solution of (1.1) for $\varepsilon>0$. Then Corollary 3.2 combined with (4.38) and (4.39) says that

$$\int_\Omega u_\varepsilon(\cdot,t) \le \max\left\{ \int_\Omega u_0\, , \frac{r|\Omega|}{\mu}\right\} \le c_3(D) \qquad \text{for all } t>0, \qquad (4.42)$$

which together with (4.37) in particular implies that

$$\begin{aligned} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le c_1\|u_{\varepsilon x}(\cdot,t)\|_{L^q(\Omega)} + c_1c_3(D) \\ &\le c_1\bigg(\int_\Omega (u_{\varepsilon x}^2+\eta)^{\frac{q}{2}}\bigg)^{\frac{1}{q}} + c_1c_3(D) \qquad \text{for all } t>0 \end{aligned} \qquad (4.43)$$

whenever $\eta>0$. Now as a consequence of (4.42) and (4.43), Lemma 3.5 entails that for all $\eta>0$,

$$\begin{aligned} \frac{d}{dt}\int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} &\le q\Big(4\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}+r\Big)\cdot\int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} + q\bigg(\int_\Omega u_\varepsilon(\cdot,t)\bigg)^{q+1} \\ &\le 4qc_1\bigg(\int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}}\bigg)^{1+\frac{1}{q}} \\ &\quad + q(4c_1c_3(D)+r)\int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} + qc_3^{q+1}(D) \quad \text{for all } t>0. \end{aligned} \qquad (4.44)$$

Moreover, by the rough estimate $(a+b)^{\frac{q}{2}} \le 2(a^{\frac{q}{2}}+b^{\frac{q}{2}})$, valid for all nonnegative a and b, we see that if $\eta < \eta_0 := (2|\Omega|)^{-\frac{2}{q}}$ then

$$\int_\Omega (u_{0x}^2+\eta)^{\frac{q}{2}} \le 2\int_\Omega |u_{0x}|^q + 2\eta^{\frac{q}{2}}|\Omega| \le 2D^q + 1 \tag{4.45}$$

thanks to (4.32). According to (4.44) and (4.45), an ODE comparison therefore ensures that for any such η we have

$$\int_\Omega (u_{\varepsilon x}^2(\cdot,t)+\eta)^{\frac{q}{2}} \le y_D(t) \qquad \text{for all } t\in(0,T(D)),$$

which by(4.41) implies that

$$\int_\Omega |u_{\varepsilon x}(\cdot,t)|^q \le 2D^q + 2 \qquad \text{for all } t\in(0,T(D)). \tag{4.46}$$

Along with (4.37) and (4.42), this shows that for all $\varepsilon>0$,

$$u_\varepsilon(x,t) \le c_1(2D^q+2)^{\frac{1}{q}} + c_1c_3(D) \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T(D)),$$

whence Lemma 4.3 applies to provide a strong $W^{1,q}$-solution (u,v) of (1.5) in $\Omega\times(0,T(D))$ with the approximation properties (4.33)-(4.35). Thereupon, the inequality (4.36) results from Corollary 3.6, (4.33) and (4.34). □

On the basis of (4.36), upon twice applying the above lemma we can now verify our main result on existence and extensibility of strong $W^{1,q}$-solutions of (1.5).

Proof of Theorem 1.2. When applied to $D := \|u_0\|_{W^{1,q}(\Omega)}$, Lemma 4.5 provides $T>0$ and a strong $W^{1,q}$-solution (u,v) of (1.5) in $\Omega\times(0,T)$ fulfilling

$$\int_\Omega |u_x(\cdot,t)|^q \le \left\{\int_\Omega |u_{0x}|^q + q\int_0^t \Big(\int_\Omega u(\cdot,s)\Big)^{q+1} ds\right\}\cdot \exp\left(4q\int_0^t \|u(\cdot,s)\|_{L^\infty(\Omega)}ds + qrt\right) \tag{4.47}$$

for a.e. $t\in(0,T)$. Accordingly, the set

$$S \;:=\; \left\{ \widetilde{T}>0 \;\middle|\; \begin{array}{l}\text{There exists a strong } W^{1,q}\text{-solution of (1.5) in } \Omega\times(0,\widetilde{T}) \\ \qquad\text{which satisfies (4.47) for a.e. } t\in(0,\widetilde{T})\end{array} \right\}$$

is not empty and thus $T_{max} := \sup S$ well-defined. Clearly, (1.5) possesses a strong $W^{1,q}$-solution (u,v) in $\Omega\times(0,T_{max})$ which is unique according to Lemma 4.2, so that it remains to verify (1.6). To this end, we assume on the contrary that $T_{max}<\infty$ but $\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$, which would imply the existence of $M>0$ such that

$$u(x,t) \le M \qquad \text{for all } (x,t)\in\Omega\times(0,T_{max}). \tag{4.48}$$

Then by (4.47) we could find a null set $N\subset(0,T_{max})$ such that

$$\int_\Omega |u_x(\cdot,t)|^q \le \left\{\int_\Omega |u_{0x}|^q + q(|\Omega|M)^{q+1}T_{max}\right\}\cdot e^{(4qM+qr)T_{max}} \qquad \text{for all } t\in(0,T_{max})\setminus N,$$

which, again thanks to (4.48), would entail that

$$\|u(\cdot,t)\|_{W^{1,q}(\Omega)} \le D_1 \qquad \text{for all } t \in (0,T_{max}) \setminus N$$

with some $D_1 > 0$. Now a second application of Lemma 4.5 would show that with $T(D_1)$ provided by the latter, for each $t_0 \in (0,T_{max}) \setminus N$ the problem

$$\begin{cases} \hat{u}_t = -(\hat{u}\hat{v}_x)_x + r\hat{u} - \mu\hat{u}^2, \qquad x \in \Omega, t \in (0,T(D_1)), \\ 0 = \hat{v}_{xx} - \hat{v} + \hat{u}, \qquad x \in \Omega, t \in (0,T(D_1)), \\ \hat{v}_x = 0, \qquad x \in \partial\Omega, t \in (0,T(D_1)), \\ \hat{u}(x,0) = u(x,t_0) \qquad x \in \Omega, \end{cases}$$

admits a strong $W^{1,q}$-solution $(\hat{u},\hat{v})$ in $\Omega \times (0,T(D_1))$ fulfilling

$$\int_\Omega |\hat{u}_x(\cdot,t)|^q \le \Big\{ \int_\Omega |u_x(\cdot,t_0)|^q + q\int_0^t \Big(\int_\Omega \hat{u}(\cdot,s)\Big)^{q+1} ds \Big\} \cdot \exp\Big(4q\int_0^t \|\hat{u}(\cdot,s)\|_{L^\infty(\Omega)} ds + qrt \Big) \quad (4.49)$$

for a.e. $t \in (0,T(D_1))$.
Thus, choosing any $t_0 \in (0,T_{max}) \setminus N$ such that $t_0 > T_{max} - \frac{T(D_1)}{2}$ here we would infer that

$$(\tilde{u},\tilde{v})(\cdot,t) := \begin{cases} (u,v)(x,t) & \text{if } (x,t) \in \Omega \times (0,t_0), \\ (\hat{u},\hat{v})(x,t-t_0) & \text{if } (x,t) \in \Omega \times [t_0,t_0+T(D_1)), \end{cases}$$

would define a strong $W^{1,q}$-solution of (1.5) in $\Omega \times (0,t_0+T(D_1))$ which clearly would satisfy (4.47) for a.e. $t < t_0$. As for larger t, combining (4.49) with (4.47) we would obtain

$$\begin{aligned} &\int_\Omega |\tilde{u}_x(\cdot,t)|^q \\ \le\ & \Big\{ \int_\Omega |u_x(\cdot,t_0)|^q + q\int_{t_0}^t \Big(\int_\Omega \hat{u}(\cdot,s-t_0)\Big)^{q+1} ds \Big\} \cdot \exp\Big(4q\int_{t_0}^t \|\hat{u}(\cdot,s-t_0)\|_{L^\infty(\Omega)} ds + qr(t-t_0)\Big) \\ =\ & \Big\{ \int_\Omega |u_x(\cdot,t_0)|^q + q\int_{t_0}^t \Big(\int_\Omega \tilde{u}(\cdot,s)\Big)^{q+1} ds \Big\} \cdot \exp\Big(4q\int_{t_0}^t \|\tilde{u}(\cdot,s)\|_{L^\infty(\Omega)} ds + qr(t-t_0)\Big) \\ \le\ & \Big\{ \int_\Omega |u_{0x}|^q + q\int_0^{t_0} \Big(\int_\Omega u(\cdot,s)\Big)^{q+1} ds \Big\} \cdot \exp\Big(4q\int_0^{t_0} \|u(\cdot,s)\|_{L^\infty(\Omega)} ds + qrt_0\Big) \times \\ & \qquad \times \exp\Big(4q\int_{t_0}^t \|\tilde{u}(\cdot,s)\|_{L^\infty(\Omega)} ds + qr(t-t_0)\Big) \\ & +q\Big(\int_{t_0}^t \Big(\int_\Omega \tilde{u}(\cdot,s)\Big)^{q+1} ds \Big) \cdot \exp\Big(4q\int_{t_0}^t \|\tilde{u}(\cdot,s)\|_{L^\infty(\Omega)} ds + qr(t-t_0)\Big) \end{aligned} \quad (4.50)$$

for a.e. $t \in (t_0,t_0+T(D_1))$. Since

$$\begin{aligned} &\exp\Big(4q\int_0^{t_0} \|u(\cdot,s)\|_{L^\infty(\Omega)} ds + qrt_0\Big) \cdot \exp\Big(4q\int_{t_0}^t \|\tilde{u}(\cdot,s)\|_{L^\infty(\Omega)} ds + qr(t-t_0)\Big) \\ &\quad = \exp\Big(4q\int_0^t \|\tilde{u}(\cdot,s)\|_{L^\infty(\Omega)} ds + qrt\Big) \qquad \text{for all } t \in (t_0,t_0+T(D_1)) \end{aligned}$$

and, trivially,

$$4q\int_{t_0}^t \|\tilde u(\cdot,s)\|_{L^\infty(\Omega)}ds + qr(t-t_0) \le 4q\int_0^t \|\tilde u(\cdot,s)\|_{L^\infty(\Omega)}ds + qrt \qquad \text{for all } t\in(t_0,t_0+T(D_1)),$$

from (4.50) we would thus gain the inequality

$$\int_\Omega |\tilde u_x(\cdot,t)|^q \le \left\{ \int_\Omega |u_{0x}|^q + q\int_0^{t_0} \Big(\int_\Omega u(\cdot,s)\Big)^{q+1}ds + q\int_{t_0}^t \Big(\int_\Omega \tilde u(\cdot,s)\Big)^{q+1}ds \right\} \times$$
$$\times \exp\left(4q\int_0^t \|\tilde u(\cdot,s)\|_{L^\infty(\Omega)}ds + qrt\right) \qquad \text{for a.e. } t\in(t_0,t_0+T(D_1)),$$

which would therefore show that $(\tilde u,\tilde v)$ in fact satisfies (4.47) for a.e. $t\in(0,t_0+T(D_1))$. Since $t_0+T(D_1) > T_{max}+\frac{T(D_1)}{2}$, however, this would contradict the definition of T_{max}. ☐

4.5 Global solvability when $\mu\ge 1$. Proof of Proposition 1.3

According to the extensibility criterion (1.6), in order to prove global existence in (1.5) we only need to control the norm of the first solution component u with respect to the norm in $L^\infty(\Omega)$. For $\mu\ge 1$, this can readily be achieved by means of a simple parabolic comparison applied to the approximate problems (1.1).

Lemma 4.6 *Let $r\ge 0,\mu\ge 1$ and $u_0\in C^0(\bar\Omega)$ be nonnegative and such that $u_0\not\equiv 0$, and let $(u_\varepsilon,v_\varepsilon)$ denote the classical solution of (1.1) in $\Omega\times(0,\infty)$ for $\varepsilon>0$.*

Then for all $t>0$,

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \begin{cases} \frac{r}{\mu-1}\cdot\left\{1+\Big(\frac{r}{(\mu-1)\|u_0\|_{L^\infty(\Omega)}}-1\Big)e^{-rt}\right\}^{-1} & \text{if } r>0 \text{ and } \mu>1,\\ \frac{\|u_0\|_{L^\infty(\Omega)}}{1+(\mu-1)\|u_0\|_{L^\infty(\Omega)}\cdot t} & \text{if } r=0 \text{ and } \mu>1,\\ \|u_0\|_{L^\infty(\Omega)}\cdot e^{rt} & \text{if } r>0 \text{ and } \mu=1,\\ \|u_0\|_{L^\infty(\Omega)} & \text{if } r=0 \text{ and } \mu=1. \end{cases} \qquad (4.51)$$

Proof. Since $v_\varepsilon\ge 0$, we have

$$\begin{aligned} u_{\varepsilon t} &= \varepsilon u_{\varepsilon xx} - u_{\varepsilon x}v_{\varepsilon x} - u_\varepsilon v_\varepsilon + ru_\varepsilon - (\mu-1)u_\varepsilon^2 \\ &\le \varepsilon u_{\varepsilon xx} - u_{\varepsilon x}v_{\varepsilon x} + ru_\varepsilon - (\mu-1)u_\varepsilon^2 \qquad \text{in } \Omega\times(0,\infty). \end{aligned}$$

Thus, if we let y denote the solution of the initial-value problem

$$\begin{cases} y'(t) = ry(t)-(\mu-1)y^2(t), & t>0,\\ y(0)=\|u_0\|_{L^\infty(\Omega)}, \end{cases}$$

then the comparison principle asserts that $u_\varepsilon(x,t)\le y(t)$ for all $(x,t)\in\Omega\times(0,\infty)$. Explicitly computing y in the respective cases addressed in i)-iv), we easily derive (4.51). ☐

Taking $\varepsilon\searrow 0$ thus yields the following.

Corollary 4.7 *Let $r \geq 0$ and $\mu \geq 1$. Then for each nonnegative u_0 belonging to $W^{1,q}(\Omega)$ for some $q > 1$, the problem (1.5) possesses a unique global strong $W^{1,q}$-solution (u, v). Furthermore, if $u_0 \not\equiv 0$, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \begin{cases} \frac{r}{\mu-1} \cdot \left\{1 + \left(\frac{r}{(\mu-1)\|u_0\|_{L^\infty(\Omega)}} - 1\right)e^{-rt}\right\}^{-1} & \text{if } r > 0 \text{ and } \mu > 1, \\ \frac{\|u_0\|_{L^\infty(\Omega)}}{1+(\mu-1)\|u_0\|_{L^\infty(\Omega)} \cdot t} & \text{if } r = 0 \text{ and } \mu > 1, \\ \|u_0\|_{L^\infty(\Omega)} \cdot e^{rt} & \text{if } r > 0 \text{ and } \mu = 1, \\ \|u_0\|_{L^\infty(\Omega)} & \text{if } r = 0 \text{ and } \mu = 1. \end{cases} \tag{4.52}$$

Proof. This is a direct consequence of Lemma 4.3 because of the uniform estimates provided by Lemma 4.6. □

Proof of Proposition 1.3. The statement is immediate from Corollary 4.7. □

4.6 Blow-up for $\mu < 1$. Proof of Theorem 1.4

The cornerstone of our analysis in the case $\mu < 1$ is formed by an integral inequality for the functional $\int_\Omega u^p(\cdot,t)$ with a superlinear production term. The main step towards this inequality, to be formulated in (4.67) below, is the objective of the next lemma.

Lemma 4.8 *Let $r \geq 0$ and $\mu > 0$. Then for all $p > 1$ and each $\eta > 0$ one can find $B(p,\eta) > 0$ such that whenever $q > 1$ and (u, v) is a strong $W^{1,q}$-solution of (1.5) in $\Omega \times (0,T)$ with some $T > 0$ and some nonnegative $u_0 \in W^{1,q}(\Omega)$, we have*

$$\int_\Omega u^p(\cdot,t) \geq \int_\Omega u_0^p + \Big\{(1-\mu)p - 1 - \eta\Big\} \cdot \int_0^t \int_\Omega u^{p+1} - B(p,\eta) \int_0^t \Big(\int_\Omega u\Big)^{p+1} \quad \text{for all } t \in (0,T). \tag{4.53}$$

Proof. We shall perform an variant of the testing procedure used in Lemma 4.2. To this end, for fixed $T_0 \in (0,T)$, arbitrary $t_0 \in (0,T_0)$ and any $\delta \in (0, T_0 - t_0)$ we let $\chi_\delta \in W^{1,\infty}(\mathbb{R})$ be as defined in (4.9). We next note that according to (4.1), for each $\xi > 0$ the function $(u+\xi)^{p-1}$ belongs to $L^\infty_{loc}([0,T); W^{1,q}(\Omega))$ with $\Big((u+\xi)^{p-1}\Big)_x = (p-1)(u+\xi)^{p-2}u_x$ a.e. in $\Omega \times (0,T)$. Therefore, if we extend u so as to become a continuous function on $\bar\Omega \times [-1,T)$ by letting

$$u(x,t) := u_0(x) \qquad \text{if } (x,t) \in \bar\Omega \times [-1,0], \tag{4.54}$$

then according to the remark following Definition 4.1, for all $\delta \in (0, T_0 - t_0)$, $h \in (0,1)$ and $\xi > 0$,

$$\varphi(x,t) := \chi_\delta(t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s) + \xi\Big)^{p-1} ds, \qquad (x,t) \in \Omega \times (0,T),$$

defines an admissible test function for (4.3), whence the latter yields the identity

$$-\int_0^T \int_\Omega \chi_\delta'(t) u(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s) + \xi\Big)^{p-1} dsdxdt$$

$$
\begin{aligned}
& -\int_0^T \int_\Omega \chi_\delta(t) u(x,t) \cdot \frac{\Big(u(x,t)+\xi\Big)^{p-1} - \Big(u(x,t-h)+\xi\Big)^{p-1}}{h} dxdt \\
& -\int_\Omega u_0(x) \Big(u_0(x)+\xi\Big)^{p-1} dx \\
= \ & (p-1) \int_0^T \int_\Omega \chi_\delta(t) u(x,t) v_x(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-2} u_x(x,s) dsdxdt \\
& + r \int_0^T \int_\Omega \chi_\delta(t) u(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-1} dsdxdt \\
& - \mu \int_0^T \int_\Omega \chi_\delta(t) u^2(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-1} dsdxdt
\end{aligned}
\tag{4.55}
$$

for any such δ, h and ξ. Here by continuity of u, using the explicit form of χ'_δ induced by (4.9) we find that

$$
\begin{aligned}
& -\int_0^T \int_\Omega \chi'_\delta(t) u(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-1} dsdxdt \\
& \qquad = \frac{1}{\delta} \int_{t_0}^{t_0+\delta} \int_\Omega u(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-1} dsdxdt \\
& \qquad \to \int_\Omega u(x,t_0) \cdot \frac{1}{h} \int_{t_0-h}^{t_0} \Big(u(x,s)+\xi\Big)^{p-1} dsdx \qquad \text{as } \delta \searrow 0.
\end{aligned}
$$

Thus, upon four applications of the dominated convergence theorem in the second and the three rightmost integrals in (4.55) we infer that in the limit $\delta \searrow 0$, the latter implies that

$$
\begin{aligned}
& \int_\Omega u(x,t_0) \cdot \frac{1}{h} \int_{t_0-h}^{t_0} \Big(u(x,s)+\xi\Big)^{p-1} dsdx \\
& \qquad -\int_0^{t_0} \int_\Omega u(x,t) \cdot \frac{\Big(u(x,t)+\xi\Big)^{p-1} - \Big(u(x,t-h)+\xi\Big)^{p-1}}{h} dxdt \\
& \qquad -\int_\Omega u_0(x) \Big(u_0(x)+\xi\Big)^{p-1} dx \\
& = \ (p-1) \int_0^{t_0} \int_\Omega u(x,t) v_x(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-2} u_x(x,s) dsdxdt \\
& \qquad + r \int_0^{t_0} \int_\Omega u(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-1} dsdxdt \\
& \qquad - \mu \int_0^{t_0} \int_\Omega u^2(x,t) \cdot \frac{1}{h} \int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-1} dsdxdt
\end{aligned}
\tag{4.56}
$$

for all $h \in (0,1)$ and $\xi > 0$. We now rewrite the second term on the left according to

$$
-\int_0^{t_0} \int_\Omega u(x,t) \cdot \frac{\Big(u(x,t)+\xi\Big)^{p-1} - \Big(u(x,t-h)+\xi\Big)^{p-1}}{h} dxdt
$$

$$
\begin{aligned}
= & -\frac{1}{h}\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^p dxdt + \frac{1}{h}\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)\cdot\Big(u(x,t-h)+\xi\Big)^{p-1} dxdt \\
& +\frac{\xi}{h}\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^{p-1} dxdt - \frac{\xi}{h}\int_0^{t_0}\int_\Omega \Big(u(x,t-h)+\xi\Big)^{p-1} dxdt,
\end{aligned} \tag{4.57}
$$

where by Young's inequality and (4.54),

$$
\begin{aligned}
-\frac{1}{h}&\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^p dxdt + \frac{1}{h}\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)\cdot\Big(u(x,t-h)+\xi\Big)^{p-1} dxdt \\
\le & -\frac{1}{h}\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^p dxdt \\
& +\frac{1}{ph}\int_0^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^p dxdt + \frac{p-1}{ph}\int_0^{t_0}\Big(u(x,t-h)+\xi\Big)^p dxdt \\
= & \ \frac{p-1}{ph}\int_{-h}^0 \Big(u(x,t)+\xi\Big)^p dxdt - \frac{p-1}{ph}\int_{t_0-h}^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^p dxdt.
\end{aligned}
$$

Similarly joining the last two terms in (4.57), we infer that

$$
\begin{aligned}
-&\int_0^{t_0}\int_\Omega u(x,t)\cdot\frac{\Big(u(x,t)+\xi\Big)^{p-1}-\Big(u(x,t-h)+\xi\Big)^{p-1}}{h} dxdt \\
&= \frac{p-1}{p}\int_\Omega \Big(u_0(x)+\xi\Big)^p dx - \frac{p-1}{ph}\int_{t_0-h}^{t_0}\Big(u(x,t)+\xi\Big)^p dxdt \\
&\quad -\xi\int_\Omega \Big(u_0(x)+\xi\Big)^{p-1} dx + \frac{\xi}{h}\int_{t_0-h}^{t_0}\int_\Omega \Big(u(x,t)+\xi\Big)^{p-1} dxdt \\
&\to \frac{p-1}{p}\int_\Omega \Big(u_0(x)+\xi\Big)^p dx - \frac{p-1}{p}\int_\Omega \Big(u(x,t_0)+\xi\Big)^p dx \\
&\quad -\xi\int_\Omega \Big(u_0(x)+\xi\Big)^{p-1} dx + \xi\int_\Omega \Big(u(x,t_0)+\xi\Big)^{p-1} dx \qquad \text{as } h\searrow 0,
\end{aligned}
$$

where we again have used the continuity of u. By the same token and the fact that

$$
\Omega\times(0,T_0)\ni(x,t)\mapsto \frac{1}{h}\int_{t-h}^t \Big(u(x,s)+\xi\Big)^{p-2} u_x(x,s)ds \ \rightharpoonup\ \Big(u(x,t)+\xi\Big)^{p-2} u_x(x,t) \quad \text{in } L^q(\Omega\times(0,T_0))
$$

as $h\searrow 0$ according to (4.1) and a standard result on Steklov averages, taking $h\searrow 0$ in (4.56) we readily obtain that

$$
\begin{aligned}
\int_\Omega & u(x,t_0)\Big(u(x,t_0)+\xi\Big)^{p-1} dx - \int_\Omega u_0(x)\Big(u_0(x)+\xi\Big)^{p-1} dx \\
& +\frac{p-1}{p}\int_\Omega \Big(u_0(x)+\xi\Big)^p dx - \frac{p-1}{p}\int_\Omega \Big(u(x,t_0)+\xi\Big)^p dx \\
& -\xi\int_\Omega \Big(u_0(x)+\xi\Big)^{p-1} dx + \xi\int_\Omega \Big(u(x,t_0)+\xi\Big)^{p-1} dx
\end{aligned}
$$

$$\geq \quad (p-1)\int_0^{t_0}\int_\Omega u(x,t)v_x(x,t)\Big(u(x,t)+\xi\Big)^{p-2}u_x(x,t)dxdt$$
$$+r\int_0^{t_0}\int_\Omega u^p(x,t)dxdt-\mu\int_0^{t_0}\int_\Omega u^{p+1}(x,t)dxdt \tag{4.58}$$

holds for all $\xi>0$. Now once more by continuity of u we have $u(u+\xi)^{p-2}\to u^{p-1}$ uniformly in $\Omega\times(0,T_0)$ as $\xi\searrow 0$, because $p>1$. We therefore may let $\xi\searrow 0$ in (4.58) to conclude that

$$\frac{1}{p}\int_\Omega u^p(x,t_0)dx-\frac{1}{p}\int_\Omega u_0^p(x)dx\geq(p-1)\int_0^{t_0}\int_\Omega u^{p-1}u_xv_x+r\int_0^{t_0}\int_\Omega u^p-\mu\int_0^{t_0}\int_\Omega u^{p+1}. \tag{4.59}$$

Here we integrate by parts and use that $v_{xx}=v-u$ and $v_x|_{\partial\Omega}=0$ to see that

$$(p-1)\int_0^{t_0}\int_\Omega u^{p-1}u_xv_x=-\frac{p-1}{p}\int_0^{t_0}\int_\Omega u^pv_{xx}=-\frac{p-1}{p}\int_0^{t_0}\int_\Omega u^pv+\frac{p-1}{p}\int_0^{t_0}\int_\Omega u^{p+1}. \tag{4.60}$$

By Young's inequality and Lemma 2.2, given $\eta>0$ we can now find $c_1=c_1(p,\eta)>0$ and $c_2=c_2(p,\eta)>0$ such that

$$\begin{aligned}\frac{p-1}{p}\int_0^{t_0}\int_\Omega u^pv &\leq \frac{\eta}{2p}\int_0^{t_0}\int_\Omega u^{p+1}+c_1\int_0^{t_0}\int_\Omega v^{p+1}\\ &\leq \frac{\eta}{2p}\int_0^{t_0}\int_\Omega u^{p+1}+\frac{\eta}{2p}\int_0^{t_0}\int_\Omega u^{p+1}+c_2\int_0^{t_0}\Big(\int_\Omega u(\cdot,t)\Big)^{p+2}dt,\end{aligned}$$

whence (4.59) and (4.60) yield after dropping a nonnegative term that

$$\frac{1}{p}\int_\Omega u^p(x,t_0)dx-\frac{1}{p}\int_\Omega u_0^p(x)dx\geq\Big(-\frac{\eta}{p}+\frac{p-1}{p}-\mu\Big)\cdot\int_0^{t_0}\int_\Omega u^{p+1}-c_2\int_0^{t_0}\Big(u(\cdot,t)\Big)^{p+1}dt.$$

Since $t_0\in(0,T_0)$ and $T_0\in(0,T)$ were arbitrary, this proves (4.53). ☐

In deriving Theorem 1.4 from this, we shall rely on a variant of Grønwall's lemma.

Lemma 4.9 *Let $a>0, b\geq 0, d>0$ and $\kappa>1$ be such that*

$$a>\Big(\frac{2b}{d}\Big)^{\frac{1}{\kappa}}. \tag{4.61}$$

Then if for some $T>0$, the function $y\in C^0([0,T))$ is nonnegative and satisfies

$$y(t)\geq a-bt+d\int_0^t y^\kappa(s)ds \qquad \textit{for all } t\in(0,T), \tag{4.62}$$

we necessarily have

$$T\leq\frac{2}{(\kappa-1)a^{\kappa-1}d}. \tag{4.63}$$

PROOF. For $\delta \in (0, \delta_0)$ with $\delta_0 := a - (\frac{2b}{d})^{\frac{1}{\kappa}}$ we let

$$T_\delta := \frac{2}{(\kappa - 1)(a - \delta)^{\kappa - 1} d}$$

and

$$z_\delta(t) := \Big\{ (a - \delta)^{1-\kappa} - \frac{(\kappa - 1)d}{2} t \Big\}^{-\frac{1}{\kappa - 1}}, \qquad t \in [0, T_\delta);$$

that is, we define z_δ to be the solution of

$$\begin{cases} z_\delta'(t) = \frac{d}{2} z^\kappa(t), & t \in (0, T_\delta), \\ z_\delta(0) = a - \delta. \end{cases} \tag{4.64}$$

Then z_δ increases and hence satisfies $z_\delta \geq a - \delta > a - \delta_0 > (\frac{2b}{d})^{\frac{1}{\kappa}}$ on $[0, T_\delta)$ by (4.61), so that

$$z_\delta'(t) = -\frac{d}{2} z_\delta^\kappa(t) + d z_\delta^\kappa(t) \leq -\frac{d}{2} \cdot \frac{2b}{d} + d z_\delta^\kappa(t) = -b + d z_\delta^\kappa(t) \qquad \text{for all } t \in (0, T_\delta). \tag{4.65}$$

Furthermore, from (4.62) we know that $y(0) > z_\delta(0)$, so that

$$S_\delta := \Big\{ t \in (0, T_\delta) \ \Big| \ y > z_\delta \text{ in } [0, t] \Big\}$$

is not empty and hence $t_\delta := \sup S_\delta$ well-defined. Now if we had $t_\delta < T_\delta$ for some $\delta \in (0, \delta_0)$, then clearly $y > z_\delta$ on $[0, t_\delta)$ and $y(t_\delta) = z_\delta(t_\delta)$, so that by (4.62) and (4.65) we would infer that

$$z_\delta(t_\delta) = y(t_\delta) \geq a - b t_\delta + d \int_0^{t_\delta} y^\kappa(s) ds > a - \delta - b t_\delta + d \int_0^{t_\delta} z_\delta^\kappa(s) ds \geq z_\delta(t_\delta).$$

This absurd conclusion shows that actually $y > z_\delta$ throughout $[0, T_\delta)$, which since $z_\delta(t) \nearrow \infty$ as $t \nearrow T_\delta$ entails that $T < T_\delta$. In the limit $\delta \searrow 0$ this implies (4.63). $\Box$

PROOF of Theorem 1.4. With $\eta := \frac{(1-\mu)p-1}{2} > 0$, we let $B(p, \eta)$ denote the constant provided by Lemma 4.8. We claim that then the above statement holds if we let

$$C(p) := \left(\frac{4B(p, \eta)}{(1 - \mu)p - 1} \right)^{\frac{1}{p-1}} \cdot |\Omega|^{1 + \frac{1}{p(p+1)}}. \tag{4.66}$$

Indeed, suppose on the contrary that with this choice, (1.7) holds for some nonnegative $u_0 \in W^{1,q}(\Omega)$, $q > 1$, but that the corresponding strong $W^{1,q}$-solution from Theorem 1.2 be global in time. Then $y(t) := \int_\Omega u^p(x, t) dx$, $t \geq 0$, would define a continuous function on $[0, \infty)$ which according to Lemma 4.8 and our choice of η would satisfy

$$y(t) \geq y(0) + \frac{(1 + \mu)p - 1}{2|\Omega|^{\frac{1}{p}}} \int_0^t y^{\frac{p+1}{p}}(s) ds - B(p, \eta)|\Omega|^{p+1} \widehat{m}^{p+1} t \qquad \text{for all } t > 0 \tag{4.67}$$

with $\widehat{m} := \max\{\frac{1}{|\Omega|}\int_\Omega u_0, \frac{r}{\mu}\}$, where we have employed Lemma 4.1 in estimating

$$\int_0^t \Big(\int_\Omega u\Big)^{p+1} ds \le |\Omega|^{p+1}\widehat{m}^{p+1} t \qquad \text{for all } t>0,$$

and where we have used the Hölder inequality to see that

$$\int_0^t \Big(\int_\Omega u^p\Big)^{\frac{p+1}{p}} \le |\Omega|^{\frac{1}{p}} \int_0^t \int_\Omega u^{p+1}.$$

Since the numbers $a := y(0), b := B(p,\eta)|\Omega|^{p+1}\widehat{m}^{p+1}$, $d := \frac{(1-\mu)p-1}{2|\Omega|^{\frac{1}{p}}}$ and $\kappa := \frac{p+1}{p}$ satisfy

$$\Big\{a\cdot\Big(\frac{2b}{d}\Big)^{-\frac{1}{\kappa}}\Big\}^{\frac{1}{p}} = \|u_0\|_{L^p(\Omega)}\cdot\Big(\frac{(1-\mu)p-1}{4B(p,\eta)|\Omega|^{p+1+\frac{1}{p}}\widehat{m}^{p+1}}\Big)^{\frac{1}{p+1}} > 1$$

by (1.7) and (4.66), an application of Lemma 4.9, however, says that y and hence (u,v) cannot be global in time. The conclusion (1.8) is then an immediate consequence of (1.6). □

5 Exceeding carrying capacities in (1.1). Proof of Theorem 1.1

With all the above preparations at hand, our main result concerning the solution behavior in the problem (1.1) with small diffusion actually reduces to a straightforward consequence.

Proof of Theorem 1.1. Given u_0 fulfilling (1.7), we let $T>0$ denote the maximal existence time of the strong $W^{1,q}$-solution (u,v) of (1.5), whence from Theorem 1.4 we know that $T<\infty$ and

$$\limsup_{t\nearrow T} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{5.1}$$

Now if the claim was false, then for some $M>0$ we could find a sequence $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j\to\infty$ and

$$u_{\varepsilon_j}(x,t) \le M \qquad \text{for all } (x,t)\in\Omega\times(0,T) \text{ and each } j\in\mathbb{N}. \tag{5.2}$$

Then Lemma 4.3 would ensure that

$$u_\varepsilon \to \tilde{u} \qquad \text{in } C^0(\bar\Omega\times[0,T]) \tag{5.3}$$

and

$$v_\varepsilon \to \tilde{v} \qquad \text{in } C^{2,0}(\bar\Omega\times[0,T])$$

as $\varepsilon=\varepsilon_j\searrow 0$, where $(\tilde{u},\tilde{v})$ is a strong $W^{1,q}$-solution of (1.5) in $\Omega\times(0,T)$. Thanks to the uniqueness property of such solutions, as stated in Lemma 4.2, we thus infer that actually $(\tilde{u},\tilde{v})=(u,v)$, so that in particular $u\le M$ in $\Omega\times(0,T)$ according to (5.2) and (5.3). This, however, would contradict (5.1), whereby the proof is completed. □

6 Numerical illustrations: Large densities as a transient phenomenon

In this section we shall present some numerical experiments in order to illustrate the solution behavior in (1.5). The simulations were carried out using a time-explicit finite difference scheme on an equidistant spatial grid on the unit interval $\Omega := (0, 1)$, with grid size 0.01 and time step size 10^{-6}.

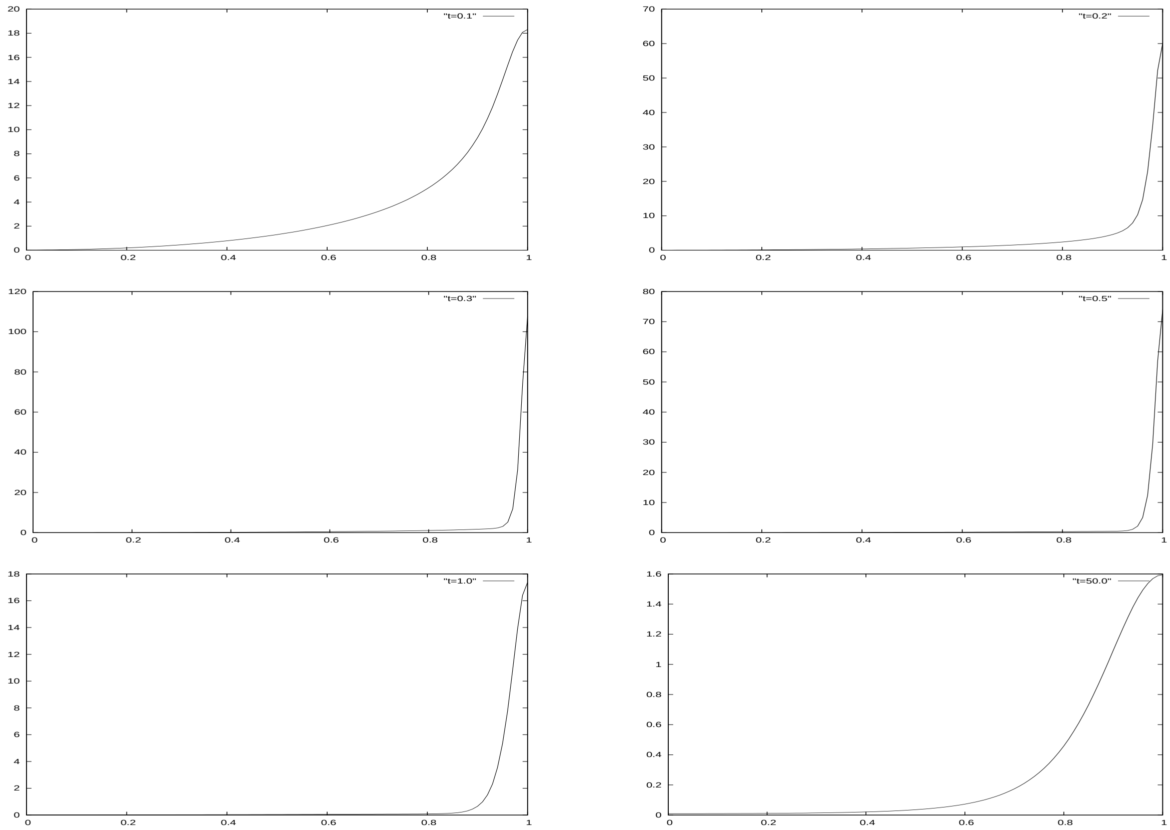

Figure 1: Solution behavior for $\varepsilon = 0.0148, r = 0.1, \mu = 0.1$ and $u_0(x) := 10x^2$ at times $t = 0.1, t = 0.2, t = 0.3, t = 0.5, t = 1.0$ and $t = 50$, respectively.
Vertical axis: x; horizontal axis: $u(x, t)$

In Figure 1, the diffusion constant and the rates of reproduction and death are fixed according to $\varepsilon = 0.0148$, $r = 0.1$ and $\mu = 0.1$, and the spatial profiles of the respective component u of the solution emanating from $u_0(x) = 10x^2$, $x \in \Omega$, are shown at various times.
The first three pictures show how the solution dynamically forms an aggregate near $x = 1$ around time $t = 0.3$, at its spatial maximum exceeding the carrying capacity $u_c = \frac{r}{\mu} = 1$ and the initial upper bound $\|u_0\|_{L^\infty(\Omega)} = 10$ by factors larger than 100 and 10, respectively. However, the further evolution clearly reveals that this is a transient phenomenon: Afterwards, namely, this accumulation relaxes and the solution appears to stabilize toward a spatially inhomogeneous equilibrium shown in the last picture.
The degree of excession of u_c is additionally underlined by Figure 2, which traces the behavior of

$\|u(\cdot,t)\|_{L^\infty(\Omega)}$ over the time interval $t \in [0,5]$ for the above solution.

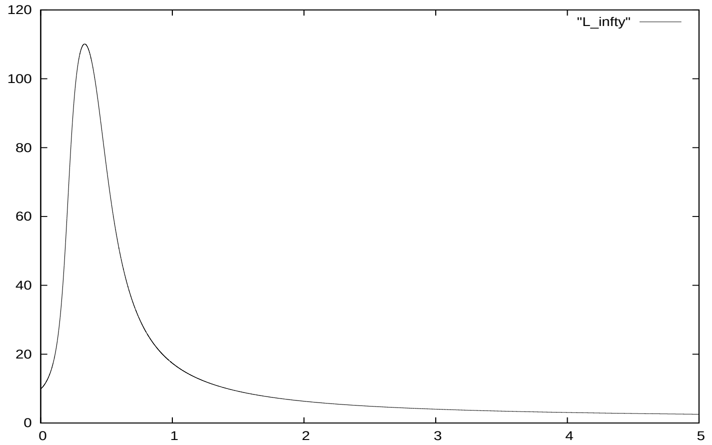

Figure 2: Time evolution of the spatial norm of $u(\cdot t)$ in $L^\infty(\Omega)$ of the solution in Figure 1.
Vertical axis: t; horizontal axis: $\|u(\cdot,t)\|_{L^\infty(\Omega)}$

Finally, Figure 3 illustrates that the potential to generate such temporary clusters does not rely on any production of cells, but that it sharply depends on their diffusivity ε: Indeed, even in the borderline case $r=0$, still with $\mu = 0.1$ and $u_0(x) = 10x^2$, large values of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ do occur if ε is small enough; for $\varepsilon = 0.5$, however, the solution remains bounded from above by $\|u_0\|_{L^\infty(\Omega)}$.

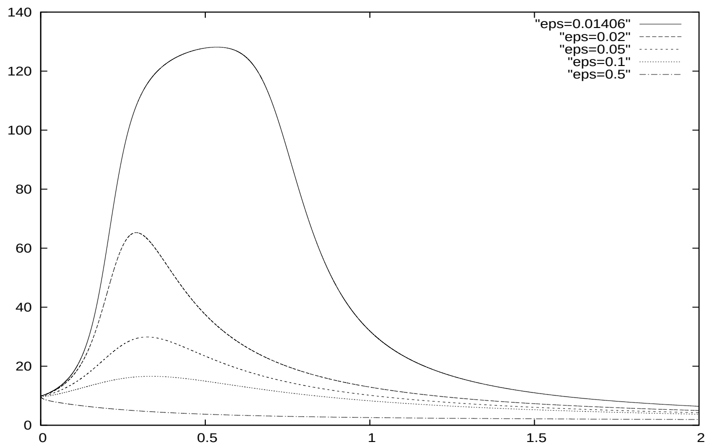

Figure 3: Behavior of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ over $t \in [0,2]$ in dependence of $\varepsilon \in \{0.01406, 0.02, 0.05, 0.1, 0.5\}$ when $r=0$, $\mu = 0.1$ and $u_0(x) = 10x^2$.
Vertical axis: t; horizontal axis: $\|u(\cdot,t)\|_{L^\infty(\Omega)}$

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