A degenerate fourth-order parabolic equation modeling Bose-Einstein condensation. Part I: Local existence of solutions

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Abstract

A degenerate fourth-order parabolic equation modeling condensation phenomena related to Bose-Einstein particles is analyzed. The model is a Fokker-Planck-type approximation of the Boltzmann-Nordheim equation, only keeping the leading order term. It maintains some of the main features of the kinetic model, namely mass and energy conservation and condensation at zero energy. The existence of a local-in-time nonnegative continuous weak solution is proven. If the solution is not global, it blows up with respect to the L^∞ norm in finite time. The proof is based on approximation arguments, interpolation inequalities in weighted Sobolev spaces, and suitable a priori estimates for a weighted gradient L^2 norm.

Key words: Degenerate parabolic equation, fourth-order parabolic equation, existence of weak solutions, Bose-Einstein condensation, weighted spaces.

MSC 2010: 35K35, 35K65, 35B09, 35Q40.

1 Introduction

The dynamics of weakly interacting quantum particles like bosons can be described by the homogeneous Boltzmann-Nordheim equation for the distribution function $f(x,t)$ depending on the energy $x = x_1 \geq 0$ and time $t > 0$ [17],

$$\begin{aligned} f_t(x_1,t) &= \frac{1}{\sqrt{x_1}} \int_D S[f(x_3,t)f(x_4,t)(1+f(x_1,t))(1+f(x_2,t)) \\ &\quad - f(x_1,t)f(x_2,t)(1+f(x_3,t))(1+f(x_4,t))]dx_3dx_4, \end{aligned} \tag{1.1}$$

where $x_2 = x_3 + x_4 - x_1$, $D = \{x_3 + x_4 > x_1\}$, and the transition rate S in the energy space depends on $x_1, \ldots, x_4$. Nordheim [23] proposed 1928 a Boltzmann-like quantum kinetic theory for Bosons and

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Fermions, describing the dynamics of the momentum distribution. Equation (1.1) holds under the simplifying assumptions that the Boson fluid is spatially homogeneous, its momentum distribution is isotropic, and the Bose particles interact only through s-wave scattering [17, 25].

The main feature of (1.1) is the existence of finite-time blow-up solutions if the initial density is sufficiently dense, modeling the condensation process [12]. The post-nucleation self-similar solution was investigated in detail by Spohn [25]. Due to the high complexity of the Boltzmann-Nordheim equation, approximate Fokker-Planck-type equations modeling condensation phenomena related to Bose-Einstein particles were studied in the literature.

For instance, if the energy exchange of each collision is small, the Fokker-Planck approximation of the Nordheim equation in the non-relativistic regime leads to the so-called Kompaneets equation [22]. It was originally suggested to describe the evolution of a homogeneous plasma when radiation interacts with matter via Compton scattering. Escobedo et al. [11] showed that this equation develops singularites at zero energy.

Another Fokker-Planck model was studied by Kaniadakis and Quarati [20, 21], proposing a nonlinear correction to the linear drift term to account for the presence of quantum indistinguishable particles (bosons and fermions). The model was derived in [1] from a Boltzmann Bose-Einstein model in the crazing collision limit. Toscani [26] proved that the limit equation possesses global-in-time solutions if the initial mass is sufficiently small and the solutions blow up in finite time if the initial mass is large enough.

A Fokker-Planck-type equation, only containing the superlinear drift term, was analyzed recently by Carrillo et al. [9]. The existence of a unique measure-valued solution, which concentrates the mass at the origin, was proven. Moreover, all mass concentrates in the long-time limit $t \to \infty$.

All these Fokker-Planck equations are of first or second order. A *higher-order* Fokker-Planck approximation of the Boltzmann-Nordheim equation was motivated by Josserand et al. [17]. This model is the subject of this paper. Assuming that the main contribution to the collision operator on the right-hand side of (1.1) comes from the neighborhood of $x \approx x_1 \approx x_2 \approx x_3 \approx x_4$, the integrand of the collision operator can be expanded to second order, leading to the fourth-order parabolic equation

$$u_t = x^{-1/2}\Big(x^{13/2}(u^4(u^{-1})_{xx} - u^2(\log u)_{xx})\Big)_{xx}, \quad x \in (0,\infty),\ t > 0, \tag{1.2}$$

where $u(x,t)$ denotes the energy distribution. This approximation maintains some of the features of the original Boltzmann equation. Indeed, assuming no-flux-type boundary conditions at $x = 0$ and $x \to \infty$, this equation conserves the total mass $N = \int_0^\infty x^{1/2}u dx$ and the kinetic energy $E = \int_0^\infty x^{3/2}u dx$. Furthermore, the entropy $S = \int_0^\infty((1+u)\log(1+u) - u\log u)x^{1/2}dx$ is nondecreasing, and the equilibrium is reached at the Bose-Einstein distribution $u = (e^{(x-\mu)/T} - 1)^{-1}$, where μ and T are some parameters [17].

We expect that the local approximation (1.2) contains the relevant information on the finite-time collapse of the distribution function. For such a study, it is reasonable to keep only the leading-order cubic term in (1.2). Furthermore, we restrict ourselves to the finite energy interval $(0, L)$ for an arbitrarily large $L > 0$ to avoid some technicalities due to infinite domains. In fact, some of our estimates will depend on the interval length $L > 0$, and it is not clear if the limit $L \to \infty$ can be taken. However, the probability of finding particles with extremely large energy will be arbitrarily low, and we believe that the restriction to a finite energy interval is not essential from a modeling view point. (It is important from a view point of mathematical analysis.) Because of the condensation at energy $x = 0$, we expect that the density essentially vanishes at large energy values which makes Neumann-type boundary conditions at $x = L$ plausible.

More precisely, in this paper we shall subsequently consider the slightly generalized problem given by

$$\begin{cases} u_t = x^{-\beta}\Big(x^\alpha u^{n+2}(u^{-1})_{xx}\Big)_{xx}, & x \in \Omega,\ t > 0, \\ x^\alpha u^{n+2}(u^{-1})_{xx} = \Big(x^\alpha u^{n+2}(u^{-1})_{xx}\Big)_{xx} = 0, & x = 0,\ t > 0, \\ u_x = u_{xxx} = 0, & x = L,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where $\alpha \geq 0$, $\beta \in \mathbb{R}$, $n > 0$, and $\Omega = (0, L) \subset \mathbb{R}$, with a given nonnegative function u_0.
The boundary conditions at $x = 0$ correspond to those imposed in [17, Formulas (13)-(14)]. In the original equation, we have $\alpha = 13/2$, $\beta = 1/2$, and $n = 2$. The approximate equation in (1.3) still conserves mass and energy. Moreover, it admits the stationary solutions $u(x) = x^{-\sigma}$ with $\sigma \in \{0, 1, \frac{7}{6}, \frac{3}{2}\}$, containing the same Kolmogorov-Zkharov spectra as the full Boltzmann-Nordheim equation [17, Section 3.3]. This indicates that there is condensation at zero energy $x = 0$.

From a mathematical point of view, significant challenges for the analysis stem from the fact that the parabolic equation in (1.3) degenerates both at $u = 0$ and at $x = 0$; accordingly, the literature does not yet provide any result for this equation, except for the heuristic study on self-similar solutions in [17]. It will turn out that this double degeneracy drastically distinguishes the solution behavior in (1.3) from that in related well-studied degenerate fourth-order parabolic equations such as the thin-film equation $u_t + (u^n u_{xxx})_x = 0$ [3, 2, 10]. Whereas e.g. the Neumann problem for the latter equation always possesses a globally defined continuous weak solution which remains bounded [5, 6], we shall see in the forthcoming paper [19] that the particular interplay of degeneracies in (1.3) can enforce solutions to blow up with respect to their spatial norm in $L^\infty(\Omega)$ within finite time. More generally, quite various types of higher-order diffusion equations such as e.g. the quantum diffusion or Derrida-Lebowitz-Speer-Spohn equation [15, 18], equations of epitaxial thin-film growth [27], or also some nonlinear sixth-order equations [8, 13, 24] have recently attracted considerable interest. To the best of our knowledge, however, such effects of spontaneous singularity formation, only due to a pure diffusion mechanism without any presence of external forces, have not been detected in any of these examples.

Against this background, the furthest conceivable outcome of any existence theory can only address *local* solvability. The goal of the present work is to establish an essentially optimal result in this direction, asserting local existence of a continuous weak solution u that conserves mass and that can be extended up to a maximal existence time $T_{max} \in (0, \infty]$ at which $\|u(\cdot, t)\|_{L^\infty(\Omega)}$ must blow up whenever $T_{max} < \infty$.

Before we state our main result, we introduce some notation. We define for $\gamma \in \mathbb{R}$ the weighted Sobolev space

$$W^{1,2}_\gamma(\Omega) = \{v \in W^{1,2}_{\mathrm{loc}}(\Omega) : \|v\|_{L^2(\Omega)} + \|x^{\gamma/2} v_x\|_{L^2(\Omega)} < \infty\}$$

with norm $\|v\|_\gamma = (\|v\|^2_{L^2(\Omega)} + \|x^{\gamma/2}v_x\|_{L^2(\Omega)})^{1/2}$. We denote by χ_Q the characteristic function on the set $Q \subset \mathbb{R}^n$. The space $C^{4,1}(\bar{\Omega} \times (0,T))$ consists of all functions u such that u_{xxxx} and u_t exist and are continuous on $\bar{\Omega} \times (0,T)$. Furthermore, for any (not necessarily open) subset $Q \subset \mathbb{R}^n$, $C_0^\infty(Q)$ is the space of all functions such that $\mathrm{supp}(f) \subset Q$ is compact.

Definition 1.1 *Let n, α, $\beta \in \mathbb{R}$, and $T > 0$, and suppose that $u_0 \in C^0(\bar{\Omega})$ is nonnegative. Then by a* continuous weak solution *of (1.3) in $\Omega \times (0,T)$ we mean a nonnegative function $u \in C^0(\bar{\Omega} \times [0,T))$ with the properties $u \in C^{4,1}(((0,L] \times (0,T)) \cap \{u > 0\})$ as well as*

$$\chi_{\{u>0\}} x^\alpha u^n u_{xx} \in L^1_{loc}(\bar{\Omega} \times [0,T)) \qquad \text{and} \qquad \chi_{\{u>0\}} x^\alpha u^{n-1} u_x^2 \in L^1_{loc}(\bar{\Omega} \times [0,T)), \tag{1.4}$$

for which $u(\cdot,t)$ is differentiable with respect to x at $x = L$ for a.e. $t \in (0,T)$ with

$$u_x(L,t) = 0 \qquad \text{for a.e. } t \in (0,T), \tag{1.5}$$

and which satisfies the integral identity

$$-\int_0^T \int_\Omega x^\beta u \phi_t dxdt - \int_\Omega x^\beta u_0 \phi(\cdot,0)dx = \int_0^T \int_\Omega \chi_{\{u>0\}}[-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2]\phi_{xx} dxdt \tag{1.6}$$

for all $\phi \in C_0^\infty(\bar{\Omega} \times [0,T))$ fulfilling $\phi_x(L,t) = 0$ for all $t \in (0,T)$.

Note that if u is a positive classical solution in the sense of this definition and $\alpha > 1$, then partial integration in (1.6) shows that u satisfies the boundary conditions in (1.3). Our main result reads as follows.

Theorem 1.1 (Local existence of solutions) *Let $n \in (n^*, 3)$, where $n^* = 1.5361\ldots$ is the unique positive root of the polynomial $n \mapsto n^3 + 5n^2 + 16n - 40$. Let $\alpha > 3$ and $\beta \in (-1, \alpha - 4)$. Then for any $\gamma \in (5 - \alpha + \beta, 1)$ and each nonnegative function $u_0 \in W^{1,2}_\gamma(\Omega)$, there exists $T_{max} \in (0,\infty]$ such that (1.3) possesses a continuous weak solution $u \in L^\infty_{loc}([0,T_{max}); W^{1,2}_\gamma(\Omega))$. Furthermore,*

$$\text{if } T_{max} < \infty \text{ then } \limsup_{t \to T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty, \tag{1.7}$$

and the solution conserves the mass in the sense that

$$\int_\Omega x^\beta u(x,t)dx = \int_\Omega x^\beta u_0(x)dx \quad \text{for a.e. } t \in (0,T_{max}). \tag{1.8}$$

Note that the physical values $\alpha = \frac{13}{2}$, $\beta = \frac{1}{2}$, and $n = 2$ are admissible choices in the theorem. We do not know whether the existence of local-in-time nonnegative solutions can be shown for $n < n^* = 1,5361\ldots$ or $n > 3$. One crucial question is whether the equation still preserves nonnegativity of solutions in these parameter regimes. This question, supported by numerical experiments, is the subject of future work.

We also note that Lemma 3.7 shows that if $T_{max} < \infty$ then

$$\limsup_{t \to T_{max}} \|x^{\gamma/2} u_x(\cdot,t)\|_{L^2(\Omega)} = \infty,$$

which along with (1.8) implies that our solution is maximal also in the weighted space $W^{1,2}_\gamma(\Omega)$.

The existence analysis of (1.3) is facing two main challenges. The first challenge is to find suitable a priori estimates for u. In contrast to other nonlinear fourth-order equations, like the thin-film [5] or DLSS equation [18], we cannot expect to find Lyapunov-type estimates for all times because of blow-up phenomena [19]. After all, it turns out that at least a local existence analysis can be based on tracking the time evolution of the functional $\int_\Omega x^\gamma u_x^2(x,t)dx$, at a first stage resulting in an a priori estimate of the form

$$\begin{aligned}
\frac{d}{dt}\int_\Omega x^\gamma u_x^2 dx \quad &+ \quad c\int_\Omega x^{\alpha-\beta+\gamma}u^n u_{xxx}^2 dx + c\int_\Omega x^{\alpha-\beta+\gamma}u^{n-2}u_x^2 u_{xx}^2 dx \\
&+ \quad c\int_\Omega x^{\alpha-\beta+\gamma-2}u^n u_{xx}^2 dx + c\int_\Omega x^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4 dx \\
&\le \quad C\int_\Omega x^{\alpha-\beta+\gamma-6}u^{n+2}dx \qquad (1.9)
\end{aligned}$$

for suitable $c > 0$ and $C > 0$, which can formally be derived from (1.3) for $n \in (n_\star, 3), \alpha > 0, \beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$. Approaches of this type have been pursued in related problems such as, e.g., the thin film equation $u_t + (u^n u_{xxx})_x = 0$, where a basic solution theory could be built on a uniform-in-time bound obtained for $\int_\Omega u_x^2$ by a exploiting a corresponding entropy-like inequality [5]. In the present case, however, the appearance of a superlinear source term on the right of (1.9) seems not only technically inevitable, but in view of the announced result on the existence of exploding solutions [19] this might reflect a substantially destabilizing feature naturally inherent to the nonlinear diffusion mechanism in (1.3). Accordingly, a particular challenge will consist in estimating the right-hand side in (1.9) adequately, and indeed it turns out that under the restrictions on α, β and γ made in Theorem 1.1, by means of interpolation arguments, it is possible to derive from (1.9) the autonomous ordinary differential inequality for $\int_\Omega x^\gamma u_x^2 dx$. Despite a superlinear source term appearing therein, upon integration this will imply appropriate weighted integral estimates for u and its derivatives at least on small time intervals, and inter alia yield the inequality

$$\int_\Omega x^\gamma u_x^2(x,t)dx \le \tilde{C} \qquad \text{for all } t \in (0,T) \qquad (1.10)$$

for some $\tilde{C} > 0$ and suitably *small* $T > 0$. A rigorous variant of (1.9) is shown in Lemmas 4.1 and 4.2. In view of the degeneracies in (1.3), our analysis will rely on a suitable regularization. To achieve this, we shall replace $x^{-\beta}$ and x^α by $(x+\varepsilon)^{-\beta}$ and $g_\varepsilon(x)$, respectively, where $\varepsilon > 0$, g_ε is positive in Ω, and $g_{\varepsilon,x}$ vanishes on the boundary. The latter condition ensures that the approximate flux $J = -g_\varepsilon(x)(-u^n u_{xx} + 2u^{n-1}u_x^2)$ vanishes on the boundary as well.

The second challenge is to show that these regularized problems preserve positivity of the solutions u_ε. This is achieved by proving the estimate

$$\sup_{0<t<T}\int_\Omega \frac{dx}{u_\varepsilon^2(x,t)} \le C(\varepsilon,T,L); \qquad (1.11)$$

see Lemma 5.2. We note that a related approach was employed for the thin-film equation, where solutions u_ε of appropriately regularized non-degenerate equations $u_t+(f_\varepsilon(u)u_{xxx})_x = 0$, with suitably chosen uniformly positive f_ε, enjoy an entropy-type estimate of the form $\sup_{t>0}\int_\Omega G_\varepsilon(u)dx \le C_\varepsilon$ with $G''_\varepsilon(y) = 1/f_\varepsilon(y)$ [5, 7]. A subtle use of this inequality, in the formal limit $\varepsilon \to 0$ reducing to the estimate $\sup_{t>0}\int_\Omega u^{2-n}dx \le C$ similar to (1.11), yields various results on nonnegativity and also on positivity of u when $n > 2$ [5, 3]. We emphasize that in contrast to this thin-film problem, the situation is significantly more involved in the present context, which is mainly due to the additional x-dependent degeneracy at zero energy $x = 0$, but also due to the different structure in the nonlinear dependence of the diffusivity in (1.3) on the solution u. In consequence, our approach toward the local existence result in Theorem 1.1, in particular with regard to the derivation of our entropy-type estimate (1.9), requires arguments which substantially differ from those in [5, 7].

The limit process $\varepsilon \to 0$ will finally be carried out on the basis of a spatio-temporal Hölder estimate for the approximate solutions, which thanks to the fact that $\gamma < 1$ can be derived from (1.10) along with the adaptation of a well-known argument from parabolic theory, which turns this into an appropriate Hölder estimate with respect to time (Lemma 6.1). Unfortunately, the constant $C(\varepsilon, T, L)$ depends on the interval length L such that $C(\varepsilon, T, L) \to \infty$ as $L \to \infty$. Consequently, the limit $L \to \infty$ might require additional arguments.

We do not pursue here the mathematically very interesting question whether for suitably chosen initial data the problem (1.3) possesses nontrivial solutions which exist even for all positive times. It is conceivable that such global solutions might be found near the constant functions $u \equiv c > 0$ which evidently solve (1.3). However, a corresponding perturbation analysis apparently requires a technical setup different from the one considered here, and thereby goes beyond the scope of this work; in addition, keeping in mind that (1.3) is intended to model condensation phenomena we do not know about the physical relevance of such near-constant solutions, reflecting the presence of significantly many large-energy particles for all times.

The paper is organized as follows. In Section 2, we introduce the family of approximate problems. Interpolation inequalities in weighted spaces, which are needed for the existence analysis, are shown in Section 3. The proof of the a priori estimates (Lemmas 4.1 and 4.2) is the subject of Section 4. Then Section 5 is concerned with the local existence for the approximate problems and the absence of dead core formation. A Hölder estimate for the approximate solutions is derived in Section 6. Finally, the proof of Theorem 1.1 is presented in Section 7.

2 A family of approximate problems

We formulate a family of approximate problems in which the singularity at $x = 0$ is removed but the boundary conditions in (1.3) hold at $x = L$ and $x = 0$. To this end, we let $\varepsilon_0 = \min\{1, \sqrt{L/2}\}$, and for $\varepsilon \in (0, \varepsilon_0)$, we choose $\zeta_\varepsilon \in C_0^\infty(\Omega)$ satisfying $0 \le \zeta_\varepsilon \le 1$ and $\zeta_\varepsilon(y) = 1$ for $y \in (\varepsilon^2, L - \varepsilon^2)$. Furthermore, we set

$$z_\varepsilon(x) = \varepsilon + \int_0^x \zeta_\varepsilon(y)dy, \quad x \in [0, L].$$

Then the function z_ε belongs to $C^\infty([0, L])$, $z_\varepsilon(x) \ge \varepsilon$ for all $x \in [0, L]$, and it satisfies homogeneous Neumann boundary conditions, $z_{\varepsilon,x}(0) = z_{\varepsilon,x}(L) = 0$. Then $g_\varepsilon := z_\varepsilon^\alpha$ belongs to $C^\infty([0, L])$ and satisfies $g_\varepsilon \ge \varepsilon^\alpha$ on $[0, L]$ and $g_{\varepsilon,x}(0) = g_{\varepsilon,x}(L) = 0$. Further pointwise estimates for g_ε are summarized in the following lemma.

Lemma 2.1 (Properties of g_ε) *Let $\alpha > 0$. Then the following properties hold:*
(i) There exists a positive decreasing function $\Lambda : [0, \varepsilon_0) \to (0, 1)$ such that $\inf_{(0,\varepsilon_0)} \Lambda > 0$, $\Lambda(0) = 1$, and for all $\varepsilon \in (0, \varepsilon_0)$,

$$\Lambda(\varepsilon)(x + \varepsilon)^\alpha \le g_\varepsilon(x) \le (x + \varepsilon)^\alpha, \quad x \in [0, L].$$

(ii) There exists $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$0 \le g_{\varepsilon x}(x) \le c(x + \varepsilon)^{\alpha-1}, \quad x \in [0, L].$$

(iii) There exists $c > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$,

$$\frac{g_{\varepsilon x}(x)^2}{g_\varepsilon(x)} \le c(x + \varepsilon)^{\alpha-2}, \quad x \in [0, L].$$

Proof. (i) Since $\zeta_\varepsilon \le 1$, we have $z_\varepsilon(x) \le \varepsilon + x$ for $x \in [0, L]$. This yields the second inequality, $g_\varepsilon(x) = z_\varepsilon(x)^\alpha \le (x + \varepsilon)^\alpha$. To prove the first one, we divide $[0, L]$ into three subintervals. First, for $x \in [0, \varepsilon^2]$ the property $z_\varepsilon(x) \ge \varepsilon$ yields

$$\frac{z_\varepsilon(x)}{x + \varepsilon} \ge \frac{\varepsilon}{x + \varepsilon} \ge \frac{\varepsilon}{\varepsilon^2 + \varepsilon} = \frac{1}{1 + \varepsilon}.$$

Next, if $x \in (\varepsilon^2, L - \varepsilon^2)$ then $\zeta_\varepsilon(x) = 1$, whence using that $z_\varepsilon(\varepsilon^2) \ge \varepsilon$, we obtain

$$\frac{z_\varepsilon(x)}{x + \varepsilon} = \frac{1}{x + \varepsilon}\left(z_\varepsilon(x) + \int_{\varepsilon^2}^x \zeta_\varepsilon(y)dy\right) \ge \frac{\varepsilon + (x - \varepsilon^2)}{x + \varepsilon} \ge 1 - \frac{\varepsilon^2}{\varepsilon^2 + \varepsilon} = \frac{1}{1 + \varepsilon}.$$

We finally consider the case $x \in [L - \varepsilon^2, L]$, in which because of the nonnegativity of z_ε and the fact that $\zeta_\varepsilon = 1$ on $[\varepsilon^2, L - \varepsilon^2]$, we infer that

$$\frac{z_\varepsilon(x)}{x + \varepsilon} \ge \frac{z_\varepsilon(L - \varepsilon^2)}{x + \varepsilon} = \frac{\varepsilon + (L - 2\varepsilon^2)}{x + \varepsilon} \ge \frac{L + \varepsilon - 2\varepsilon^2}{L + \varepsilon} = 1 - \frac{2\varepsilon^2}{L + \varepsilon}.$$

The claim hence follows by defining $\Lambda(\varepsilon) = \min\{1/(1 + \varepsilon), 1 - 2\varepsilon^2/(L + \varepsilon)\}$.

(ii) As $0 \le z_{\varepsilon,x} \le 1$, we have $g_{\varepsilon,x} = \alpha z_\varepsilon^{\alpha-1} z_{\varepsilon,x} \le \alpha z_\varepsilon^{\alpha-1}$ in $[0, L]$. Thus, (i) implies (ii).

(iii) This follows directly from (i) and (ii). □

With the above choices of $\varepsilon_0 > 0$ and g_ε, we proceed to regularize the original problem appropriately. The idea is to replace in the first equation in (1.3), rewritten in the form $u_t = x^{-\beta}(-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2)_{xx}$, the coefficients $x^{-\beta}$ and x^α by $(x + \varepsilon)^{-\beta}$ and $g_\varepsilon(x)$, respectively. Accordingly, for $\varepsilon \in (0, \varepsilon_0)$, we shall consider the approximate problem

$$\begin{cases} u_t = \frac{1}{(x+\varepsilon)^\beta} \cdot \Big\{ - g_\varepsilon(x) u^n u_{xx} + 2g_\varepsilon(x) u^{n-1} u_{xx} \Big\}_{xx}, & x \in \Omega,\ t > 0, \\ u_x = u_{xxx} = 0, \quad x \in \partial\Omega,\ t > 0. \end{cases} \tag{2.12}$$

The boundary behavior of g_ε guarantees that the flux

$$J(x, t) = -g_\varepsilon(x) u^n u_{xx} + 2g_\varepsilon(x) u^{n-1} u_x^2 \tag{2.13}$$

vanishes on $\partial\Omega = \{0, L\}$. This results upon expanding J_x according to

$$\begin{aligned} J_x &= -g_\varepsilon(x)u^n u_{xxx} + (4-n)g_\varepsilon(x)u^{n-1}u_x u_{xx} + 2(n-1)g_\varepsilon(x)u^{n-2}u_x^3 \\ &\quad -g_{\varepsilon,x}(x)u^n u_{xx} + 2g_{\varepsilon,x}(x)u^{n-1}u_x^2, \end{aligned} \tag{2.14}$$

and evaluating this expression on $\partial\Omega$:

Lemma 2.2 (Boundary flux vanishes) *Let $n > 0$, $\alpha > 0$, $\beta \in \mathbb{R}$, $T > 0$, and $\varepsilon \in (0, \varepsilon_0)$, and let $u \in C^{4,1}(\overline{\Omega} \times (0, T))$ be a positive classical solution of (2.12). Then $J_x(x, t) = 0$ for all $x \in \partial\Omega$ and $t \in (0, T)$, where J is defined in (2.13).*

Proof. The statement is a consequence of (2.14) and the identities $u_x = u_{xxx} = g_{\varepsilon,x} = 0$ on $\partial\Omega$. □

The above choice of boundary conditions ensures that the total mass is preserved.

Lemma 2.3 (Conservation of total mass) *Under the assumptions of Lemma 2.2, we have*

$$\frac{d}{dt}\int_\Omega (x+\varepsilon)^\beta u(x,t)dx = 0 \qquad \text{for all } t \in (0,T).$$

Proof. The claim immediately results by integrating (2.12) over Ω and using that $J_x = 0$ on $\partial\Omega$. □

3 Some interpolation inequalities

As a preparation for our subsequent analysis, let us collect some interpolation inequalities in weighted spaces. The first of these reads as follows.

Lemma 3.1 *Let $n \in \mathbb{R} \setminus \{-1, 1\}$, $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$. Then for any $\eta > 0$, one can find $C(\eta) > 0$ such that for all positive functions $u \in C^2(\bar{\Omega})$ satisfying $u_x = 0$ on $\partial\Omega$, we have*

$$\begin{aligned} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^n u_{xx}^2 dx &+ \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4 dx + \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4}u^n u_x^2 dx \\ &\le \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^n u_{xxx}^2 dx + \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2 u_{xx}^2 dx \\ &\quad + C(\eta)\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}dx \end{aligned} \tag{3.1}$$

for all $\varepsilon > 0$.

The proof of Lemma 3.1 will be achieved in a series of steps to be presented separately in Lemmas 3.2-3.4. We first estimate the last integral on the left-hand side of (3.1) by a sum involving a small portion of the first term in (3.1).

Lemma 3.2 *Let $n \in \mathbb{R} \setminus \{-1\}$ and α, β, and γ be arbitrary real numbers. Then for all $\eta > 0$, there exists $C(\eta) > 0$ such that whenever $\varepsilon > 0$ and $u \in C^2(\bar{\Omega})$ is positive with $u_x = 0$ on $\partial\Omega$, the inequality*

$$\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4}u^n u_x^2 dx \le \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^n u_{xx}^2 dx + C(\eta)\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}dx \tag{3.2}$$

holds.

PROOF. Using $u_x = 0$ on $\partial\Omega$, we may integrate by parts and use Young's inequality to find that

$$\begin{aligned}
\Gamma &:= \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \\
&= -\frac{1}{n+1}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^{n+1} u_{xx} dx - \frac{\alpha-\beta+\gamma-4}{n+1}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-5} u^{n+1} u_x dx \\
&\le \frac{\eta}{2}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + \frac{1}{2(n+1)^2\eta}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \\
&\quad + \frac{1}{2}\Gamma + \frac{(\alpha-\beta+\gamma-4)^2}{2(n+1)^2}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx.
\end{aligned}$$

Rearranging yields (3.2). □

Using the above preparation, we can control the first term in (3.1) as desired:

Lemma 3.3 *Let $n \in \mathbb{R} \setminus \{-1\}$ and $\alpha, \beta, \gamma \in \mathbb{R}$. Then for all $\eta > 0$, one can find $C(\eta) > 0$ with the property that any positive function $u \in C^3(\bar\Omega)$ with $u_x = 0$ on $\partial\Omega$ satisfies*

$$\begin{aligned}
\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx &\le \eta\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + \eta\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&\quad + C(\eta)\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \qquad (3.3)
\end{aligned}$$

for each $\varepsilon > 0$.

PROOF. Since $u_x = 0$ on $\partial\Omega$, an integration by parts shows that

$$\begin{aligned}
\Gamma &:= \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx = -\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_x u_{xxx} dx \qquad (3.4)\\
&\quad - n\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-1} u_x^2 u_{xx} dx - (\alpha-\beta+\gamma-2)\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^n u_x u_{xx} dx,
\end{aligned}$$

where by Young's inequality we find that

$$-\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_x u_{xxx} dx \le \frac{\eta}{2}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c_1\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx$$

and

$$\begin{aligned}
-n\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-1} u_x^2 u_{xx} dx &\le \frac{\eta}{4}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&\quad + c_2\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx
\end{aligned}$$

as well as

$$\begin{aligned}
-(\alpha-\beta+\gamma-2)\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^n u_x u_{xx} dx &\le \frac{\eta}{4}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&\quad + c_3\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx
\end{aligned}$$

with $c_1 := \frac{1}{2\eta}$, $c_2 := \frac{n^2}{\eta}$ and $c_3 := \frac{(\alpha-\beta+\gamma-2)^2}{\eta}$. Since Lemma 3.2 provides $c_4 > 0$ such that

$$(c_1 + c_2) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx \le \frac{1}{2}\Gamma + c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx,$$

(3.4) thereby proves (3.3). $\square$

Now the latter allows us to also estimate the second term in (3.1) in the claimed manner.

Lemma 3.4 *Let $n \in \mathbb{R} \setminus \{-1, 1\}$ and α, β, $\gamma \in \mathbb{R}$. Then for all $\eta > 0$, we can pick $C(\eta) > 0$ such that if $u \in C^3(\bar{\Omega})$ is positive and satisfies $u_x = 0$ on $\partial\Omega$, then for all $\varepsilon > 0$, we have*

$$\begin{aligned} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \ &\le \ \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + \eta \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + C(\eta) \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx. \end{aligned} \tag{3.5}$$

Proof. Once more integrating by parts and using Young's inequality, we see that

$$\begin{aligned} \Gamma \ &:= \ \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\ &= \ -\frac{3}{n-1} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-1} u_x^2 u_{xx} dx - \frac{\alpha-\beta+\gamma-2}{n-1} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^{n-1} u_x^3 dx \\ &\le \ \frac{1}{4}\Gamma + c_1 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + \frac{1}{4}\Gamma + c_2 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \end{aligned}$$

with $c_1 := \frac{9}{(n-1)^2}$ and $c_2 := \frac{(\alpha-\beta+\gamma-4)^2}{(n-1)^2}$. Thus,

$$\Gamma \le 2c_1 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + 2c_2 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx,$$

whence invoking Lemma 3.3, we readily arrive at (3.5). $\square$

Proof of Lemma 3.1. We only need to combine Lemmas 3.2, 3.3, and 3.4. $\square$

The following inequality is closely related to those used in the context of the thin-film equation $u_t + (u^n u_{xxx})_x = 0$ [4].

Lemma 3.5 *Let $n \in \mathbb{R} \setminus \{3\}$ and α, β, $\gamma \in \mathbb{R}$. Then for all $\eta \in (0,1)$ and any positive $u \in C^2(\bar{\Omega})$ fulfilling $u_x = 0$ on $\partial\Omega$, the inequality*

$$\begin{aligned} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \ &\le \ \frac{25}{(1-\eta)(n-3)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &\quad + \frac{(\alpha-\beta+\gamma)^2}{\eta(1-\eta)(n-3)^2} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \end{aligned} \tag{3.6}$$

is valid for all $\varepsilon > 0$.

Proof. We integrate by parts using $u_x = 0$ on $\partial\Omega$ and apply Young's inequality to obtain the estimate

$$\begin{aligned}
\Gamma &:= \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \\
&= -\frac{5}{n-3}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-3} u_x^4 u_{xx} dx - \frac{\alpha-\beta+\gamma}{n-3}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-1} u^{n-3} u_x^5 dx \\
&\leq \frac{1}{2}\Gamma + \frac{1}{2}\cdot\frac{25}{(n-3)^2}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&\quad + \frac{\eta}{2}\Gamma + \frac{1}{2\eta}\cdot\frac{(\alpha-\beta+\gamma)^2}{(n-3)^2}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx,
\end{aligned}$$

which can readily be checked to be equivalent to (3.6). □

The following two lemmas are concerned with estimates on the Hölder and L^∞ norms of functions in $W^{1,2}_{loc}(\Omega)$.

Lemma 3.6 *Let $\gamma \in (-\infty, 1)$. Then there exists $c = c(\gamma) > 0$ such that for any $\varepsilon \in [0,1)$ and any $u \in W^{1,2}_{loc}(\Omega)$,*

$$|u(x_2) - u(x_1)| \leq c\left(\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx\right)^{\frac{1}{2}} |x_2 - x_1|^\theta \quad \text{for all } x_1, x_2 \in \Omega,$$

where $\theta := \min\{\frac{1}{2}, \frac{1-\gamma}{2}\}$.

Proof. Let $0 < x_1 < x_2 < L$ and suppose first that $\gamma \in [0,1)$. Then by the Cauchy-Schwarz inequality,

$$|u(x_2) - u(x_1)| = \left|\int_{x_1}^{x_2} u_x(x) dx\right| \leq \left(\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx\right)^{\frac{1}{2}} \left(\int_{x_1}^{x_2} (x+\varepsilon)^{-\gamma} dx\right)^{\frac{1}{2}}.$$

Employing the Hölder continuity of $x \mapsto x^{1-\gamma}$, we obtain

$$\int_{x_1}^{x_2} (x+\varepsilon)^{-\gamma} dx = \frac{1}{1-\gamma}((x_2+\varepsilon)^{1-\gamma} - (x_1+\varepsilon)^{1-\gamma}) \leq \frac{c_1}{1-\gamma}|x_2 - x_1|^{1-\gamma}.$$

The result thus follows with $c = (c_1/(1-\gamma))^{\frac{1}{2}}$.
If $\gamma \in (-\infty, 0)$, we replace γ by $-\gamma$ in the above arguments and use the Lipschitz continuity of $x \mapsto x^{1+|\gamma|}$. □

Lemma 3.7 *Let $\gamma \in (-\infty, 1)$ and $\beta \in \mathbb{R}$. Then there exists $c = c(\beta, L) > 0$ such that for all $\varepsilon \in [0,1)$ and any $u \in W^{1,2}_{loc}(\Omega)$,*

$$\|u\|_{L^\infty(\Omega)} \leq c\left(\int_\Omega (x+\varepsilon)^\beta |u| dx + \left(\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx\right)^{1/2}\right).$$

PROOF. Assuming that $B = \int_\Omega (x+\varepsilon)^\beta |u| dx$ is finite, we see that there exists $x_0 \in (\frac{L}{2}, L)$ such that $(x_0+\varepsilon)^\beta |u(x_0)| \le \frac{2B}{L}$, for otherwise the inequality $B \ge \int_{\frac{L}{2}}^{L} (x+\varepsilon)^\beta |u| dx > \frac{L}{2} \cdot \frac{2B}{L} = B$ gives a contradiction. Since $\frac{L}{2} \le x_0 + \varepsilon \le L+1$, we infer that

$$|u(x_0)| \le c_1 \int_\Omega (x+\varepsilon)^\beta |u| dx,$$

where $c_1 = \frac{2}{L} \cdot \max\{(\frac{L}{2})^{-\beta}, (L+1)^{-\beta}\}$. The conclusion thus follows from Lemma 3.6. $\square$

Note that if $\beta > -1$, which will be assumed in all our subsequent applications of this lemma, the above constant $c(\beta, L)$ can be chosen to be bounded as $L \to \infty$.

4 A differential inequality for $\int_\Omega x^\gamma u_x^2$

A key role in our analysis will be played by the following a priori estimate for the functional $y(t) := \int_\Omega (x+\varepsilon)^\beta u_x^2 dx$ in terms of a weighted norm of u in $L^{n+2}(\Omega)$. In Lemma 4.2 below, we shall turn this into an autonomous differential equation for $y(t)$, which will be essential for our local existence proof.

Lemma 4.1 (A priori estimate in terms of a weighted L^{n+2} norm) *Let $n_\star = 1.5361\ldots$ be the unique positive root of $n \mapsto P(n) := n^3 + 5n^2 + 16n - 40$, and let $n \in (n_\star, 3)$, $\alpha > 0$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$. Then there exist $\varepsilon_\star \in (0, \varepsilon_0)$, $c > 0$ (independent of L), and $K > 0$ such that if for some $T > 0$ and $\varepsilon \in (0, \varepsilon_\star)$, $u \in C^{4,1}(\bar\Omega \times (0,T))$ is a positive classical solution to (2.12), then*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \quad &+ \quad c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\
&+ \quad c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + c\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\
&\le \quad K\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \qquad \textit{for all } t \in (0,T). \qquad (4.1)
\end{aligned}$$

PROOF. With the notation (2.13), we can write the first equation in (2.12) as $u_t = (x+\varepsilon)^{-\beta} J_{xx}$. Since $u_x = J_x = 0$ on $\partial\Omega$ by Lemma 2.2, an integration by parts gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx &= -\int_\Omega ((x+\varepsilon)^\gamma u_x)_x u_t dx = -\int_\Omega ((x+\varepsilon)^\gamma u_x)_x (x+\varepsilon)^{-\beta} J_{xx} dx \\
&= \int_\Omega \Big[(x+\varepsilon)^{-\beta+\gamma} u_{xx} + \gamma(x+\varepsilon)^{-\beta+\gamma-1} u_x\Big]_x J_x dx
\end{aligned}$$

for all $t \in (0,T)$. Computing

$$\begin{aligned}
&\Big[(x+\varepsilon)^{-\beta+\gamma} u_{xx} + \gamma(x+\varepsilon)^{-\beta+\gamma-1} u_x\Big]_x \\
&\quad = (x+\varepsilon)^{-\beta+\gamma} u_{xxx} + (2\gamma-\beta)(x+\varepsilon)^{-\beta+\gamma-1} u_{xx} + \gamma(\gamma-\beta-1)(x+\varepsilon)^{-\beta+\gamma-2} u_x
\end{aligned}$$

and expanding J_x by means of (2.14), we thus obtain the identity

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \quad = \quad & -\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx \\
& + (4-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-1} u_x u_{xx} u_{xxx} dx \\
& + 2(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^3 u_{xxx} dx \\
& - \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^n u_{xx} u_{xxx} dx \\
& + 2\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-1} u_x^2 u_{xxx} dx \\
& - (2\gamma-\beta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^n u_{xx} u_{xxx} dx \\
& + (2\gamma-\beta)(4-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-1} u_x u_{xx}^2 dx \\
& + 2(2\gamma-\beta)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-2} u_x^3 u_{xx} dx \\
& - (2\gamma-\beta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_{\varepsilon x}(x) u^n u_{xx}^2 dx \\
& + 2(2\gamma-\beta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_{\varepsilon x}(x) u^{n-1} u_x^2 u_{xx} dx \\
& - \gamma(\gamma-\beta-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^n u_x u_{xxx} dx \\
& + \gamma(\gamma-\beta-1)(4-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-1} u_x^2 u_{xx} dx \\
& + 2\gamma(\gamma-\beta-1)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-2} u_x^4 dx \\
& - \gamma(\gamma-\beta-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_{\varepsilon x}(x) u^n u_x u_{xx} dx \\
& + 2\gamma(\gamma-\beta-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_{\varepsilon x}(x) u^{n-1} u_x^3 dx \\
=: \quad & I_1 + \dots + I_{15} \qquad \text{for all } t \in (0,T).
\end{aligned}
\tag{4.2}
$$

Our goal is to adequately apply the interpolation inequalities in Lemma 3.1 and Lemma 3.5 and to identify those integrals which absorb the $O(\eta)$ contributions in (3.1) and (3.6) such that finally only a possibly large multiple of the integral over $(x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}$ remains.
To achieve this, we observe that the integral I_1 is nonpositive and thus can be used to absorb positive contributions. Apart from this, the only absorptive contribution to be used in the sequel will result from I_3, which we therefore rearrange first: Namely, by two further integrations by parts, once more relying on the fact that $u_x = 0$ on $\partial\Omega$, this term can be rewritten according to

$$
I_3 \quad = \quad -6(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx
$$

$$
\begin{aligned}
& \quad -2(n-1)(n-2)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-3} u_x^4 u_{xx} dx \\
& \quad -2(\gamma-\beta)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-2} u_x^3 u_{xx} dx \\
& \quad -2(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-2} u_x^3 u_{xx} dx \\
= & \quad -6(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx \\
& \quad -\frac{2}{5}(n-1)(n-2)(3-n)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx \\
& \quad +\frac{2}{5}(\gamma-\beta)(n-1)(n-2)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-3} u_x^5 dx \\
& \quad +\frac{2}{5}(n-1)(n-2)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-3} u_x^5 dx \\
& \quad -2(\gamma-\beta)(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma-1} g_\varepsilon(x) u^{n-2} u_x^3 u_{xx} dx \\
& \quad -2(n-1)\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_{\varepsilon x}(x) u^{n-2} u_x^3 u_{xx} dx \\
=: & \quad I_{31}+\cdots+I_{36} \qquad \text{for all } t\in(0,T).
\end{aligned} \tag{4.3}
$$

In order to specify our choice of $\varepsilon_\star$, let us note that, according to our restriction on n and with P as specified in the formulation of the lemma, we have $P(n)>0$, which implies that when $n<2$,

$$
\begin{aligned}
& 4(3-n)\Big\{6(n-1)-\frac{(4-n)^2}{4}-\frac{10(n-1)(2-n)}{3-n}\Big\} \\
& \quad = \quad 24(3-n)(n-1)-(3-n)(4-n)^2-40(n-1)(2-n) \\
& \quad = \quad -24n^2+96n-72-3n^2+24n-48+n^3-8n^2+16n+40n^2-120n+80 \\
& \quad = \quad P(n)>0.
\end{aligned}
$$

Since in the case $n\in[2,3)$ we clearly have

$$
6(n-1)-\frac{(4-n)^2}{4}\ge 6(2-1)-\frac{(4-2)^2}{4}=5>0,
$$

this entails that for any choice of $n\in(n_\star,3)$,

$$
6(n-1)-\frac{(4-n)^2}{4}-\frac{10(n-1)(2-n)_+}{3-n}>0.
$$

Consequently, with $\Lambda(\varepsilon)$ as in Lemma 2.1, we can pick $\varepsilon_\star\in(0,\varepsilon_0)$ such that with $\Lambda_\star:=\Lambda(\varepsilon_\star)$, we have

$$
\Big\{6(n-1)-\frac{(4-n)^2}{4}\Big\}\Lambda_\star-\frac{10(n-1)(2-n)_+}{3-n}>0,
$$

and thereupon fix a number $\mu \in (0,1)$ sufficiently close to 1 and $\eta > 0$ suitably small such that still

$$\Big\{6(n-1) - \frac{(4-n)^2}{4\mu}\Big\}\Lambda_\star - \frac{10(n-1)(2-n)_+}{3-n} - \Big\{\Lambda_\star + \frac{50}{(3-n)^2} + 1\Big\}\eta > 0, \tag{4.4}$$

and such that moreover

$$(1-\mu-\eta)\Lambda_\star - \eta > 0. \tag{4.5}$$

Upon these choices, we first use Young's inequality to estimate I_2 according to

$$I_2 \le \mu \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + \frac{(4-n)^2}{4\mu} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx. \tag{4.6}$$

Next, recalling Lemma 2.1, we obtain

$$\begin{aligned} I_4 &\le \frac{\eta}{2}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_1 \int_\Omega (x+\varepsilon)^{-\beta+\gamma} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^n u_{xx}^2 dx \\ &\le \frac{\eta}{2}|I_1| + c_2\Gamma_1, \end{aligned} \tag{4.7}$$

where

$$\Gamma_1 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx \tag{4.8}$$

and c_1 and c_2, as all numbers $c_3, c_4, \dots$ appearing below, denote positive constants depending on n, α, β, and γ, but neither on $\varepsilon \in (0, \varepsilon_\star)$ nor on the solution u.
Similarly, we find $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned} I_5 &\le \frac{\eta}{4}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_3 \int_\Omega (x+\varepsilon)^{-\beta+\gamma} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^{n-2} u_x^4 dx \\ &\le \frac{\eta}{4}|I_1| + c_4\Gamma_2 \end{aligned} \tag{4.9}$$

with

$$\Gamma_2 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx, \tag{4.10}$$

and then $c_5 > 0$ and $c_6 > 0$ satisfying

$$\begin{aligned} I_6 &\le \frac{\eta}{8}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_5 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^n u_{xx}^2 dx \\ &\le \frac{\eta}{8}|I_1| + c_5\Gamma_1 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} I_7 &\le \frac{\eta}{2}\int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_6 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^n u_{xx}^2 dx \\ &\le \frac{\eta}{2}\tilde{I}_{31} + c_6\Gamma_1 \end{aligned} \tag{4.12}$$

where

$$\tilde{I}_{31} := \frac{I_{31}}{-6(n-1)} = \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx. \tag{4.13}$$

In much the same manner, we derive the inequalities

$$\begin{aligned} I_8 &\le \frac{\eta}{4} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_7 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-2} u_x^4 dx \\ &\le \frac{\eta}{4} \tilde{I}_{31} + c_7 \Gamma_2 \end{aligned} \tag{4.14}$$

and

$$\begin{aligned} I_{10} &\le \frac{\eta}{8} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_8 \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^n u_x^2 dx \\ &\le \frac{\eta}{8} \tilde{I}_{31} + c_9 \Gamma_3 \end{aligned} \tag{4.15}$$

for some positive c_7, c_8 and c_9 and

$$\Gamma_3 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4} u^n u_x^2 dx, \tag{4.16}$$

as well as

$$\begin{aligned} I_{11} &\le \frac{\eta}{16} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx + c_{10} \int_\Omega (x+\varepsilon)^{-\beta+\gamma-4} g_\varepsilon(x) u^n u_x^2 dx \\ &\le \frac{\eta}{16} |I_1| + c_{10} \Gamma_3 \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} I_{12} &\le \frac{\eta}{16} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_{11} \int_\Omega (x+\varepsilon)^{-\beta+\gamma-4} g_\varepsilon(x) u^n u_x^2 dx \\ &\le \frac{\eta}{16} \tilde{I}_{31} + c_{11} \Gamma_3 \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} I_{14} &\le \frac{\eta}{32} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx + c_{12} \int_\Omega (x+\varepsilon)^{-\beta+\gamma-4} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^{n+2} dx \\ &\le \frac{\eta}{32} \tilde{I}_{31} + c_{13} \Gamma_4, \end{aligned} \tag{4.19}$$

where

$$\Gamma_4 := \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \tag{4.20}$$

and c_{10}, c_{11}, c_{12}, and c_{13} are positive constants.
As for the remaining terms on the right of (4.2), we again apply Lemma 2.1 to find $c_{14} > 0$ and $c_{15} > 0$ such that

$$I_9 \le c_{14} \Gamma_1 \tag{4.21}$$

and

$$I_{13} \le c_{15} J_2, \tag{4.22}$$

whereas Young's inequality provides $c_{16} > 0$ fulfilling

$$\begin{aligned}
I_{15} &\le c_{16} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-3} u^{n-1} |u_x|^3 dx \\
&\le c_{16} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx + c_{16} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2} dx \\
&= c_{16}\Gamma_2 + c_{16}\Gamma_4.
\end{aligned} \tag{4.23}$$

Finally, abbreviating

$$\tilde{I}_{32} := \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx, \tag{4.24}$$

using Young's inequality we obtain constants $c_{17}, ..., c_{21}$ such that

$$\begin{aligned}
I_{33} &\le \frac{\eta}{2} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{17} \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n-2} u_x^4 dx \\
&\le \frac{\eta}{2} \tilde{I}_{32} + c_{17}\Gamma_2
\end{aligned} \tag{4.25}$$

and

$$\begin{aligned}
I_{34} &\le \frac{\eta}{4} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{18} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^{n-2} u_x^4 dx \\
&\le \frac{\eta}{4} \tilde{I}_{32} + c_{18}\Gamma_2
\end{aligned} \tag{4.26}$$

as well as

$$\begin{aligned}
I_{35} &\le \frac{\eta}{8} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{19} \int_\Omega (x+\varepsilon)^{-\beta+\gamma-2} g_\varepsilon(x) u^{n} u_{xx}^2 dx \\
&\le \frac{\eta}{8} \tilde{I}_{32} + c_{19}\Gamma_1
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
I_{36} &\le \frac{\eta}{16} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx + c_{20} \int_\Omega (x+\varepsilon)^{-\beta+\gamma} \frac{g_{\varepsilon x}^2(x)}{g_\varepsilon(x)} \cdot u^{n} u_{xx}^2 dx \\
&\le \frac{\eta}{16} \tilde{I}_{32} + c_{21}\Gamma_1.
\end{aligned} \tag{4.28}$$

In light of (4.6)-(4.28), (4.2) and (4.3) thus yield

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} \int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \;\le\; & -(1-\mu-\eta) \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^n u_{xxx}^2 dx \\
& - \Big\{ 6(n-1) - \frac{(4-n)^4}{4\mu} - \eta \Big\} \cdot \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-2} u_x^2 u_{xx}^2 dx \\
& + \Big\{ -\frac{2}{5}(n-1)(n-2)(3-n) + \eta \Big\} \cdot \int_\Omega (x+\varepsilon)^{-\beta+\gamma} g_\varepsilon(x) u^{n-4} u_x^6 dx
\end{aligned}$$

$$
\begin{aligned}
&+ c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^n u_{xx}^2dx + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4dx \\
&+ c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4}u^n u_x^2dx + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}dx \qquad (4.29)
\end{aligned}
$$

for all $t\in(0,T)$ with some $c_{22}>0$, where we have used that $\sum_{j=1}^N \frac{\eta}{2^j}<\eta$ for all $N\in\mathbb{N}$.

Now by means of Lemma 3.5 we can find $c_{23}>0$ such that

$$
\begin{aligned}
&\Big\{-\frac{2}{5}(n-1)(n-2)(3-n)+\eta\Big\}\int_\Omega (x+\varepsilon)^{-\beta+\gamma}g_\varepsilon(x)u^{n-4}u_x^6dx \\
&\quad\le \Big\{\frac{2}{5}(n-1)(2-n)_+(3-n)+\eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-4}u_x^6dx \\
&\quad\le \frac{25}{(1-\eta)(3-n)^2}\Big\{\frac{2}{5}(n-1)(2-n)_+(3-n)+\eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2u_{xx}^2dx \\
&\qquad + c_{23}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4dx \\
&\quad\le \frac{25}{(3-n)^2}\Big\{\frac{2}{5}(n-1)(2-n)_+(3-n)+2\eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2u_{xx}^2dx \\
&\qquad + c_{23}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4dx.
\end{aligned}
$$

The last inequality follows from the fact that for $A:=\frac{2}{5}(n-1)(2-n)_+(3-n)<1$ $(0<n<3)$, we have $(A+\eta)/(1-\eta)\le A+2\eta$ if $0<\eta<\frac{1}{2}(1-A)$. Then applying Lemma 3.1, we obtain $c_{24}>0$ satisfying

$$
\begin{aligned}
c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^n u_{xx}^2dx \quad &+ \quad (c_{22}+c_{23})\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4dx \\
&\quad + c_{22}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-4}u^n u_x^2dx \\
&\le \quad \eta\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^n u_{xxx}^2dx + \eta\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2u_{xx}^2dx \\
&\quad + c_{24}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}dx. \qquad (4.30)
\end{aligned}
$$

Therefore, (4.29) shows that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2dx \le -(1-\mu-\eta)\int_\Omega (x+\varepsilon)^{-\beta+\gamma}g_\varepsilon(x)u^n u_{xxx}^2dx \\
&-\Big\{6(n-1)-\frac{(4-n)^2}{4\mu}-\eta\Big\}\cdot\int_\Omega (x+\varepsilon)^{-\beta+\gamma}g_\varepsilon(x)u^{n-2}u_x^2u_{xx}^2dx \\
&+\eta\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^n u_{xxx}^2dx \\
&+\Big\{\frac{25}{(3-n)^2}\Big[\frac{2}{5}(n-1)(2-n)_+(3-n)+2\eta\Big]+\eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2u_{xx}^2dx \\
&+(c_{22}+c_{24})\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}dx \qquad \text{for all } t\in(0,T).
\end{aligned}
$$

Since clearly $1-\mu-\eta$ and $6(n-1)-\frac{(4-n)^2}{4\mu}-\eta$ are both positive thanks to (4.5) and (4.4), we may now use the lower estimate for g_ε established in Lemma 2.1 to infer that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \quad &\le \quad -\Big\{(1-\mu-\eta)\Lambda_\star-\eta\Big\}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^n u_{xxx}^2 dx \\
&\quad -\left\{\left[6(n-1)-\frac{(4-n)^2}{4\mu}-\eta\right]\Lambda_\star-\frac{10(n-1)(2-n)_+}{3-n}-\frac{50\eta}{(3-n)^2}-\eta\right\} \\
&\quad \times\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2u_{xx}^2 dx \\
&\quad +(c_{22}+c_{24})\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6}u^{n+2}dx \qquad \text{for all } t\in(0,T),
\end{aligned}$$

because $\varepsilon<\varepsilon_\star$ and hence $\Lambda(\varepsilon)\ge\Lambda_\star$ by the monotonicity of Λ asserted by Lemma 2.1. According to (4.5) and (4.4), after another application of (4.30), this entails (4.1). □

Under additional assumptions on the parameters α, β, and γ, we are able to derive a priori estimates for small times only depending on the initial data. More precisely, if the parameter γ is chosen large enough, then the weight in the integral on the right-hand side of (4.1) is sufficiently regular, whence from the above we can deduce a bound for $\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx$ for all sufficiently small $t>0$. Since we plan to finally achieve a boundedness property for u itself with respect to the norm in $L^\infty(\Omega)$, we require that $\gamma<1$. This explains the restriction on β in the following lemma.

Lemma 4.2 (A priori estimate for small times) *Let $n_*=1.5361\ldots$ and $\varepsilon_\star\in(0,1)$ be as in Lemma 4.1, let $\alpha>0$, $\beta\in(-1,\alpha-4)$, and*

$$\gamma\in(5-\alpha+\beta,1). \tag{4.31}$$

Then one can find $c=c(L)>0$ such that for all $A>0$ and $B>0$, there exists $T_0(A,B)\in(0,1)$ with the following property: If for some $\varepsilon\in(0,\varepsilon_\star)$ and $T\in(0,T_0(A,B))$, $u\in C^{4,1}(\bar\Omega\times[0,T))$ is positive and solves (2.12) in $\Omega\times(0,T)$ with

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,0)dx\le A \qquad \text{and} \qquad \int_\Omega (x+\varepsilon)^\beta u(x,0)dx\le B, \tag{4.32}$$

then

$$\begin{aligned}
&\sup_{t\in(0,T)}\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx\le c\int_0^T\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^n u_{xxx}^2 dxdt \qquad (4.33)\\
&+\ c\int_0^T\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-2}u_x^2u_{xx}^2dxdt+c\int_0^T\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u^{n-4}u_x^6dxdt\\
&+\ c\int_0^T\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^n u_{xx}^2dxdt+c\int_0^T\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u^{n-2}u_x^4dxdt\\
&\le\ A+1. \qquad (4.34)
\end{aligned}$$

In particular, in that case there exists $C(A,B,L)>0$ such that the flux J, defined in (2.13), satisfies

$$\int_0^T\int_\Omega (x+\varepsilon)^{-\alpha-\beta+\gamma}J_x^2dxdt\le C(A,B,L). \tag{4.35}$$

Proof. Let us first note that our hypothesis $\beta \in (-1, \alpha - 4)$ entails the inequality $5 - \alpha + \beta < 1$, whence the assumption $\gamma \in (5 - \alpha + \beta, 1)$ indeed is meaningful. Then with $T_0(A, B) \in (0, 1)$ to be fixed below, we assume that $T \in (0, T_0(A, B))$ and that u has the properties listed above. Thus, for each $t \in (0, T)$, by (4.32) and Lemma 2.3, we have $\int_\Omega (x+\varepsilon)^\beta u(x,t)dx \le B$, so that Lemma 3.7 says that

$$u(x,t) \le c_1 B + c_1 \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \Big)^{\frac{1}{2}} \qquad \text{for all } x \in \Omega \tag{4.36}$$

with some $c_1 > 0$, where we have used that $\gamma < 1$. Consequently, thanks to (4.31), the integral on the right-hand side of (4.1) can be estimated according to

$$\begin{aligned} \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} u^{n+2}(x,t)dx \quad &\le \quad 2^{n+2}(c_1 B)^{n+2} \Big(\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-6} dx \Big)^{n+2} \\ &\qquad + 2^{n+2} \cdot L \cdot c_1^{n+2} \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \Big)^{\frac{n+2}{2}} \\ &\le \quad c_2(B) + c_3 \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \Big)^{\frac{n+2}{2}} \end{aligned}$$

with appropriate constants $c_2(B) > 0$ and $c_3 > 0$. From Lemmas 4.1 and 3.5, we thus obtain $c_4 > 0$, $c_5(B) > 0$, and $c_6 > 0$ such that

$$\begin{aligned} \frac{d}{dt} \int_\Omega (x+\varepsilon)^\gamma u_x^2 dx \quad &+ \quad c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^n u_{xxx}^2 dx + c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-2} u_x^2 u_{xx}^2 dx \\ &+ \quad c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma} u^{n-4} u_x^6 dx \\ &+ \quad c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^n u_{xx}^2 dx + c_4 \int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2} u^{n-2} u_x^4 dx \\ &\le \quad c_5(B) + c_6 \Big(\int_\Omega (x+\varepsilon)^\gamma u_x^2 \Big)^{\frac{n+2}{2}} dx \qquad \text{for all } t \in (0, T). \end{aligned} \tag{4.37}$$

With the above constants being fixed, we consider the solution $y \equiv y_{A,B}$ of the initial-value problem

$$\begin{cases} y'(t) = c_5(B) + c_6 y^{\frac{n+2}{2}}(t), & t > 0, \\ y(0) = A. \end{cases}$$

It is then clearly possible to fix some sufficiently small $T_0(A, B) \in (0, 1)$ such that $y(t) \le A + 1$ for all $t \in (0, T_0(A, B))$, and a comparison argument for ordinary differential equations, applied to (4.37), shows that

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,t)dx \le A + 1 \qquad \text{for all } t \in (0, T),\ T < T_0(A, B).$$

Inserting this into (4.37) and integrating, we readily arrive at (4.33).
From this, the estimate (4.35) easily follows upon recalling (2.14), (4.36), and Lemma 2.1 and applying Lemma 3.7 and Lemma 3.1. □

5 Local existence in the approximate problems

The a priori estimate of Lemma 4.2 allows us to prove the local existence of a classical solution to the approximate problem (2.12) for smooth initial data u_0 with compactly supported derivative u_{0x}.

Lemma 5.1 (Local existence for smooth data) *Let $\varepsilon_0 = \min\{1, \sqrt{\frac{L}{2}}\}$ and $\varepsilon \in (0, \varepsilon_0)$, and let $u_0 \in C^\infty(\overline{\Omega})$ be positive and such that $u_{0x} \in C_0^\infty(\Omega)$. Then there exist $T_{max} \in (0, \infty]$ and a unique positive classical solution $u \in C^{4,1}(\bar{\Omega} \times [0, T_{max}))$ of (2.12) in $\Omega \times (0, T_{max})$ with $u(x, 0) = u_0(x)$ for all $x \in \Omega$. Moreover, T_{max} has the property that*

$$\text{if } T_{max} < \infty \text{ then either } \liminf_{t \nearrow T_{max}} \Big(\inf_{x\in\Omega} u(x,t) \Big) = 0 \quad \text{or } \limsup_{t \nearrow T_{max}} \Big(\sup_{x\in\Omega} u(x,t) \Big) = \infty. \tag{5.1}$$

Proof. For $k \in \mathbb{N}$, we let $f_k \in C^\infty(\mathbb{R})$ be a smooth nondecreasing truncation function on $\mathbb{R}$ such that $f_k(s) = s$ for all $s \in [\frac{1}{k}, k]$ and $\frac{1}{2k} \le f_k \le 2k$ on $\mathbb{R}$. Then each of the problems

$$\begin{cases} u_{kt} = \frac{1}{(x+\varepsilon)^\beta} \cdot \Big\{ - g_\varepsilon(x) f_k^n(u_k) u_{kxx} + 2 g_\varepsilon(x) f_k^{n-1}(u_k) u_{kx}^2 \Big\}_{xx}, & x \in \Omega, \ t > 0, \\ u_{kx} = u_{kxxx} = 0, \qquad x \in \partial\Omega, \ t > 0, \\ u_k(x, 0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{5.2}$$

is non-degenerate, and since u_{0x} has compact support in Ω, standard parabolic theory [14] yields a uniquely determined global solution $u \in C^{4.1}(\bar{\Omega} \times [0, \infty))$.
Now for sufficiently large $k_0 \in \mathbb{N}$ and each $k \ge k_0$, it follows from the continuity of u_k and the positivity of u_0 in $\bar{\Omega}$ that

$$T_k := \sup \Big\{ T > 0 \ \Big| \ \frac{1}{k} \le u_k \le k \text{ in } \Omega \times (0, T) \Big\}$$

is a well-defined element of $(0, \infty]$, and by uniqueness in (5.2), it is clear that the sequence $(T_k)_{k \ge k_0}$ is nondecreasing, and that $u_{k_2} \equiv u_{k_1}$ in $\Omega \times (0, T_{k_1})$ whenever $k_2 \ge k_1 \ge k_0$. Consequently, the definition $T_{max} := \lim_{k\to\infty} T_k \in (0, \infty]$ is meaningful, and the trivially existing pointwise limit $u(x, t) := \lim_{k\to\infty} u_k(x, t)$, $(x, t) \in \bar{\Omega} \times [0, T_{max})$, satisfies $u \equiv u_k$ in $\Omega \times (0, T_k)$ for each $k \ge k_0$. It is therefore evident from (5.2) and the definition of f_k that u actually solves (2.12) in $\Omega \times (0, T_{max})$ with $u|_{t=0} = u_0$ in Ω.
It remains to verify (5.1). To this end, we assume on the contrary that $T_{max} < \infty$, but that both $\liminf_{t\nearrow T_{max}} (\inf_{x\in\Omega} u(x, t)) > 0$ and $\limsup_{t\nearrow T_{max}} (\sup_{x\in\Omega} u(x, t)) < \infty$. Then for some $k \ge k_0$, we would have $\frac{2}{k} \le u \le \frac{k}{2}$ in $\Omega \times (0, T_{max})$, implying that $u \equiv u_k$ in $\Omega \times (0, T_{max})$ by uniqueness. But since u_k is continuous at $t = T_{max}$, this would entail that $T_k > T_{max}$ and hence contradict the definition of T_{max}. □

The following result rules out the occurrence of the first alternative in (5.1); that is, solutions to the approximate problem (2.12) cannot develop a dead core within finite time.

Lemma 5.2 (Absence of dead core formation) *Let $n > 1$, $\alpha > 0$, and $\beta \in \mathbb{R}$. Then for all $\varepsilon \in (0, \varepsilon_0), \delta > 0, M > 0$, and $T > 0$ there exists $C(\varepsilon, \delta, M, T, L) > 0$ such that if $u \in C^{4,1}(\bar{\Omega} \times [0, T))$ is a positive classical solution of (2.12) in $\Omega \times (0, T)$ satisfying*

$$u(x, 0) \ge \delta \qquad \text{for all } x \in \Omega \tag{5.3}$$

and

$$u(x,t) \le M \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T), \tag{5.4}$$

we have the inequality

$$\int_\Omega \frac{1}{u^2(x,t)} dx \le C(\varepsilon, \delta, M, T, L) \qquad \textit{for all } t \in (0,T). \tag{5.5}$$

PROOF. Our goal is to conclude (5.5) from a differential inequality for $\int_\Omega (x+\varepsilon)^\beta u^{-2}$ which we shall thus derive first. To this end, we twice integrate by parts over Ω to compute, using J as defined in (2.13),

$$\begin{aligned}
\frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \frac{1}{u^2} dx &= -2 \int_\Omega (x+\varepsilon)^\beta \frac{u_t}{u^2} dx = -2 \int_\Omega \frac{1}{u^3} J_{xx} dx \\
&= -6 \int_\Omega \frac{u_x}{u^4} J_x dx = 6 \int_\Omega \Big(\frac{u_x}{u^4}\Big)_x J dx \\
&= 6 \int_\Omega \Big[\frac{u_{xx}}{u^4} - 4\frac{u_x^2}{u^5}\Big]\Big[- g_\varepsilon(x) u^n u_{xx} + 2 g_\varepsilon(x) u^{n-1} u_x^2\Big] dx \\
&= -6 \int_\Omega g_\varepsilon(x) u^{n-4} u_{xx}^2 dx + 36 \int_\Omega g_\varepsilon(x) u^{n-5} u_x^2 u_{xx} dx - 48 \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx
\end{aligned}$$

for all $t \in (0,T)$, because $u_x = J_x = 0$ on $\partial\Omega$ according to Lemma 2.2. Since one more integration by parts yields

$$36 \int_\Omega g_\varepsilon(x) u^{n-5} u_x^2 u_{xx} dx = 12(5-n) \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx - 12 \int_\Omega g_{\varepsilon x}(x) u^{n-5} u_x^3 dx,$$

this shows that

$$\begin{aligned}
\frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx &= -6 \int_\Omega g_\varepsilon(x) u^{n-4} u_{xx}^2 dx - 12(n-1) \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx \\
&\quad - 12 \int_\Omega g_{\varepsilon x}(x) u^{n-5} u_x^3 dx
\end{aligned} \tag{5.6}$$

for all $t \in (0,T)$. Here, since $n > 1$, the second term on the right-hand side is nonpositive, and by means of Young's inequality and Lemma 2.1, we can find $c_1 > 0$ and $c_2 > 0$ fulfilling

$$\begin{aligned}
-12 \int_\Omega g_{\varepsilon x}(x) u^{n-5} u_x^3 dx &\le 12(n-1) \int_\Omega g_\varepsilon(x) u^{n-6} u_x^4 dx + c_1 \int_\Omega \frac{g_{\varepsilon x}^4(x)}{g_\varepsilon^3(x)} \cdot u^{n-2} dx \\
&\le c_2 \int_\Omega (x+\varepsilon)^{\alpha-4} u^{n-2} dx,
\end{aligned}$$

whence (5.6) in particular entails that

$$\frac{d}{dt} \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx \le c_2 \int_\Omega (x+\varepsilon)^{\alpha-4} u^{n-2} dx \qquad \text{for all } t \in (0,T). \tag{5.7}$$

Now if $n \geq 2$, writing $c_3(\varepsilon) := \int_\Omega (x+\varepsilon)^{\alpha-4} dx$ and using (5.4), from (5.7) we obtain

$$\frac{d}{dt}\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx \leq c_2 c_3(\varepsilon) M^{n-2} \qquad \text{for all } t \in (0,T),$$

which after integration implies that

$$\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,t)} dx \leq \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,0)} dx + c_2 c_3(\varepsilon) M^{n-2} T \qquad \text{for all } t \in (0,T).$$

As a consequence of (5.3), we thereby find that

$$\begin{aligned}
\int_\Omega \frac{1}{u^2(x,t)} dx &\leq c_4(\varepsilon) \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,t)} dx \\
&\leq c_4(\varepsilon) \Big\{ \frac{c_5(\varepsilon)}{\delta^2} + c_2 c_3(\varepsilon) M^{n-2} T \Big\} \qquad \text{for all } t \in (0,T)
\end{aligned}$$

with $c_4(\varepsilon) := \max\{\varepsilon^{-\beta}, (L+\varepsilon)^{-\beta}\}$ and $c_5(\varepsilon) := \int_\Omega (x+\varepsilon)^\beta dx$.
In the remaining case $n < 2$, we first apply the Hölder inequality with $p = \frac{2}{n}$ and $p' = \frac{2}{2-n}$ to the right-hand side in (5.7) to obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx &\leq c_2 \int_\Omega (x+\varepsilon)^{\alpha-4-(2-n)\frac{\beta}{2}} \big((x+\varepsilon)^\beta u^{-2}\big)^{\frac{2-n}{2}} dx \\
&\leq c_2 c_6(\varepsilon) \Big(\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2} dx \Big)^{\frac{2-n}{2}} \qquad \text{for all } t \in (0,T)
\end{aligned}$$

with

$$c_6(\varepsilon) := \Big(\int_\Omega (x+\varepsilon)^{\frac{2(\alpha-4)-(2-n)\beta}{n}} dx \Big)^{\frac{n}{2}}.$$

Integrating this in time shows that in this case,

$$\int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,t)} dx \leq \Big\{ \int_\Omega (x+\varepsilon)^\beta \cdot \frac{1}{u^2(x,0)} dx + \frac{n}{2} c_2 c_6(\varepsilon) T \Big\}^{\frac{2}{n}} \qquad \text{for all } t \in (0,T),$$

and hence,

$$\int_\Omega \frac{1}{u^2(x,t)} dx \leq c_4(\varepsilon) \cdot \Big\{ \frac{c_5(\varepsilon)}{\delta^2} + \frac{n}{2} c_2 c_6(\varepsilon) T \Big\}^{\frac{2}{n}} \qquad \text{for all } t \in (0,T),$$

according to (5.4). $\square$

6 Hölder continuity

We next derive a spatio-temporal Hölder estimate for the above solutions to the approximate problems. This will allow us to construct a continuous weak solution of (1.3) along a uniformly convergent sequence of appropriate solutions of (2.12) as $\varepsilon \to 0$.

Lemma 6.1 (Hölder estimate) *With $n_\star$ as in Lemma 4.1, assume that $n \in (n_\star, 3)$ and that $\alpha > 0$, $\beta < \alpha - 4$, and $\gamma < 1$ are such that $\alpha - \beta + \gamma > 5$. Moreover, let $A > 0$ and $B > 0$, and let $\varepsilon_\star$ and $T_0(A,B) \in (0,1)$ be as given by Lemma 4.1 and Lemma 4.2, respectively. Then there exists $C(A,B) > 0$ such that, whenever $u \in C^{4,1}(\bar\Omega \times [0,T))$ is a positive classical solution of (2.12) in $\bar\Omega \times (0,T)$ for some $T \in (0, T_0(A,B))$ with*

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2(x,0)dx \le A \qquad \textit{and} \qquad \int_\Omega (x+\varepsilon)^\beta u(x,0)dx \le B,$$

the estimate

$$|u(x_2,t_2) - u(x_1,t_1)| \le C(A,B) \cdot \Big(|x_2 - x_1|^\theta + |t_2 - t_1|^{\frac{\theta}{2\theta+3}} \Big) \tag{6.1}$$

holds for all x_1, $x_2 \in \Omega$ and t_1, $t_2 \in (0,T)$ with $\theta := \min\{\frac{1}{2}, \frac{1-\gamma}{2}\}$.

Proof. According to Lemma 4.2, we can pick c_1, as well as all constants $c_2, \dots$ below, possibly depending on n, α, β, γ, A, B, and L but independent from ε and u, such that

$$\int_\Omega (x+\varepsilon)^\gamma u_x^2 \le c_1 \qquad \text{for all } t \in (0,T).$$

Hence, Lemma 3.6 provides $c_2 > 0$ such that we have the spatial Hölder estimate

$$|u(x_2,t_0) - u(x_1,t_0)| \le c_2 |x_2 - x_1|^\theta \qquad \text{for all } x_1, x_2 \in \Omega \text{ and } t_0 \in (0,T). \tag{6.2}$$

Using this, a corresponding Hölder estimate with respect to the time variable, that is, the inequality

$$|u(x_0,t_2) - u(x_0,t_1)| \le M |t_2 - t_1|^{\frac{\theta}{2\theta+3}} \qquad \text{for all } x_0 \in \Omega \text{ and } t_1, t_2 \in (0,T) \tag{6.3}$$

with suitably large $M > 1$, can be derived by adapting a standard technique due to Gilding and Kružkov ([16], cf. also [5] for a related procedure in a fourth-order setting). Indeed, following [5], let us assume that (6.3) be false, meaning that for some $x_0 \in \Omega$ and $t_1, t_2 \in (0,T)$ we have

$$u(x_0,t_2) - u(x_0,t_1) > M |t_2 - t_1|^{\frac{\theta}{2\theta+3}}, \tag{6.4}$$

where for definiteness we may suppose that $t_1 < t_2$. We then fix any $\zeta \in C_0^\infty(\mathbb{R})$ such that $0 \le \zeta \le 1$ on $\mathbb{R}$, $\zeta \equiv 1$ in $[-\frac{1}{2}, \frac{1}{2}]$ and $\zeta \equiv 0$ in $\mathbb{R} \setminus [-1,1]$, and let

$$\psi(x) := \zeta\Big(\frac{x - x_0}{\eta}\Big), \qquad x \in \bar\Omega,$$

with

$$\eta := \Big(\frac{M}{16 c_2}\Big)^{\frac{1}{\theta}} (t_2 - t_1)^{\frac{1}{2\theta+3}}. \tag{6.5}$$

Furthermore, we introduce the functions ξ_δ, $\delta > 0$, given by

$$\xi_\delta(t) := \frac{1}{\delta} \int_{-\infty}^t \Big\{ \zeta\Big(\frac{s - t_2}{\delta}\Big) - \zeta\Big(\frac{s - t_1}{\delta}\Big) \Big\} ds, \qquad t \in (0,T), \tag{6.6}$$

which belong to $C_0^\infty((0,T))$ and satisfy $0 \ge \xi_\delta \ge -c_3$ with $c_3 := \int_{-1}^t \zeta(\sigma)d\sigma$, provided that $\delta < \delta_0 := \min\{t_1, T-t_2\}$ (this ensures that $\xi_\delta(0) = \xi_\delta(T) = 0$). Therefore, testing (2.12) against $\psi(x)\xi_\delta(t)$, $(x,t) \in \Omega \times (0,T)$, we obtain

$$\int_0^T \int_\Omega u(x,t)\psi(x)\xi'_\delta(t)dxdt = \int_0^T \int_\Omega \Big[(x+\varepsilon)^{-\beta}\psi(x)\xi_\delta(t)\Big]_x J_x(x,t)dxdt \qquad \text{for all } \delta \in (0,\delta_0) \quad (6.7)$$

with J as defined in (2.13), where we again have used that $J_x = 0$ on $\partial\Omega$ by (2.14).
We insert the definition of $\xi'_\delta(t)$, substitute $\sigma = \frac{t-t_i}{\delta}$, $i \in \{1,2\}$, and perform the limit $\delta \to 0$, to estimate the left-hand side in (6.7) from below according to

$$\begin{aligned}
\frac{1}{c_3}\lim_{\delta\searrow 0}\int_0^T \int_\Omega u(x,t)\psi(x)\xi'_\delta(t)dxdt &= \frac{1}{c_3}\lim_{\delta\searrow 0}\int_\Omega\int_{-1}^1 \Big[u(x,t_2+\delta\sigma) - u(x,t_1+\delta\sigma)\Big]\cdot\zeta(\sigma)d\sigma\cdot\psi(x)dx \\
&= \int_\Omega \Big[u(x,t_2) - u(x,t_1)\Big]\cdot\psi(x)dx \\
&\ge \int_\Omega \Big[u(x_0,t_2) - u(x_0,t_1)\Big]\cdot\psi(x)dx \\
&\quad - \int_\Omega \Big[|u(x,t_2) - u(x_0,t_2)| + |u(x_0,t_1) - u(x,t_1)|\Big]\cdot\psi(x)dx,
\end{aligned}$$

whence using (6.4) and (6.2) yields

$$\begin{aligned}
\frac{1}{c_3}\lim_{\delta\searrow 0}\int_0^T \int_\Omega u(x,t)\psi(x)\xi'_\delta(t)dxdt &\ge M(t_2-t_1)^{\frac{\theta}{2\theta+3}}\int_\Omega\psi(x)dx - 2c_2\int_\Omega |x-x_0|^\theta\cdot\psi(x)dx \\
&\ge M(t_2-t_1)^{\frac{\theta}{2\theta+3}}\cdot\frac{\eta}{2} - 2c_2\cdot\eta^\theta\cdot 2\eta \\
&= \frac{\eta}{2}\cdot\Big\{M(t_2-t_1)^{\frac{\theta}{2\theta+3}} - 8c_2\eta^\theta\Big\} \\
&= \frac{\eta}{2}\cdot\frac{1}{2}M(t_2-t_1)^{\frac{\theta}{2\theta+3}} \\
&= c_4 M^{1+\frac{1}{\theta}}(t_2-t_1)^{\frac{\theta+1}{2\theta+3}} \qquad (6.8)
\end{aligned}$$

with $c_4 := [4\cdot(16c_2)^{\frac{1}{\theta}}]^{-1}$. On the right-hand side of (6.7), by the Cauchy-Schwarz inequality and Lemma 4.2, we find $c_5 > 0$ fulfilling

$$\begin{aligned}
&\Big|\int_0^T\int_\Omega \Big[(x+\varepsilon)^{-\beta}\psi(x)\xi_\delta(t)\Big]_x\cdot J_x dxdt\Big| \\
&\le \Big(\int_0^T\int_\Omega (x+\varepsilon)^{-\alpha-\beta+\gamma}J_x^2 dxdt\Big)^{\frac12}\Big(\int_0^T \xi_\delta^2(t)dt\Big)^{\frac12}\Big(\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma}\Big[(x+\varepsilon)^{-\beta}\psi(x)\Big]_x^2 dx\Big)^{\frac12} \\
&\le c_5c_3(t_2-t_1+2\delta)^{\frac12}\cdot\Big(\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma}\Big[(x+\varepsilon)^{-\beta}\psi(x)\Big]_x^2 dx\Big)^{\frac12}, \qquad (6.9)
\end{aligned}$$

since $\xi_\delta(t) = 0$ for $t \le t_1-\delta$ or $t \ge t_2+\delta$ and $\xi_\delta^2 \le c_3^2$. We use $\alpha-\beta-\gamma > 5-2\gamma > 3$ according to our assumptions and $\zeta' \equiv 0$ on $\mathbb{R}\backslash[-1,1]$, and we recall the definition (6.5) of η to find $c_6 > 0$ satisfying

$$\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma}\cdot(x+\varepsilon)^{-2\beta}\psi_x^2(x)dx = \frac{1}{\eta^2}\int_\Omega (x+\varepsilon)^{\alpha-\beta-\gamma}\cdot\zeta\Big(\frac{x-x_0}{\eta}\Big)dx$$

$$
\begin{aligned}
&\le \frac{1}{\eta^2}\cdot (L+1)^{\alpha-\beta-\gamma}\cdot 2\eta\cdot \|\zeta'\|_{L^\infty(\mathbb{R})}\\
&= c_6 M^{-\frac{1}{\theta}}(t_2-t_1)^{-\frac{1}{2\theta+3}}\\
&\le c_6 (t_2-t_1)^{-\frac{1}{2\theta+3}},
\end{aligned}
$$

because of $M>1$ and thus $M^{-\frac{1}{\theta}}<1$. Similarly, with some $c_7>0$ we have

$$
\begin{aligned}
\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma}\cdot(x+\varepsilon)^{-2\beta-2}\psi^2(x)dx &= \int_\Omega (x+\varepsilon)^{\alpha-\beta-\gamma-2}\psi^2(x)dx\\
&\le (L+1)^{\alpha-\beta-\gamma-2}\cdot 2\eta\\
&= c_7\cdot M^{\frac{1}{\theta}}(t_2-t_1)^{\frac{1}{2\theta+3}},
\end{aligned}
$$

whence altogether

$$
\left(\int_\Omega (x+\varepsilon)^{\alpha+\beta-\gamma}\Big[(x+\varepsilon)^{-\beta}\psi(x)\Big]_x^2 dx\right)^{\frac{1}{2}} \le c_8\Big\{(t_2-t_1)^{-\frac{1}{2(2\theta+3)}} + M^{\frac{1}{2\theta}}(t_2-t_1)^{\frac{1}{2(2\theta+3)}}\Big\}
$$

holds with some $c_8>0$. Therefore, (6.7), (6.8), and (6.9) in the limit $\delta\searrow 0$ yield $c_9>0$ such that

$$
\begin{aligned}
\frac{c_3}{4}M^{1+\frac{1}{\theta}}(t_2-t_1)^{\frac{\theta+1}{2\theta+3}} &\le c_9\Big\{(t_2-t_1)^{\frac{1}{2}-\frac{1}{2(2\theta+3)}} + M^{\frac{1}{2\theta}}(t_2-t_1)^{\frac{1}{2}+\frac{1}{2(2\theta+3)}}\Big\}\\
&= c_9\Big\{(t_2-t_1)^{\frac{\theta+1}{2\theta+3}} + M^{\frac{1}{2\theta}}(t_2-t_1)^{\frac{\theta+2}{2\theta+3}}\Big\},
\end{aligned}
$$

which implies the inequality

$$
\begin{aligned}
c_4 M^{1+\frac{1}{\theta}} &\le c_9\Big\{1+M^{\frac{1}{2\theta}}(t_2-t_1)^{\frac{1}{2\theta+3}}\Big\}\\
&\le c_9\Big\{1+M^{\frac{1}{2\theta}}(T_0(A,B))^{\frac{1}{2\theta+3}}\Big\}.
\end{aligned}
$$

Since $1+\frac{1}{\theta}>\frac{1}{2\theta}$, this gives an upper bound for M and thus yields the desired contradiction if M has been chosen suitably large initially. This proves the Hölder estimate (6.3) in time, and combining the latter with (6.2) completes the proof. $\Box$

7 Proof of Theorem 1.1

Let $u_0\in W^{1,2}_\gamma(\Omega)$, where $\gamma<1$, and let $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,1)$ be a sequence satisfying $\varepsilon_j\to 0$ as $j\to\infty$. By a standard approximation argument, we may construct a sequence of functions $(u_{0\varepsilon_j})_{j\in\mathbb{N}}\subset C^\infty(\overline{\Omega})$ such that

$$
u_{0\varepsilon}>0 \quad \text{in } \bar\Omega \qquad \text{and} \qquad u_{0\varepsilon x}\in C_0^\infty(\Omega) \qquad\qquad \text{for all } \varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}, \tag{7.1}
$$

and such that

$$
u_{0\varepsilon}\to u_0 \quad \text{in } W^{1,2}_\gamma(\Omega) \qquad \text{as } \varepsilon=\varepsilon_j\searrow 0. \tag{7.2}
$$

The following lemma asserts that under the assumptions on n, α, β, and γ required in Lemma 4.2, the corresponding solutions of (2.12) emanating from $u_{0\varepsilon_j}$ have their maximal existence time bounded from below for all sufficiently small $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, and moreover they accumulate at some continuous weak solution of (1.3).

Lemma 7.1 *Let n, α, β, and γ be as in Theorem 1.1. Then for all $A>0$ and $B>0$, there exists $T(A,B)\in(0,1)$ such that, whenever $u_0\in W^{1,2}_{loc}(\Omega)$ is nonnegative and satisfies*

$$\int_\Omega x^\gamma u_{0x}^2(x)dx\le A \qquad \textit{and} \qquad \int_\Omega x^\beta u_0(x)dx\le B, \tag{7.3}$$

the following holds: For any $(\varepsilon_j)_{j\in\mathbb{N}}\subset(0,\varepsilon_0)$ such that $\varepsilon_j\to 0$ as $j\to\infty$ and each $(u_{0\varepsilon_j})_{j\in\mathbb{N}}\subset C^\infty(\bar\Omega)$ fulfilling (7.1) and (7.2), the problem (2.12) possesses a unique positive classical solution $u_\varepsilon\in C^{4,1}(\bar\Omega\times[0,T(A,B)])$ for all sufficiently small $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, and moreover, there exists a subsequence $(\varepsilon_{j_l})_{l\in\mathbb{N}}$ such that

$$u_\varepsilon\to u \quad \textit{in } C^0(\bar\Omega\times[0,T(A,B)]) \qquad \textit{as } \varepsilon=\varepsilon_{j_l}\to 0 \tag{7.4}$$

with some continuous weak solution u of (1.3) in $\Omega\times(0,T(A,B))$.

Proof. We claim that the statement is valid if we let $T\equiv T(A,B):=T_0(A+1,B+1)$ with T_0 as provided by Lemma 4.2. To verify this, we first note that, according to (7.3) and upon passing to subsequences, we may assume that

$$\int_\Omega (x+\varepsilon_j)^\gamma u_{0\varepsilon_j x}^2(x)\le A+1 \qquad \text{for all } j\in\mathbb{N} \tag{7.5}$$

and

$$\int_\Omega (x+\varepsilon_j)^\beta u_{0\varepsilon_j}(x)dx\le B+1 \qquad \text{for all } j\in\mathbb{N}. \tag{7.6}$$

For $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, we then let u_ε denote the corresponding positive classical solution of (2.12), that is, of the initial-boundary value problem

$$\begin{cases} u_{\varepsilon t}=\frac{1}{(x+\varepsilon)^\beta}\cdot\Big\{-g_\varepsilon(x)u_\varepsilon^n u_{\varepsilon xxx}+2g_\varepsilon(x)u_\varepsilon^{n-1}u_{\varepsilon x}^2\Big\}_{xx}, & x\in\Omega,\ t>0,\\ u_{\varepsilon x}=u_{\varepsilon xxx}=0, \qquad x\in\partial\Omega,\ t>0, & \\ u_\varepsilon(x,0)=u_{0\varepsilon}(x), \qquad x\in\Omega, & \end{cases}$$

which according to Lemma 5.1 exists up to a maximal time $T_\varepsilon\in(0,\infty]$ having the property stated in (5.1). We divide the proof into four steps.

Step 1. We first show that actually $T_\varepsilon\ge T$.
To see this, we apply Lemma 4.2 to find, upon passing to a subsequence if necessary, that for some $c_1>0$ and all $\varepsilon\in(\varepsilon_j)_{j\in\mathbb{N}}$, we have

$$\begin{aligned} &\sup_{t\in(0,\hat T_\varepsilon)}\int_\Omega (x+\varepsilon)^\gamma u_{\varepsilon x}^2(x,t)dx\\ &+\int_0^{\hat T_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u_\varepsilon^{n-2}u_{\varepsilon x}^2u_{\varepsilon xx}^2dxdt+\int_0^{\hat T_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma}u_\varepsilon^{n-4}u_{\varepsilon x}^6dxdt\\ &+\int_0^{\hat T_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u_\varepsilon^{n}u_{\varepsilon xx}^2dxdt+\int_0^{\hat T_\varepsilon}\int_\Omega (x+\varepsilon)^{\alpha-\beta+\gamma-2}u_\varepsilon^{n-2}u_{\varepsilon x}^4dxdt\\ \le\ & c_1, \end{aligned} \tag{7.7}$$

where $\hat{T}_\varepsilon := \min\{T_\varepsilon, T\}$. Since

$$\sup_{t\in(0,\hat{T}_\varepsilon)} \int_\Omega (x+\varepsilon)^\beta u_\varepsilon(x,t)dx = \int_\Omega (x+\varepsilon)^\beta u_{0\varepsilon}(x)dx \le B+1$$

by Lemma 2.3, Lemma 3.7 yields $c_2 > 0$ such that for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$,

$$\sup_{t\in(0,\hat{T}_\varepsilon)} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2. \tag{7.8}$$

Moreover, for fixed $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ we may apply Lemma 3.6 with γ replaced by 0 to see that (7.7) entails that with some $c_3(\varepsilon) > 0$, the spatial Hölder estimate

$$|u_\varepsilon(x,t) - u_\varepsilon(y,t)| \le c_3(\varepsilon)|x-y|^{\frac{1}{2}} \tag{7.9}$$

is valid for all $x, y \in \bar{\Omega}$ and any $t \in (0, \hat{T}_\varepsilon)$. Now assuming that $T_\varepsilon < T$ for some $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$, in view of the extensibility criterion in Lemma 5.1 and the inequality (7.8), we would have

$$u_\varepsilon(x_k, t_k) \to 0 \qquad \text{as } k \to \infty$$

with some $(x_k)_{k\in\mathbb{N}} \subset \Omega$ and $(t_k)_{k\in\mathbb{N}} \subset (0, T_\varepsilon)$, where we may assume that $x_k \to x_0$ and $t_k \nearrow T_\varepsilon$ as $k \to \infty$ with some $x_0 \in \bar{\Omega}$. According to (7.8), (7.9), and the Arzelà-Ascoli theorem, we may pass to subsequences to achieve that with some $v \in C^0(\bar{\Omega})$ we have

$$u_\varepsilon(\cdot, t_k) \to v \quad \text{in } C^0(\bar{\Omega}) \qquad \text{as } k \to \infty, \tag{7.10}$$

and conclude that $v(x_0) = 0$ and hence, again by (7.9), that

$$0 \le v(x) \le c_3(\varepsilon)|x - x_0|^{\frac{1}{2}} \qquad \text{for all } x \in \Omega. \tag{7.11}$$

This, however, contradicts the outcome of Lemma 5.2: The latter, namely, along with (7.8) implies that with some $c_4(\varepsilon) > 0$ we have

$$\int_\Omega \frac{1}{u_\varepsilon^2(x,t_k)} dx \le c_4(\varepsilon) \qquad \text{for all } k \in \mathbb{N},$$

so that Fatou's lemma and (7.11) give

$$\frac{1}{c_3^2(\varepsilon)} \int_\Omega \frac{1}{|x-x_0|} dx \le \int_\Omega \frac{1}{v^2(x)} dx \le c_4(\varepsilon),$$

which is impossible.

Step 2. We next construct the limit function u.
To achieve this, we observe that since $T_\varepsilon \ge T$ according to the above arguments, we may replace $\hat{T}_\varepsilon$ by T in (7.7) and (7.8) and apply Lemma 6.1 to derive the ε-independent estimate

$$\|u_\varepsilon\|_{C^{\theta, \frac{\theta}{2\theta+3}}(\bar{\Omega}\times[0,T])} \le c_5 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$$

with a certain $c_5 > 0$. Therefore, the Arzelà-Ascoli theorem yields a subsequence, again denoted by $(\varepsilon_j)_{j\in\mathbb{N}}$, and a nonnegative function $u \in C^0(\bar\Omega \times [0,T])$ such that

$$u_\varepsilon \to u \qquad \text{in } C^0(\bar\Omega \times [0,T]) \tag{7.12}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, interior parabolic regularity theory [14] shows that $(u_{\varepsilon_j})_{j\in\mathbb{N}}$ is relatively compact in $C^{4,1}_{loc}(((0,L] \times (0,T]) \cap \{u > 0\})$, and hence we may assume that as $\varepsilon = \varepsilon_j \searrow 0$, we also have

$$u_\varepsilon \to u \qquad \text{in } C^{4,1}_{loc}(\mathcal{P}), \qquad \text{where } \mathcal{P} := \Big((0,L] \times (0,T]\Big) \cap \{u > 0\}. \tag{7.13}$$

Step 3. We proceed to verify that there exists a null set $N \subset (0,T)$ such that for all $t \in (0,T) \setminus N$, $u(\cdot,t)$ is differentiable at $x = L$ with $u_x(L,t) = 0$ for all $t \in (0,T) \setminus N$.
To this end, we note that (7.7) in particular implies that for some $c_6 > 0$, we have

$$\int_0^T \int_{\frac{L}{2}}^L u_\varepsilon^{n-2} u_{\varepsilon x}^2 u_{\varepsilon xxx}^2 dxdt + \int_0^T \int_{\frac{L}{2}}^L u_\varepsilon^{n-4} u_{\varepsilon x}^6 dxdt \le c_6 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$$

and since

$$\begin{aligned}
\Big\{ \Big(u_\varepsilon^{\frac{n+2}{4}} \Big)_x^2 \Big\}_x^2 &= \Big(\frac{n+2}{4}\Big)^4 \cdot \Big\{ 2u_\varepsilon^{\frac{n-2}{2}} u_{\varepsilon x} u_{\varepsilon xxx} + \frac{n-2}{2} u_\varepsilon^{\frac{n-4}{2}} u_{\varepsilon x}^3 \Big\}^2 \\
&\le 2\Big(\frac{n+2}{4}\Big)^4 \cdot \Big\{ 4u_\varepsilon^{n-2} u_{\varepsilon x}^2 u_{\varepsilon xxx}^2 + \Big(\frac{n-2}{2}\Big)^2 u_\varepsilon^{n-4} u_{\varepsilon x}^6 \Big\} \qquad \text{in } \Omega \times (0,T),
\end{aligned}$$

we thus find $c_7 > 0$ fulfilling

$$\int_0^T \int_{\frac{L}{2}}^L \Big\{ \Big(u_\varepsilon^{\frac{n+2}{4}} \Big)_x^2 \Big\}_x^2 dxdt \le c_7 \qquad \text{for all } \varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}. \tag{7.14}$$

Since $u_{\varepsilon x}(L,t) = 0$ and hence $(u_\varepsilon^{\frac{n+2}{4}})_x(L,t) = 0$ for all $t \in (0,T)$ by (2.12), using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\Big(u_\varepsilon^{\frac{n+2}{4}} \Big)_x^2 (x,t) &= -\int_x^L \Big\{ \Big(u_\varepsilon^{\frac{n+2}{4}} \Big)_x^2 \Big\}_x (y,t)dy \\
&\le (L-x)^{\frac{1}{2}} \cdot a_\varepsilon(t) \qquad \text{for all } x \in \Big(\frac{L}{2}, L\Big) \text{ and } t \in (0,T),
\end{aligned} \tag{7.15}$$

where

$$a_\varepsilon(t) := \int_{\frac{L}{2}}^L \Big\{ \Big(u_\varepsilon^{\frac{n+2}{4}} \Big)_x^2 \Big\}_x^2 (y,t)dy, \qquad t \in (0,T).$$

Again by the Cauchy-Schwarz inequality, (7.15) in turn implies that

$$\begin{aligned}
\Big| u_\varepsilon^{\frac{n+2}{4}}(L-x,-t) - u_\varepsilon^{\frac{n+2}{4}}(x,t) \Big| &= \Big| \int_x^L \Big(u_\varepsilon^{\frac{n+2}{4}} \Big)_x (y,t)dy \Big| \\
&\le (L-x)^{\frac{1}{2}} \cdot \Big\{ \int_x^L (L-y)^{\frac{1}{2}} \cdot a_\varepsilon^{\frac{1}{2}}(t)dy \Big\}^{\frac{1}{2}} \\
&= \sqrt{\frac{2}{3}} (L-x)^{\frac{5}{4}} \cdot a_\varepsilon^{\frac{1}{4}}(t) \qquad \text{for all } x \in \Big(\frac{L}{2}, L\Big) \text{ and } t \in (0,T),
\end{aligned}$$

by (7.14) and the definition of a_ε meaning that

$$\int_0^T \sup_{x\in(\frac{L}{2},L)} \frac{\Big|u_\varepsilon^{\frac{n+2}{4}}(L,t) - u_\varepsilon^{\frac{n+2}{4}}(x,t)\Big|^4}{(L-x)^5}\,dt \le \frac{4}{9}c_7.$$

Using (7.12) and Fatou's lemma, from this we conclude that

$$\int_0^T \sup_{x\in(\frac{L}{2},L)} \frac{\Big|u^{\frac{n+2}{4}}(L,t) - u^{\frac{n+2}{4}}(x,t)\Big|^4}{(L-x)^5}\,dt \le \frac{4}{9}c_7,$$

so that in particular we can find a null set $N\subset(0,T)$ such that for all $t\in(0,T)\setminus N$,

$$b(t) := \sup_{x\in(\frac{L}{2},L)} \frac{\Big|u^{\frac{n+2}{4}}(L,t) - u^{\frac{n+2}{4}}(x,t)\Big|^4}{(L-x)^5}$$

is finite.
Now if $t\in(0,T)\setminus N$ is such that $u(L,t)>0$, then from (7.13) we clearly infer the existence of $u_x(L,t) = \lim_{\varepsilon=\varepsilon_j\searrow 0} u_{\varepsilon x}(L,t) = 0$. On the other hand, if $t\in(0,T)\setminus N$ is such that $u(L,t)=0$, then according to the definition of $b(t)$, we obtain

$$u^{n+2}(x,t) \le b(t)\cdot(L-x)^5 \qquad \text{for all } x\in\Big(\frac{L}{2},L\Big),$$

that is,

$$\begin{aligned}
\Big|\frac{u(L,t)-u(x,t)}{L-x}\Big| &= \frac{u(x,t)}{L-x} \\
&\le \frac{\Big\{b(t)\cdot(L-x)^5\Big\}^{\frac{1}{n+2}}}{L-x} \\
&= b^{\frac{1}{n+2}}(t)\cdot(L-x)^{\frac{3-n}{n+2}} \qquad \text{for all } x\in\Big(\frac{L}{2},L\Big),
\end{aligned}$$

so that, since $n<3$, we infer that also in this case $u_x(L,t)$ exists and vanishes.

Step 4. We finally show that u furthermore satisfies the integral identity (1.6).
To prepare this, let us first derive two further estimates from (7.7): Namely, for any $q\in[1,2)$, (7.7) along with Lemma 2.1 and the Hölder inequality implies that for all measurable $\Omega_0\subset\Omega$ and any measurable $Q\subset\Omega_0\times(0,T)$, we have

$$\begin{aligned}
\iint_Q \Big|g_\varepsilon(x)u_\varepsilon^n u_{\varepsilon xxx}\Big|^q dxdt &\le \iint_Q (x+\varepsilon)^{q\alpha} u_\varepsilon^{nq}|u_{\varepsilon xxx}|^q dxdt \\
&\le \Big(\iint_Q (x+\varepsilon)^{\alpha-\beta+\gamma-2} u_\varepsilon^n u_{\varepsilon xxx}^2 dxdt\Big)^{\frac{q}{2}} \Big(\iint_Q (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} u_\varepsilon^{\frac{nq}{2-q}} dxdt\Big)^{\frac{2-q}{2}} \\
&\le c_1^{\frac{q}{2}} T^{\frac{2-q}{2}}\cdot\Big(\int_{\Omega_0} (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} dx\Big)^{\frac{2-q}{2}}\cdot\|u_\varepsilon\|_{L^\infty(Q)}^{\frac{nq}{2}}, \qquad (7.16)
\end{aligned}$$

whereas similarly,

$$
\begin{aligned}
\iint_Q \Big| g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \Big|^q dxdt &\le \iint_Q (x+\varepsilon)^{q\alpha} u_\varepsilon^{(n-1)q} |u_{\varepsilon x}|^{2q} dxdt \\
&\le \Big(\iint_Q (x+\varepsilon)^{\alpha-\beta+\gamma-2} u_\varepsilon^{n-2} u_{\varepsilon x}^4 dxdt \Big)^{\frac{q}{2}} \\
&\quad \times \Big(\iint_Q (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} u_\varepsilon^{\frac{nq}{2-q}} dxdt \Big)^{\frac{2-q}{2}} \\
&\le c_1^{\frac{q}{2}} T^{\frac{2-q}{2}} \cdot \Big(\int_{\Omega_0} (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} dx \Big)^{\frac{2-q}{2}} \cdot \|u_\varepsilon\|_{L^\infty(Q)}^{\frac{nq}{2}}. \qquad (7.17)
\end{aligned}
$$

In view of our assumptions $\alpha > 3$, $\beta > -1$, and $\gamma < 1$, we infer that $\alpha+\beta-\gamma+2 > 3$. Hence, picking any $q \in (1,2)$ we know that

$$
c_8 := \sup_{\varepsilon\in(0,1)} \Big(\int_\Omega (x+\varepsilon)^{\frac{q(\alpha+\beta-\gamma+2)}{2-q}} dx \Big)^{\frac{2-q}{2}}
$$

is finite. Then (7.16) and (7.17), applied to $Q := Q_T := \Omega \times (0,T)$, show that because of $q > 1$, we may pass to a further subsequence to achieve that

$$
g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} \rightharpoonup w \qquad \text{in } L^q(\Omega\times(0,T))
$$

and

$$
g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \rightharpoonup z \qquad \text{in } L^q(\Omega\times(0,T))
$$

as $\varepsilon = \varepsilon_j \to 0$ with some w and z belonging to $L^q(\Omega\times(0,T))$. In view of the pointwise convergence properties $u_{\varepsilon x} \to u_x$ and $u_{\varepsilon xxx} \to u_{xxx}$ inside $\mathcal{P}$, as guaranteed by (7.13), we may identify these limits to obtain that actually

$$
g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} \rightharpoonup x^\alpha u^n u_{xxx} \qquad \text{in } L^q(\mathcal{P}) \tag{7.18}
$$

and

$$
g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \rightharpoonup x^\alpha u^{n-1} u_x^2 \qquad \text{in } L^q(\mathcal{P}), \tag{7.19}
$$

because $n > 1$.

Next, outside the set $\mathcal{P}$, we may use that $u_\varepsilon \to 0$ uniformly in $Q_T \setminus \mathcal{P}$ to infer upon another application of (7.13) and (7.16) to $q := 1$ that

$$
\iint_{Q_T\setminus\mathcal{P}} g_\varepsilon(x) u_\varepsilon^n |u_{\varepsilon xxx}| dxdt \to 0 \tag{7.20}
$$

and

$$
\iint_{Q_T\setminus\mathcal{P}} g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 dxdt \to 0 \tag{7.21}
$$

as $\varepsilon = \varepsilon_j \to 0$, noting here again that $\alpha+\beta-\gamma+2 > 0$ by assumption.

Now for the verification of (1.6), we fix any $\phi \in C_0^\infty(\bar\Omega \times [0,T))$ such that $\phi_x = 0$ at $x = L$. We then approximate ϕ by letting

$$\phi_\delta(x,t) := \phi(0,t) + \int_0^x \zeta_\delta(y)\phi_x(y,t)dy, \qquad (x,t) \in \bar\Omega \times [0,T), \tag{7.22}$$

for $\delta \in (0, \frac{L}{2})$, where

$$\zeta_\delta(x) := \zeta\Big(\frac{x}{\delta}\Big), \qquad x \in \bar\Omega,$$

with a fixed cut-off function $\zeta \in C^\infty\mathbb{R})$ such that $\zeta \equiv 0$ in $(-\infty, 1]$, $\zeta \equiv 1$ in $[2, \infty)$ and $0 \le \zeta' \le 2$ on $\mathbb{R}$. This construction ensures that $\phi_{\delta x}$ vanishes at both $x = L$ and $x = 0$, so that upon multiplying (2.12) by $(x+\varepsilon)^\beta \phi_\delta$, we may integrate by parts, again using Lemma 2.2, to obtain

$$\begin{aligned} -\int_0^T \int_\Omega (x+\varepsilon)^\beta u_\varepsilon \phi_{\delta t} dxdt \quad &- \quad \int_\Omega (x+\varepsilon)^\beta u_{0\varepsilon}(x)\phi_\delta(x,0)dx \\ &= \quad \int_0^T \int_\Omega \Big[- g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} + 2g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \Big] \cdot \phi_{\delta xx} dxdt \end{aligned} \tag{7.23}$$

for all $\varepsilon \in (\varepsilon_j)_{j\in\mathbb{N}}$ and all $\delta \in (0, \frac{L}{2})$. Here, from (7.12) and the fact that $u_{0\varepsilon} \to u_0$ in $C^0(\bar\Omega)$ by (7.2) and the restriction $\gamma < 1$, it is clear that

$$-\int_0^T \int_\Omega (x+\varepsilon)^\beta u_\varepsilon \phi_{\delta t} dxdt \to -\int_0^T \int_\Omega x^\beta u \phi_t dxdt$$

and

$$-\int_\Omega (x+\varepsilon)^\beta u_{0\varepsilon}(x)\phi_\delta(x,0)dx \to -\int_\Omega x^\beta u_0(x)\phi_\delta(x,0)dx$$

as $\varepsilon = \varepsilon_j \to 0$, whereas (7.18)-(7.21) warrant that

$$\int_0^T \int_\Omega \Big[- g_\varepsilon(x) u_\varepsilon^n u_{\varepsilon xxx} + 2g_\varepsilon(x) u_\varepsilon^{n-1} u_{\varepsilon x}^2 \Big] \cdot \phi_{\delta xx} dxdt \to \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{\delta xx} dxdt$$

as $\varepsilon = \varepsilon_j \to 0$; hence, (7.23) yields

$$-\int_0^T \int_\Omega x^\beta u \phi_{\delta t} dxdt - \int_\Omega x^\beta u_0(x)\phi_\delta(x,0)dx = \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{\delta xx} dxdt \tag{7.24}$$

for all $\delta \in (0, \frac{L}{2})$. Now, taking $\delta \searrow 0$, we observe that by (7.22),

$$\phi_{\delta xx}(x,t) = \zeta_\delta(x) \cdot \phi_{xx}(x,t) + \frac{1}{\delta}\zeta'\Big(\frac{x}{\delta}\Big) \cdot \phi_x(x,t) \qquad \text{for all } (x,t) \in \Omega \times (0,T),$$

so that since $0 \le \zeta' \le 2$ we find that

$$\begin{aligned}
&\left| \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{\delta xx} dxdt - \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot \phi_{xx} dxdt \right| \\
\le \quad & \|\phi_{xx}\|_{L^\infty(\Omega\times(0,T))} \cdot \left| \iint_{\mathcal{P}} [-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2] \cdot (1-\zeta_\delta(x)) dxdt \right| \\
& + \frac{2}{\delta} \|\phi_x\|_{L^\infty(\Omega\times(0,T))} \cdot \iint_{S_\delta} x^\alpha u^n |u_{xx}| dxdt + \frac{4}{\delta} \|\phi_x\|_{L^\infty(\Omega\times(0,T))} \cdot \iint_{S_\delta} x^\alpha u^{n-1} u_x^2 dxdt \\
=: \quad & I_1(\delta) + I_2(\delta) + I_3(\delta),
\end{aligned}$$

where $S_\delta := ((0,2\delta)\times(0,T)) \cap \mathcal{P}$. Clearly,

$$I_1(\delta) \to 0 \qquad \text{as } \delta \searrow 0$$

by the dominated convergence theorem in conjunction with the integrability property $-x^\alpha u^n u_{xx} + 2x^\alpha u^{n-1} u_x^2 \in L^1(\mathcal{P})$ asserted by (7.18) and (7.19). Moreover, applying (7.13) and (7.16) to $Q := ((0,2\delta)\times(0,T)) \cap \mathcal{P}$ and $q := 1$ and once more recalling (7.18) and (7.19), we see that

$$\iint_{S_\delta} x^\alpha u^n |u_{xx}| dxdt \le c_9 \Big(\int_0^{2\delta} x^{\alpha+\beta-\gamma+2} dx \Big)^{\frac{1}{2}} \le c_{10} \delta^{\frac{\alpha+\beta-\gamma+3}{2}}$$

and similarly

$$\iint_{S_\delta} x^\alpha u^{n-1} u_x^2 dxdt \le c_{11} \delta^{\frac{\alpha+\beta-\gamma+3}{2}}$$

for all $\delta \in (0, \frac{L}{2})$ with positive constants c_9, c_{10}, and c_{11}. As our hypotheses $\alpha > 3$, $\beta > -1$, and $\gamma < 1$ guarantee that $\frac{\alpha+\beta-\gamma+3}{2} > 1$, we thus obtain that also

$$I_2(\delta) + I_3(\delta) \to 0 \qquad \text{as } \delta \searrow 0,$$

so that, since clearly $\phi_\delta \to \phi$ and $\phi_{\delta t} \to \phi_t$ uniformly in $\Omega \times (0,t)$, we conclude from (7.24) that indeed (1.6) is valid. $\Box$

We can now prove our main result.

Proof of Theorem 1.1. According to Lemma 7.1 with $K > 0$ as in (4.1), we know that there exists $T > 0$ and a continuous weak solution u of (1.3) in $\Omega \times (0,T)$, which due to Lemma 4.1 and the approximation statement in Lemma 7.1 has the additional regularity property

$$u \in L^\infty((0,T); W^{1,2}_\gamma(\Omega)) \tag{7.25}$$

and satisfies

$$\int_\Omega x^\gamma u_x^2(x,t) dx \le \int_\Omega x^\gamma u_{0x}^2(x) dx + K \int_0^t \int_\Omega x^{\alpha-\beta+\gamma-6} u^{n+2} dxds \qquad \text{for a.e. } t \in (0,T). \tag{7.26}$$

From Lemma 2.3 combined with Lemma 7.1, we infer that moreover

$$\int_\Omega x^\beta u(x,t)dx = B_0 := \int_\Omega x^\beta u_0(x)dx \qquad \text{for all } t \in (0,T). \tag{7.27}$$

Therefore,

$$T_{max} := \sup\Big\{T>0 \;\Big|\; \text{There exists a continuous weak solution } u \text{ of (1.3) in } \Omega\times(0,T) \text{ which satisfies (7.25), (7.26) and (7.27)}\Big\} \le \infty$$

is well-defined, and it remains to show that (1.7) holds.
Indeed, let us assume on the contrary that $T_{max} < \infty$ but $u \le M$ in $\Omega \times (0, T_{max})$ for some $M > 0$. Then (7.26) would imply that

$$\int_\Omega x^\gamma u_x^2(x,t)dx \le A_0 := \int_\Omega x^\gamma u_{0x}^2(x)dx + KM^{n+2}T_{max}\int_\Omega x^{\alpha-\beta+\gamma-6}dx \quad \text{for a.e. } t\in(0,T_{max}),$$

where our assumption $\gamma > 5 - \alpha + \beta$ ensures that $\alpha - \beta + \gamma - 6 > -1$ and hence $A_0 < \infty$. We could thus pick some $t_0 \in (0, T_{max})$ such that

$$t_0 > T_{max} - \frac{1}{2}T(A_0,B_0) \qquad \text{and} \qquad \int_\Omega x^\gamma u_x^2(x,t_0)dx \le A_0,$$

to see upon another application of Lemma 7.1 to $A := A_0$, $B := B_0$ and

$$v_0(x) := u(x,t_0), \qquad x\in\Omega,$$

that the problem

$$\begin{cases} v_t = \frac{1}{x^\beta}\cdot\Big\{x^\alpha[-v^n v_{xx} + 2v^{n-1}v_x^2]\Big\}_{xx}, & x\in\Omega,\ t>0,\\ x^\alpha[-v^n u_{xx} + 2v^{n-1}u_x^2] = x^\alpha[-v^n v_{xx} + 2v^{n-1}v_x^2]_x = 0, & x=0,\ t>0,\\ v_x = v_{xxx} = 0, & x=L,\ t>0,\\ v(x,0) = v_0(x), & x\in\Omega, \end{cases}$$

would possess a continuous weak solution v in $\Omega \times (0, T(A_0, B_0))$ which, again by Lemma 4.1, Lemma 2.3, and Lemma 7.1, would satisfy $v \in L^\infty((0, T(A_0, B_0)); W^{1,2}_\gamma(\Omega))$ and

$$\int_\Omega x^\gamma v_x^2(x,t)dx \le \int_\Omega x^\gamma v_{0x}^2(x)dx + K\int_0^t\int_\Omega x^{\alpha-\beta+\gamma-6}v^{n+2}dxds \quad \text{for a.e. } t\in(0,T(A_0,B_0))$$

as well as

$$\int_\Omega x^\beta v(x,t)dx = B_0 \qquad \text{for all } t\in(0,T(A_0,B_0)).$$

It can therefore easily be checked that

$$\tilde{u}(x,t) := \begin{cases} u(x,t) & \text{if } x\in\Omega \text{ and } t\in(0,t_0),\\ v(x,t-t_0) & \text{if } x\in\Omega \text{ and } t\in[t_0,t_0+T(A_0,B_0)), \end{cases}$$

would define a continuous weak solution $\tilde{u}$ of (1.3) in $\Omega \times (0, t_0 + T(A_0, B_0))$, yet fulfilling (7.25), (7.26), and (7.27). As $t_0 + T(A_0, B_0) > T_{max}$, this contradicts the definition of T_{max}. □

Acknowledgment. The first author acknowledges partial support from the Austrian Science Fund (FWF), grants P20214, P22108, I395, and W1245. He thanks Miquel Escobedo for the hint on the paper [17], which initiated this study.

References

[1] T. Allemand and G. Toscani. The grazing collision limit of Kac caricature of Bose-Einstein particles. *Asympt. Anal.* 72 (2011), 201-229.

[2] J. Becker and G. Grün. The thin-film equation: recent advances and some new perspectives. *J. Phys. Condensed Matter* 17 (2005), 291-307.

[3] E. Beretta, M. Bertsch, and R. Dal Passo. Nonnegative solutions of a fourth-order nonlinear degenerate parabolic equation. *Arch. Rat. Mech. Anal.* 129 (1995), 175-200.

[4] F. Bernis. Finite speed of propagation and continuity of the interface for thin viscous flows. *Adv. Differ. Eq.* 1 (1996), 337-368.

[5] F. Bernis and A. Friedman. Higher order nonlinear degenerate parabolic equations. *J. Diff. Eqs.* 83 (1990), 179-206.

[6] A. Bertozzi. The mathematics of moving contact lines in thin liquid films. *Notices Amer. Math. Soc.* 45 (1998), 689-697.

[7] A. Bertozzi and M. Pugh. The lubrication approximation for thin viscous films: regularity and long-time behavior of weak solutions. *Commun. Pure Appl. Math.* 49 (1996), 85-123.

[8] M. Bukal, A. Jüngel, and D. Matthes. A multidimensional nonlinear sixth-order quantum diffusion equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire* 30 (2013), 337-365.

[9] J. A. Carrillo, M. Di Francesco, G. Toscani. Condensation phenomena in nonlinear drift equations. Preprint, 2013. `arXiv:1307.2275.`

[10] R. Dal Passo, H. Garcke, and G. Grün. On a fourth order degenerate parabolic equation: global entropy estimates and qualitative behavior of solutions. *SIAM J. Math. Anal.* 29 (1998), 321-342.

[11] M. Escobedo, M. Herrero, and J. Velázquez. A nonlinear Fokker-Planck equation modelling the approach to thermal equilibrium in a homogeneous plasma. *Trans. Amer. Math. Soc.* 350 (1998), 3837-3901.

[12] M. Escobedo and J. Velázquez. Finite time blow-up for the bosonic Nordheim equation. Preprint, 2012. `arXiv:1206.5410.`

[13] J. Evans, V. Galaktionov, and J. King. Unstable sixth-order thin film equation: I. Blow-up similarity solutions. *Nonlinearity* 20 (2007), 1799-1841.

[14] A. Friedman. *Partial Differential Equations of Parabolic Type*. Prentice-Hall, Englewood Cliffs, 1964.

[15] U. Gianazza, G. Savaré, and G. Toscani. The Wasserstein gradient flow of the Fisher information and the quantum drift-diffusion equation. *Arch. Rat. Mech. Anal.* 194 (2009), 133-220.

[16] B. Gilding, Hölder continuity of solutions of parabolic equations. *J. London Math. Soc.* 13 (1976), 103-106.

[17] C. Josserand, Y. Pomeau, and S. Rica. Self-similar singularities in the kinetics of condensation. *J. Low Temp. Phys.* 145 (2006), 231-265.

[18] A. Jüngel and D. Matthes. The Derrida-Lebowitz-Speer-Spohn equation: existence, non-uniqueness, and decay rates of the solutions. *SIAM J. Math. Anal.* 39 (2008), 1996-2015.

[19] A. Jüngel and M. Winkler. A degenerate fourth-order parabolic equation modeling Bose-Einstein condensation. Part II: Finite-time blow-up. Preprint, 2013.

[20] G. Kaniadakis and P. Quarati. Kinetic equation for classical particles obeying an exclusion principle. *Phys. Rev. E* 48 (1993), 4263-4270.

[21] G. Kaniadakis and P. Quarati. Classical model of bosons and fermions. *Phys. Rev. E* 49 (1994), 5103-5110.

[22] A. Kompaneets. The establishment of thermal equilibrium between quanta and electrons. *Soviet Physics JETP* 4 (1957), 730-737.

[23] L. Nordheim. On the kinetic method in the new statistics and its application in the electron theory of conductivity, *Proc. R. Soc. Lond. A* 119 (1928), 689-698.

[24] I. Pawlow and W. Zajaczkowski. On a class of sixth order viscous Cahn-Hilliard type equations. *Discrete Contin. Dyn. Syst. S* 6 (2013), 517-546.

[25] H. Spohn. Kinetics of the Bose-Einstein condensation. *Physica D* 239 (2010), 627-634.

[26] G. Toscani. Finite time blow up in Kaniadakis-Quarati model of Bose-Einstein particles. *Commun. Part. Diff. Eqs.* 37 (2012), 77-87.

[27] M. Winkler. Global solutions in higher dimensions to a fourth-order parabolic equation modeling epitaxial thin-film growth. *Z. Angew. Math. Phys.* 62 (2011), 575-608.