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SHARP RATE OF CONVERGENCE TO BARENBLATT PROFILES FOR A CRITICAL FAST DIFFUSION EQUATION

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Abstract. We study the asymptotic behaviour near extinction of positive solutions of the Cauchy problem for the fast diffusion equation with a critical exponent. We improve a previous result on slow convergence to Barenblatt profiles.

1. Introduction.

We consider the Cauchy problem for the fast diffusion equation,

$$\begin{cases} u_\tau = \nabla \cdot (u^{m-1}\nabla u), & y \in \mathbb{R}^n,\ \tau \in (0,T), \\ u(y,0) = u_0(y) \ge 0, & y \in \mathbb{R}^n, \end{cases} \tag{1}$$

where $n \ge 3$, $T > 0$ and $m = (n-4)/(n-2)$. It is known that for $m < m_c := (n-2)/n$ all solutions with initial data in some suitable space, such as $L^p(\mathbb{R}^n)$ with $p = n(1-m)/2$, extinguish in finite time. We shall consider solutions which vanish in a finite time $\tau = T$ and study their behaviour near $\tau = T$.

For the extinction range $m < m_c$ there are (infinite-mass) solutions of the self-similar form

$$U_{D,T}(y,\tau) := \frac{1}{R(\tau)^n}\left(D + \frac{\beta(1-m)}{2}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}, \tag{2}$$

where $D \ge 0$ and

$$R(\tau) := (T-\tau)^{-\beta}, \qquad \beta := \frac{1}{n(1-m)-2} = \frac{1}{n\,(m_c - m)} > 0.$$

We will call these solutions *Barenblatt solutions.*

Many papers ([2, 3, 4, 6, 9], for example) are concerned with the convergence of solutions of (1) to the Barenblatt solutions as $\tau \to T$. More precisely, the decay rates of

$$R(\tau)^n(u(\tau,y) - U_{D,T}(y,\tau))$$

as $\tau \to T$ are discussed there when $D > 0$. The case when $D = 0$ has been considered in [7, 8, 10].

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The reasons why the critical exponent

$$m_* := \frac{n-4}{n-2} < m_c \,,$$

plays a very important role in the results of [2, 3, 4, 6, 7, 8, 9, 10] will be explained below. If $n = 3, 4$ then $m_* \le 0$ which is a case treated in some more detail in [3].

To study the asymptotic profile as $\tau \to T$, it is useful to rewrite (1) in similarity variables:

$$t := \frac{1}{\mu} \ln\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{\beta}{\mu}}\, \frac{y}{R(\tau)}\,, \quad \mu := \frac{2}{1-m},$$

with R as above, and the rescaled function

$$v(x,t) := R(\tau)^n\, u(y,\tau)$$

satisfies then the nonlinear Fokker-Planck equation

$$v_t = \nabla \cdot (v^{m-1} \nabla v) + \mu\, \nabla \cdot (x\, v), \qquad x \in \mathbb{R}^n,\, t > 0. \tag{3}$$

The Barenblatt solutions $U_{D,T}(y,\tau)$ are mapped onto *Barenblatt profiles* $V_D(x)$, which are stationary solutions of (3):

$$V_D(x) := (D + |x|^2)^{-1/(1-m)}, \quad x \in \mathbb{R}^n\,. \tag{4}$$

The convergence of solutions of (1) to $U_{D,T}$ corresponds to the stabilization of solutions of (3) to nontrivial equilibria V_D.

The critical exponent m_* has the property that the difference of two Barenblatt profiles is integrable for $m \in (m_*, m_c)$, while it is not integrable for $m \le m_*$. Furthermore, m_* is the unique value of m such that the linearization of the operator $\nabla\cdot(v^{m-1}\,\nabla v) + \mu\,\nabla\cdot(x\,v)$ around V_D (on a natural weighted L^2-space) has no spectral gap, see [3]. This is the reason why one can expect that the rate of convergence to V_D is exponential for $m \ne m_*$ and algebraic for $m = m_*$.

In [2, 3, 8, 9] one can find several sufficient conditions under which $v(\cdot,t)$ converges to v_D exponentially if $m < m_c$, $m \ne m_*$. The case $m = m_*$ was first treated in [4] by functional analytic methods. A suitable linearization of the nonlinear Fokker-Planck equation (3) was viewed as the plain heat flow on a suitable Riemannian manifold and then nonlinear stability was studied by entropy methods. Later, the case $m = m_*$ was considered in [6, 7]. One of the main results in [6] says that convergence to V_D from below cannot occur at any rate faster than $t^{-1/2}$ which is the fastest decay rate of positive solutions of the linear one-dimensional heat equation. Upper bounds of the rate of convergence to V_D were also established in [6]. More precisely, the following was shown there:

Theorem 1.1. *Let $n > 2$, $m = m_\star$ and $D > 0$. Assume that ψ_0 is a continuous nonnegative function on $\mathbb{R}^n$, $\psi_0 \not\equiv 0$. Let v be the solution of (3) with the initial condition*

$$v(x,0) = v_0(x) := \Big(|x|^2 + D + \psi_0(x)\Big)^{-\frac{n-2}{2}}, \qquad x \in \mathbb{R}^n. \tag{5}$$

If there are $B > 0$ and $\gamma \ge 1$ such that

$$\psi_0(x) \le B \ln^{-\gamma} |x|, \qquad |x| > 2, \tag{6}$$

then there is $c > 0$ and for any $\varepsilon \in (0,1)$ there exists $C_\varepsilon > 0$ such that

$$ct^{-\frac{1}{2}} \le \|V_D - v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le C_\varepsilon t^{-\frac{1-\varepsilon}{2}}, \qquad t \ge 1.$$

The aim of this paper is to show that the rate $t^{-1/2}$ indeed occurs if $\gamma > 1$.

Theorem 1.2. *Let $n > 2$, $m = m_\star$ and $D > 0$. Assume that ψ_0 is a continuous nonnegative function on $\mathbb{R}^n$, $\psi_0 \not\equiv 0$. Let v be the solution of (3) with the initial condition (5). If there are $B > 0$ and $\gamma > 1$ such that (6) holds then there are $C > c > 0$ such that*

$$ct^{-\frac{1}{2}} \le \|V_D - v(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \le Ct^{-\frac{1}{2}}, \qquad t \ge 1.$$

We prove our result by constructing a suitable supersolution. In Section 2 we give some estimates for solutions of the linear heat equation. In Sections 3 and 4 we construct suitable supersolutions in an outer and inner region, respectively. In Section 5 we prove the upper bound from Theorem 1.2.

2. **Some estimates for solutions of the one-dimensional heat equation.**

In this section we collect some properties of positive solutions χ to the Cauchy problem for the one-dimensional heat equation,

$$\begin{cases} \chi_t = \chi_{\xi\xi}, & \xi \in \mathbb{R},\ t > 0, \\ \chi(\xi,0) = \chi_0(\xi), & \xi \in \mathbb{R}, \end{cases} \tag{7}$$

where the initial data χ_0 are nonnegative and integrable on $\mathbb{R}$, with our main focus being on the particular case when

$$\chi_0(\xi) := (1 + a\xi^2)^{-\frac{\gamma}{2}}, \qquad \xi \in \mathbb{R}, \tag{8}$$

with $\gamma > 1$ and appropriate $a > 0$.

Let us first invoke a simple comparison argument to make sure that in this framework, at any positive time the spatial decay of the solution of (7) cannot be significantly faster than that of the initial data.

Lemma 2.1. *Let $a > 0$ and $\gamma > 0$, and let χ denote the solution of (7) with χ_0 given by (8). Then for all $t_0 > 0$ there exists $c(t_0) > 0$ such that*

$$\chi(\xi,t_0) \ge c(t_0)(1+\xi)^{-\gamma} \qquad \textit{for all } \xi \in \mathbb{R}. \tag{9}$$

Proof. We let

$$\underline{\chi}(\xi,t) = (1 + a\xi^2 + 2at)^{-\frac{\gamma}{2}} \qquad \text{for } \xi \in \mathbb{R} \text{ and } t \ge 0.$$

Then

$$\underline{\chi}_t(\xi,t) - \underline{\chi}_{\xi\xi}(\xi,t) = -a^2\gamma(\gamma+2)\xi^2(1 + a\xi^2 + 2at)^{-\frac{\gamma}{2}-2} \le 0$$

for $\xi \in \mathbb{R}$ and $t > 0$. Since clearly $\underline{\chi}(\xi,0) = \chi_0(\xi)$ for all $\xi \in \mathbb{R}$, the comparison principle guarantees that $\chi \ge \underline{\chi}$ on $\mathbb{R} \times (0,\infty)$, whence in particular

$$\chi(\xi,t_0) \ge \underline{\chi}(\xi,t_0) = (1 + 2at_0 + a\xi^2)^{-\frac{\gamma}{2}} \qquad \text{for all } \xi \in \mathbb{R}.$$

Writing $c_1(t_0) := \max\{a, 1 + 2at_0\}$ and estimating

$$1 + 2at_0 + a\xi^2 \le c_1(t_0) + c_1(t_0)\xi^2 \le c_1(t_0)(1+\xi^2) \qquad \text{for all } \xi \in \mathbb{R},$$

we therefore obtain (9) by choosing $c(t_0) := c_1^{-\gamma/2}(t_0)$, for instance. □

Next, for suitably small $a > 0$ in (8), another comparison argument yields nonpositivity of the term $\chi_{\xi\xi} + (n-2)\chi_\xi$ appearing in (24).

Lemma 2.2. *Let $\gamma > 1$ and $a > 0$ be such that*

$$a \le \frac{4(n-2)^2}{(\gamma+1)^2}, \tag{10}$$

and let χ_0 be as given by (8). The the solution χ of (7) satisfies

$$\chi_{\xi\xi}(\xi,t) + (n-2)\chi_\xi(\xi,t) \le 0 \qquad \text{for all } \xi > 0 \text{ and } t > 0. \tag{11}$$

Proof. From (8) we first obtain that for each $t > 0$, the function $\chi(\cdot,t)$ is symmetric with respect to $\xi = 0$ and nonincreasing for $\xi \ge 0$. In particular, this implies that $\chi_\xi(0,t) = 0$ and $\chi_{\xi\xi}(0,t) \le 0$ for all $t > 0$, so that

$$z(\xi,t) := \chi_{\xi\xi}(\xi,t) + (n-2)\chi_\xi(\xi,t), \qquad \xi \ge 0,\ t \ge 0,$$

satisfies

$$z(0,t) \le 0 \qquad \text{for all } t > 0. \tag{12}$$

Moreover, since clearly $\chi \in C^\infty(\mathbb{R} \times [0,\infty))$, we may use (8) to compute the initial distribution of z: In fact, since

$$\chi_{0\xi}(\xi) = -a\gamma\xi(1+a\xi^2)^{-\frac{\gamma}{2}-1}$$

and

$$\chi_{0\xi\xi}(\xi) = -a\gamma(1+a\xi^2)^{-\frac{\gamma}{2}-1} + a^2\gamma(\gamma+2)\xi^2(1+a\xi^2)^{-\frac{\gamma}{2}-2}$$

for all $\xi \in \mathbb{R}$, we have

$$\begin{aligned} z(\xi,0) &= \chi_{0\xi\xi}(\xi) + (n-2)\chi_{0\xi}(\xi) \\ &= a\gamma(1+a\xi^2)^{-\frac{\gamma}{2}-2}\Big\{-1-(n-2)\xi + a(\gamma+1)\xi^2 - (n-2)a\xi^3\Big\} \end{aligned} \tag{13}$$

for all $\xi > 0$. Here the positive term in brackets can be estimated using Young's inequality according to

$$a(\gamma+1)\xi^2 \le (n-2)\xi + \frac{1}{4(n-2)}[a(\gamma+1)]^2\xi^3 \qquad \text{for all } \xi > 0, \tag{14}$$

where our smallness assumption (10) on a guarantees that

$$\frac{1}{4(n-2)}[a(\gamma+1)]^2 \le (n-2)a.$$

Therefore, (13) and (14) show that

$$z(\xi,0) \le 0 \qquad \text{for all } \xi > 0, \tag{15}$$

so that since clearly $z_t = z_{\xi\xi}$ for $\xi > 0$ and $t > 0$, the comparison principle asserts that the ordering properties in (12) and (15) indeed extend to all $\xi > 0$ and $t > 0$, as claimed. □

The proof of Lemma 5.1 below will essentially rely on the fact that the solution of (7) with χ_0 as in (8) has its spatial gradient decaying in time at least as fast as specified in the following one-sided estimate. Of particular importance for us will be the circumstance that by choosing ξ_0 large we can generate an arbitrarily large factor in (16). In fact, the following statement is valid for rather general integrable initial data having some symmetry and monotonicity properties.

Lemma 2.3. *Let $\chi_0 \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ be symmetric with respect to $\xi = 0$ and decreasing on $[0,\infty)$. Then there exists $K > 0$ such that the solution of (7) satisfies*

$$\chi_\xi(\xi_0, t) \le -K\xi_0 t^{-\frac{3}{2}} \qquad \text{for all } \xi_0 \ge 2 \text{ and } t \ge \frac{\xi_0^2}{4}, \tag{16}$$

where

$$K := \frac{\chi_0(1) - \chi_0(2)}{8\sqrt{\pi} e}. \tag{17}$$

Proof. We differentiate the identity

$$\chi(\xi_0, t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(\xi_0-\xi)^2}{4t}} \chi_0(\xi) d\xi, \qquad \xi_0 \in \mathbb{R},$$

to see that

$$-\chi_\xi(\xi_0, t) = \frac{1}{4\sqrt{\pi} t^{\frac{3}{2}}} \int_{-\infty}^{\infty} (\xi_0 - \xi) e^{-\frac{(\xi_0-\xi)^2}{4t}} \chi_0(\xi) d\xi.$$

Here we split the integral on the right and substitute $z = \xi_0 - \xi$ when $\xi < \xi_0$, and $z = \xi - \xi_0$ when $\xi > \xi_0$, to obtain

$$-\chi_\xi(\xi_0, t) = \frac{1}{4\sqrt{\pi} t^{\frac{3}{2}}} \int_0^{\infty} z e^{-\frac{z^2}{4t}} \Big[\chi_0(\xi_0 - z) - \chi_0(\xi_0 + z)\Big] dz. \tag{18}$$

Now our symmetry and monotonicity assumptions on χ_0 ensure that

$$\chi_0(\xi_0 - z) - \chi_0(\xi_0 + z) \ge 0 \qquad \text{for all } \xi_0 \ge 0 \text{ and } z \ge 0; \tag{19}$$

indeed, in the case $z \le \xi_0$ we have $0 \le \xi_0 - z \le \xi_0 + z$ and hence $\chi_0(\xi_0 - z) \le \chi_0(\xi_0 + z)$ by monotonicity of χ_0 on $[0,\infty)$, whereas if $z > \xi_0$ we first use the symmetry of χ_0 to infer that $\chi_0(\xi_0 - z) = \chi_0(z - \xi_0) \le \chi_0(z + \xi_0)$, the latter inequality again relying on the nonincreasing of χ_0 on $[0,\infty)$ and the fact that $\xi_0 \ge 0$.

In view of (19), for each fixed $\xi_0 \ge 2$ we may estimate the right-hand side in (18) from below by restricting the integration interval to find that

$$-\chi_\xi(\xi_0, t) \ge \frac{1}{4\sqrt{\pi} t^{\frac{3}{2}}} \int_{\xi_0-1}^{\xi_0} z e^{-\frac{z^2}{4t}} \Big[\chi_0(\xi_0 - z) - \chi_0(\xi_0 + z)\Big] dz. \tag{20}$$

Here once more by monotonicity, we can refine (19) according to

$$\chi_0(\xi_0 - z) - \chi_0(\xi_0 + z) \ge \chi_0(1) - \chi_0(2) \qquad \text{for all } z \in [\xi_0 - 1, \xi_0], \tag{21}$$

and moreover eliminate the time variable in the integral in (20) in estimating

$$e^{-\frac{z^2}{4t}} \ge e^{-\frac{\xi_0^2}{4t}} \ge e^{-1} \qquad \text{for all } z \in [0, \xi_0] \text{ and } t \ge \frac{\xi_0^2}{4}.$$

From (20) and (21) we thus conclude that

$$-\chi_\xi(\xi_0, t) \ge \quad \tfrac{\chi_0(1)-\chi_0(2)}{4\sqrt{\pi} e t^{3/2}} \int_{\xi_0-1}^{\xi_0} z dz = \tfrac{\chi_0(1)-\chi_0(2)}{8\sqrt{\pi} e t^{3/2}} (2\xi_0 - 1).$$

Since $2\xi_0 - 1 \ge \xi_0$ for any such ξ_0, this proves (16) with K as in (17). ∎

We finally assert that nonnegativity of χ_0 ensures nonnegativity of the sum $\chi_t + \frac{1}{2t}\chi$. This information will be useful in Lemma 4.2.

Lemma 2.4. *Suppose that $\chi_0 \in C^0(\mathbb{R}) \cap L^1(\mathbb{R})$ is nonnegative. Then the solution χ of (7) satisfies*

$$\chi_t(\xi_0, t) + \frac{1}{2t}\chi(\xi_0, t) \ge 0 \qquad \text{for all } \xi_0 \in \mathbb{R} \text{ and } t > 0. \tag{22}$$

Proof. This follows from the Aronson-Bénilan inequality. We give a simple proof here for readers' convenience.

Abbreviating the Gaussian kernel as

$$G(\eta,t) := (4\pi t)^{-\frac{1}{2}} \exp\Big(-\frac{\eta^2}{4t}\Big)$$

for $\eta \in \mathbb{R}$ and $t>0$, we represent χ according to the formula

$$\chi(\xi_0,t) = \int_{-\infty}^{\infty} G(\xi-\xi_0,t)\chi_0(\xi)d\xi$$

for $\xi_0 \in \mathbb{R}$ and $t>0$. A time differentiation thereof shows that

$$\chi_t(\xi_0,t) + \frac{1}{2t}\chi(\xi_0,t) = \int_{-\infty}^{\infty} \Big\{G_t(\xi-\xi_0,t) + \frac{1}{2t}G(\xi-\xi_0,t)\Big\}\chi_0(\xi)d\xi$$

for all $\xi_0 \in \mathbb{R}$ and $t>0$. Since

$$G_t(\eta,t) + \frac{1}{2t}G(\eta,t) = \frac{1}{8\sqrt{\pi}} t^{-\frac{5}{2}} \eta^2 \, e^{-\frac{\eta^2}{4t}} > 0$$

for all $\eta \in \mathbb{R}$, $t>0$, and $\chi_0 \ge 0$ this implies (22). □

3. **Supersolution in an outer region.** For $m=m_*$ and radial solutions $v = v(r,t)$, (3) becomes

$$v_t = (v^{-\frac{2}{n-2}}v_r)_r + \frac{n-1}{r}v^{-\frac{2}{n-2}}v_r + (n-2)(rv_r+nv), \qquad r>0,\, t>0.$$

If we further transform v via

$$v(r,t) = \Big(r^2 + D + \varphi(r,t)\Big)^{-\frac{n-2}{2}}, \qquad r\ge 0,\, t\ge 0,$$

then φ satisfies for $r>0$ and $t>0$ the equation

$$\mathcal{P}\varphi := \varphi_t - (r^2+D+\varphi)\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) + (n-2)r\varphi_r + \frac{n-2}{2}\varphi_r^2 = 0. \tag{23}$$

The one-dimensional structure is then reflected in the equation

$$\chi_t = \chi_{\xi\xi} + e^{-2\xi}\Big\{(D+\chi)\big(\chi_{\xi\xi} + (n-2)\chi_\xi\big) - \frac{n-2}{2}\chi_\xi^2\Big\} \tag{24}$$

obtained upon the further transformation

$$\chi(\xi,t) := \varphi(r,t), \quad \xi = \ln r, \qquad r>0,\ t>0.$$

To begin our construction of a supersolution of (23), let us first consider the region where $r>1$ and define a family of corresponding supersolutions, yet involving the free parameters ξ_0, t_0 and A which will be fixed step by step in the sequel. The following lemma accomplishes this by using Lemma 2.2.

Lemma 3.1. *Let $\gamma>1$ and $a>0$ be such that (10) holds, and let χ denote the solution of (7) with χ_0 as given by (8). Then for any choice of $D>0$ and all $\xi_0>0$, $t_0>0$ and $A>0$, the function $\varphi_{out}^{(\xi_0,t_0,A)}$ defined by*

$$\varphi_{out}^{(\xi_0,t_0,A)}(r,t) := A\chi(\ln r + \xi_0, t+t_0), \qquad r>0,\ t\ge 0, \tag{25}$$

satisfies

$$\mathcal{P}\varphi_{out}^{(\xi_0,t_0,A)}(r,t) \ge 0 \qquad \textit{for all } r>1 \textit{ and } t>0. \tag{26}$$

Proof. We write $\varphi := \varphi_{out}^{(\xi_0,t_0,A)}$ and compute

$$\varphi_t(r,t) = A\chi_t(\ln r+\xi_0,t+t_0), \qquad \varphi_r(r,t) = \frac{A}{r}\chi_\xi(\ln r+\xi_0,t+t_0),$$
$$\varphi_{rr}(r,t) = \frac{A}{r^2}\Big\{\chi_{\xi\xi}(\ln r+\xi_0,t+t_0) - \chi_\xi(\ln r+\xi_0,t+t_0)\Big\}$$

for $r>0$ and $t>0$. Recalling that $\mu=n-2$ and omitting the arguments $\ln r+\xi_0$ and $t+t_0$, we thus obtain

$$\begin{aligned}\mathcal{P}\varphi(r,t) &= A\chi_t - (r^2+D+\varphi)\Big\{\frac{A}{r^2}(\chi_{\xi\xi}-\chi_\xi) + \frac{n-1}{r}\frac{A}{r}\chi_\xi\Big\} \\ &\quad +\mu A\chi_\xi + \frac{\mu}{2}\Big\{\frac{A}{r}\chi_\xi\Big\}^2 \\ &= A\Big\{\chi_t - \chi_{\xi\xi} - \frac{D+\varphi}{r^2}\Big[\chi_{\xi\xi}+(n-2)\chi_\xi\Big]\Big\} + \frac{n-2}{2}A^2\chi_\xi^2\end{aligned}$$

for all $r>0$ and $t>0$, where the last term is nonnegative since $n>2$. As by definition of χ we have $\chi_t=\chi_{\xi\xi}$ for all $\xi\in\mathbb{R}$ and $t>0$, and since Lemma 2.2 warrants that

$$\chi_{\xi\xi}(\ln r+\xi_0,t+t_0) + (n-2)\chi_\xi(\ln r+\xi_0,t+t_0) \le 0$$

for all $r>1$ and $t>0$ because we assume that $\xi_0>0$. □

4. **Supersolution in an inner region.** In the corresponding inner region where $r<1$, our supersolution will essentially be of self-similar structure. In order to warrant compatibility with the above outer supersolution at the matching boundary $r=1$, similar to the procedure in [6] we shall introduce a correcting factor $f^{(\xi_0,t_0)} = f^{(\xi_0,t_0)}(t)$ which at each fixed $t\ge 0$ is adjusted properly so as to yield continuity of the composed global supersolution. Specifically, our inner supersolution will be of the form

$$\varphi_{in}^{(\xi_0,t_0,A)}(r,t) := Af^{(\xi_0,t_0)}(t)(t+t_0)^{-\frac12}\rho\Big(r(t+t_0)^{-\frac12}\Big), \qquad r\in[0,1],\ t\ge 0, \quad (27)$$

with $\xi_0 >, t_0>0$ and $A>0$ to be fixed below. Here our choice of the profile function ρ is described in the following lemma containing an evident observation which is essentially the same as formulated in [6, Lemma 2.4].

Lemma 4.1. *For $D>0$ and $\gamma>0$, let ρ denote the solution of*

$$\begin{cases} \rho''(\sigma) + \frac{1}{\sigma}\rho'(\sigma) + \frac{1}{D}\rho(\sigma) = 0, & \sigma>0, \\ \rho(0)=1, \quad \rho'(0)=0. \end{cases} \quad (28)$$

Then there exists $\sigma_0\in(0,1)$ such that $\rho>0$ on $[0,\sigma_0]$ and $\rho'<0$ on $(0,\sigma_0]$.

Now our correcting factor $f^{(\xi_0,t_0)}$ is defined and characterized in the following lemma.

Lemma 4.2. *Let $\gamma>1$ and $a>0$ be such that (10) holds, and let χ denote the solution of (7) with χ_0 given by (8). Moreover, let $D>0$ and ρ and σ_0 be as in Lemma 4.1. Then the function $f^{(\xi_0,t_0)}$, as defined for $\xi_0>0$ and $t_0>\sigma_0^{-2}$ by setting*

$$f^{(\xi_0,t_0)}(t) := \frac{(t+t_0)^{\frac12}\chi(\xi_0,t+t_0)}{\rho\Big((t+t_0)^{-\frac12}\Big)}, \qquad t\ge 0, \quad (29)$$

has the following properties:

i) For all $\xi_0 > 0$ one can find $k_0(\xi_0) > 0$ such that whenever $t_0 > 0$, we have

$$f^{(\xi_0,t_0)}(t) \ge k_0(\xi_0) \qquad \text{for all } t \ge 0. \tag{30}$$

ii) There exists $k_1 > 0$ such that

$$f^{(\xi_0,t_0)}(t) \le k_1 \qquad \text{for all } t \ge 0 \tag{31}$$

holds for any choice of ξ_0 and $t_0 > 0$.

iii) There exists $C > 0$ such that for each $\xi_0 > 0$ and $t_0 > 0$ we have the one-sided estimate

$$(f^{(\xi_0,t_0)})'(t) \ge -\frac{C}{(t+t_0)^2} \qquad \text{for all } t > 0. \tag{32}$$

Proof. i) Since $\chi_0 \in L^1(\mathbb{R})$ due to the fact that $\gamma > 1$, according to well-known decay properties of solutions to the one-dimensional heat equation we can fix $c_1 > 0$ such that

$$\chi(\xi,s) \le c_1 s^{-\frac{1}{2}} \qquad \text{for all } \xi \in \mathbb{R} \text{ and } s > 0. \tag{33}$$

Moreover, writing $c_2 := \rho(\sigma_0) > 0$, we know from the monotonicity of ρ on $[0,\sigma_0]$ that

$$\rho(\sigma) \ge c_2 \qquad \text{for all } \sigma \in [0\sigma_0]. \tag{34}$$

Therefore,

$$f^{(\xi_0,t_0)}(t) \le \frac{c_1}{c_2} \qquad \text{for all } t > 0,$$

because $t_0 \ge \sigma_0^{-2}$.

ii) Using that χ_0 is positive on $\mathbb{R}$, for given $\xi_0 > 0$ a standard argument based on the positivity of the Gaussian heat kernel provides $c_3(\xi_0) > 0$ such that

$$\chi(\xi_0,s) \ge c_3 s^{-\frac{1}{2}} \qquad \text{for all } s \ge 1.$$

Again by monotonicity of ρ, this implies the lower estimate

$$f^{(\xi_0,t_0)}(t) \ge c_3(\xi_0) \qquad \text{for all } t > 0,$$

because $\rho(0) = 1$ and $t_0 \ge \sigma_0^{-2} \ge 1$.

iii) By differentiation,

$$\begin{aligned}(f^{(\xi_0,t_0)})'(t) \quad &= \quad \frac{(t+t_0)^{\frac{1}{2}}}{\rho\big((t+t_0)^{-\frac{1}{2}}\big)}\Big\{\chi_t(\xi_0,t+t_0) + \frac{1}{2(t+t_0)}\chi(\xi_0,t+t_0)\Big\} \\ &\quad + \frac{\chi(\xi_0,t+t_0)\rho'\big((t+t_0)^{-\frac{1}{2}}\big)}{2(t+t_0)\rho^2\big((t+t_0)^{-\frac{1}{2}}\big)} \qquad \text{for all } t > 0,\end{aligned} \tag{35}$$

where Lemma 2.4 asserts that

$$\chi_t(\xi_0,t+t_0) + \frac{1}{2(t+t_0)}\chi(\xi_0,t+t_0) \ge 0 \qquad \text{for all } t > 0. \tag{36}$$

In order to estimate the last term in (35), we use the fact that $\rho'(0) = 0$ to find $c_4 > 0$ such that

$$|\rho'(\sigma)| \le c_4\sigma \qquad \text{for all } \sigma \in [0,\sigma_0].$$

Again using (33), (34) and our restriction $t_0 \ge \sigma_0^{-2}$, we therefore obtain

$$\left| \frac{\chi(\xi_0, t+t_0)\rho'\big((t+t_0)^{-\frac{1}{2}}\big)}{2(t+t_0)\rho^2\big((t+t_0)^{-\frac{1}{2}}\big)} \right| \le \frac{c_1 c_4}{2c_2^2(t+t_0)^2} \qquad \text{for all } t>0. \tag{37}$$

Combining (35) with (36) and (37) thus establishes (32). □

After these preparations, we can now verify that the function defined in (27) indeed has the desired supersolution property if the parameter t_0 therein is chosen suitably large.

Lemma 4.3. *Let $D>0$ and $\gamma>1$, and let ρ and σ_0 be as introduced in Lemma 4.1. Then for each $\xi_0>0$ one can find $t_\star = t_\star(\xi_0) > \sigma_0^{-2}$ with the property that whenever $t_0 > t_\star$ and $A>0$, the function $\varphi_{in}^{(\xi_0,t_0,A)}$ defined by (27), with $f^{(\xi_0,t_0)}$ given by (29), satisfies*

$$\mathcal{P}\varphi_{in}^{(\xi_0,t_0,A)}(r,t) \ge 0 \qquad \textit{for all } r\in(0,1) \textit{ and } t>0. \tag{38}$$

Proof. We fix $\xi_0>0$ and let $C>0$ and $k_0(\xi_0)$ be as provided by Lemma 4.2. We moreover pick $t_\star > \sigma_0^{-2}$ such that

$$t_\star \ge \frac{C}{2k_0(\xi_0)}, \tag{39}$$

and let $t_0 > t_\star$ and $A>0$. Then for $\varphi = \varphi_{in}^{(\xi_0,t_0,A)}$ as given by (27), writing $f = f^{(\xi_0,t_0)}$ and $\sigma = \sigma(r,t) = r(t+t_0)^{-1/2}$ we compute

$$\varphi_t(r,t) = Af(t)(t+t_0)^{-\frac{3}{2}}\left\{ -\frac{\sigma}{2}\rho'(\sigma) - \frac{1}{2}\rho(\sigma) + \frac{f'(t)}{f(t)}(t+t_0)\rho(\sigma)\right\}$$

and

$$\varphi_r(r,t) = Af(t)(t+t_0)^{-1}\rho'(\sigma)$$

as well as

$$\varphi_{rr}(r,t) = Af(t)(t+t_0)^{-\frac{3}{2}}\rho''(\sigma)$$

for $r\in(0,1)$ and $t>0$. Since $\mu = n-2>0$, we thus obtain

$$\begin{aligned}\mathcal{P}\varphi(r,t) \quad = \quad & Af(t)(t+t_0)^{-\frac{3}{2}}\bigg\{ -\frac{\sigma}{2}\rho'(\sigma) - \frac{1}{2}\rho(\sigma) + \frac{f'(t)}{f(t)}(t+t_0)\rho(\sigma) \\ & -(r^2+D+\varphi)\Big(\rho''(\sigma) + \frac{n-1}{\sigma}\rho'(\sigma)\Big) + (n-2)r^2\frac{1}{\sigma}\rho'(\sigma)\bigg\} \\ & + \frac{n-2}{2}\varphi_r^2(r,t) \qquad \text{for all } r\in(0,1) \text{ and } t>0. \end{aligned} \tag{40}$$

Here in the sum of the terms in the second line of (40) we recall the identity

$$\rho''(\sigma) = -\frac{1}{\sigma}\rho'(\sigma) - \frac{1}{D}\rho(\sigma)$$

for $\sigma \in (0, \sigma_0)$ to see that with $\sigma = \sigma(r,t)$ we have

$$\begin{aligned} -(r^2+D+\varphi)\Big(\rho''(\sigma)+\frac{n-1}{\sigma}\rho'(\sigma)\Big)+(n-2)r^2\frac{1}{\sigma}\rho'(\sigma) \\ &= -(r^2+D+\varphi)\Big(-\frac{2}{D}\rho(\sigma)+\frac{n-2}{\sigma}\rho'(\sigma)\Big)+(n-2)r^2\frac{1}{\sigma}\rho'(\sigma) \\ &= \frac{2(r^2+D+\varphi)}{D}\rho(\sigma)-\frac{(n-2)(D+\varphi)}{\sigma}\rho'(\sigma) \\ &\geq \rho(\sigma) \qquad \text{for all } r\in(0,1) \text{ and } t>0, \end{aligned} \tag{41}$$

because according to the fact that $t_\star > \sigma_0^{-2}$, for any such r and t we have $\sigma(r,t) = r(t+t_0)^{-1/2} < t_0^{-1/2} < t_\star^{-1/2} < \sigma_0$ and hence $\rho(\sigma) \geq 0$ and $\rho'(\sigma) \leq 0$ by Lemma 4.1.

Moreover, in the first line in (40) we can use Lemma 4.2 i) and iii) along with our restriction (39) on $t_\star$ to estimate

$$\begin{aligned} \frac{f'(t)}{f(t)}(t+t_0)\rho(\sigma) &\geq -\frac{C}{k_0(\xi_0)(t+t_0)}\rho(\sigma) \geq -\frac{C}{k_0(\xi_0)t_\star}\rho(\sigma) \\ &\geq -\frac{1}{2}\rho(\sigma) \qquad \text{for all } r\in(0,1) \text{ and } t>0. \end{aligned} \tag{42}$$

Once more since $\rho' \leq 0$ on $(0,\sigma_0)$, upon dropping two nonnegative terms in (40) we infer using (41) and (42) that

$$\mathcal{P}\varphi(r,t) \geq \frac{1}{2}Af(t)(t+t_0)^{-\frac{3}{2}}\rho(\sigma) \geq 0 \qquad \text{for all } r\in(0,1) \text{ and } t>0,$$

as desired. ☐

5. **Construction of a global supersolution.** We next glue together the above inner and outer functions in order to obtain a globally defined supersolution of (23) in the Nagumo sense. To accomplish this, we need to fix ξ_0 and $t_0 > 0$ conveniently large, but the parameter A is still at our disposal.

Lemma 5.1. *Let $D > 0$, and suppose that χ solves (7) with χ_0 given by (8) for some $\gamma > 1$ and some $a > 0$ complying with (10). Then with σ_0 as in Lemma 4.1, there exist $\xi_0 > 0$ and $t_0 > \sigma_0^{-2}$ such that for any choice of $A > 0$, the function $\varphi^{(A)}$ defined by*

$$\varphi^{(A)}(r,t) := \begin{cases} \varphi_{in}^{(\xi_0,t_0,A)}(r,t), & r\in[0,1],\ t\geq 0, \\ \varphi_{out}^{(\xi_0,t_0,A)}(r,t), & r>1,\ t\geq 0, \end{cases} \tag{43}$$

with $\varphi_{in}^{(\xi_0,t_0,A)}$ and $\varphi_{out}^{(\xi_0,t_0,A)}$ as given by (27) and (25), respectively, is continuous for $r \geq 0$ and $t \geq 0$ and satisfies

$$\mathcal{P}\varphi^{(A)}(r,t) \geq 0 \qquad \textit{for all } r\in(0,\infty)\setminus\{1\} \textit{ and } t>0 \tag{44}$$

as well as

$$\liminf_{r\nearrow 1}\varphi_r^{(A)}(r,t) > \limsup_{r\searrow 1}\varphi_r^{(A)}(r,t) \qquad \textit{for all } t>0. \tag{45}$$

Proof. We let $k_1 > 0$ and $K > 0$ be as found in Lemma 4.2 ii) and Lemma 2.3, respectively. Furthermore, taking ρ from Lemma 4.1, from the fact that $\rho'(0) = 0$ we obtain the existence of $c_1 > 0$ such that

$$\rho'(\sigma) \leq -c_1\sigma \qquad \text{for all } \sigma\in(0,\sigma_0). \tag{46}$$

We thereupon fix $\xi_0 > 0$ large enough fulfilling

$$\xi_0 > \frac{c_1 k_1}{K}, \tag{47}$$

and after that apply Lemma 4.3 to find $t_\star = t_\star(\xi_0) > \sigma_0^{-2}$ with the properties listed there. We finally choose any

$$t_0 > \max\left\{t_\star(\xi_0)\,,\,\frac{\xi_0^2}{4}\right\} \tag{48}$$

and let $\varphi^{(A)}$ be given by (43) with arbitrary $A > 0$.

It is then immediate from Lemma 4.3 and Lemma 3.1 that the parabolic inequality in (44) is valid both for $(r,t) \in (0,1) \times (0,\infty)$ and for $(r,t) \in (1,\infty) \times (0,\infty)$, so that it remains to verify (45). For this purpose, we first evaluate

$$\liminf_{r \nearrow 1} \varphi_{in,r}^{(\xi_0,t_0,A)}(r,t) = Af(t)(t+t_0)^{-1}\rho'\big((t+t_0)^{-\frac{1}{2}}\big) \qquad \text{for all } t > 0$$

and

$$\limsup_{r \searrow 1} \varphi_{out,r}^{(\xi_0,t_0,A)}(r,t) = A\chi_\xi(\xi_0, t+t_0) \qquad \text{for all } t > 0.$$

Here due to the fact that $(t+t_0)^{-1/2} < t_0^{-1/2} < t_\star^{-1/2}(\xi_0) < \sigma_0$ for $t > 0$ by (48), we may use (46) which in conjunction with Lemma 4.2 ii) implies that

$$\liminf_{r \nearrow 1} \varphi_{in,r}^{(\xi_0,t_0,A)}(r,t) \ge -Ak_1c_1(t+t_0)^{-\frac{3}{2}} \qquad \text{for all } t > 0. \tag{49}$$

On the other hand, (48) moreover warrants that for $t > 0$ we have $t+t_0 > t_0 > \xi_0^2/4$, whence Lemma 2.3 applies to show that

$$\limsup_{r \nearrow 1} \varphi_{out,r}^{(\xi_0,t_0,A)}(r,t) \le -AK\xi_0(t+t_0)^{-\frac{3}{2}} \qquad \text{for all } t > 0. \tag{50}$$

In light of our restriction (47) on ξ_0, combining (49) with (50) yields (45). $\square$

Finally, by choosing A appropriately large we can achieve that $\varphi^{(A)}$ initially dominates any function φ_0 with the decay as in Theorem 1.2.

Lemma 5.2. *Let $D > 0$. Suppose that φ_0 is continuous on $[0,\infty)$ and such that we have*

$$\varphi_0(r) \le B(\ln r)^{-\gamma} \qquad \text{for all } r > 2 \tag{51}$$

with some $\gamma > 1$ and $B > 0$. Then there exists $C > 0$ such that the solution φ of (23) emanating from $\varphi(\cdot,0) = \varphi_0$ satisfies

$$\varphi(r,t) \le C(t+1)^{-\frac{1}{2}} \qquad \text{for all } r \ge 0 \text{ and } t \ge 0. \tag{52}$$

Proof. We pick any $a > 0$ such that (10) holds, and let χ solve (7) with χ_0 given by (8). Moreover, we fix ρ and $\sigma_0 \in (0,1)$ as in Lemma 4.1, take $\xi_0 > 0$ and $t_0 > \sigma_0^{-2}$ as provided by Lemma 5.1, and let $f = f^{(\xi_0,t_0)}$ be as introduced in Lemma 4.2. Then Lemma 2.1 guarantees that for some $c_1(t_0) > 0$ we have

$$\chi(\xi,t_0) \ge c_1(t_0)(1+\xi)^{-\gamma} \qquad \text{for all } \xi \in \mathbb{R}, \tag{53}$$

whereas by continuity of φ_0 we obtain $c_2 > 0$ such that

$$\varphi_0(r) \le c_2 \qquad \text{for all } r \in [0,2]. \tag{54}$$

We claim that if we fix $A > 0$ large enough fulfilling

$$A \ge \max\left\{\frac{Bc_3^\gamma}{c_1(t_0)}\,,\,\frac{c_2}{\chi(\ln 2 + \xi_0, t_0)}\right\} \tag{55}$$

with $c_3 := (1 + \ln 2 + \xi_0)/\ln 2$, then the function $\varphi^{(A)}$ defined in (43) satisfies

$$\varphi^{(A)}(r,0) \ge \varphi_0(r) \qquad \text{for all } r \ge 0. \tag{56}$$

To verify this, we first consider the case $r > 2$, in which from (43) and (53) we know that

$$\varphi^{(A)}(r,0) = \varphi_{out}^{(\xi_0,t_0,A)}(r,0) = A\chi(\ln r + \xi_0, t_0) \ge Ac_1(t_0)(1 + \ln r + \xi_0)^{-\gamma}.$$

Since

$$\frac{1 + \ln r + \xi_0}{\ln r} \le 1 + \frac{1 + \xi_0}{\ln 2} = c_3 \qquad \text{for all } r > 2,$$

this entails that

$$\varphi^{(A)}(r,0) \ge Ac_1(t_0)(c_3 \ln r)^{-\gamma} \ge B(\ln r)^{-\gamma} \ge \varphi_0(r) \qquad \text{for all } r > 2 \tag{57}$$

because of the first restriction on A implied by (55). We next observe that since ρ is nonincreasing on $(0, \sigma_0)$ and $t_0 > \sigma_0^{-2}$, and since χ is nonincreasing with respect to $\xi \in (0,\infty)$ by the maximum principle, for each $t > 0$ we know that

$$\varphi^{(A)}(\cdot, t) \quad \text{is nonincreasing on } [0,\infty). \tag{58}$$

In particular, by (43) we thus have

$$\begin{aligned} \varphi^{(A)}(r,0) &\ge \varphi^{(A)}(2,0) = \varphi_{out}^{(\xi_0,t_0,A)}(2,0) = A\chi(\ln 2 + \xi_0, t_0) \\ &\ge c_2 \ge \varphi_0(r) \qquad \text{for all } r \in [0,2], \end{aligned}$$

which combined with (57) establishes (56).
By parabolic comparison based on Lemma 5.1, we hence conclude that $\varphi^{(A)}(r,t) \ge \varphi(r,t)$ for all $r \ge 0$ and $t \ge 0$, and that therefore, again in view of (58),

$$\varphi(r,t) \le \varphi^{(A)}(0,t) = \varphi_{in}^{(\xi_0,t_0,A)}(0,t) = Af^{(\xi_0,t_0)}(t)(t+t_0)^{-\frac{1}{2}}\rho(0)$$

for all $r \ge 0$ and $t \ge 0$. With k_1 as in Lemma 4.2, since $\rho(0) = 1$ we thus infer that

$$\varphi(r,t) \ge Ak_1(t+t_0)^{-\frac{1}{2}} \qquad \text{for all } r \ge 0 \text{ and } t \ge 0,$$

which readily yields (52). □

Proof of the upper bound from Theorem 1.2. If we choose a continuous function φ_0 satisfying (51) such that $\psi_0(x) \le \varphi_0(|x|)$ for $x \in \mathbb{R}^n$, we obtain by comparison that

$$\big(|x|^2 + D + \varphi(|x|,t)\big)^{-\frac{n-2}{2}} \le v(x,t) \le V_D(x), \qquad x \in \mathbb{R}^n, \quad t \ge 0.$$

Lemma 5.2 and the Mean Value Theorem yield then the result. □

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