J. London Math. Soc. (2) 90 (2014) 167–183
doi:10.1112/jlms/jdu025

Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent

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Abstract

We study the asymptotic behaviour near extinction of positive solutions of the Cauchy problem for the fast diffusion equation with a critical exponent. After a suitable rescaling that yields a nonlinear Fokker–Planck equation, we find a continuum of algebraic rates of convergence to a self-similar profile. These rates depend explicitly on the spatial decay rates of initial data. This improves a previous result on slow convergence for the critical fast diffusion equation and provides answers to some open problems.

1. *Introduction*

We consider the Cauchy problem for the fast diffusion equation,

$$\begin{cases} u_\tau = \nabla \cdot (u^{m-1}\nabla u), & y \in \mathbb{R}^n,\ \tau \in (0,T), \\ u(y,0) = u_0(y) \geqslant 0, & y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geqslant 3$, $T > 0$ and $m = (n-4)/(n-2)$. It is known that, for $m < m_c := (n-2)/n$, all solutions with initial data in some suitable space, such as $L^p(\mathbb{R}^n)$ with $p = n(1-m)/2$, extinguish in finite time. We shall consider solutions that vanish in a finite time $\tau = T$ and study their behaviour near $\tau = T$.

For the extinction range $m < m_c$ there are (infinite-mass) solutions of the self-similar form

$$U_{D,T}(y,\tau) := \frac{1}{R(\tau)^n}\left(D + \frac{\beta(1-m)}{2}\left|\frac{y}{R(\tau)}\right|^2\right)^{-1/(1-m)}, \tag{1.2}$$

where $D \geqslant 0$ and

$$R(\tau) := (T-\tau)^{-\beta}, \quad \beta := \frac{1}{n(1-m)-2} = \frac{1}{n(m_c - m)} > 0.$$

We will call these solutions *Barenblatt solutions*.

Many papers ([**3–7**], for example) are concerned with the convergence of solutions of (1.1) to the Barenblatt solutions as $\tau \to T$. More precisely, the decay rates of

$$R(\tau)^n(u(\tau,y) - U_{D,T}(y,\tau))$$

as $\tau \to T$ are discussed there.

The reasons why the critical exponent

$$m_* := \frac{n-4}{n-2} < m_c,$$

Received 22 February 2013; revised 27 November 2013; published online 29 May 2014.

2010 *Mathematics Subject Classification* 35K55 (primary), 35B40 (secondary).

The first author was supported in part by the Slovak Research and Development Agency under the contract No. APVV-0134-10 and by the VEGA grant 1/0711/12. The second author gratefully acknowledges the support of the Royal Society, the ESF network HCAA and Wolfson Foundation.

plays a very important role in the results of [3–7] will be explained below. If $n = 3, 4$, then $m_* \leqslant 0$, which is a case treated in some more detail in [4].

To study the asymptotic profile as $\tau \to T$, it is convenient to rewrite (1.1) in similarity variables:

$$t := \frac{1}{\mu} \ln\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{\beta}{\mu}}\, \frac{y}{R(\tau)}, \quad \mu := \frac{2}{1-m},$$

with R as above, and the rescaled function

$$v(x, t) := R(\tau)^n u(y, \tau)$$

satisfies then the nonlinear Fokker–Planck equation

$$v_t = \nabla \cdot (v^{m-1} \nabla v) + \mu \nabla \cdot (xv), \quad x \in \mathbb{R}^n, \ \ t > 0. \tag{1.3}$$

The Barenblatt solutions $U_{D,T}(y, \tau)$ are thereby transformed into *Barenblatt profiles* $V_D(x)$, which have the advantage of being stationary:

$$V_D(x) := (D + |x|^2)^{-1/(1-m)}, \quad x \in \mathbb{R}^n. \tag{1.4}$$

In the new variables, the convergence of solutions of (1.1) to $U_{D,T}$ takes the form of stabilization of solutions of (1.3) to non-trivial equilibria V_D.

The critical exponent m_* has the property that the difference of two Barenblatt profiles is integrable for $m \in (m_*, m_c)$, while it is not integrable for $m \leqslant m_*$. Furthermore, m_* is the unique value of m such that the linearization of the operator $\nabla \cdot (v^{m-1} \nabla v) + \mu \nabla \cdot (xv)$ around V_D (on a natural weighted L^2-space) has no spectral gap, see [4]. This is why one can expect that the rate of convergence to V_D is exponential for $m \neq m_*$ and algebraic for $m = m_*$.

In [3, 4, 6, 7], one can find several sufficient conditions under which $v(\cdot, t)$ converges to v_D exponentially if $m < m_c$, $m \neq m_*$. The case $m = m_*$ was treated in [5] by functional analytic methods. A suitable linearization of the nonlinear Fokker–Planck equation (1.3) was viewed as the plain heat flow on a suitable Riemannian manifold, and then nonlinear stability was studied by entropy methods. Theorem 3.1 in [5] (which can be viewed as the main result of [5]) gives algebraic upper bounds for the decay rate of the entropy functional and for the convergence rate to V_D. One can expect the rates to be sharp since the linearization decays at those rates, but in [5] there is no rigorous proof of optimality. In fact, no lower bounds for the rates are established in [5]. One of the main aims of the present paper is to prove optimal lower bounds for the convergence rates for a large class of initial data. Our first main result says that convergence to V_D from below cannot occur at any rate faster than $t^{-1/2}$, which is the fastest decay rate of positive solutions of the linear one-dimensional heat equation.

THEOREM 1.1. *Let $n > 2$, $m = m_\star$ and $D > 0$. Assume that v_0 is continuous and non-negative on $\mathbb{R}^n$, $v_0 \leqslant V_D$, $v_0 \not\equiv V_D$, with V_D given by (1.4). Then there exists $c > 0$ such that the solution v of (1.3) with the initial condition $v(\cdot, 0) = v_0$ satisfies*

$$v(0, t) \leqslant V_D(0) - ct^{-1/2} \quad \text{for } t > 1.$$

If v_0 intersects V_D, then we expect that a faster rate of convergence may occur, similarly as for sign-changing solutions of the linear heat equation.

Next, we discuss upper bounds for the convergence rate. Corollary 3.2 in [5] says (among other things) that if $0 < D_1 < D_0$, $D \in [D_1, D_0]$ and

$$V_{D_0}(x) \leqslant v_0(x) \leqslant V_{D_1}(x), \quad x \in \mathbb{R}^n, \tag{1.5}$$

$$|v_0(x) - V_D(x)| \leqslant f(|x|), \quad x \in \mathbb{R}^n, \ \ f(|\cdot|) \in L^1(\mathbb{R}^n), \tag{1.6}$$

then, for the solution v of (1.3) with the initial condition $v(x,0) = v_0(x)$, it holds that

$$\|v(\cdot,t) - V_D\|_{L^\infty(\mathbb{R}^n)} \leqslant Kt^{-1/4}, \quad t \geqslant 1, \tag{1.7}$$

for some $K > 0$.

The question of whether the rates obtained in [**5**] are optimal for a class of data was posed in [**5**] as an open problem together with the question of whether one can prove convergence, maybe with worse rates or without rates, for more general initial data (see [**5**, Subsection 8.2]).

Our first step in answering these questions is the following:

THEOREM 1.2. *Assume that $n > 2$, $m = m_*$ and $D > 0$, and that V_D is as defined in (1.4). Let v be the solution of (1.3) with the initial condition*

$$v(x,0) = v_0(x) := (|x|^2 + D + \psi_0(x))^{-(n-2)/2}, \quad x \in \mathbb{R}^n, \tag{1.8}$$

where ψ_0 is continuous and non-negative on $\mathbb{R}^n$, $\psi_0 \not\equiv 0$. If there are $B > 0$ and $\gamma \in (0,1)$ such that

$$\psi_0(x) \leqslant B \ln^{-\gamma} |x|, \quad |x| > 2, \tag{1.9}$$

then there exists $C > 0$ such that

$$V_D(x)(1 - CV_D^{2/(n-2)}(x)t^{-\gamma/2}) \leqslant v(x,t) \leqslant V_D(x), \quad x \in \mathbb{R}^n, \ t \geqslant 1.$$

This theorem yields convergence with rates for a class of data that do not satisfy (1.6), but also for data that belong to the class considered in [**5**]. Namely, if ψ_0 satisfies (1.9) with $\gamma > 1$, then (1.9) also holds with $\gamma = 1 - \varepsilon$, $\varepsilon \in (0,1)$, and some $B = B(\varepsilon) > 0$.

As an immediate consequence of Theorems 1.1 and 1.2 we obtain:

COROLLARY 1.3. *Let $n > 2$, $m = m_\star$ and $D > 0$. Assume that ψ_0 is continuous and non-negative on $\mathbb{R}^n$, $\psi_0 \not\equiv 0$. Let v be the solution of (1.3) with the initial condition (1.8). If there are $B > 0$ and $\gamma \geqslant 1$ such that (1.9) holds, then there is $c > 0$ and, for any $\varepsilon \in (0,1)$, there exists $C_\varepsilon > 0$ such that*

$$ct^{-1/2} \leqslant \|V_D - v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leqslant C_\varepsilon t^{-(1-\varepsilon)/2}, \quad t \geqslant 1.$$

If $\gamma > 1$, then the initial data from Corollary 1.3 satisfy (1.5) and (1.6), and fill a large part of the range of applicability of the entropy method from [**5**]. The wrong power of time appearing in (1.7) is due to interpolation. It was shown in [**5**] that, in the linearized situation, the heat kernel decay has a one-dimensional behaviour in the sense that its rate is $t^{-1/2}$ (see [**5**, Corollaries 4.4 and 4.5]), but the consequent smoothing effect between L^1 and L^2 yields a $t^{-1/4}$ decay only (see [**5**, Section 4.4]). The $L^1 - L^2$ bounds allow one to recover the correct $L^1 - L^\infty$ decay in the linear situation, but the lack of such functional analytic tools in the nonlinear situation causes the appearance of the wrong power of time for the L^∞-norm.

In this paper, we work with the PDE directly, without any use of functional analysis. Our next result implies that Theorem 1.2 is sharp.

THEOREM 1.4. *Assume that $n > 2$, $m = m_*$ and $D > 0$. Let V_D be as defined in (1.4) and let v be the solution of (1.3) with the initial condition (1.8). If there are $b > 0$ and $\gamma \in (0,1)$ such that*

$$\psi_0(x) \geqslant b \ln^{-\gamma} |x|, \quad |x| > 2,$$

then there exists $c > 0$ such that

$$v(0,t) \leqslant V_D(0) - ct^{-\gamma/2}, \quad t > 1.$$

Theorems 1.2 and 1.4 yield that if $V_D(x) - v_0(x)$ behaves like $|x|^{-n} \ln^{-\gamma} |x|$ for $|x|$ large and some $\gamma \in (0,1)$, then $\|v(\cdot,t) - V_D\|_{L^\infty(\mathbb{R}^n)}$ behaves like $t^{-\gamma/2}$ for t large. Hence, we obtain a continuum of algebraic rates for initial data that do not satisfy (1.6). These rates are the same as for $u_t = u_{xx}$, $x \in \mathbb{R}$, with positive initial data decaying as $|x|^{-\gamma}$. Hence, the long-time behaviour of solutions of (1.3) is one-dimensional, while the short-time behaviour of solutions of the linearized equation is n-dimensional (cf. [**5**, Corollary 4.4]).

We prove our results by constructing suitable sub- and super-solutions. In order not to make the paper unnecessarily long, we consider only initial data below V_D, but one can modify the arguments to prove analogous results for initial data above V_D.

In Section 2, we establish the lower bound from Theorem 1.2, and in Section 3 the upper bound from Theorem 1.4. Section 4 is devoted to the proof of Theorem 1.1.

2. *Lower bound. Proof of Theorem* 1.2

To construct a suitable super-solution, we shall use the following:

LEMMA 2.1. *Let $\gamma \in (0,1)$. Then the solution of the problem*

$$\begin{cases} \Phi''(z) + \dfrac{z}{2}\Phi'(z) + \dfrac{\gamma}{2}\Phi(z) = 0, & z > 0, \\ \Phi(0) = 1, \quad \Phi'(0) = 0, \end{cases} \tag{2.1}$$

is positive and decreasing on $[0,\infty)$, and there exist $c > 0$ and $C > 0$ such that

$$cz^{-\gamma} \leqslant \Phi(z) \leqslant Cz^{-\gamma} \quad \text{for all } z \geqslant 1 \tag{2.2}$$

and

$$-Cz^{-\gamma-1} \leqslant \Phi'(z) \leqslant -cz^{-\gamma-1} \quad \text{for all } z \geqslant 1 \tag{2.3}$$

as well as

$$|\Phi''(z)| \leqslant Cz^{-\gamma-2} \quad \text{for all } z \geqslant 1. \tag{2.4}$$

Proof. The solution Φ of (2.1) can be written explicitly in the form

$$\Phi(z) = e^{-\zeta}\mathcal{M}\left(\frac{1-\gamma}{2}, \frac{1}{2}, \zeta\right), \quad \zeta := \frac{z^2}{4},$$

where $\mathcal{M}$ is Kummer's function (see [**1**])

$$\mathcal{M}(a,b,\zeta) := 1 + \frac{a}{b}\zeta + \cdots + \frac{a(a+1)\cdots(a+k)}{b(b+1)\cdots(b+k)k!}\zeta^k + \cdots$$

and

$$\zeta^{b-a}\, e^{-\zeta}\mathcal{M}(a,b,\zeta) \longrightarrow \frac{\Gamma(b)}{\Gamma(a)} \quad \text{as } \zeta \longrightarrow \infty, \tag{2.5}$$

which yields (2.2).

If we now rewrite the equation in (2.1) as

$$\Phi''(z) + \tfrac{1}{2}z^{1-\gamma}(z^\gamma\Phi(z))' = 0$$

and use the identity

$$\zeta\frac{d}{d\zeta}(\zeta^{b-a}\,e^{-\zeta}\mathcal{M}(a,b,\zeta)) = (b-a)\zeta^{b-a}\,e^{-\zeta}\mathcal{M}(a-1,b,\zeta)$$

(see [1]) together with (2.5), then we obtain that

$$|\tfrac{1}{2}z^{1-\gamma}(z^{\gamma}\Phi(z))'| \leqslant Cz^{-\gamma-2}, \quad z \geqslant 1,$$

which implies (2.4).

Since Φ cannot have any local minimum, one can see that Φ' is negative and (2.3) follows from (2.4). □

For $m = m_*$ and radial solutions $v = v(r,t)$, (1.3) becomes

$$v_t = (v^{-2/(n-2)}v_r)_r + \frac{n-1}{r}v^{-2/(n-2)}v_r + (n-2)(rv_r + nv), \quad r > 0,\ t > 0.$$

If we further transform v via

$$v(r,t) = (r^2 + D + \varphi(r,t))^{-(n-2)/2}, \quad r \geqslant 0,\ t \geqslant 0,$$

then, after some computation, it can be checked that φ satisfies, for $r > 0$ and $t > 0$, the equation

$$\mathcal{P}\varphi := \varphi_t - (r^2 + D + \varphi)\left(\varphi_{rr} + \frac{n-1}{r}\varphi_r\right) + (n-2)r\varphi_r + \frac{n-2}{2}\varphi_r^2 = 0. \qquad (2.6)$$

The change of variables

$$\chi(\xi,t) := \varphi(r,t), \quad \xi := \ln r,\ r > 0,\ t \geqslant 0,$$

yields that

$$\mathcal{Q}\chi := \chi_t - \chi_{\xi\xi} - e^{-2\xi}\left\{(D+\chi)[\chi_{\xi\xi} + (n-2)\chi_\xi] - \frac{n-2}{2}\chi_\xi^2\right\} = 0 \qquad (2.7)$$

for $\xi \in \mathbb{R}$ and $t > 0$.

In a region where r is appropriately large, we shall use functions of the form

$$\chi^{(\xi_0,t_0,A)}(\xi,t) := A(t+t_0)^{-\gamma/2}\Phi((\xi+\xi_0)(t+t_0)^{-1/2}), \quad \xi \geqslant 0,\ t \geqslant 0, \qquad (2.8)$$

as (upper) comparison functions. For clarity of notation, we consider $\xi_0 > 0$, $t_0 \geqslant 1$ and $A > 0$ as free parameters here. We shall fix ξ_0, t_0 in Lemma 2.7 and $A > 0$ in the proof of Lemma 2.8.

LEMMA 2.2. *Let $\gamma \in (0,1)$. For $t_0 \geqslant 1$, $\xi_0 \in \mathbb{R}$ and $A > 0$, the function $\chi = \chi^{(\xi_0,t_0,A)}$ defined in (2.8) satisfies*

$$\chi_t = \chi_{\xi\xi} \quad \text{for } \xi > 0 \text{ and } t > 0. \qquad (2.9)$$

Moreover, there exists $t_\star > 1$ with the property that, whenever $t_0 > t_\star$, for any choice of $\xi_0 > 0$ and $A > 0$ we have

$$\chi_{\xi\xi} + (n-2)\chi_\xi \leqslant 0 \quad \text{for all } \xi > 0 \text{ and } t > 0. \qquad (2.10)$$

Proof. Since

$$\chi_\xi = A(t+t_0)^{-\gamma/2-1/2}\Phi'(z), \quad \chi_{\xi\xi} = A(t+t_0)^{-\gamma/2-1}\Phi''(z) \qquad (2.11)$$

and

$$\chi_t = -\frac{1}{2}A(t+t_0)^{-\gamma/2-1} z\Phi'(z) - \frac{\gamma}{2}A(t+t_0)^{-\gamma/2-1}\Phi(z)$$

with $z := (\xi+\xi_0)(t+t_0)^{-1/2}$, the identity (2.9) is immediate from (2.1).

To verify (2.10), we observe that, since $\Phi''(0) < 0$ by (2.1), there exists $z_0 > 0$ such that

$$\Phi''(z) \leqslant 0 \quad \text{for all } z \in [0, z_0]. \tag{2.12}$$

Then (2.3) and (2.4) ensure that, with some $c_1 > 0$ and $c_2 > 0$, we have

$$\Phi'(z) \leqslant -c_1 z^{-\gamma-1} \quad \text{for all } z > z_0 \tag{2.13}$$

and

$$\Phi''(z) \leqslant c_2 z^{-\gamma-2} \quad \text{for all } z > z_0. \tag{2.14}$$

We now let $t_\star > 1$ be large enough such that

$$t_\star \geqslant \left(\frac{c_2}{(n-2)c_1 z_0}\right)^2, \tag{2.15}$$

and claim that (2.10) holds whenever $t_0 > t_\star$, $\xi_0 > 0$ and $A > 0$. Indeed, recalling (2.11), (2.12) and the monotonicity of Φ, we easily see that in the region where $z = (\xi+\xi_0)(t+t_0)^{-1/2} \leqslant z_0$, both $\chi_{\xi\xi}$ and χ_ξ are non-positive, and hence clearly $\chi_{\xi\xi} + (n-2)\chi_\xi \leqslant 0$. On the other hand, if $z > z_0$, then from (2.11), (2.13) and (2.14) it follows that

$$\frac{\chi_{\xi\xi}(\xi,t)}{-(n-2)\chi_\xi(\xi,t)} = \frac{\Phi''(z)}{-(n-2)\sqrt{t+t_0}\,\Phi'(z)} \leqslant \frac{c_2}{(n-2)c_1(\xi+\xi_0)}.$$

Since $\xi+\xi_0 > z_0\sqrt{t+t_0}$, (2.15) implies that

$$\frac{\chi_{\xi\xi}(\xi,t)}{-(n-2)\chi_\xi(\xi,t)} < \frac{c_2}{(n-2)c_1 z_0\sqrt{t+t_0}} < \frac{c_2}{(n-2)c_1 z_0 \sqrt{t_\star}} \leqslant 1$$

holds at any such point, as claimed. □

Lemma 2.3. *Let $D > 0$ and $\gamma \in (0,1)$. Then there exists $t_\star > 1$ such that, for any choice of $t_0 > t_\star$, $\xi_0 > 0$ and $A > 0$, the function $\chi^{(\xi_0,t_0,A)}$ in (2.8) satisfies*

$$\mathcal{Q}\chi^{(\xi_0,t_0,A)} \geqslant 0 \quad \textit{for all } \xi > 0 \textit{ and } t > 0, \tag{2.16}$$

where $\mathcal{Q}$ is the operator defined in (2.7).

Proof. We take $t_\star$ as given by Lemma 2.2 and assume that $t_0 > t_\star$. Then, writing $\chi := \chi^{(\xi_0,t_0,A)}$ and using (2.9) and (2.10), we obtain

$$\begin{aligned}\mathcal{Q}\chi &= -e^{-2\xi}\left\{(D+\chi)[\chi_{\xi\xi} + (n-2)\chi_\xi] - \frac{n-2}{2}\chi_\xi^2\right\} \\ &\geqslant e^{-2\xi}\frac{n-2}{2}\chi_\xi^2 \geqslant 0 \quad \text{for all } \xi > 0 \text{ and } t > 0,\end{aligned}$$

because $D + \chi \geqslant 0$ according to the non-negativity of χ asserted by Lemma 2.1, and because $n \geqslant 3$. □

The function we shall use as a super-solution near the origin (cf. (2.22) below) will have a certain self-similar structure. As a preparation, let us state the following lemma.

LEMMA 2.4. *Let $D > 0$ and $\gamma > 0$. For $\lambda := (1/D)(\gamma/2 + 1)$, let ρ denote the solution of*

$$\begin{cases} \rho''(\sigma) + \frac{1}{\sigma}\rho'(\sigma) + \lambda\rho(\sigma) = 0, \quad \sigma > 0, \\ \rho(0) = 1, \quad \rho'(0) = 0. \end{cases} \tag{2.17}$$

Then there exists $\sigma_0 \in (0,1)$ such that ρ is positive and decreasing on $[0,\sigma_0]$.

Proof. Both statements are obvious from (2.17). □

In order to match inner and outer functions appropriately, we shall need a correcting factor that is time-dependent, but approaches one in the large time limit.

LEMMA 2.5. *Given $D > 0$ and $\gamma \in (0,1)$, let Φ, ρ and σ_0 be as in Lemmas 2.1 and 2.4. Then, for $\xi_0 > 0$ and $t_0 > \sigma_0^{-2}$, the function $f^{(\xi_0,t_0)}$ defined by*

$$f^{(\xi_0,t_0)}(t) := \frac{\Phi(\xi_0(t+t_0)^{-1/2})}{\rho((t+t_0)^{-1/2})}, \quad t \geqslant 0, \tag{2.18}$$

satisfies

$$f^{(\xi_0,t_0)}(t) \longrightarrow 1 \quad \text{as } t \longrightarrow \infty. \tag{2.19}$$

Furthermore, for any $\xi_0 > 0$ there exists $C(\xi_0) > 0$ such that whenever $t_0 > 1$, we have

$$|(f^{(\xi_0,t_0)})'(t)| \leqslant \frac{C(\xi_0)}{(t+t_0)^2} \quad \text{for all } t > 0. \tag{2.20}$$

Proof. Since $\Phi(0) = \rho(0) = 1$, (2.19) is obvious. As for (2.20), we for $t > 0$ compute

$$(f^{(\xi_0,t_0)})'(t) = \frac{1}{2}(t+t_0)^{-3/2}\left(-\xi_0\frac{\Phi'(\xi_0(t+t_0)^{-1/2})}{\rho((t+t_0)^{-1/2})} + \frac{\Phi(\xi_0(t+t_0)^{-1/2})\rho'((t+t_0)^{-1/2})}{\rho^2((t+t_0)^{-1/2})}\right). \tag{2.21}$$

Since ρ is positive on $[0,\sigma_0]$ and $\Phi'(0) = \rho'(0) = 0$, we can choose $c_1 > 0, c_2 > 0$ and $c_3 > 0$ such that

$$\rho(\sigma) \geqslant c_1 \quad \text{for all } \sigma \in [0,\sigma_0]$$

as well as

$$|\Phi'(z)| \leqslant c_2 z \;\text{ for all } z \in [0,\xi_0] \quad \text{and} \quad |\rho'(\sigma)| \leqslant c_3\sigma \;\text{ for all } \sigma \in [0,\sigma_0].$$

We thereby obtain from (2.21) that, for any choice of $t_0 > \sigma_0^{-2}$, one has

$$|(f^{(\xi_0,t_0)})'(t)| \leqslant \left(\frac{c_2\xi_0^2}{2c_1} + \frac{c_3}{2c_1^2}\right)(t+t_0)^{-2} \quad \text{for all } t > 0,$$

because $\Phi \leqslant 1$ on $[0,\infty)$ by Lemma 2.1. □

We can now introduce a family of functions, one of which will serve as a super-solution in the region where $r < 1$. To this end, for $D > 0$ and $\gamma \in (0,1)$ we let ρ, σ_0 and $f^{(\xi_0,t_0)}$ as in Lemmas 2.4 and 2.5, and given $\xi_0 > 0$, $t_0 > \sigma_0^{-2}$ and $A > 0$, we define, for $r \in [0,1]$ and $t \geqslant 0$, the function

$$\varphi^{(\xi_0,t_0,A)}(r,t) := Af^{(\xi_0,t_0)}(t)(t+t_0)^{-\gamma/2}\rho(r(t+t_0)^{-1/2}). \tag{2.22}$$

We then have the following lemma.

Lemma 2.6. *Let $D > 0$ and $\gamma \in (0,1)$, and let ρ and σ_0 be as in Lemma 2.4. Then, for each $\xi_0 > 0$ there exists $t^\star > \sigma_0^{-2}$ such that, for any choice of $t_0 > t^\star$ and any $A > 0$, the function $\varphi^{(\xi_0,t_0,A)}$ given by (2.22) satisfies*

$$\mathcal{P}\varphi^{(\xi_0,t_0,A)} \geqslant 0 \quad \textit{for all } r \in (0,1) \textit{ and } t > 0. \tag{2.23}$$

Proof. Given $\xi_0 > 0$, we take $C(\xi_0)$ as provided by Lemma 2.5, and claim that (2.23) is valid whenever $t_0 > t^\star$ and

$$t^\star > \max\left\{\frac{1}{\sigma_0^2}, \frac{C(\xi_0)}{\Phi(\xi_0)}\right\}, \tag{2.24}$$

where Φ is from Lemma 2.1.

To see this, we fix any such t_0 and, writing $\varphi = \varphi^{(\xi_0,t_0,A)}$, $f = f^{(\xi_0,t_0)}$ and $\sigma = r(t+t_0)^{-1/2}$, compute

$$\varphi_r = Af(t)(t+t_0)^{-\gamma/2-1/2}\rho'(\sigma), \quad \varphi_{rr} = Af(t)(t+t_0)^{-\gamma/2-1}\rho''(\sigma) \tag{2.25}$$

and

$$\varphi_t = -\tfrac{1}{2}Af(t)(t+t_0)^{-\gamma/2-1}(\sigma\rho'(\sigma) + \gamma\rho(\sigma)) - Af'(t)(t+t_0)^{-\gamma/2}\rho(\sigma)$$

for $r \in (0,1)$ and $t > 0$. Now, since $t_0 > t^\star > \sigma_0^{-2}$, in the region where $r < 1$ and $t > 0$ we have $\sigma < t_0^{-1/2} < \sigma_0$, so that Lemma 2.4 guarantees that $\rho(\sigma) > 0$ and $\rho'(\sigma) \leqslant 0$, and hence $\rho''(\sigma) + (1/\sigma)\rho'(\sigma) = -\lambda\rho(\sigma) < 0$. In particular, if we write (2.6) as

$$\mathcal{P}\varphi = \varphi_t - (D+\varphi)\left(\varphi_{rr} + \frac{n-1}{r}\varphi_r\right) - r^2\varphi_{rr} - r\varphi_r + \frac{n-2}{2}\varphi_r^2,$$

and use (2.25), we obtain

$$\begin{aligned}
-(D+\varphi)\left(\varphi_{rr} + \frac{n-1}{r}\varphi_r\right) &= -(D+\varphi)Af(t)(t+t_0)^{-\gamma/2-1}\left(\rho''(\sigma) + \frac{n-1}{\sigma}\rho'(\sigma)\right) \\
&= -(D+\varphi)Af(t)(t+t_0)^{-\gamma/2-1}\left(-\lambda\rho(\sigma) + \frac{n-2}{\sigma}\rho'(\sigma)\right) \\
&\geqslant \lambda(D+\varphi)Af(t)(t+t_0)^{-\gamma/2-1}\rho(\sigma) \\
&\geqslant \lambda DAf(t)(t+t_0)^{-\gamma/2-1}\rho(\sigma),
\end{aligned}$$

because $n \geqslant 3$. Moreover,

$$\begin{aligned}
-r^2\varphi_{rr} - r\varphi_r &= -Af(t)(t+t_0)^{-\gamma/2-1}r^2\left(\rho''(\sigma) + \frac{1}{\sigma}\rho'(\sigma)\right) \\
&= \lambda Af(t)(t+t_0)^{-\gamma/2-1}r^2 \geqslant 0
\end{aligned}$$

for $r < 1$ and $t > 0$. Since $((n-2)/2)\varphi_r^2 \geqslant 0$, we therefore have

$$\begin{aligned}
\mathcal{P}\varphi &\geqslant \varphi_t + \lambda DAf(t)(t+t_0)^{-\gamma/2-1}\rho(\sigma) \\
&= Af(t)(t+t_0)^{-\gamma/2-1}\left\{-\frac{\sigma}{2}\rho'(\sigma) - \frac{\gamma}{2}\rho(\sigma) - \frac{f'(t)}{f(t)}(t+t_0)\rho(\sigma) + \lambda D\rho(\sigma)\right\} \\
&\quad \text{for } r \in (0,1) \text{ and } t > 0.
\end{aligned} \tag{2.26}$$

Now, the monotonicity properties of Φ and ρ imply that, since $t_0 > 1$, we obtain

$$f(t) \geqslant \frac{1}{\rho(0)}\Phi(\xi_0 t_0^{-1/2}) \geqslant \Phi(\xi_0) \quad \text{for all } t > 0,$$

so that, using (2.20), we obtain

$$\left|\frac{f'(t)}{f(t)}(t+t_0)\right| \leqslant \frac{C(\xi_0)}{\Phi(\xi_0)(t+t_0)} \leqslant \frac{C(\xi_0)}{\Phi(\xi_0)t_0} \quad \text{for all } t > 0.$$

Thus, according to the fact that $t^\star > C(\xi_0)/\Phi(\xi_0)$ by (2.24), we have

$$\left|\frac{f'(t)}{f(t)}(t+t_0)\right| \leqslant 1 \quad \text{for all } t>0.$$

Hence, (2.26) entails that

$$\begin{aligned}\mathcal{P}\varphi &\geqslant Af(t)(t+t_0)^{-\gamma/2-1}\left\{-\frac{\sigma}{2}\rho'(\sigma)+\left(\lambda D-\frac{\gamma}{2}-1\right)\rho(\sigma)\right\}\\ &= -Af(t)(t+t_0)^{-\gamma/2-1}\frac{\sigma}{2}\rho'(\sigma)\geqslant 0 \quad \text{for } r\in(0,1) \text{ and } t>0,\end{aligned}$$

because of our choice of λ in (2.17) and, again, the monotonicity of ρ on $(0,\sigma_0)$. This completes the proof. □

LEMMA 2.7. *Let $D>0$ and $\gamma\in(0,1)$. Then, with σ_0 as in Lemma 2.4, there exist $\xi_0>0$ and $t_0>\sigma_0^{-2}$ such that, for any $A>0$, the function $\bar\varphi^{(A)}$ defined by*

$$\bar\varphi^{(A)}(r,t) := \begin{cases}\varphi^{(\xi_0,t_0,A)}(r,t), & r\in[0,1],\ t\geqslant 0,\\ \chi^{(\xi_0,t_0,A)}(\ln r,t), & r>1,\ t\geqslant 0,\end{cases} \tag{2.27}$$

is continuous in $[0,\infty)^2$ and satisfies

$$\mathcal{P}\bar\varphi^{(A)}\geqslant 0 \quad \textit{for all } r\in(0,\infty)\setminus\{1\} \textit{ and } t>0, \tag{2.28}$$

where $\mathcal{P}$ is as in (2.6), and such that

$$\liminf_{r\nearrow 1}\bar\varphi_r^{(A)}(r,t) > \limsup_{r\searrow 1}\bar\varphi_r^{(A)}(r,t) \quad \textit{for all } t>0. \tag{2.29}$$

Proof. Given $D>0$ and $\gamma\in(0,1)$, we let ρ and Φ be as defined by (2.17) and (2.1). Then, since $\rho'(0)=\Phi'(0)=0$ and $\Phi''(0)=-\gamma/2<0$, we can find $c_1>0$ and $c_2>0$ fulfilling

$$\rho'(\sigma)\geqslant -c_1\sigma \quad \text{for all } \sigma\in(0,\sigma_0) \tag{2.30}$$

and

$$\Phi'(z)\leqslant -c_2 z \quad \text{for all } z\in(0,1). \tag{2.31}$$

We now first fix $\xi_0>0$ large such that

$$\xi_0 > \frac{c_1}{c_2\rho(\sigma_0)} \tag{2.32}$$

and then take $t_\star$ and $t^\star$ as provided by Lemmas 2.3 and 2.6, respectively, when applied to this particular choice of ξ_0. We finally pick some $t_0>\sigma_0^{-2}$ satisfying

$$t_0 > \max\{t_\star, t^\star, \xi_0^2\} \tag{2.33}$$

and claim that these choices ensure that $\bar\varphi^{(A)}$ is continuous, and that (2.28) and (2.29) are valid whenever $A>0$.

In fact, (2.28) is an immediate consequence of Lemmas 2.3 and 2.6, while the continuity of $\bar\varphi^{(A)}$ directly results from the definitions of $\varphi^{(\xi_0,t_0,A)}$, $\chi^{(\xi_0,t_0,A)}$ and the function $f^{(\xi_0,t_0)}$ introduced in Lemma 2.5. To verify (2.29), we recall (2.22) and (2.8) in computing

$$\begin{aligned}I_1(t) &:= \liminf_{r\nearrow 1}\bar\varphi_r^{(A)}(r,t) = \varphi_r^{(\xi_0,t_0,A)}(1,t)\\ &= Af^{(\xi_0,t_0)}(t)(t+t_0)^{-\gamma/2-1/2}\rho'((t+t_0)^{-1/2}), \quad t>0,\end{aligned} \tag{2.34}$$

and

$$I_2(t) := \limsup_{r\searrow 1} \bar{\varphi}_r^{(A)}(r,t) = \chi_\xi^{(\xi_0,t_0,A)}(0,t)$$
$$= A(t+t_0)^{-\gamma/2-1/2}\Phi'(\xi_0(t+t_0)^{-1/2}), \quad t>0. \tag{2.35}$$

Here, we note that, by (2.18) and the monotonicity of Φ and ρ,

$$f^{(\xi_0,t_0)}(t) \leqslant \Phi(0)\rho^{-1}(t_0^{-1/2}) = \rho^{-1}(t_0^{-1/2}) \leqslant \rho^{-1}(\sigma_0) \quad \text{for all } t>0, \tag{2.36}$$

because $t_0 > \sigma_0^{-2}$. Furthermore, (2.30) and (2.31) assert that

$$\rho'((t+t_0)^{-1/2}) \geqslant -c_1(t+t_0)^{-1/2} \quad \text{for all } t>0 \tag{2.37}$$

and

$$\Phi'(\xi_0(t+t_0)^{-1/2}) \leqslant -c_2\xi_0(t+t_0)^{-1/2} \quad \text{for all } t>0, \tag{2.38}$$

again since $(t+t_0)^{-1/2} < \sigma_0$, and since $\xi_0(t+t_0)^{-1/2} < 1$ due to (2.33). Using (2.36)–(2.38), we obtain from (2.34) and (2.35) that

$$I_1(t) - I_2(t) \geqslant A(t+t_0)^{-\gamma/2-1}(-c_1\rho^{-1}(\sigma_0) + c_2\xi_0) \quad \text{for all } t>0,$$

so that our requirement (2.32) guarantees that (2.29) holds. □

Lemma 2.8. *Let $D>0$. Assume that φ_0 is continuous and non-negative on $[0,\infty)$, and there exist $\gamma \in (0,1)$ and $B>0$ such that*

$$\varphi_0(r) \leqslant B\ln^{-\gamma} r \quad \textit{for all } r>2. \tag{2.39}$$

Then there exists $C>0$ such that the solution φ of (2.6) with $\varphi(\cdot,0)=\varphi_0$ satisfies

$$\varphi(r,t) \leqslant C(t+1)^{-\gamma/2} \quad \textit{for all } r\geqslant 0 \textit{ and } t \geqslant 0. \tag{2.40}$$

Proof. Given $D>0$ and $\gamma\in(0,1)$, we fix $\sigma_0\in(0,1)$, $\xi_0>0$ and $t_0>\sigma_0^{-2}$ as in Lemmas 2.4 and 2.7, and take $f=f^{(\xi_0,t_0)}$ from Lemma 2.5. In order to define, with some specific $A>0$, a super-solution of the form (2.27) which initially dominates φ, we set $z_0 := (\ln 2 + \xi_0)t_0^{-1/2}$ and then obtain from Lemma 2.1 that, for some $c_1>0$, the function Φ in (2.1) satisfies

$$\Phi(z) \geqslant c_1 z^{-\gamma} \quad \text{for all } z\geqslant z_0. \tag{2.41}$$

Moreover, since φ_0 is bounded, we can pick $c_2>0$ such that

$$\varphi_0(r) \leqslant c_2 \quad \text{for all } r\in[0,2]. \tag{2.42}$$

We now fix any $A>0$ fulfilling

$$A > \max\left\{ \frac{c_2 t_0^{\gamma/2}}{f(0)\rho(t_0^{-1/2})}, \frac{c_2 t_0^{\gamma/2}}{\Phi(z_0)}, \frac{B}{c_1}\left(1+\frac{\xi_0}{\ln 2}\right)^\gamma \right\} \tag{2.43}$$

and claim that then the function $\bar{\varphi}^{(A)}$ in (2.27) has the property

$$\bar{\varphi}^{(A)}(r,0) > \varphi_0(r) \quad \text{for all } r\geqslant 0. \tag{2.44}$$

To prove this, we first observe that, for small r, by (2.42) and (2.43) it holds that

$$\frac{\bar{\varphi}^{(A)}(r,0)}{\varphi_0(r)} \geqslant \frac{\bar{\varphi}^{(A)}(r,0)}{c_2} = \frac{1}{c_2}Af(0)t_0^{-\gamma/2}\rho(rt_0^{-1/2})$$
$$\geqslant \frac{1}{c_2}Af(0)t_0^{-\gamma/2}\rho(t_0^{-1/2}) > 1, \quad r\in[0,1],$$

because $\rho' \leqslant 0$ on $(0, \sigma_0)$ and $t_0^{-1/2} < \sigma_0$. Similarly, in the intermediate region where $1 < r \leqslant 2$, (2.42), (2.43) and the monotonicity of Φ yield

$$\frac{\bar{\varphi}^{(A)}(r,0)}{\varphi_0(r)} \geqslant \frac{1}{c_2} A t_0^{-\gamma/2} \Phi((\ln 2 + \xi_0)(t_0^{-1/2})) > 1, \quad r \in (1,2].$$

Finally, for large r we apply (2.41) to estimate

$$\begin{aligned}\bar{\varphi}^{(A)}(r,0) &= A t_0^{-\gamma/2} \Phi((\ln r + \xi_0)(t_0^{-1/2})) \geqslant c_1 A (\ln r + \xi_0)^{-\gamma} \\ &\geqslant c_1 A \left(1 + \frac{\xi_0}{\ln 2}\right)^{-\gamma} (\ln r)^{-\gamma}, \quad r > 2,\end{aligned}$$

because $\ln r + \xi_0 \leqslant \ln r + \xi_0 \ln r / \ln 2$ for such r. Along with (2.43) and (2.39), this guarantees that also

$$\bar{\varphi}^{(A)}(r,0) > \varphi_0(r), \quad r > 2.$$

Having thus found that (2.44) is true, we may invoke Lemma 2.7 combined with the comparison principle to infer that $\varphi \leqslant \bar{\varphi}^{(A)}$ in $[0,\infty)^2$. In particular, since $\rho \leqslant 1, \Phi \leqslant 1$ and $f \leqslant \rho^{-1}(t_0^{-1/2})$ by monotonicity, this means that

$$\varphi(r,t) \leqslant A f(t)(t + t_0)^{-\gamma/2} \leqslant A \rho^{-1}(t_0^{-1/2})(t + t_0)^{-\gamma/2}, \quad r \in [0,1], \ t \geqslant 0,$$

as well as

$$\varphi(r,t) \leqslant A(t + t_0)^{-\gamma/2} \quad \text{for all } r > 1 \text{ and } t \geqslant 0,$$

from which (2.40) clearly follows. □

Proof of Theorem 1.2. If we choose φ_0 satisfying (2.39) such that $\psi_0(x) \leqslant \varphi_0(|x|)$ for $x \in \mathbb{R}^n$, then we obtain by comparison that

$$(|x|^2 + D + \varphi(|x|,t))^{-(n-2)/2} \leqslant v(x,t) \leqslant V_D(x), \quad x \in \mathbb{R}^n, \ t \geqslant 0.$$

Lemma 2.8 and the Mean Value Theorem yield then the result. □

3. *Upper bound. Proof of Theorem* 1.4

LEMMA 3.1. *For* $\gamma > 0$, *let*

$$\hat{\Phi}(z) := \left(1 + \frac{z^2}{4}\right)^{-\gamma/2}, \quad z \geqslant 0. \tag{3.1}$$

Then,

$$\hat{\Phi}''(z) + \frac{z}{2}\hat{\Phi}'(z) + \frac{\gamma}{2}\hat{\Phi}(z) \geqslant \frac{\gamma}{4}\left(1 + \frac{z^2}{4}\right)^{-\gamma/2-1} \quad \textit{for all } z > 0. \tag{3.2}$$

Moreover,

$$\hat{\Phi}'(z) = -\frac{\gamma}{4} z \left(1 + \frac{z^2}{4}\right)^{-\gamma/2-1} \quad \textit{for all } z > 0 \tag{3.3}$$

and

$$|\hat{\Phi}''(z)| \leqslant \frac{\gamma(\gamma+1)}{4}\left(1 + \frac{z^2}{4}\right)^{-\gamma/2-1} \quad \textit{for all } z > 0. \tag{3.4}$$

Proof. The statements can be verified by straightforward computations. □

LEMMA 3.2. *Let $D > 0$ and $\gamma > 0$. Then there exists $\xi_0 > 1$ such that, for any choice of $a \in (0,1)$, the function $\hat{\chi}^{(a)}$ given by*

$$\hat{\chi}^{(a)}(\xi,t) := a(t+1)^{-\gamma/2}\hat{\Phi}\left(\frac{\xi-\xi_0}{\sqrt{t+1}}\right), \quad \xi \geqslant \xi_0,\ t \geqslant 0, \tag{3.5}$$

satisfies

$$\mathcal{Q}\hat{\chi}^{(a)} \leqslant 0 \quad \text{for all } \xi > \xi_0 \text{ and } t > 0, \tag{3.6}$$

where $\mathcal{Q}$ is as defined in (2.7).

Proof. We abbreviate

$$z := \frac{\xi-\xi_0}{\sqrt{t+1}}$$

and calculate

$$\hat{\chi}^{(a)}_\xi = a(t+1)^{-\gamma/2-1/2}\hat{\Phi}'(z), \quad \hat{\chi}^{(a)}_{\xi\xi} = a(t+1)^{-\gamma/2-1}\hat{\Phi}''(z)$$

and

$$\hat{\chi}^{(a)}_t = -\frac{1}{2}a(t+1)^{-\gamma/2-1}z\hat{\Phi}'(z) - \frac{\gamma}{2}a(t+1)^{-\gamma/2-1}\hat{\Phi}(z).$$

Therefore,

$$\begin{aligned}\mathcal{Q}\hat{\chi}^{(a)} = a(t+1)^{-\gamma/2-1}\Big\{ &-\frac{z}{2}\hat{\Phi}'(z) - \frac{\gamma}{2}\hat{\Phi}(z) - \hat{\Phi}''(z) \\ &- e^{-2\xi}[D + a(t+1)^{-\gamma/2}\hat{\Phi}(z)][\hat{\Phi}''(z) + (n-2)\sqrt{t+1}\hat{\Phi}'(z)] \\ &+\frac{n-2}{2}\,e^{-2\xi}(t+1)^{-\gamma/2}\hat{\Phi}'^2(z)\Big\} \quad \text{for } \xi > \xi_0 \text{ and } t > 0.\end{aligned}$$

Here, we recall (3.1) and our assumption $a < 1$ in estimating

$$|D + a(t+1)^{-\gamma/2}\hat{\Phi}(z)| \leqslant D + 1$$

and use (3.2) to see that

$$-\frac{z}{2}\hat{\Phi}'(z) - \frac{\gamma}{2}\hat{\Phi}(z) - \hat{\Phi}''(z) \leqslant -\frac{\gamma}{4}\left(1+\frac{z^2}{4}\right)^{-\gamma/2-1}$$

at any point $(\xi,t) \in (\xi_0,\infty)\times(0,\infty)$. Thus,

$$\begin{aligned}\frac{\mathcal{Q}\hat{\chi}^{(a)}}{a(t+1)^{-\gamma/2-1}} &\leqslant -\frac{\gamma}{4}\left(1+\frac{z^2}{4}\right)^{-\gamma/2-1} + (D+1)\,e^{-2\xi}|\hat{\Phi}''(z)| \\ &\quad + (n-2)\,e^{-2\xi}\sqrt{t+1}|\hat{\Phi}'(z)| + \frac{n-2}{2}\,e^{-2\xi}\hat{\Phi}'^2(z) \\ &=: -I_1 + I_2 + I_3 + I_4 \quad \text{for } \xi > \xi_0 \text{ and } t > 0,\end{aligned}$$

and we claim that this implies (3.6) if we pick $\xi_0 > 1$ large enough such that

$$(\gamma+1)(D+1)\,e^{-2\xi} < \tfrac{1}{3} \quad \text{for all } \xi > \xi_0 \tag{3.7}$$

and

$$(n-2)(D+1)(\xi-\xi_0)\,e^{-2\xi} < \tfrac{1}{3} \quad \text{for all } \xi > \xi_0 \tag{3.8}$$

as well as

$$\frac{(n-2)\gamma}{2}\,e^{-2\xi} < \frac{1}{3} \quad \text{for all } \xi > \xi_0. \tag{3.9}$$

Indeed, in conjunction with (3.4), (3.7) implies that

$$\begin{aligned}\frac{I_2}{I_1} &= \frac{4(D+1)}{\gamma}\, e^{-2\xi} \left(1+\frac{z^2}{4}\right)^{\gamma/2+1} |\hat{\Phi}''(z)| \\ &\leqslant \frac{4(D+1)}{\gamma}\, e^{-2\xi}\,\frac{\gamma(\gamma+1)}{4} = (\gamma+1)(D+1)\,e^{-2\xi} \\ &< \frac{1}{3} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0. \end{aligned} \tag{3.10}$$

Since $\sqrt{t+1} = (\xi-\xi_0)/z$, (3.3) and (3.8) next guarantee that

$$\begin{aligned}\frac{I_3}{I_1} &= \frac{4(n-2)(D+1)}{\gamma}\, e^{-2\xi}\sqrt{t+1}\left(1+\frac{z^2}{4}\right)^{\gamma/2+1} |\hat{\Phi}'(z)| \\ &= \frac{4(n-2)(D+1)}{\gamma}(\xi-\xi_0)\, e^{-2\xi}\,\frac{(1+z^2/4)^{\gamma/2+1}}{z}|\hat{\Phi}'(z)| \\ &= (n-2)(D+1)(\xi-\xi_0)\,e^{-2\xi} \\ &< \frac{1}{3} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0. \end{aligned} \tag{3.11}$$

Finally, again by (3.3),

$$\begin{aligned}\frac{I_4}{I_1} &= \frac{2(n-2)}{\gamma}\, e^{-2\xi}(t+1)^{\gamma/2+1}\hat{\Phi}'^2(z) \\ &= \frac{(n-2)\gamma}{8}\, e^{-2\xi} z^2 \left(1+\frac{z^2}{4}\right)^{-\gamma/2-1}, \end{aligned}$$

so that, since clearly $z^2(1+z^2/4)^{-\gamma/2-1} \leqslant 4$, from (3.9) we infer that

$$\frac{I_4}{I_1} \leqslant \frac{(n-2)\gamma}{2}\, e^{-2\xi} < \frac{1}{3} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0.$$

Combined with (3.10) and (3.11), this establishes (3.6). □

In view of the explicit definition (3.1) of $\hat{\Phi}$, the above function $\hat{\chi}^{(a)}$ can alternatively be written in the fully explicit form

$$\hat{\chi}^{(a)}(\xi,t) = a\left(t+1+\frac{(\xi-\xi_0)^2}{4}\right)^{-\gamma/2}, \quad \xi \geqslant \xi_0, \ \ t \geqslant 0.$$

Lemma 3.3. *Let $D > 0$ and $\gamma > 0$. Then there exists $r_0 > e$ such that, for all $a \in (0,1)$,*

$$\underline{\varphi}^{(a)}(r,t) := \begin{cases} a(t+1)^{-\gamma/2}, & r \in [0,r_0],\ t \geqslant 0, \\ \hat{\chi}^{(a)}(\ln r, t), & r > r_0,\ t \geqslant 0, \end{cases} \tag{3.12}$$

defines a continuous function $\underline{\varphi}^{(a)}$ on $[0,\infty)^2$ such that also $\underline{\varphi}^{(a)}_r$ is continuous on $[0,\infty)^2$, and such that

$$\mathcal{P}\underline{\varphi}^{(a)} \leqslant 0 \quad \textit{for all } r \in (0,\infty)\setminus\{r_0\} \textit{ and } t > 0. \tag{3.13}$$

Here, $\hat{\chi}^{(a)}$ is as defined in Lemma 3.2 with $\xi_0 := \ln r_0$, and $\mathcal{P}$ is as in (2.6).

Proof. With $\xi_0 > 1$ as provided by Lemma 3.2, we let $r_0 := e^{\xi_0} > e$ and thereupon obtain that (3.6) precisely yields $\mathcal{P}\underline{\varphi}^{(a)} \leqslant 0$ for $r > r_0$ and $t > 0$. To see the same for small r, we only need to note that clearly

$$\underline{\varphi}_r^{(a)} = \underline{\varphi}_{rr}^{(a)} \equiv 0 \quad \text{for } r < r_0 \text{ and } t > 0, \tag{3.14}$$

so that

$$\mathcal{P}\underline{\varphi}^{(a)} = \underline{\varphi}_t^{(a)} = -\frac{a\gamma}{2}(t+1)^{-\gamma/2-1} < 0 \quad \text{for } r < r_0 \text{ and } t > 0.$$

Having thus established (3.13), we are left with proving the continuity of $\underline{\varphi}_r^{(a)}$. In view of (3.14), however, this immediately follows from the observation that

$$\lim_{r\searrow r_0} \underline{\varphi}_r^{(a)}(r,t) = \frac{1}{r_0}\hat{\chi}^{(a)}(\xi_0,t) = \frac{a}{r_0}(t+1)^{-\gamma/2}\hat{\Phi}'(0) = 0 \quad \text{for all } t > 0,$$

whereby the proof is completed. □

Lemma 3.4. *Let $D > 0$. Suppose that $\varphi_0 \in C^0([0,\infty))$ is positive and such that*

$$\varphi_0(r) \geqslant b \ln^{-\gamma} r \quad \textit{for all } r > 2 \tag{3.15}$$

with some positive constants b and γ. Then there exists $c > 0$ such that the solution φ of (2.6) fulfilling $\varphi(\cdot,0) = \varphi_0$ satisfies

$$\varphi(0,t) \geqslant c(t+1)^{-\gamma/2} \quad \textit{for all } t > 0. \tag{3.16}$$

Proof. We let $r_0 > e$ be as given by Lemma 3.3. Then, since φ_0 is continuous and positive, we can find $c_1 > 0$ such that

$$\varphi_0(r) \geqslant c_1 \quad \text{for all } r \in [0, r_0], \tag{3.17}$$

and fix $a \in (0,1)$ small enough fulfilling

$$a < \min\{c_1, bc_2^{\gamma/2}\}, \tag{3.18}$$

where

$$c_2 := \min\left\{\frac{1}{16}, \frac{1}{4\xi_0^2}\right\}$$

with $\xi_0 := \ln r_0 > 1$. We claim that this choice ensures that, with $\underline{\varphi}^{(a)}$ defined by (3.12), we have

$$\varphi_0(r) \geqslant \underline{\varphi}^{(a)}(r,0) \quad \text{for all } r \geqslant 0. \tag{3.19}$$

In fact, if r is small, then by (3.17) and (3.18),

$$\varphi_0(r) \geqslant c_1 > a = \underline{\varphi}^{(a)}(r,0) \quad \text{for all } r \in [0, r_0].$$

In order to show (3.19) for large r, we observe that, by (3.12), (3.5) and (3.1),

$$\underline{\varphi}^{(a)}(r,0) = a\hat{\Phi}(\ln r - \xi_0) = a\left(1 + \frac{(\ln r - \xi_0)^2}{4}\right)^{-\gamma/2}, \quad r > r_0,$$

because $r_0 > e > 2$. Here, we estimate

$$1 + \frac{(\ln r - \xi_0)^2}{4} \geqslant \frac{(\ln r - \xi_0)^2}{4} \geqslant \frac{(\ln r)^2}{16} \quad \text{if } \ln r \geqslant 2\xi_0$$

and

$$1 + \frac{(\ln r - \xi_0)^2}{4} \geqslant 1 \geqslant \left(\frac{\ln r}{2\xi_0}\right)^2 \quad \text{if } \ln r < 2\xi_0,$$

whence, by definition of c_2, it follows that

$$\underline{\varphi}^{(a)}(r,0) \leqslant a(c_2(\ln r)^2)^{-\gamma/2} < b(\ln r)^{-\gamma} \leqslant \varphi_0(r) \quad \text{for all } r > 2.$$

We have thereby verified (3.19), which in turn, on an application of the comparison principle, entails that $\varphi \geqslant \underline{\varphi}^{(a)}$ in $[0,\infty)^2$. Evaluated at $r = 0$, this implies in particular that

$$\varphi(0,t) \geqslant \underline{\varphi}^{(a)}(0,t) = a(t+1)^{-\gamma/2} \quad \text{for all } t \geqslant 0,$$

and hence proves (3.16). □

Proof of Theorem 1.4. We choose φ_0 satisfying (3.15) such that $\psi_0(x) \geqslant \varphi_0(|x|)$ for $x \in \mathbb{R}^n$. Then, we obtain by comparison that

$$(|x|^2 + D + \varphi(|x|,t))^{-(n-2)/2} \geqslant v(x,t), \quad x \in \mathbb{R}^n, \ t \geqslant 0.$$

Lemma 3.4 and the Mean Value Theorem yield then the result. □

4. *Universal upper bound. Proof of Theorem* 1.1

LEMMA 4.1. *Let $\xi_1 \in \mathbb{R}$, and suppose that α and β are smooth functions on $(\xi_1,\infty) \times (0,\infty)$, for which there exist $k > 0$ and $K > 0$ such that*

$$k \leqslant \alpha(\xi,t) \leqslant K \quad \text{and} \quad |\beta(\xi,t)| \leqslant K \ \text{ for all } \xi > \xi_1 \text{ and } t > 0.$$

Then, for any non-negative solution

$$0 \not\equiv w \in C^{2,1}((\xi_1,\infty) \times (0,\infty)) \cap C^0([\xi_1,\infty) \times [0,\infty))$$

of

$$w_t = \alpha(\xi,t) w_{\xi\xi} + \beta(\xi,t) w_\xi, \quad \xi > \xi_1, \ t > 0,$$

one can find $c > 0$ such that

$$\sup_{\xi > \xi_1} w(\xi,t) \geqslant c(t+1)^{-1/2} \quad \text{for all } t > 0.$$

Proof. This lower bound follows from [**2**], for example. □

LEMMA 4.2. *Let $D > 0$ and assume that φ_0 is continuous and non-negative on $[0,\infty)$, $\varphi_0 \not\equiv 0$. Then there exists $c > 0$ such that the solution φ of* (2.6) *with $\varphi(\cdot,0) = \varphi_0$ satisfies*

$$\sup_{r>0} \varphi(r,t) \geqslant c(t+1)^{-1/2} \quad \text{for all } t > 0. \tag{4.1}$$

Proof. Passing to a suitable minorant of φ_0 if necessary, in view of the comparison principle we may assume that, for some $r_0 > 0$, we have $0 \not\equiv \varphi_0 \in C_0^\infty((r_0,\infty))$ with $0 \leqslant \varphi_0 \leqslant 1$. Now, conveniently rewritten in terms of $\chi(\xi,t) = \varphi(r,t)$, $\xi = \ln r$, (2.6) becomes (cf. also (2.7))

$$\begin{aligned}
\chi_t &= \chi_{\xi\xi} + e^{-2\xi}\left\{(D+\chi)[\chi_{\xi\xi} + (n-2)\chi_\xi] - \frac{n-2}{2}\chi_\xi^2\right\} \\
&= [1 + (D+\chi)\,e^{-2\xi}]\chi_{\xi\xi} + e^{-2\xi}\left[(n-2)(D+\chi) - \frac{n-2}{2}\chi_\xi\right]\chi_\xi \\
&=: \alpha(\xi,t)\chi_{\xi\xi} + \beta(\xi,t)\chi_\xi, \quad \xi \in \mathbb{R}, \ t > 0.
\end{aligned}$$

Let us next choose $\xi_0 \in \mathbb{R}$ such that $\xi_0 < \ln r_0 - 2$. Then, since $0 \leqslant \chi \leqslant 1$ in $\mathbb{R} \times (0,\infty)$, we have

$$1 \leqslant \alpha(\xi,t) \leqslant 1 + (D+1)\,e^{-2\xi_0} \quad \text{for all } \xi > \xi_0 \text{ and } t > 0.$$

Therefore, due to the fact that φ_0 is smooth with compact support, interior parabolic Schauder estimates [**8**] provide $c_1 > 0$ such that

$$|\chi_\xi(\xi,t)| \leqslant c_1 \quad \text{for all } \xi > \xi_1 \text{ and } t > 0,$$

so that

$$|\beta(\xi,t)| \leqslant e^{-2(\xi_0+1)}\left[(n-2)(D+1) + \frac{n-2}{2}c_1\right] \quad \text{for all } \xi > \xi_1 \text{ and } t > 0.$$

Since we already know that $\chi \geqslant 0$ and that $\chi(\cdot,0) \not\equiv 0$ in (ξ_0+1,∞) according to our choices of r_0 and ξ_0, we may now invoke Lemma 4.1 to conclude that there exists $c_2 > 0$ such that

$$\sup_{\xi>\xi_0+1} \chi(\xi,t) \geqslant c_2(t+1)^{-1/2} \quad \text{for all } t > 0.$$

Restated using the variable φ, this immediately yields (4.1). □

Proof of Theorem 1.1. We write the initial function v_0 as

$$v_0(x) = (|x|^2 + D + \psi_0(x))^{-(n-2)/2}, \quad x \in \mathbb{R}^n,$$

where ψ_0 is continuous and non-negative on $\mathbb{R}^n$, $\psi_0 \not\equiv 0$. We can assume, without loss of generality, that $\psi_0(0) > 0$. We choose φ_0 such that $\psi_0(x) \geqslant \varphi_0(|x|)$ for $x \in \mathbb{R}^n$ and $\varphi_0 \not\equiv 0$ is non-increasing. We then obtain by comparison that

$$(|x|^2 + D + \varphi(|x|,t))^{-(n-2)/2} \geqslant v(x,t), \quad x \in \mathbb{R}^n, \ t \geqslant 0.$$

Since $\sup_{r>0} \varphi(r,t) = \varphi(0,t)$, the result follows from Lemma 4.2 and the Mean Value Theorem. □

Acknowledgement. We are grateful to the referee for comments that have helped us improve the presentation. This work was completed while the first author was visiting the Interactive Research Center of Science, Tokyo Institute of Technology. He is grateful for their warm hospitality.

References

1. M. Abramowitz and I. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables* (Nat. Bureau of Standards, Washington, DC, 1964).

2. D. G. Aronson, 'Bounds for the fundamental solution of a parabolic equation', *Bull. Amer. Math. Soc.* 73 (1967) 890–896.

3. A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, 'Asymptotics of the fast diffusion equation via entropy estimates', *Arch. Rat. Mech. Anal.* 191 (2009) 347–385.

4. M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez, 'Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities', *Proc. Natl Acad. Sci.* 107 (2010) 16459–16464.

5. M. Bonforte, G. Grillo and J. L. Vázquez, 'Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold', *Arch. Rat. Mech. Anal.* 196 (2010) 631–680.

6. M. Fila, J. L. Vázquez and M. Winkler, 'A continuum of extinction rates for the fast diffusion equation', *Comm. Pure Appl. Anal.* 10 (2011) 1129–1147.

7. M. Fila, J. L. Vázquez, M. Winkler and E. Yanagida, 'Rate of convergence to Barenblatt profiles for the fast diffusion equation', *Arch. Rat. Mech. Anal.* 204 (2012) 599–625.

8. O. A. Ladyzenskaja, V. A. Solonnikov and N. N. Ural'ceva, *Linear and quasi-linear equations of parabolic type* (American Mathematical Society, Providence, RI, 1968).

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