Global Existence and Uniform Boundedness of Smooth Solutions to a Cross-Diffusion System with Equal Diffusion Rates

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Abstract

We consider the Shigesada-Kawasaki-Teramoto cross-diffusion model for two competing species. If both species have the same random diffusion coefficients and the space dimension is less than or equal to three, we establish the global existence and uniform boundedness of smooth solutions to the model in convex domains. This extends some previous works of Kim [12] and Shim [21] in one dimensional space.

Key words: Cross diffusion system, global existence, smooth solution, uniform boundedness
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1 Introduction

Cross-diffusion system is an important class of reaction-diffusion problems, and it has received considerable attentions for several decades [17, 18, 27]. At the individual level, the basic underlying assumption for cross-diffusion is that the transition probability from the departure point to the arrival point only depends upon departure conditions, e.g., population density and environmental condition at the departure location [19]. To model the spatial segregation of interacting species, Shigesada *et al.* [22] proposed the following cross-diffusion model for two competing species (abbreviated as S-K-T model henceforth):

$$\begin{cases} u_t = \Delta\left[(d_1 + \alpha v)u\right] + u(a_1 - b_1 u - c_1 v), & x \in \Omega,\ t > 0, \\ v_t = \Delta\left[(d_2 + \beta u)v\right] + v(a_2 - b_2 u - c_2 v), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $u(x,t), v(x,t)$ represent the density of two species at location x and time t, d_1, d_2 are their random diffusion rates, α and β are cross-diffusion coefficients, Ω is an open bounded domain in $\mathbb{R}^n$ with smooth boundary, denoted by $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, and the homogeneous Neumann boundary conditions mean that species do not cross the boundary. The parameters a_i $(i = 1, 2)$ are the intrinsic growth rates for two species, respectively, b_1, c_2 account for the intraspecific competition, and c_1, b_2 are the interspecific competition coefficients. We assume that $a_i, b_i, c_i,\ \alpha, \beta$ are positive constants. The initial data u_0 and v_0 are sufficiently smooth nonnegative functions satisfying $u_0 \not\equiv 0$ and $v_0 \not\equiv 0$. One interesting feature of the S-K-T model is that the highest order of derivatives for the densities of both species appear in each of the two equations. The original S-K-T model also includes self-diffusion and biased movement along the environmental gradient. We are neglecting those terms as they do not cause any additional technical difficulty on the global existence of smooth solutions to (1.1), so that we can focus on the effect of cross-diffusion.

In a series of works [1, 2, 3], Amann proved the local existence of smooth solutions to the S-K-T model. However, the global existence of smooth solutions has not yet been fully established, except the following cases:

For the case $\alpha, \beta > 0$, Kim [12] established the global existence of smooth solutions when $n = 1$ and $d_1 = d_2$. Later on Shim [21] proved the uniform boundedness of smooth solutions when $n = 1$ and $d_1 = d_2$. For any $n \geq 1$, Deuring [8] showed the existence of smooth solutions to the S-K-T model if the cross-diffusion coefficients are small relative to the initial data.

In the case $\alpha = 0$ or $\beta = 0$, there have been a series of development: When $n = 2$, Lou *et al.* [16] proved the existence of smooth solutions to the S-K-T model. The method in [16] can also be modified to cover the case $n = 1$. Choi *et al.* [7] and Le *et al.* [13] independently settled the case $n \leq 5$. Tuoc [25] proved the existence of smooth solutions to the S-K-T model when $n \leq 9$. Recently, Hoang *et al.* [10] established the global existence of smooth solutions for any space dimension.

We refer to [14, 24, 26] for related works where the self-diffusion plays an important role. The existence of global weak solutions are considered in [5, 6].

The goal of this paper is to extend the results of Kim [12] and Shim [21] to the case $d_1 = d_2$ and $n \le 3$ in convex domains. As $d_1 = d_2$, after suitable scaling we can rewrite the S-K-T model as

$$\begin{cases} u_t = \Delta u + a\Delta(uv) + \mu u(1 - \alpha_1 u - \alpha_2 v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + b\Delta(uv) + \nu u(1 - \beta_1 u - \beta_2 v), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded convex domain, while $a, b, \mu, \nu, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive parameters.

Theorem 1.1 *Let $n \le 3$ and $\Omega \subset \mathbb{R}^n$ be a bounded convex domain with smooth boundary, and suppose that $a, b, \mu, \nu, \alpha_1, \alpha_2, \beta_1$ and β_2 are positive. Then for any choice of nonnegative functions u_0 and v_0 belonging to $W^{2,\infty}(\Omega)$, the problem (1.2) possesses a global classical solution $(u, v) \in (C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)))^2$ which is bounded in the sense that there exists $C > 0$ fulfilling*

$$0 \le u(x,t) \le C \quad \text{and} \quad 0 \le v(x,t) \le C \qquad \text{for all } x \in \Omega \text{ and } t > 0. \tag{1.3}$$

This paper is organized as follows: In Section 2 we present some basic materials concerning the local existence of smooth solutions and establish some *a priori* estimates concerning u, v, u^2 and v^2. Section 3 is devoted to establishing the space-time bounds on u, v in L^3 and $\nabla u, \nabla v$ in L^2. The assumption that both diffusion coefficients are equal allows us to find a suitable linear combination of u and v which solves a scalar parabolic equation. In Section 4 we deploy a delicate bootstrap argument to establish the L^p bounds of u and v for any p. The assumption $n \le 3$ enables us to close a circle of arguments so that the integrability powers of u and v can be improved in each iterative step. In Section 5 we establish the time-independent bounds for ∇u and ∇v with respect to the $L^4(\Omega)$ norm, in which the convexity of Ω is used. Finally we employ Amann's extensibility result to complete the proof of the global existence of smooth solutions to (1.2).

2 Local existence, extensibility and basic estimates

The following has been obtained in [1], [2] and [3] (cf. also [14]).

Lemma 2.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, let $a, b, \mu, \nu, \alpha_1, \alpha_2, \beta_1$ and β_2 be positive, and assume that u_0 and v_0 are nonnegative functions from $W^{1,\infty}(\Omega)$. Then there exist $T_{max} \in (0, \infty]$ and a pair $(u, v) \in (C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})))^2$ of nonnegative functions u and v which solve (1.2) classically in $\bar\Omega \times [0, T_{max})$, and which are such that*

$$\text{if } T_{max} < \infty, \text{ then } \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{W^{1,p}(\Omega)} + \|v(\cdot,t)\|_{W^{1,p}(\Omega)} \Big) = \infty \quad \text{for all } p > n. \tag{2.1}$$

Let us state some immediate basic properties of these solutions.

Lemma 2.2 *We have*

$$\int_\Omega u(\cdot,t) \le m_1 := \max\left\{ \int_\Omega u_0, \frac{|\Omega|}{\alpha_1} \right\} \qquad \text{for all } t \in (0, T_{max}) \tag{2.2}$$

and

$$\int_\Omega v(\cdot,t) \le m_2 := \max\left\{ \int_\Omega v_0, \frac{|\Omega|}{\beta_1} \right\} \qquad \textit{for all } t \in (0, T_{max}) \tag{2.3}$$

as well as

$$\int_t^{t+\tau} \int_\Omega u^2 \le \frac{(1+\mu\tau)m_1}{\mu\alpha_1} \qquad \textit{for all } t \in (0, \widehat{T}_{max}) \tag{2.4}$$

and

$$\int_t^{t+\tau} \int_\Omega v^2 \le \frac{(1+\nu\tau)m_2}{\nu\beta_1} \qquad \textit{for all } t \in (0, \widehat{T}_{max}), \tag{2.5}$$

where

$$\tau := \min\left\{1\,, \frac{1}{2}T_{max}\right\} \qquad \textit{and} \qquad \widehat{T}_{max} := \begin{cases} T_{max} - \tau & \textit{if } T_{max} < \infty, \\ \infty & \textit{if } T_{max} = \infty. \end{cases} \tag{2.6}$$

Proof. As u and v are nonnegative and $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, in integration of the first equation in (1.2) over Ω yields

$$\frac{d}{dt}\int_\Omega u = \mu \int_\Omega u - \mu\alpha_1 \int_\Omega u^2 - \mu\alpha_2 \int_\Omega uv \le \mu \int_\Omega u - \mu\alpha_1 \int_\Omega u^2 \qquad \text{for all } t \in (0, T_{max}), \tag{2.7}$$

so that since $(\int_\Omega u)^2 \le |\Omega| \int_\Omega u^2$ by the Cauchy-Schwarz inequality, $y(t) := \int_\Omega u(\cdot,t)$, $t \in [0, T_{max})$, satisfies

$$y'(t) \le \mu y(t) - \frac{\mu\alpha_1}{|\Omega|} y^2(t) \qquad \text{for all } t \in (0, T_{max}).$$

By an ODE comparison, this firstly shows that

$$y(t) \le \max\left\{ y(0)\,, \frac{\mu}{\frac{\mu\alpha_1}{|\Omega|}} \right\} = m_1 \qquad \text{for all } t \in (0, T_{max})$$

and thereby implies (2.2). Thereupon, by integration of (2.7) in time we obtain

$$\begin{aligned} \int_\Omega u(\cdot, t+\tau) + \mu\alpha_1 \int_t^{t+\tau} \int_\Omega u^2 &\le \int_\Omega u(\cdot,t) + \mu \int_t^{t+\tau} \int_\Omega u \\ &\le m_1 + \mu\tau m_1 \qquad \text{for all } t \in (0, \widehat{T}_{max}), \end{aligned}$$

which proves (2.4). The inequalities (2.3) and (2.5) can be derived in much the same manner. □

Throughout the sequel, τ and $\widehat{T}_{max}$ are as defined in (2.6).

3 Further integrability properties

In the presently considered case when the diffusion coefficients in both evolution equations in (1.2) coincide, a suitable linear combination w of u and v solves an inhomogeneous scalar parabolic equation (cf. (3.8) and (3.9) below). In order to provide integrability properties of the source term f appearing therein which go beyond those implied by the L^2 bounds for u and v in (2.4) and (2.5), as the main goal of this section we will seek for bounds on u and v in L^p spaces involving space and time, with $p > 2$. Based on an analysis involving a lifting argument which transports (1.2) to H^{-1}-type spaces in Lemma 3.1, Lemma 3.4 will reveal that such estimates can indeed be found for $p = 3$.

3.1 Integral bounds for $u^2 v$ and $v^2 u$

The following lemma contains the announced lifting procedure and thereby provides a key estimate for the proof of Lemma 3.4.

Lemma 3.1 *There exists $C > 0$ such that*

$$\int_t^{t+\tau} \int_\Omega u^2 v \le C \qquad \text{for all } t \in (0, \widehat{T}_{max}) \tag{3.1}$$

and

$$\int_t^{t+\tau} \int_\Omega u v^2 \le C \qquad \text{for all } t \in (0, \widehat{T}_{max}). \tag{3.2}$$

Proof. We pick any $\lambda > 0$ such that $\lambda \le \frac{\mu\alpha_2}{a}$ and let $\mathcal{A}$ denote the self-adjoint realization of $-\Delta + \lambda$ under homogeneous Neumann boundary conditions in $L^2(\Omega)$. Then since it is well-known from the theory of elliptic equations that as λ is positive, $\mathcal{A}$ possesses an order-preserving bounded inverse $\mathcal{A}^{-1}$ on $L^2(\Omega)$, for which we can thus fix $c_1 > 0$ fulfilling

$$\|\mathcal{A}^{-1}\varphi\|_{L^2(\Omega)} \le c_1 \|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in L^2(\Omega), \tag{3.3}$$

which by self-adjointness of the fractional power $\mathcal{A}^{-\frac{1}{2}}$ also implies that

$$\|\mathcal{A}^{-\frac{1}{2}}\varphi\|^2_{L^2(\Omega)} = \int_\Omega \varphi \cdot \mathcal{A}^{-1}\varphi \le c_1 \|\varphi\|^2_{L^2(\Omega)} \qquad \text{for all } \varphi \in L^2(\Omega). \tag{3.4}$$

We now rewrite the first equation in (1.2) in the form

$$\begin{aligned} u_t + \mathcal{A}(u + auv) &= \lambda u + \lambda auv + \mu u(1 - \alpha_1 u - \alpha_2 v) \\ &= (\lambda + \mu)u - \mu\alpha_1 u^2 - (\mu\alpha_2 - \lambda a)uv \qquad \text{in } \Omega \times (0, T_{max}), \end{aligned} \tag{3.5}$$

where since $\mu\alpha_2 - \lambda a$ is nonnegative, using Young's inequality we see that

$$\begin{aligned} (\lambda + \mu)u - \mu\alpha_1 u^2 - (\mu\alpha_2 - \lambda a)uv &\le (\lambda + \mu)u - \mu\alpha_1 u^2 \\ &\le \Big(\mu\alpha_1 u^2 + \frac{(\lambda + \mu)^2}{4\mu\alpha_1}\Big) - \mu\alpha_1 u^2 \\ &\le c_2 := \frac{(\lambda + \mu)^2}{4\mu\alpha_1} \qquad \text{in } \Omega \times (0, T_{max}). \end{aligned}$$

Therefore, multiplying (3.5) by $\mathcal{A}^{-1}u \ge 0$ yields the inequality

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\mathcal{A}^{-\frac{1}{2}}u|^2 + \int_\Omega u^2 + a\int_\Omega u^2 v \le c_2 \int_\Omega \mathcal{A}^{-1}u \qquad \text{for all } t \in (0, T_{max}), \tag{3.6}$$

in which by the Hölder inequality, (3.3) and Young's inequality we can estimate

$$c_2 \int_\Omega \mathcal{A}^{-1}u \le c_2 |\Omega|^{\frac{1}{2}} \|\mathcal{A}^{-1}u\|_{L^2(\Omega)} \le c_1 c_2 |\Omega|^{\frac{1}{2}} \|u\|_{L^2(\Omega)} \le \frac{1}{2}\int_\Omega u^2 + c_3 \qquad \text{for all } t \in (0, T_{max})$$

with $c_3 := \frac{c_1^2 c_2^2 |\Omega|}{2}$. In light of (3.4), we thus obtain from (3.6) that

$$y(t) := \int_\Omega |\mathcal{A}^{-\frac{1}{2}} u(\cdot,t)|^2 \quad \text{and} \quad g(t) := a \int_\Omega u^2(\cdot,t) v(\cdot,t), \qquad t \in [0, T_{max}),$$

satisfy

$$\frac{1}{2} y'(t) + \frac{1}{2c_1} y(t) + g(t) \le c_3 \qquad \text{for all } t \in (0, T_{max}), \tag{3.7}$$

whence in particular, by an ODE comparison,

$$y(t) \le c_4 := \max\left\{ \int_\Omega |\mathcal{A}^{-\frac{1}{2}} u_0|^2 \,,\, 2c_1 c_3 \right\} \qquad \text{for all } t \in (0, T_{max}),$$

because g is nonnegative. Using this information, going back to (3.7) we see upon dropping two nonnegative terms that

$$\int_t^{t+\tau} g(\sigma) d\sigma \le \frac{1}{2} y(t) + c_3 \tau \le \frac{c_4}{2} + c_3 \qquad \text{for all } t \in (0, \widehat{T}_{max}),$$

because $\tau \le 1$. As a is positive, this implies (3.1), and (3.2) can be shown quite similarly. □

3.2 Space-time bounds for (u, v) in L^3 and for $(\nabla u, \nabla v)$ in L^2

For the derivation of the desired L^3 estimate from Lemma 3.1, but also for frequent use throughout the sequel, let us state the following observation relying on our overall assumption that both diffusion coefficients in (1.2) are equal.

Lemma 3.2 *Let*

$$w(x,t) := bu(x,t) - av(x,t) \qquad \textit{for } x \in \bar{\Omega} \textit{ and } t \in [0, T_{max}). \tag{3.8}$$

Then w is a solution of

$$\begin{cases} w_t = \Delta w + f(x,t), & x \in \Omega, t \in (0, T_{max}), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = bu_0(x) - av_0(x), & x \in \Omega, \end{cases} \tag{3.9}$$

where

$$f(x,t) := b\mu u(1 - \alpha_1 u - \alpha_2 v) - a\nu v(1 - \beta_1 v - \beta_2 u) \qquad \textit{for } x \in \Omega \textit{ and } t \in (0, T_{max}). \tag{3.10}$$

Proof. This follows in a straightforward manner by taking an evident linear combination of the equations satisfied by u and v. □

The next auxiliary lemma on a boundedness property in a linear absorptive ODI is a straightforward generalization of the corresponding statement focusing on the special case $\tau = 1$ which has been proved in [23, Lemma 3.4], and thus we may omit the elementary proof here.

Lemma 3.3 *Let $T > 0$ and $\tau \in (0, T)$, and suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous and such that with some $\lambda > 0$ we have*

$$y'(t) + \lambda y(t) \le h(t) \qquad \text{for a.e. } t \in (0, T),$$

where $h \in L^1_{loc}([0, T))$ is nonnegative and such that

$$\int_t^{t+\tau} h(\sigma) d\sigma \le K \qquad \text{for all } t \in [0, T - \tau)$$

with some $K > 0$. Then

$$y(t) \le \max\left\{ y(0) + K \, , \frac{K}{\lambda\tau} + 2K \right\} \qquad \text{for all } t \in (0, T).$$

We can thereby assert the following.

Lemma 3.4 *There exists $C > 0$ such that*

$$\int_t^{t+\tau} \int_\Omega u^3 \le C \quad \text{and} \quad \int_t^{t+\tau} \int_\Omega v^3 \le C \qquad \text{for all } t \in (0, \widehat{T}_{max}) \tag{3.11}$$

as well as

$$\int_t^{t+\tau} \int_\Omega |\nabla w|^2 \le C \qquad \text{for all } t \in (0, \widehat{T}_{max}). \tag{3.12}$$

Proof. We test (3.9) by w and recall the definition (3.10) of f to see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega |\nabla w|^2 &= \int_\Omega fw \\ &= b\mu \int_\Omega uw - b\mu\alpha_1 \int_\Omega u^2 w - b\mu\alpha_2 \int_\Omega uvw \\ &\quad -a\nu \int_\Omega vw + a\nu\beta_2 \int_\Omega v^2 w + a\nu\beta_2 \int_\Omega uvw \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{3.13}$$

Using that $-av \le w = bu - av \le bu$, we can herein estimate

$$b\mu \int_\Omega uw \le \mu b^2 \int_\Omega u^2$$

and

$$-b\mu\alpha_2 \int_\Omega uvw \le \mu\alpha_2 ab \int_\Omega uv^2$$

as well as

$$-a\nu \int_\Omega vw \le \nu a^2 \int_\Omega v^2$$

and

$$a\nu\beta_2 \int_\Omega uvw \le \nu\beta_2 ab \int_\Omega u^2 v,$$

while

$$-b\mu\alpha_1 \int_\Omega u^2 w = -\mu\alpha_1 b^2 \int_\Omega u^3 + \mu\alpha_1 ab \int_\Omega u^2 v$$

and

$$a\nu\beta_1 \int_\Omega v^2 w = -\nu\beta_1 a^2 \int_\Omega v^3 + \nu\beta_1 ab \int_\Omega uv^2$$

for $t \in (0, T_{max})$. As furthermore $\int_\Omega w^2 \le b^2 \int_\Omega u^2 + a^2 \int_\Omega v^2$, (3.13) therefore implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega w^2 + \int_\Omega w^2 + \int_\Omega |\nabla w|^2 + c_1 \int_\Omega u^3 + c_1 \int_\Omega v^3 \le c_2 \int_\Omega u^2 v + c_2 \int_\Omega uv^2 + c_2 \int_\Omega u^2 + c_2 \int_\Omega v^2$$

for all $t \in (0, T_{max})$ with $c_1 := \min\{\mu\alpha_1 b^2, \nu\beta_1 a^2\} > 0$ and $c_2 := \max\{\nu\beta_2 ab + \mu\alpha_1 ab, \mu\alpha_2 ab + \nu\beta_1 ab, \mu b^2 + b^2, \nu a^2 + a^2\}$. Rewritten in terms of

$$y(t) := \int_\Omega w^2(\cdot,t), \quad g(t) := \int_\Omega |\nabla w(\cdot,t)|^2 + c_1 \int_\Omega u^3(\cdot,t) + c_1 \int_\Omega v^3(\cdot,t) \qquad \text{and}$$
$$h(t) := c_2 \int_\Omega u^2(\cdot,t)v(\cdot,t) + c_2 \int_\Omega u(\cdot,t)v^2(\cdot,t) + c_2 \int_\Omega u^2(\cdot,t) + c_2 \int_\Omega v^2(\cdot,t), \qquad t \in [0, T_{max}),$$

the ODI

$$\frac{1}{2}y'(t) + y(t) + g(t) \le h(t) \qquad \text{for } t \in (0, T_{max}) \tag{3.14}$$

thereby obtained allows us to apply Lemma 3.3, because from Lemma 2.2 and Lemma 3.1 we know that there exists $c_3 > 0$ fulfilling

$$\int_t^{t+\tau} h(\sigma)d\sigma \le c_3 \qquad \text{for all } t \in (0, \widehat{T}_{max}).$$

In consequence, on dropping the nonnegative summand $g(t)$ we firstly infer from (3.14) that

$$y(t) \le c_4 \qquad \text{for all } t \in (0, T_{max})$$

with some $c_4 > 0$, whereupon an integration in (3.14) shows that since also y is nonnegative, we have

$$\begin{aligned} \int_t^{t+\tau} g(\sigma)d\sigma &\le \frac{1}{2}y(t) + \int_t^{t+\tau} h(\sigma)d\sigma \\ &\le \frac{c_4}{2} + c_3 \qquad \text{for all } t \in (0, \widehat{T}_{max}). \end{aligned}$$

By definition of g, this entails both (3.11) and (3.12). □

The latter lemma allows us to draw the following further consequence on L^2 regularity of ∇u and ∇v. A similar statement can alternatively be obtained by making use of a entropy-dissipation inequality associated with (1.2) (cf. [6] for details in this respect). The present approach allows for a derivation independent of this subtle structure, and thereby may be adapted so as to apply also in more general frameworks, e.g. involving modifications in the cross-diffusive interaction such as obtained when $a\Delta(uv)$ and $b\Delta(uv)$ in (1.2) are replaced by $a\Delta(u^\gamma v)$ and $b\Delta(u^\gamma v)$ for some $\gamma \neq 1$.

Lemma 3.5 *There exists $C > 0$ such that*

$$\int_t^{t+\tau} \int_\Omega |\nabla u|^2 \le C \qquad \text{for all } t \in (0, \widehat{T}_{max}) \tag{3.15}$$

and

$$\int_t^{t+\tau} \int_\Omega |\nabla v|^2 \le C \qquad \text{for all } t \in (0, \widehat{T}_{max}). \tag{3.16}$$

PROOF. As u is positive in $\bar{\Omega} \times (0, T_{max})$ by the strong maximum principle, we may multiply the first equation in (1.2) by the smooth function $\ln u + 1$ and integrate by parts to see that

$$\begin{aligned} \frac{d}{dt} \int_\Omega u \ln u + \mu \int_\Omega u \ln u &= \int_\Omega (\ln u + 1) u_t + \mu \int_\Omega u \ln u \\ &= -\int_\Omega \frac{|\nabla u|^2}{u} - a \int_\Omega \frac{1}{u} \nabla u \cdot \nabla (uv) \\ &\quad + \mu \int_\Omega (2u - \alpha_1 u^2) \ln u + \mu \int_\Omega u - \mu\alpha_1 \int_\Omega u^2 \\ &\quad - \mu\alpha_2 \int_\Omega uv \ln u - \mu\alpha_2 \int_\Omega uv \end{aligned} \tag{3.17}$$

for all $t \in (0, T_{max})$. Here since

$$\xi \ln \xi \ge -\frac{1}{e} \qquad \text{for all } \xi > 0, \tag{3.18}$$

using (2.3) we can estimate

$$-\mu\alpha_2 \int_\Omega uv \ln u - \mu\alpha_2 \int_\Omega uv \le \frac{\mu\alpha_2}{e} \int_\Omega v \le c_1 := \frac{\mu\alpha_2 m_2}{e} \qquad \text{for all } t \in (0, T_{max}), \tag{3.19}$$

whereas employing (2.2) we obtain

$$\mu \int_\Omega u - \mu\alpha_1 \int_\Omega u^2 \le \mu \int_\Omega u \le c_2 := \mu m_1 \qquad \text{for all } t \in (0, T_{max}). \tag{3.20}$$

Furthermore, with $\xi_0 := \max\{1, \frac{2}{\alpha_1}\}$ we have

$$\mu \int_\Omega (2u - \alpha_1 u^2) \ln u \le c_3 := \mu|\Omega| \cdot \Big(2\xi_0 \ln \xi_0 + \frac{\alpha_1}{2e}\Big) \qquad \text{for all } t \in (0, T_{max}), \tag{3.21}$$

due to the fact that $\varphi(\xi) := (2\xi - \alpha_1 \xi^2) \ln \xi$, $\xi > 0$, satisfies $\varphi \le 0$ on $[\xi_0, \infty)$ and

$$\varphi(\xi) \le 2\xi_0 \ln \xi_0 - \alpha_1 \xi^2 \ln \xi \le 2\xi_0 \ln \xi_0 + \frac{\alpha_1}{2e} \qquad \text{for all } \xi \in (0, \xi_0),$$

because $\xi^2 \ln \xi \geq -\frac{1}{2e}$ for all $\xi > 0$.
As for the second integral on the right of (3.17), we first use the positivity of u and v in estimating

$$
\begin{aligned}
-a\int_\Omega \frac{1}{u}\nabla u\cdot\nabla(uv) &= -a\int_\Omega \frac{v}{u}|\nabla u|^2 - a\int_\Omega \nabla u\cdot\nabla v \\
&\leq -a\int_\Omega \nabla u\cdot\nabla v \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}
$$

and thereafter we recall the definition (3.8) of w to substitute $v = \frac{b}{a}u - \frac{1}{a}w$. On using Young's inequality, we hence infer that

$$
\begin{aligned}
-a\int_\Omega \frac{1}{u}\nabla u\cdot\nabla(uv) &\leq -a\int_\Omega \nabla u\cdot\Big(\frac{b}{a}\nabla u - \frac{1}{a}\nabla w\Big) \\
&= -b\int_\Omega |\nabla u|^2 + \int_\Omega \nabla u\cdot\nabla w \\
&\leq -\frac{b}{2}\int_\Omega |\nabla u|^2 + \frac{1}{2b}\int_\Omega |\nabla w|^2 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}
$$

In conjunction with (3.19), (3.20) and (3.21), this shows that for

$$
y(t) := \int_\Omega u(\cdot,t)\ln u(\cdot,t) \quad \text{and} \quad g(t) := \frac{b}{2}\int_\Omega |\nabla u(\cdot,t)|^2, \qquad t\in[0,T_{max}),
$$

on omitting a nonpositive term on the right we obtain from (3.17) the inequality

$$
y'(t) + y(t) + g(t) \leq c_4 := c_1 + c_2 + c_3 \qquad \text{for all } t\in(0,T_{max}).
$$

By a comparison relying on the nonnegativity of g, this first implies that

$$
y(t) \leq c_5 := \max\Big\{\int_\Omega u_0\ln u_0\,,\, c_4\Big\} \qquad \text{for all } t\in(0,T_{max}),
$$

and thereupon, by integration, entails that

$$
\begin{aligned}
\int_t^{t+\tau} g(\sigma)d\sigma &\leq y(t) - y(t+\tau) - \int_t^{t+\tau} y(\sigma)d\sigma + c_4\tau \\
&\leq c_5 + \frac{|\Omega|}{e} + \frac{|\Omega|}{e}\tau + c_4\tau \\
&\leq c_5 + \frac{|\Omega|}{e} + \frac{|\Omega|}{e} + c_4 \qquad \text{for all } t\in(0,\widehat{T}_{max}),
\end{aligned}
$$

because (3.18) entails that $y(t) \geq -\frac{|\Omega|}{e}$ for all $t\in(0,T_{max})$, and because $\tau\leq 1$. This establishes (3.15), whilst (3.16) can be proved analogously. □

4 L^p bounds for u and v via maximal Sobolev regularity of w

Our next goal consists in further improving our knowledge on regularity of u and v by successively establishing estimates for w from known properties of u and v, and to use these, in a style parallel to that of Lemma 3.5, for the derivation of bounds for u and v which involve integrability powers higher than those occurring in the previous step.

The implications of regularity properties of u and v on w will be studied in the framework of maximal Sobolev regularity estimates applied to the linear equation (3.9). In our first step toward this we estimate the inhomogeneity f appearing therein in terms of supposedly known properties of u and v.

Lemma 4.1 *Suppose that for some $p \ge 1$ there exists $K > 0$ such that*

$$\int_\Omega u^p(\cdot,t) \le K \quad \textit{and} \quad \int_\Omega v^p(\cdot,t) \le K \qquad \textit{for all } t \in (0,T_{max}) \tag{4.1}$$

as well as

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le K \quad \textit{and} \quad \int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla v|^2 \le K \qquad \textit{for all } t \in (0,\widehat{T}_{max}). \tag{4.2}$$

Then for any choice of $r > 1$ and each $q \in (1,\infty)$ fulfilling

$$q \le q_0(n,p,r) := \min\left\{\frac{npr}{(2nr-2p-n)_+}\,,\,\frac{n(p+1)}{2(n-2)_+}\right\}, \tag{4.3}$$

one can find $C = C(p,q,r,K) > 0$ such that the function f defined in (3.10) *satisfies*

$$\int_t^{t+\tau} \|f(\cdot,\sigma)\|^r_{L^q(\Omega)} d\sigma \le C \qquad \textit{for all } t \in (0,\widehat{T}_{max}). \tag{4.4}$$

Proof. We first note that regardless of the sign of $2nr-2p-n$ we have

$$\frac{npr}{(2nr-2p-n)_+} > \frac{npr}{2nr} = \frac{p}{2},$$

and that in both cases $n \le 2$ and $n \ge 3$,

$$\frac{n(p+1)}{2(n-2)_+} > \frac{np}{2n} = \frac{p}{2},$$

so that we may assume that besides (4.3), q satisfies $q > \frac{p}{2}$.

We next observe that by Young's inequality we can fix $c_1 > 0$ such that

$$|f(x,t)| \le c_1u^2(x,t) + c_1v^2(x,t) + c_1 \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T_{max}),$$

and that hence for some $c_2 = c_2(q,r) > 0$,

$$\int_t^{t+\tau} \|f(\cdot,\sigma)\|^r_{L^q(\Omega)} d\sigma \le c_2 \int_t^{t+\tau} \|u^2(\cdot,\sigma)\|^r_{L^q(\Omega)} d\sigma + c_2 \int_t^{t+\tau} \|v^2(\cdot,\sigma)\|^r_{L^q(\Omega)} d\sigma + c_2 \tag{4.5}$$

for all $t \in (0,\widehat{T}_{max})$. Here since $q > \frac{p}{2}$, and since (4.3) implies that $1-\frac{n}{2} \ge -\frac{n(p+1)}{4q}$, we have $W^{1,2}(\Omega) \hookrightarrow L^{\frac{4q}{p+1}}(\Omega) \hookrightarrow L^{\frac{2p}{p+1}}(\Omega)$, so that by means of the Gagliardo-Nirenberg inequality we can find $c_3 = c_3(q,r) > 0$ fulfilling

$$\begin{aligned} \|u^2(\cdot,\sigma)\|_{L^q(\Omega)}^r &= \Big\|u^{\frac{p+1}{2}}(\cdot,\sigma)\Big\|_{L^{\frac{4q}{p+1}}(\Omega)}^{\frac{4r}{p+1}} \\ &\le c_3\Big\|\nabla u^{\frac{p+1}{2}}(\cdot,\sigma)\Big\|_{L^2(\Omega)}^{\frac{4r}{p+1}\cdot\theta}\Big\|u^{\frac{p+1}{2}}(\cdot,\sigma)\Big\|_{L^{\frac{2p}{p+1}}(\Omega)}^{\frac{4r}{p+1}\cdot(1-\theta)} + c_3\Big\|u^{\frac{p+1}{2}}(\cdot,\sigma)\Big\|_{L^{\frac{2p}{p+1}}(\Omega)}^{\frac{4r}{p+1}} \end{aligned} \quad (4.6)$$

for all $\sigma \in (0,T_{max})$, where $\theta \in (0,1]$ is determined by the relation

$$-\frac{n(p+1)}{4q} = \Big(1-\frac{n}{2}\Big)\theta - \frac{n(p+1)}{2p}\cdot(1-\theta),$$

that is, where

$$\theta = \frac{n(p+1)(2q-p)}{2q(2p+n)}.$$

Straightforward computations yield

$$\begin{aligned} \frac{4r}{p+1}\cdot\theta = \frac{2nr}{2p+n}\cdot\Big(2-\frac{p}{q}\Big) &\le \frac{2nr}{2p+n}\cdot\Big(2-\frac{p}{q_0(p,r)}\Big) = \frac{2nr}{2p+n}\cdot\Big(2-\frac{(2nr-2p-n)_+}{nr}\Big) \\ &\le \frac{2nr}{2p+n}\cdot\Big(2-\frac{2nr-2p-n}{nr}\Big) = 2 \end{aligned}$$

by definition of $q_0(n,p,r)$, and since

$$\Big\|u^{\frac{p+1}{2}}(\cdot,\sigma)\Big\|_{L^{\frac{2p}{p+1}}(\Omega)}^{\frac{2p}{p+1}} = \int_\Omega u^p(\cdot,\sigma) \le K \qquad \text{for all } \sigma \in (0,T_{max}),$$

by (4.1), in view of Young's inequality we may combine (4.6) with (4.2) to infer the existence of positive constants $c_4 = c_4(q,r,K)$ and $c_5 = c_5(p,q,r,K)$ such that

$$\begin{aligned} \int_t^{t+\tau} \|u^2(\cdot,\sigma)\|_{L^q(\Omega)}^r d\sigma &\le c_4\int_t^{t+\tau}\Big\|\nabla u^{\frac{p+1}{2}}(\cdot,\sigma)\Big\|_{L^2(\Omega)}^2 d\sigma + c_4 \\ &= \frac{p^2c_4}{4}\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 + c_4 \\ &\le c_5 \qquad \text{for all } t \in (0,\widehat{T}_{max}). \end{aligned}$$

Along with a similar estimate for the corresponding integral involving v, inserted into (4.5) this proves (4.4). □

Now maximal Sobolev regularity theory turns the latter into the following.

Lemma 4.2 *Assume that*

$$\int_\Omega u^p(\cdot,t) \le K \quad \textit{and} \quad \int_\Omega v^p(\cdot,t) \le K \qquad \textit{for all } t \in (0,T_{max}) \quad (4.7)$$

as well as

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le K \quad \textit{and} \quad \int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla v|^2 \le K \qquad \textit{for all } t\in(0,\widehat{T}_{max}) \tag{4.8}$$

with some $p \ge 1$ and $K > 0$. Then for all $r > 1$ and each $q \in (1,\infty)$ fulfilling $q \le q_0(n,p,r)$ with $q_0(n,p,r)$ given by (4.3)*, there exists $C = C(p,q,r,K) > 0$ with the property that the function w defined in* (3.8) *satisfies*

$$\int_t^{t+\tau}\|w(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)}d\sigma + \int_t^{t+\tau}\|w_t(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma \le C \qquad \textit{for all } t\in(0,\widehat{T}_{max}). \tag{4.9}$$

Proof. We first consider the case of small t, in which we invoke well-known results on maximal Sobolev regularity properties of the Neumann heat semigroup $(e^{\sigma\Delta})_{\sigma\ge0}$ in Ω [9] to find $c_1 = c_1(q,r) > 0$ such that

$$\int_0^{2\tau}\|w(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)}d\sigma + \int_0^{2\tau}\|w_t(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma \le c_1\|bu_0-av_0\|^r_{W^{2,q}(\Omega)} + c_1\int_0^{2\tau}\|f(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma,$$

where f is as in (3.10). Here we apply Lemma 4.1 to see that since u_0 and v_0 belong to $W^{2,\infty}(\Omega)$, there exists $c_2 = c_2(p,q,r,K) > 0$ such that

$$\int_0^{2\tau}\|w(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)}d\sigma + \int_0^{2\tau}\|w_t(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma \le c_2, \tag{4.10}$$

which in particular establishes (4.9) for any $t \in [0,\tau]$. We next pick any $t_0 \in (\tau,\widehat{T}_{max})$ and let

$$z(\cdot,t) := w(\cdot,t) - z_1(\cdot,t), \qquad t\in[t_0-\tau,T_{max}),$$

where

$$z_1(\cdot,t) := e^{(t-t_0+\tau)\Delta}w(\cdot,t_0-\tau), \qquad t\ge t_0-\tau.$$

Then since $\partial_t z_1 = \Delta z_1$ in $\Omega\times(t_0-\tau,\infty)$ and $z_1(\cdot,t_0-\tau) = w(\cdot,t_0-\tau)$, it follows that z solves

$$\begin{cases} z_t = \Delta z + f(x,t), & x\in\Omega,\ t\in(t_0-\tau,T_{max}),\\ \frac{\partial z}{\partial\nu} = 0, & x\in\partial\Omega,\ t\in(t_0-\tau,T_{max}),\\ z(x,t_0-\tau) = 0, & x\in\Omega. \end{cases}$$

Accordingly, maximal Sobolev regularity estimates yield $c_3 = c_3(q,r) > 0$ fulfilling

$$\int_{t_0-\tau}^{t_0+\tau}\|z(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)}d\sigma + \int_{t_0-\tau}^{t_0+\tau}\|z_t(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma \le c_3\int_{t_0-\tau}^{t_0+\tau}\|f(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma,$$

so that in particular we may use Lemma 4.1 to obtain $c_4 = c_4(p,q,r,K) > 0$ such that

$$\int_{t_0-\tau}^{t_0+\tau}\|z(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)}d\sigma + \int_{t_0-\tau}^{t_0+\tau}\|z_t(\cdot,\sigma)\|^r_{L^q(\Omega)}d\sigma \le c_4. \tag{4.11}$$

Now thanks to the fact that

$$\|w(\cdot,\sigma)\|_{L^1(\Omega)} \le bm_1 + am_2 \qquad \text{for all } \sigma \in (0,T_{max})$$

by Lemma 2.2, standard $L^p - L^q$ estimates for the Neumann heat semigroup provide $c_5 = c_5(q,r) > 0$ such that

$$\int_{t_0}^{t_0+\tau} \|z_1(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)} d\sigma + \int_{t_0}^{t_0+\tau} \|\partial_t z_1(\cdot,\sigma)\|^r_{L^q(\Omega)} d\sigma \le c_5,$$

so that estimating

$$\|w(\cdot,\sigma)\|_{W^{2,q}(\Omega)} \le \|z(\cdot,\sigma)\|_{W^{2,q}(\Omega)} + \|z_1(\cdot,\sigma)\|_{W^{2,q}(\Omega)} \qquad \text{for all } \sigma \in (t_0, t_0+\tau)$$

and

$$\|w_t(\cdot,\sigma)\|_{L^q(\Omega)} \le \|z_t(\cdot,\sigma)\|_{L^q(\Omega)} + \|\partial_t z_1(\cdot,\sigma)\|_{L^q(\Omega)} \qquad \text{for all } \sigma \in (t_0, t_0+\tau)$$

we readily infer from (4.11) that (4.9) is also valid for any $t \in (\tau, \widehat{T}_{max})$ with some $C > 0$ possibly depending on p, q, r and K but not on t. □

As a first consequence thereof, let us establish an estimate for w in $L^\infty((0,T_{max}); L^\kappa(\Omega))$, provided that $\kappa > 1$ is suitably small. In a second application of Lemma 4.2 to be performed in Lemma 4.9 below, this will become part of an interpolation argument providing a space-time integral bound for ∇w.

Lemma 4.3 *For all $\kappa \in (1, \frac{n}{(n-2)_+})$ there exists $C = C(\kappa) > 0$ such that for w as in (3.8) we have*

$$\|w(\cdot,t)\|_{L^\kappa(\Omega)} \le C \qquad \text{for all } t \in (0,T_{max}). \tag{4.12}$$

Proof. Combining Lemma 3.5 with Lemma 2.2 we can find $c_1 > 0$ such that

$$\int_\Omega u(\cdot,t) \le c_1 \quad \text{and} \quad \int_\Omega v(\cdot,t) \le c_1 \qquad \text{for all } t \in (0,T_{max}) \tag{4.13}$$

as well as

$$\int_t^{t+\tau}\int_\Omega |\nabla u|^2 \le c_1 \quad \text{and} \quad \int_t^{t+\tau}\int_\Omega |\nabla v|^2 \le c_1 \qquad \text{for all } t \in (0,\widehat{T}_{max}). \tag{4.14}$$

To see that this implies (4.12), we observe that with $q_0(\cdot,\cdot,\cdot)$ as defined in Lemma 4.1 we have

$$q_0(n,1,r) = \min\left\{\frac{nr}{(2nr-n-2)_+}, \frac{n}{(n-2)_+}\right\} \to \frac{n}{(n-2)_+} \qquad \text{as } r \searrow 1,$$

whence given $\kappa \in (1, \frac{n}{(n-2)_+})$ we can fix $r = r(\kappa) > 1$ sufficiently close to 1 such that $\kappa < q_0(n,1,r)$. Consequently, Lemma 4.2 provides $c_2 = c_2(\kappa) > 0$ such that

$$\int_t^{t+\tau} \|w(\cdot,\sigma)\|^r_{W^{2,\kappa}(\Omega)} d\sigma + \int_t^{t+\tau} \|w_t(\cdot,\sigma)\|^r_{L^\kappa(\Omega)} d\sigma \le c_2 \qquad \text{for all } t \in (0,\widehat{T}_{max}), \tag{4.15}$$

which implies that for each $t \in (0, T_{max})$, in both cases $t \le \tau$ and $t > \tau$ we can pick $t_0(t) \in [0, T_{max})$ such that $t_0(t) \ge t - \tau$ and

$$\|w(\cdot, t_0(t))\|_{L^\kappa(\Omega)} \le \|w(\cdot, t_0(t))\|_{W^{2,\kappa}(\Omega)} \le c_3 \equiv c_3(\kappa) := \max\left\{c_2^{\frac{1}{r}}, \|bu_0 + av_0\|_{L^\kappa(\Omega)}\right\}.$$

Thereupon, (4.15) along with the Hölder inequality entails that

$$\begin{aligned} \|w(\cdot, t)\|_{L^\kappa(\Omega)} &= \left\| w(\cdot, t_0(t)) + \int_{t_0(t)}^t w_t(\cdot, \sigma) d\sigma \right\|_{L^\kappa(\Omega)} \\ &\le c_3 + \int_{t_0(t)}^t \|w_t(\cdot, \sigma)\|_{L^\kappa(\Omega)} d\sigma \\ &\le c_3 + \int_{t_0(t)}^t \|w_t(\cdot, \sigma)\|_{L^\kappa(\Omega)}^r d\sigma \\ &\le c_3 + c_2, \end{aligned}$$

because $t - t_0(t) \le \tau \le 1$. □

4.1 Implications of space-time bounds for ∇w on boundedness of u and v in $L^p(\Omega)$

We next examine possible implications of certain regularity properties of w on (u, v). The starting point for this will be the following result of a standard testing procedure in (1.2).

Lemma 4.4 *Let $p > 1$. Then*

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega u^p &+ (p-1)\int_\Omega u^{p-2}|\nabla u|^2 + (p-1)b\int_\Omega u^{p-1}|\nabla u|^2 + (p-1)a\int_\Omega u^{p-2}v|\nabla u|^2 \\ &= (p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla w + \mu\int_\Omega u^p - \mu\alpha_1\int_\Omega u^{p+1} - \mu\alpha_2\int_\Omega u^p v \\ &\qquad \textit{for all } t \in (0, T_{max}), \end{aligned} \tag{4.16}$$

where w is as defined in (3.8).

PROOF. The identity (4.16) can be derived in a straightforward way by testing the first equation in (1.2) against u^{p-1} and substituting $v = \frac{b}{a}u - \frac{1}{a}w$ wherever convenient. □

In a first application of this, we can now identify a relation between the exponents p and s ensuring that the L^s bound (4.17) on ∇w implies the integrability properties that have been supposed as hypotheses in Lemma 4.2.

Lemma 4.5 *Suppose that for some $s > 2$ and $K > 0$, the function w defined in* (3.8) *satisfies*

$$\int_t^{t+\tau}\int_\Omega |\nabla w|^s \le K \qquad \textit{for all } t \in (0, \widehat{T}_{max}). \tag{4.17}$$

Then for each $p > 1$ such that

$$p < \frac{(n+1)s - n - 2}{n}, \tag{4.18}$$

there exists $C = C(p, s, K) > 0$ with the property that

$$\int_\Omega u^p(\cdot,t) \le C \quad \textit{and} \quad \int_\Omega v^p(\cdot,t) \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{4.19}$$

as well as

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le C \quad \textit{and} \quad \int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla v|^2 \le C \qquad \textit{for all } t \in (0, \widehat{T}_{max}). \tag{4.20}$$

Proof. We first note that since $s > 2$, a straightforward computation shows that

$$2 - \frac{2}{s} < \frac{(n+1)s - n - 2}{n},$$

so that for proving the lemma it is sufficient to consider only such $p > 1$ which besides (4.18) also satisfy

$$p > 2 - \frac{2}{s}. \tag{4.21}$$

To achieve this, we invoke (4.16), from which on dropping three integrals therein we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p \quad &+ \quad (p-1)b\int_\Omega u^{p-1}|\nabla u|^2 \\ &\le \quad (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla w + \mu\int_\Omega u^p - \mu\alpha_1\int_\Omega u^{p+1} \qquad \text{for all } t \in (0, T_{max}),\end{aligned} \tag{4.22}$$

where twice applying Young's inequality we see that

$$\begin{aligned}(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla w \quad &\le \quad \frac{(p-1)b}{2}\int_\Omega u^{p-1}|\nabla u|^2 \\ &\quad + \frac{p-1}{2b}\int_\Omega u^{p-1}|\nabla w|^2 \qquad \text{for all } t \in (0, T_{max}),\end{aligned} \tag{4.23}$$

and that with $c_1 := \frac{p-1}{2b}$ we have

$$\frac{p-1}{2b}\int_\Omega u^{p-1}|\nabla w|^2 \le \int_\Omega |\nabla w|^s + c_1\int_\Omega u^{\frac{(p-1)s}{s-2}} \qquad \text{for all } t \in (0, T_{max}). \tag{4.24}$$

In order to control the latter integral appearing here, let us first make sure that our choices of s and p ensure that

$$W^{1,2}(\Omega) \hookrightarrow L^{\frac{2(p-1)s}{(p+1)(s-2)}}(\Omega) \hookrightarrow L^{\frac{2}{p+1}}(\Omega). \tag{4.25}$$

In fact, the second of these inclusions follows from the inequality

$$\frac{\frac{2(p-1)s}{(p+1)(s-2)}}{\frac{2}{p+1}} = \frac{(p-1)s}{s-2} > \frac{(2-\frac{2}{s}-1)\cdot s}{s-2} = 1$$

asserted by (4.21); the first embedding property in (4.25) is evident when $n \le 2$, and in the case $n \ge 3$ it is a consequence of the assumption (4.18) which ensures that then

$$\begin{aligned} \frac{2(p-1)s}{(p+1)(s-2)} &= \frac{2s}{s-2} \cdot \Big(1 - \frac{2}{p+1}\Big) < \frac{2s}{s-2} \cdot \Big(1 - \frac{2}{\frac{(n+1)s-n-2}{n} + 1}\Big) \\ &= \frac{2(n+1)}{n+1-\frac{2}{s}} < \frac{2(n+1)}{n} < \frac{2n}{n-2} \end{aligned}$$

due to the fact that $(n+1)(n-2) < n^2$.

Having thus verified (4.25), we may invoke the Gagliardo-Nirenberg inequality and (2.2) to find $c_2 = c_2(p,s) > 0$ and $c_3 = c_3(p,s) > 0$ satisfying

$$\begin{aligned} c_1 \int_\Omega u^{\frac{(p-1)s}{s-2}} &= c_1 \|u^{\frac{p+1}{2}}\|_{L^{\frac{2(p-1)s}{(p+1)(s-2)}}(\Omega)}^{\frac{2(p-1)s}{(p+1)(s-2)}} \\ &\le c_2 \|\nabla u^{\frac{p+1}{2}}\|_{L^2(\Omega)}^{\frac{2(p-1)s}{(p+1)(s-2)}\cdot\theta} \|u^{\frac{p+1}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^{\frac{2(p-1)s}{(p+1)(s-2)}\cdot(1-\theta)} + \|u^{\frac{p+1}{2}}\|_{L^{\frac{2}{p+1}}(\Omega)}^{\frac{2(p-1)s}{(p+1)(s-2)}} \\ &\le c_3 \cdot \Big\{ \int_\Omega u^{p-1}|\nabla u|^2 \Big\}^{\frac{(p-1)s}{(p+1)(s-2)}\cdot\theta} + c_3 \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{4.26}$$

where

$$-\frac{n(p+1)(s-2)}{2(p-1)s} = \Big(1 - \frac{n}{2}\Big)\theta - \frac{n(p+1)}{2}(1-\theta),$$

that is,

$$\theta = \frac{n(p+1)}{np+2} \cdot \frac{(p-2)s+2}{(p-1)s}$$

satisfies $\theta \in (0,1)$ thanks to (4.25). Moreover, making full use of (4.18) now we see that actually

$$\frac{(p-1)s}{(p+1)(s-2)} \cdot \theta - 1 < 0,$$

and that hence the exponent appearing on the right of (4.26) satisfies $\frac{(p-1)s}{(p+1)(s-2)} \cdot \theta < 1$. Therefore, one further application of Young's inequality provides $c_4 = c_4(p,s) > 0$ such that

$$c_1 \int_\Omega u^{\frac{(p-1)s}{s-2}} \le \frac{(p-1)b}{4} \int_\Omega u^{p-1}|\nabla u|^2 + c_4 \qquad \text{for all } t \in (0, T_{max}),$$

which combined with (4.23) and (4.24) shows that

$$(p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla w \le \frac{(p-1)b}{4} \int_\Omega u^{p-1}|\nabla u|^2 + \int_\Omega |\nabla w|^s + c_4 \qquad \text{for all } t \in (0, T_{max}).$$

Since clearly, again by Young's inequality,

$$(\mu+1)\int_\Omega u^p \le \mu\alpha_1 \int_\Omega u^{p+1} + c_5 \qquad \text{for all } t \in (0, T_{max})$$

with some $c_5 = c_5(p) > 0$, from (4.22) we thus infer that

$$y(t) := \int_\Omega u^p(\cdot,t), \quad g(t) := \frac{(p-1)b}{4}\int_\Omega u^{p-1}(\cdot,t)|\nabla u(\cdot,t)|^2 \quad \text{and}$$
$$h(t) := \int_\Omega |\nabla w(\cdot,t)|^s + c_4 + c_5, \qquad t \in [0, T_{max}),$$

satisfy

$$\frac{1}{p}y'(t) + y(t) + g(t) \le h(t) \qquad \text{for all } t \in (0, T_{max}). \tag{4.27}$$

As the hypothesis (4.17) yields $c_6 = c_6(K) > 0$ such that

$$\int_t^{t+\tau} h(\sigma)d\sigma \le c_6 \qquad \text{for all } t \in (0, \widehat{T}_{max}),$$

by Lemma 3.3 and the nonnegativity of g this first implies the existence of $c_7 = c_7(p, s, K) > 0$ such that

$$y(t) \le c_7 \qquad \text{for all } t \in (0, T_{max}), \tag{4.28}$$

whereafter an integration of (4.27) shows that

$$\int_t^{t+\tau} g(\sigma)d\sigma \le \frac{1}{p}y(t) + \int_t^{t+\tau} h(\sigma)d\sigma \le \frac{c_7}{p} + c_6 \qquad \text{for all } t \in (0, \widehat{T}_{max}). \tag{4.29}$$

The inequalities in (4.28) and (4.29) prove the estimates in (4.19) and (4.20) involving u, whereas the corresponding bounds for v can be obtained in quite a similar way. □

4.2 Adjusting some parameters

In order to close the above circle of arguments, we need to adjust the exponents p, q, κ and r in Lemma 4.2, Lemma 4.3 and Lemma 4.5 properly. In the course of our preparations for this, during the next three lemmata we shall encounter the core of our restriction $n \le 3$ in Theorem 1.1.
The first two lemmata in this direction characterize the expressions $\overline{q}_0(n,p)$ and $s_0(n,\kappa,p,q)$ which will be used as upper bounds for q and s, respectively, in combining Lemma 4.2 with Lemma 4.3 during the proof of Lemma 4.9 below.

Lemma 4.6 *Let*

$$\overline{q}_0(n,p) := \min\left\{\frac{np}{(n-2p)_+}, \frac{n(p+1)}{2(n-2)_+}\right\} \qquad \text{for } n \in \{1,2,3,...\} \text{ and } p \ge 1. \tag{4.30}$$

Then

$$\overline{q}_0(n,p) = \begin{cases} \infty & \text{if } n \le 2 \text{ and } p \ge 1, \\ \frac{n(p+1)}{2(n-2)} & \text{if } n \ge 3 \text{ and } p \ge 1. \end{cases} \tag{4.31}$$

Moreover,

$$\overline{q}_0(n,p) > \frac{p}{2} \qquad \text{for all } n \ge 1 \text{ and } p \ge 1, \tag{4.32}$$

and furthermore

$$\overline{q}_0(n,p) \ge n \qquad \text{whenever } p \ge 1 \text{ and } n \le 4. \tag{4.33}$$

PROOF. For the proof of (4.31) it is evidently sufficient to observe that whenever $n \ge 3$ and $p \in [1, \frac{n}{2})$, we have

$$\frac{(n-2)(n-2p)}{n} \cdot \left\{ \frac{np}{n-2p} - \frac{n(p+1)}{2(n-2)} \right\} = (n-2)p - \frac{1}{2}(p+1)(n-2p) = (p-1)\Big(p + \frac{n}{2}\Big) \ge 0.$$

In verifying (4.32) and (4.33) we only need to consider the case when $\overline{q}_0(n,p)$ is finite. But then, clearly,

$$\overline{q}_0(n,p) = \frac{n(p+1)}{2(n-2)} = \frac{n}{n-2} \cdot \frac{p+1}{2} > 1 \cdot \frac{p}{2} = \frac{p}{2},$$

and if moreover $n \le 3$ then this decomposition yields $\overline{q}_0(n,p) \ge n \cdot 1$, because then $\frac{n}{n-2} \ge n$. □

Now the main reason for our restriction $n \le 3$ in Theorem 1.1 is closely linked to the observation (4.36) in the following.

Lemma 4.7 *Let*

$$s_0(n,\kappa,p,q) := \frac{n}{1+\frac{n}{\kappa}} + \frac{(2p+n)\cdot[(2+\frac{n}{\kappa})q - n]}{n(1+\frac{n}{\kappa})(2q-p)} \qquad \text{for } n \in \{1,2,3,...\}, \kappa \ge 1, p \ge 1 \text{ and } q > \frac{p}{2}. \tag{4.34}$$

Then as $(\kappa, q) \to \Big(\frac{n}{(n-2)_+}, \overline{q}_0(n,p)\Big)$ *with* $\overline{q}_0(n,p)$ *taken from Lemma 4.6, we have*

$$s_0(n,\kappa,p,q) \to \overline{s}_0(n,p) := \begin{cases} \frac{2p+n^2+n}{n} & \text{if } n \le 2 \text{ and } p \ge 1, \\ \frac{np+n+4}{2(n-1)} & \text{if } n \ge 3 \text{ and } p \ge 1, \end{cases} \tag{4.35}$$

and moreover $[1,\infty) \ni p \mapsto \overline{s}_0(n,p)$ *is increasing with*

$$\begin{cases} \overline{s}_0(n,1) > 2 & \text{if } n \le 3, \\ \overline{s}_0(n,1) < 2 & \text{if } n \ge 4. \end{cases} \tag{4.36}$$

PROOF. In the case $n \le 2$, using (4.31) and the fact that then $\frac{n}{(n-2)_+} = \infty$, from (4.34) we immediately obtain that

$$s_0(n,\kappa,p,q) \to n + \frac{(2p+n)\cdot 2}{n \cdot 2} = \frac{2p+n^2+n}{n} \qquad \text{as } (\kappa,q) \to \Big(\frac{n}{(n-2)_+}, \overline{q}_0(n,p)\Big) = (\infty,\infty).$$

Likewise, if $n \ge 3$ then (4.31) along with a straightforward computation entails that

$$\begin{aligned} s_0(n,\kappa,p,q) \quad &\to \quad \frac{n}{1+(n-2)} + \frac{(2p+n)\cdot\Big[\Big(2+(n-2)\Big)\cdot\frac{n(p+1)}{2(n-2)} - n\Big]}{n\cdot\Big(1+(n-2)\Big)\cdot\Big(2\cdot\frac{n(p+1)}{2(n-2)} - p\Big)} \\ &= \quad \frac{np+n+4}{2(n-1)} \qquad \text{as } (\kappa,q) \to \Big(\frac{n}{(n-2)_+}, \overline{q}_0(n,p)\Big) = \Big(\frac{n}{n-2}, \frac{n(p+1)}{2(n-2)}\Big). \end{aligned}$$

This proves (4.35), which in turn readily implies the claimed monotonicity property as well as (4.36). □

A final preparation provides a sequence of numbers $p_j \ge 1$ which diverge to ∞ in the case $n \le 3$ and thus can be used to determine appropriate integrability exponents in a bootstrap procedure in Lemma 4.10.

Lemma 4.8 *With $\overline{s}_0(\cdot,\cdot)$ as given by (4.35), let*

$$p_1 := 1 \qquad \textit{and} \qquad p_{j+1} := \max\left\{\frac{(n+1)\overline{s}_0(n,p_j)-n-2}{n}, 1\right\} \quad \textit{for } j \in \{1,2,3,...\}. \tag{4.37}$$

Then

$$(p_j)_{j\in\{1,2,3,...\}} \textit{ is increasing with } \; p_j \to +\infty \textit{ as } j \to \infty \qquad \textit{if and only if } n \le 3. \tag{4.38}$$

PROOF. The statements contained in (4.38) for $n \le 2$ directly follow from the observations that combining (4.37) with (4.35), in the case $n = 1$ we have

$$\overline{s}_0(1,p) = 2p + 2 \qquad \text{for all } p \ge 1$$

and hence

$$p_{j+1} = 4p_j + 1 \qquad \text{for all } j \ge 1,$$

whereas if $n = 2$ then

$$\overline{s}_0(2,p) = p + 3 \qquad \text{for all } p \ge 1$$

and thus

$$p_{j+1} = \frac{3p_j + 5}{2} \qquad \text{for all } j \ge 1.$$

When $n = 3$, (4.35) reduces to

$$\overline{s}_0(3,p) = \frac{6p^2 + 23p + 21}{4(2p+3)} = \frac{6\left(p + \frac{3}{2}\right)\left(p + \frac{7}{3}\right)}{4(2p+3)} = \frac{3\left(p + \frac{7}{3}\right)}{4} = \frac{3p+7}{4} \qquad \text{for all } p \ge 1,$$

so that then (4.37) yields

$$p_{j+1} = p_j + \frac{2}{3}. \qquad \text{for all } j \ge 1,$$

still asserting monotone divergence of $(p_j)_{j\in\{1,2,3,...\}}$ to $+\infty$.
In the remaining case $n \ge 4$, however, (4.36) says that $\overline{s}_0(n,1) < 2$, which means that whenever $p_j = 1$, we have

$$\frac{(n+1)\overline{s}_0(n,p_j)-n-2}{n} < \frac{(n+1)\cdot 2 - n - 2}{n} = 1,$$

by (4.37) and an inductive argument implying that actually $p_j = 1$ for all $j \ge 1$. □

4.3 Spatio-temporal L^s bounds for ∇w implied by regularity properties of u and v

We can proceed to turn Lemma 4.2 and Lemma 4.3 into the following statement on regularity of ∇w.

Lemma 4.9 *Let $n \le 3$, and assume that for some $p \ge 1$ and $K > 0$ we have*

$$\int_\Omega u^p(\cdot,t) \le K \quad \textit{and} \quad \int_\Omega v^p(\cdot,t) \le K \qquad \textit{for all } t \in (0,T_{max}) \tag{4.39}$$

as well as

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le K \quad \textit{and} \quad \int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla v|^2 \le K \qquad \textit{for all } t \in (0,\widehat{T}_{max}). \tag{4.40}$$

Then for any $s > 2$ fulfilling $s < \overline{s}_0(n,p)$, with $\overline{s}_0(n,p) > 2$ as given by Lemma 4.7, we can find $C = C(p,s,K) > 0$ such that the function w in (3.8) satisfies

$$\int_t^{t+\tau}\int_\Omega |\nabla w|^s \le C \qquad \textit{for all } t \in (0,\widehat{T}_{max}). \tag{4.41}$$

Proof. We first invoke Lemma 4.6 to see that $\overline{q}_0(n,p)$ as defined in (4.30) satisfies $\overline{q}_0(n,p) > \frac{p}{2}$ and $\overline{q}_0(n,p) \ge n$, whence in particular

$$\frac{n}{\left(\frac{n}{\overline{q}_0(n,p)} - 1\right)_+} = \infty > \overline{s}_0(n,p).$$

By a continuous dependence argument, we can thus pick a number $q_0 < \overline{q}_0(n,p)$ such that still $q_0 > p$ and

$$\frac{n}{(\frac{n}{q}-1)_+} > \overline{s}_0(n,p) \qquad \text{for all } q \in (q_0, \overline{q}_0(n,p)). \tag{4.42}$$

In view of our hypothesis $s < \overline{s}_0(n,p)$ and the convergence statement (4.35) in Lemma 4.7, we can thereupon fix $q \in (q_0, \overline{q}_0(n,p))$ sufficiently close to $\overline{q}_0(n,p)$ and $\kappa \in (1, \frac{n}{(n-2)_+})$ suitably close to $\frac{n}{(n-2)_+}$ such that with $s_0(n,\kappa,p,q)$ as defined in (4.34) we have

$$s \le s_0(n,\kappa,p,q). \tag{4.43}$$

Then, particularly, $q > q_0 > \frac{p}{2}$ and hence $2q - p > 0$, so that

$$r := \frac{(2p+n)q}{n(2q-p)} \tag{4.44}$$

is positive and finite, and moreover the inequality $q < \overline{q}_0(n,p)$ asserts that $2nq - (2p+n)q = (n-2p)q < np$ and therefore $(2p+n)q < 2nq - np$, that is, $r > 1$.

Next, this definition (4.44) of r is equivalent to the identity $(2nr - 2p - n)q = npr$ and hence, together with the fact that $q < \overline{q}_0(n,p) \le \frac{n(p+1)}{2(n-2)_+}$, implies that $q \le q_0(n,p,r)$ with $q_0(n,p,r)$ as in Lemma 4.1. Therefore, Lemma 4.2 becomes applicable so as to yield $c_1 = c_1(p,s,K) > 0$ such that

$$\int_t^{t+\tau} \|w(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)} d\sigma \le c_1 \qquad \text{for all } t \in (0,\widehat{T}_{max}). \tag{4.45}$$

Apart from that, since $\kappa < \frac{n}{(n-2)_+}$ we may employ Lemma 4.3 to find $c_2 > 0$ such that

$$\|w(\cdot,\sigma)\|_{L^\kappa(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max}). \tag{4.46}$$

In order to prepare an appropriate interpolation between (4.45) and (4.46), let us first make sure that our above choices warrant that

$$W^{2,q}(\Omega) \hookrightarrow W^{1,s}(\Omega) \hookrightarrow L^\kappa(\Omega). \tag{4.47}$$

Indeed, the second statement herein results from the fact that $\kappa < \frac{n}{(n-2)_+}$, which together with our restriction $s > 2$ entails that

$$\Big(1 - \frac{n}{s}\Big) - \Big(-\frac{n}{\kappa}\Big) > 1 - \frac{n}{2} + (n-2)_+ \ge 0 \tag{4.48}$$

in both cases $n = 1$ and $n \ge 2$. The first inclusion in (4.47), however, is a direct consequence of (4.42): Since $s < \overline{s}_0(n,p)$, namely, this guarantees that $\frac{n}{(\frac{n}{q}-1)_+} > s$ and hence

$$2 - \frac{n}{q} > 1 - \frac{n}{s}. \tag{4.49}$$

Having thereby asserted (4.47), we can invoke the Gagliardo-Nirenberg inequality to obtain $c_3 = c_3(p,s) > 0$ such that

$$\int_t^{t+\tau} \|\nabla w(\cdot,\sigma)\|^s_{L^s(\Omega)} d\sigma \le c_3 \int_t^{t+\tau} \|w(\cdot,\sigma)\|^{s\theta}_{W^{2,q}(\Omega)} \|w(\cdot,\sigma)\|^{s(1-\theta)}_{L^\kappa(\Omega)} d\sigma + c_3 \int_t^{t+\tau} \|w(\cdot,\sigma)\|^s_{L^\kappa(\Omega)} d\sigma \tag{4.50}$$

for all $t \in (0, \widehat{T}_{max})$, with $\theta \in (0,1)$ given by

$$1 - \frac{n}{s} = \Big(2 - \frac{n}{q}\Big)\theta - \frac{n}{\kappa}(1-\theta)$$

and hence

$$\theta = \frac{1 - \frac{n}{s} + \frac{n}{\kappa}}{2 - \frac{n}{q} + \frac{n}{\kappa}}.$$

According to the restriction (4.43), thanks to (4.48) and (4.49) we thus have

$$s\theta = \frac{(1+\frac{n}{\kappa})s - n}{2 - \frac{n}{q} + \frac{n}{\kappa}} \le \frac{(1+\frac{n}{\kappa}) \cdot s_0(n,\kappa,p,q) - n}{2 - \frac{n}{q} + \frac{n}{\kappa}} = \frac{(2p+n)q}{n(2q-p)},$$

which by (4.44) means that $s\theta \le r$. As a consequence of (4.46) and (4.45), from (4.50) we conclude that indeed

$$\begin{aligned}
\int_t^{t+\tau} \|\nabla w(\cdot,\sigma)\|^s_{L^s(\Omega)} d\sigma &\le c_2 c_3^{s(1-\theta)} \int_t^{t+\tau} \|w(\cdot,\sigma)\|^{s\theta}_{W^{2,q}(\Omega)} d\sigma + c_3 c_2^s \\
&\le c_2 c_3^{s(1-\theta)} \Big(\int_t^{t+\tau} \|w(\cdot,\sigma)\|^r_{W^{2,q}(\Omega)} d\sigma \Big)^{\frac{s\theta}{r}} + c_3 c_2^s \\
&\le c_3 c_2^{s(1-\theta)} c_1^{\frac{s\theta}{r}} + c_3 c_2^s \qquad \text{for all } t \in (0, \widehat{T}_{max}),
\end{aligned}$$

because $\tau \le 1$. □

4.4 Unconditional L^p bounds for u and v

Now in the case $n \le 3$, successively applying Lemma 4.9 and Lemma 4.5, by using the sequence $(p_j)_{j\in\{1,2,3,\dots\}}$ provided by Lemma 4.8 we can now assert boundedness of u and v with respect to the norm in $L^p(\Omega)$ for any finite p, without any further hypotheses involving w.

Lemma 4.10 *Let $n \le 3$. Then for all $p \ge 1$ there exists $C = C(p) > 0$ satisfying*

$$\int_\Omega u^p(\cdot,t) \le C \quad \textit{and} \quad \int_\Omega v^p(\cdot,t) \le C \qquad \textit{for all } t \in (0,T_{max}). \tag{4.51}$$

PROOF. We let $(p_j)_{j\in\{1,2,3,\dots\}} \subset [1,\infty)$ be defined by (4.37), that is, we let $p_1 := 1$ and

$$p_{j+1} := \frac{(n+1)\overline{s}_0(n,p_j) - n - 2}{n}, \qquad j \ge 1, \tag{4.52}$$

where $\overline{s}_0(\cdot,\cdot) > 2$ is taken from Lemma 4.7. Since Lemma 4.8 asserts that thanks to our assumption $n \le 3$ we then have $p_j \to \infty$ as $j \to \infty$, for the proof of the lemma it is sufficient to show that for all $j \ge 1$ and each $p \in \{1\} \cup [1,p_j)$ we can find $c_1(p) > 0$ satisfying

$$\int_\Omega u^p(\cdot,t) \le c_1(p) \quad \text{and} \quad \int_\Omega v^p(\cdot,t) \le c_1(p) \qquad \text{for all } t \in (0,T_{max}) \tag{4.53}$$

and moreover

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le c_1(p) \quad \text{and} \quad \int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla v|^2 \le c_1(p) \qquad \text{for all } t \in (0,\widehat{T}_{max}). \tag{4.54}$$

Indeed, for $j = 1$ this directly results from combining (2.2) and (2.3) with Lemma 3.5, so that proceeding inductively we may assume that this statement holds for some $j \ge 1$ and then only need to show that the bounds in (4.53) and (4.54) then extend to any fixed $p \in (1,p_{j+1})$.
To see this, given $p \in (1,p_{j+1})$ we can pick $s \in (2,\overline{s}_0(n,p_j))$ sufficiently close to $\overline{s}_0(n,p_j)$ such that

$$p < \frac{(n+1)s - n - 2}{n} \tag{4.55}$$

and then use the continuity of $1 \le \tilde{p} \mapsto \overline{s}_0(n,\tilde{p})$ to fix $\tilde{p} \in \{1\} \cup [1,p_j)$ suitably large fulfilling

$$s < \overline{s}_0(n,\tilde{p}).$$

Thanks to the latter inequality, (4.53) and (4.54) apply to ensure in conjunction with Lemma 4.9 that

$$\int_t^{t+\tau}\int_\Omega |\nabla w|^s \le c_2 \qquad \text{for all } t \in (0,\widehat{T}_{max})$$

with some $c_2 > 0$ whereupon (4.55) enables us to invoke Lemma 4.5, providing $c_3 > 0$ such that

$$\int_\Omega u^p(\cdot,t) \le c_3 \quad \text{and} \quad \int_\Omega v^p(\cdot,t) \le c_3 \qquad \text{for all } t \in (0,T_{max})$$

and

$$\int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla u|^2 \le c_3 \quad \text{and} \quad \int_t^{t+\tau}\int_\Omega u^{p-1}|\nabla v|^2 \le c_3 \qquad \text{for all } t \in (0,\widehat{T}_{max}).$$

The proof is thus complete. □

4.5 L^∞ and Hölder estimates for u and v

In order to derive estimates for u and v also with respect to the norm in $L^\infty(\Omega)$, a useful ingredient in another application of Lemma 4.4 will consist in a uniform bound for ∇w which in view of Lemma 4.2 is implied by Lemma 4.10.

Lemma 4.11 *There exists $C > 0$ such that the function w from (3.8) satisfies*

$$\|\nabla w(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.56}$$

PROOF. Thanks to Lemma 4.10, Lemma 4.2 says that for each $q \in (1,\infty)$ we can find $c_1 = c_1(q) > 0$ such that

$$\int_t^{t+\tau} \|w(\cdot,\sigma)\|^q_{W^{2,q}(\Omega)} d\sigma + \int_t^{t+\tau} \|w_t(\cdot,\sigma)\|^q_{L^q(\Omega)} d\sigma \le c_1 \qquad \text{for all } t \in (0, \widehat{T}_{max}).$$

According to a well-known embedding property [4, Theorem 5.2], an application of this to suitably large q readily yields (4.56). □

Thereupon, a Moser-type iteration indeed yields boundedness of (u, v).

Lemma 4.12 *There exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{and} \quad \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.57}$$

PROOF. We only prove boundedness of u, omitting minor adaptations necessary for estimating v along the same lines.
To this end, we let $p_j := 2^j$ for $j \in \{0, 1, 2, ...\}$, and in order to estimate the finite numbers

$$M_j := \max\left\{1\,, \sup_{t\in(0,T_{max})} \int_\Omega u^{p_j}(\cdot,t)\right\}, \qquad j \ge 0, \tag{4.58}$$

we first invoke Lemma 4.11 to fix $c_1 > 0$ such that

$$|\nabla w| \le c_1 \qquad \text{in } \Omega \times (0, T_{max}).$$

Therefore, (4.16) along with Young's inequality implies that for each $j \ge 1$ we have

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^{p_j} + \int_\Omega u^{p_j} + p_j(p_j-1)\int_\Omega u^{p_j-2}|\nabla u|^2 &\le p_j(p_j-1)\int_\Omega u^{p_j-1}\nabla u\cdot\nabla w + (\mu p_j+1)\int_\Omega u^{p_j} \\
&\le \frac{p_j(p_j+1)}{2}\int_\Omega u^{p_j-2}|\nabla u|^2 \\
&\quad + \Big\{\frac{p_j(p_j-1)c_1^2}{2} + \mu p_j + 1\Big\}\cdot\int_\Omega u^{p_j}
\end{aligned}$$

for all $t \in (0, T_{max})$, and hence

$$\frac{d}{dt}\int_\Omega u^{p_j} + \int_\Omega u^{p_j} + \int_\Omega |\nabla u^{\frac{p_j}{2}}|^2 \le c_2 p_j^2 \int_\Omega u^{p_j} \qquad \text{for all } t \in (0, T_{max}) \tag{4.59}$$

with some $c_2 > 0$ independent of j, because $\frac{p_j(p_j-1)}{2} \cdot (\frac{2}{p_j})^2 = \frac{2(p_j-1)}{p_j} \ge 1$ for all $j \ge 1$. Here we apply that Gagliardo-Nirenberg inequality and Young's inequality to find j-independent positive constants c_3 and c_4 fulfilling

$$\begin{aligned} c_2 p_j^2 \int_\Omega u^{p_j} &= c_2 p_j^2 \|u^{\frac{p_j}{2}}\|_{L^2(\Omega)}^2 \le c_3 p_j^2 \cdot \Big\{ \|\nabla u^{\frac{p_j}{2}}\|_{L^2(\Omega)}^{\frac{2n}{n+2}} \|u^{\frac{p_j}{2}}\|_{L^1(\Omega)}^{\frac{4}{n+2}} + \|u^{\frac{p_j}{2}}\|_{L^1(\Omega)}^2 \Big\} \\ &\le c_3 p_j^2 \cdot \Big\{ \|\nabla u^{\frac{p_j}{2}}\|_{L^2(\Omega)}^{\frac{2n}{n+2}} \cdot M_{j-1}^{\frac{4}{n+2}} + M_{j-1}^2 \Big\} \\ &\le \|\nabla u^{\frac{p_j}{2}}\|_{L^2(\Omega)}^2 + c_4 p_j^{n+2} M_{j-1}^2 + c_3 p_j^2 M_{j-1}^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

From (4.59) we therefore obtain $B > 1$ independent of j such that

$$\frac{d}{dt} \int_\Omega u^{p_j} + \int_\Omega u^{p_j} \le (c_3 + c_4) p_j^{n+2} M_{j-1}^2 \le B^j M_{j-1}^2 \qquad \text{for all } t \in (0, T_{max}),$$

which by an ODE comparison entails that

$$\sup_{t \in (0, T_{max})} \int_\Omega u^{p_j}(\cdot, t) \le \max \Big\{ \int_\Omega u_0^{p_j} \, , \, B^j M_{j-1}^2 \Big\}. \tag{4.60}$$

Since boundedness of u is evident in the case when $M_j \le \max\{1, \int_\Omega u_0^{p_j}\}$ for infinitely many $j \ge 1$, we may assume that $M_j > \max\{1, \int_\Omega u_0^{p_j}\}$ and thus, by (4.58) and (4.60), that

$$M_j \le B^j M_{j-1}^2 \tag{4.61}$$

for all sufficiently large $j \in \mathbb{N}$, whence upon enlarging B if necessary we can achieve that (4.61) actually holds for all $j \ge 1$. By induction, this yields that

$$M_j \le B^{\sum_{i=0}^{j-1} (j-i) \cdot 2^i} \cdot M_0^{2^j} = B^{2^{j+1} - j - 2} M_0^{2^j} \le B^{2^{j+1}} M_0^{2^j} \qquad \text{for all } j \ge 1,$$

and that hence

$$M_j^{\frac{1}{2^j}} \le B^2 M_0 \qquad \text{for all } j \ge 1,$$

which implies that u indeed belongs to $L^\infty(\Omega \times (0, T_{max}))$. □

Inter alia, this allows for applying standard theory on Hölder estimates for scalar parabolic equations in deriving the following.

Lemma 4.13 *There exist $\gamma \in (0,1)$ and $C > 0$ such that*

$$\|u\|_{C^{\gamma, \frac{\gamma}{2}}(\bar{\Omega} \times [t, t+\tau])} \le C \quad \text{and} \quad \|v\|_{C^{\gamma, \frac{\gamma}{2}}(\bar{\Omega} \times [t, t+\tau])} \le C \qquad \text{for all } t \in [0, \widehat{T}_{max}). \tag{4.62}$$

Proof. According to Lemma 4.12 and Lemma 4.11, we can fix positive constants c_1, c_2 and c_3 such that with w as defined in (3.8) we have

$$u(x,t) \le c_1, \quad v(x,t) \le c_2 \quad \text{and} \quad |\nabla w(x,t)| \le c_3 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}).$$

Introducing

$$A(x,t,\xi) := \Big(1 + bu(x,t) + av(x,t)\Big) \cdot \xi - u(x,t)\nabla w(x,t)$$

and

$$B(x,t) := \mu u(x,t)\Big(1 - \alpha_1 u(x,t) - \alpha_2 v(x,t)\Big)$$

for $(x,t,\xi) \in \Omega \times (0, T_{max}) \times \mathbb{R}^n$, using Young's inequality we can thereby estimate

$$\begin{aligned} A(x,t,\nabla u) &= (1+bu+av)|\nabla u|^2 - u\nabla u \cdot \nabla w \\ &\geq (1+bu+av)|\nabla u|^2 - u \cdot |\nabla u| \cdot |\nabla w| \\ &\geq \frac{1}{2}|\nabla u|^2 - \frac{1}{2}u^2|\nabla w|^2 \\ &\geq \frac{1}{2}|\nabla u|^2 - \frac{c_1^2 c_3^2}{2} \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}) \end{aligned} \tag{4.63}$$

and

$$\Big|A(x,t,\nabla u)\Big| \leq (1 + bc_1 + ac_2)|\nabla u| + c_1 c_3 \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}) \tag{4.64}$$

as well as

$$|B(x,t)| \leq \mu c_1(1 + \alpha_1 c_1 + \alpha_2 c_2) \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}). \tag{4.65}$$

We now recall that $v = \frac{b}{a}u - \frac{1}{a}w$ to rewrite the first equation in (1.2) according to

$$\begin{aligned} u_t &= \nabla \cdot (\nabla u + av\nabla u + au\nabla v) + \mu u(1 - \alpha_1 u - \alpha_2 v) \\ &= \nabla \cdot (\nabla u + av\nabla u + bu\nabla u - u\nabla w) + \mu u(1 - \alpha_1 u - \alpha_2 v) \\ &= \nabla \cdot A(x,t,\nabla u) + B(x,t) \qquad \text{for } x \in \Omega \text{ and } t \in (0, T_{max}). \end{aligned}$$

Therefore, (4.63), (4.64) and (4.65) allow us to apply a well-known result on Hölder regularity in quasilinear parabolic equations [20, Theorem 1.3 and Remark 1.4] to conclude that u satisfies the estimate of the form in (4.62). Along with an analogous argument for v, this proves the claim. □

5 Gradient estimates. Proof of Theorem 1.1

In our final step toward proving that $T_{max} = \infty$, we will establish time-independent bounds for ∇u and ∇v with respect to the norm in $L^4(\Omega)$. In achieving this, we shall make use of the following Ehrling-type lemma which allows for additive interpolation of $\int_\Omega |\nabla \varphi|^6$ between a certain second-order and some zero-order expression, quantitatively involving a modulus of continuity of the function φ to be estimated.

Lemma 5.1 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\omega : (0,\infty) \to (0,\infty)$ be nondecreasing. Then for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\|\nabla\varphi\|_{L^6(\Omega)}^6 \leq \varepsilon \int_\Omega |\nabla\varphi|^2 |D^2\varphi|^2 + c(\varepsilon)\|\varphi\|_{L^\infty(\Omega)}^6 \tag{5.1}$$

is valid for any

$$\varphi \in \mathcal{F}_\omega := \Big\{ \phi : \bar\Omega \to \mathbb{R} \ \Big| \ \textit{For all } \varepsilon' > 0, \textit{ we have } |\phi(x) - \phi(y)| < \varepsilon' \\ \textit{whenever } x, y \in \bar\Omega \textit{ are such that } |x-y| < \omega(\varepsilon') \Big\} \tag{5.2}$$

which additionally satisfy $\varphi \in C^2(\bar\Omega)$ *and* $\frac{\partial\varphi}{\partial\nu} = 0$ *on* $\partial\Omega$.

PROOF. Given $\varepsilon > 0$, we write

$$\varepsilon' := \sqrt{\frac{\varepsilon}{4n+128}} \tag{5.3}$$

and let $\delta := \omega(\varepsilon')$. Then by compactness of $\bar\Omega$, the open covering $(B_\delta(x))_{x\in\bar\Omega}$ of $\bar\Omega$ already contains a finite covering, meaning that we can pick $N \in \mathbb{N}$ and $\{x_1, ..., x_N\} \subset \bar\Omega$ such that $\bar\Omega \subset \bigcup_{j=1}^N B_\delta(x_j)$. We now take an associated partition of unity by fixing $(\zeta_j)_{j\in\{1,...,N\}} \subset C^1(\bar\Omega)$ such that $\zeta_j \ge 0$ in $\bar\Omega$ and $\operatorname{supp}\zeta_j \subset B_\delta(x_j)$ for all $j \in \{1, ..., N\}$ and $\sum_{j=1}^N \zeta_j \equiv 1$ in $\bar\Omega$. Finally, in accordance with Young's inequality we can choose $c_1 > 0$ fulfilling

$$AB \le \frac{1}{8N} A^{\frac{6}{5}} + c_1 B^6 \qquad \text{for all } A \ge 0 \text{ and } B \ge 0, \tag{5.4}$$

and claim that then (5.1) holds for all $\varphi \in C^2(\bar\Omega) \cap \mathcal{F}_\omega$ with $\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$ if we let

$$C(\varepsilon) := 512 N c_1 c_2^6 \tag{5.5}$$

with

$$c_2 := \max_{j\in\{1,...,N\}} \|\nabla\zeta_j\|_{L^6(\Omega)}. \tag{5.6}$$

To verify this, given any such φ we abbreviate

$$I := \int_\Omega |\nabla\varphi|^6 \quad \text{and} \quad J := \int_\Omega |\nabla\varphi|^2 |D^2\varphi|^2$$

as well as

$$\overline{\varphi}_j := \varphi(x_j), \quad I_j := \int_\Omega |\nabla\varphi|^6 \zeta_j \quad \text{and} \quad J_j := \int_\Omega |\nabla\varphi|^2 |D^2\varphi|^2 \zeta_j \qquad \text{for } j \in \{1, ..., N\}.$$

Then integrating by parts we firstly obtain that for each $j \in \{1, ..., N\}$,

$$\begin{aligned} I_j &= \int_\Omega |\nabla\varphi|^4 \nabla\varphi \cdot \nabla(\varphi - \overline{\varphi}_j)\zeta_j \\ &= -\int_\Omega (\varphi - \overline{\varphi}_j)|\nabla\varphi|^4 \Delta\varphi\zeta_j - \int_\Omega (\varphi - \overline{\varphi}_j)\nabla\varphi \cdot \nabla|\nabla\varphi|^4 \zeta_j \\ &\quad - \int_\Omega (\varphi - \overline{\varphi}_j)|\nabla\varphi|^4 \nabla\varphi \cdot \nabla\zeta_j \\ &=: I_{j1} + I_{j2} + I_{j3}, \end{aligned} \tag{5.7}$$

where boundary integrals do not appear due to the fact that $\frac{\partial\varphi}{\partial\nu}|_{\partial\Omega} = 0$. Here in the first two integrals on the right we use the inclusion $\varphi \in \mathcal{F}_\omega$ which along with our choice of δ ensures that

$$|\varphi - \overline{\varphi}_j| \le \varepsilon' \qquad \text{in } \operatorname{supp} \zeta_j.$$

Since $|\Delta\varphi| \le \sqrt{n}|D^2\varphi|$ in Ω, by means of the Cauchy-Schwarz inequality and Young's inequality we can therefore estimate

$$\begin{aligned}
I_{j1} &\le \varepsilon' \int_\Omega |\nabla\varphi|^4 |\Delta\varphi| \zeta_j \\
&\le \varepsilon' \cdot \Big(\int_\Omega |\nabla\varphi|^6 \zeta_j \Big)^{\frac{1}{2}} \Big(\int_\Omega |\nabla\varphi|^2 |\Delta\varphi|^2 \zeta_j \Big)^{\frac{1}{2}} \\
&\le \sqrt{n}\varepsilon' \sqrt{I_j}\sqrt{J_j} \\
&\le \frac{1}{2} I_j + \frac{n\varepsilon'^2}{2} J_j. \qquad (5.8)
\end{aligned}$$

Similarly, in view of the identity $\nabla|\nabla\varphi|^4 = 4|\nabla\varphi|^2 D^2\varphi \cdot \nabla\varphi$ we have

$$\begin{aligned}
I_{j2} &\le 4\varepsilon' \int_\Omega |\nabla\varphi|^2 \Big| \nabla\varphi \cdot (D^2\varphi \cdot \nabla\varphi) \Big| \zeta_j \\
&\le 4\varepsilon' \int_\Omega |\nabla\varphi|^4 |D^2\varphi|^2 \zeta_j \\
&\le 4\varepsilon' \sqrt{I_j}\sqrt{J_j} \\
&\le \frac{1}{4} I_j + 16\varepsilon'^2 J_j. \qquad (5.9)
\end{aligned}$$

In the rightmost summand in (5.7), we rather use the rough estimate $|\varphi - \overline{\varphi}_j| \le 2\|\varphi\|_{L^\infty(\Omega)}$ to infer from the Hölder inequality, (5.6) and (5.4) that

$$\begin{aligned}
I_{j3} &\le 2\|\varphi\|_{L^\infty(\Omega)} \Big(\int_\Omega |\nabla\varphi|^6 \Big)^{\frac{5}{6}} \Big(\int_\Omega |\nabla\zeta_j|^6 \Big)^{\frac{1}{6}} \\
&\le 2c_2 \|\varphi\|_{L^\infty(\Omega)} I^{\frac{5}{6}} \\
&\le \frac{1}{8N} I + c_1 \cdot \Big(2c_2 \|\varphi\|_{L^\infty(\Omega)} \Big)^6. \qquad (5.10)
\end{aligned}$$

In consequence, (5.7)-(5.10) show that

$$\frac{1}{4} I_j \le \Big(\frac{n}{2} + 16 \Big) \varepsilon'^2 J_j + \frac{1}{8N} I + 2^6 c_1 c_2^6 \|\varphi\|^6_{L^\infty(\Omega)},$$

which on summation over $j \in \{1, ..., N\}$ yields

$$\begin{aligned}
I &= \sum_{j=1}^N I_j \\
&\le (2n+64)\varepsilon'^2 \cdot \sum_{j=1}^N J_j + 4N \cdot \Big(\frac{1}{8N} I + 2^6 c_1 c_2^6 \|\varphi\|^6_{L^\infty(\Omega)} \Big) \\
&= (2n+64)\varepsilon'^2 J + \frac{1}{2} I + 256 N c_1 c_2^6 \|\varphi\|^6_{L^\infty(\Omega)}.
\end{aligned}$$

As thus

$$I \le (4n+128)\varepsilon'^2 J + 512 N c_1 c_2^6 \|\varphi\|_{L^\infty(\Omega)}^6,$$

in light of the definitions (5.3) and (5.5) of ε' and $C(\varepsilon)$ this precisely proves (5.1). □

Along with the uniform continuity property of the trajectory $((u(\cdot,t),v(\cdot,t))_{t\in[0,T_{max})}$ implied by Lemma 4.13, the latter enables us to suitably estimate certain first-order terms obtained during a standard testing procedure applied to track the evolution of $\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla v|^4$. The following statement on this is the only place where the convexity of Ω is explicitly needed. We remark that by means of suitable embeddings involving trace Sobolev estimates in a manner demonstrated e.g. in [11] for a related framework, this additional requirement can be removed; in order not to further complicate our presentation, we refrain from addressing this more general setting here.

Lemma 5.2 *Let $n \le 3$, and suppose that Ω is convex. Then there exists $C>0$ such that*

$$\int_\Omega |\nabla u|^4 \le C \quad \textit{and} \quad \int_\Omega |\nabla v|^4 \le C \qquad \textit{for all } t\in(0,T_{max}). \tag{5.11}$$

PROOF. By direct computation using the first equation in (1.2), we see that

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla u|^4 &= \int_\Omega |\nabla u|^2 \nabla u\cdot\nabla u_t + \int_\Omega |\nabla u|^4 \\
&= \int_\Omega |\nabla u|^2\nabla u\cdot\nabla\Delta u + a\int_\Omega |\nabla u|^2\nabla u\cdot\nabla\Delta(uv) \\
&\quad +(\mu+1)\int_\Omega |\nabla u|^4 - \mu\alpha_1\int_\Omega u|\nabla u|^4 - \mu\alpha_2\int_\Omega |\nabla u|^2\nabla u\cdot\nabla(uv) \\
&=: I_1+...+I_5 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{5.12}$$

Here we use the pointwise identity

$$\nabla u\cdot\nabla\Delta u = \frac{1}{2}\Delta|\nabla u|^2 - |D^2u|^2 \qquad \text{in } \Omega\times(0,T_{max}) \tag{5.13}$$

and integrate by parts to obtain

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_\Omega |\nabla u|^2\Delta|\nabla u|^2 - \int_\Omega |\nabla u|^2|D^2u|^2 \\
&= -\frac{1}{2}\int_\Omega \Big|\nabla|\nabla u|^2\Big|^2 + \frac{1}{2}\int_{\partial\Omega}|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu} - \int_\Omega |\nabla u|^2|D^2u|^2 \\
&\le -\int_\Omega |\nabla u|^2|D^2u|^2 \qquad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{5.14}$$

because $\frac{\partial|\nabla u|^2}{\partial\nu} \le 0$ on $\partial\Omega$ due to the convexity of Ω [15].
As for I_2, we once more integrate by parts to compute

$$\begin{aligned}
I_2 &= -a\int_\Omega |\nabla u|^2\Delta u\Delta(uv) - a\int_\Omega \Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta(uv) \\
&=: I_{21}+I_{22} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{5.15}$$

where since $\Delta(uv) = v\Delta u + 2\nabla u \cdot \nabla v + u\Delta v$ and $v = \frac{b}{a}u - \frac{1}{a}w$ by (3.8) we have

$$\begin{aligned} I_{21} &= -a\int_\Omega v|\nabla u|^2|\Delta u|^2 - 2a\int_\Omega |\nabla u|^2(\nabla u\cdot\nabla v)\Delta u \\ &\quad -a\int_\Omega u|\nabla u|^2\Delta u\Delta v \\ &= -a\int_\Omega v|\nabla u|^2|\Delta u|^2 - 2a\int_\Omega |\nabla u|^2(\nabla u\cdot\nabla v)\Delta u \\ &\quad -b\int_\Omega u|\nabla u|^2|\Delta u|^2 + \int_\Omega u|\nabla u|^2\Delta u\Delta w \qquad \text{for all } t\in(0,T_{max}). \end{aligned} \tag{5.16}$$

Proceeding similarly, for the second integral on the right of (5.15) we obtain

$$\begin{aligned} I_{22} &= -a\int_\Omega v\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta u - 2a\int_\Omega(\nabla u\cdot\nabla v)\Big(\nabla u\cdot\nabla|\nabla u|^2\Big) \\ &\quad -a\int_\Omega u\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta v \\ &= -a\int_\Omega v\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta u - 2a\int_\Omega(\nabla u\cdot\nabla v)\Big(\nabla u\cdot\nabla|\nabla u|^2\Big) \\ &\quad -b\int_\Omega u\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta u + \int_\Omega u\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta w \\ &=: I_{221} + \dots + I_{224} \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{5.17}$$

where two integrations by parts using (5.13) yield

$$\begin{aligned} -a\int_\Omega v\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta u &= a\int_\Omega v|\nabla u|^2|\Delta u|^2 + a\int_\Omega|\nabla u|^2(\nabla u\cdot\nabla v)\Delta u \\ &\quad +a\int_\Omega v|\nabla u|^2\nabla u\cdot\nabla\Delta u \qquad \text{for all } t\in(0,T_{max}) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} a\int_\Omega v|\nabla u|^2\nabla u\cdot\nabla\Delta u &= \frac{a}{2}\int_\Omega v|\nabla u|^2\Delta|\nabla u|^2 - a\int_\Omega v|\nabla u|^2|D^2u|^2 \\ &= -\frac{a}{2}\int_\Omega v\Big|\nabla|\nabla u|^2\Big|^2 - \frac{a}{2}\int_\Omega|\nabla u|^2\nabla v\cdot\nabla|\nabla u|^2 \\ &\quad +\frac{a}{2}\int_{\partial\Omega} v|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu} - a\int_\Omega v|\nabla u|^2|D^2u|^2 \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{5.19}$$

again because $\frac{\partial|\nabla u|^2}{\partial\nu} \le 0$ on $\partial\Omega$, so that

$$I_{221} \le a\int_\Omega v|\nabla u|^2|\Delta u|^2 + a\int_\Omega|\nabla u|^2(\nabla u\cdot\nabla v)\Delta u - \frac{a}{2}\int_\Omega|\nabla u|^2\nabla v\cdot\nabla|\nabla u|^2 \qquad \text{for all } t\in(0,T_{max}). \tag{5.20}$$

In quite a similar way, replacing av by bu in (5.18) and (5.19) we obtain

$$
\begin{aligned}
I_{223} &= b\int_\Omega u|\nabla u|^2|\Delta u|^2 + b\int_\Omega |\nabla u|^4\Delta u \\
&\quad -\frac{b}{2}\int_\Omega u\Big|\nabla|\nabla u|^2\Big|^2 - \frac{b}{2}\int_\Omega |\nabla u|^2\nabla u\cdot\nabla|\nabla u|^2 \\
&\quad +\frac{b}{2}\int_{\partial\Omega} u|\nabla u|^2\frac{\partial|\nabla u|^2}{\partial\nu} - b\int_\Omega u|\nabla u|^2|D^2u|^2 \\
&\le b\int_\Omega u|\nabla u|^2|\Delta u|^2 + b\int_\Omega |\nabla u|^4\Delta u \\
&\quad -\frac{b}{2}\int_\Omega |\nabla u|^2\nabla u\cdot\nabla|\nabla u|^2 \qquad \text{for all } t\in(0,T_{max}). \qquad (5.21)
\end{aligned}
$$

Favorably, the nonnegative terms on the right-hand sides of (5.20) and (5.21) are canceled by the respective leading term on the right-hand sides of (5.16) and (5.17), whence in summary we infer from (5.15), (5.16), (5.17), (5.20) and (5.21) that

$$
\begin{aligned}
I_2 &\le -2a\int_\Omega |\nabla u|^2(\nabla u\cdot\nabla v)\Delta u - b\int_\Omega u|\nabla u|^2|\Delta u|^2 \\
&\quad +\int_\Omega u|\nabla u|^2\Delta u\Delta w \\
&\quad +a\int_\Omega |\nabla u|^2(\nabla u\cdot\nabla v)\Delta u - \frac{a}{2}\int_\Omega |\nabla u|^2\nabla v\cdot\nabla|\nabla u|^2 \\
&\quad -2a\int_\Omega (\nabla u\cdot\nabla v)\nabla u\cdot\nabla|\nabla u|^2 \\
&\quad +b\int_\Omega |\nabla u|^4\Delta u - \frac{b}{2}\int_\Omega |\nabla u|^2\nabla u\cdot\nabla|\nabla u|^2 \\
&\quad +\int_\Omega u\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta w \\
&\le -a\int_\Omega |\nabla u|^2(\nabla u\cdot\nabla v)\Delta u + \int_\Omega u|\nabla u|^2\Delta u\Delta w \\
&\quad -\frac{a}{2}\int_\Omega |\nabla u|^2\nabla v\cdot\nabla|\nabla u|^2 \\
&\quad -2a\int_\Omega (\nabla u\cdot\nabla v)\nabla u\cdot\nabla|\nabla u|^2 \\
&\quad +b\int_\Omega |\nabla u|^4\Delta u - \frac{b}{2}\int_\Omega |\nabla u|^2\nabla u\cdot\nabla|\nabla u|^2 \\
&\quad +\int_\Omega u\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta w \qquad \text{for all } t\in(0,T_{max}). \qquad (5.22)
\end{aligned}
$$

Here, recalling that $c_1 := \|u\|_{L^\infty(\Omega\times(0,T_{max}))}$ is finite by Lemma 4.12, and that $|\Delta u| \le \sqrt{n}|D^2u|$, upon several applications of Young's inequality we see that

$$
-a\int_\Omega |\nabla u|^2(\nabla u\cdot\nabla v)\Delta u \le a\sqrt{n}\int_\Omega |\nabla u|^2|\nabla v||D^2u|
$$

$$\begin{aligned}
&\le \frac{1}{2}\int_\Omega |\nabla u|^2|D^2u|^2 + \frac{a^2n}{2}\int_\Omega |\nabla u|^4|\nabla v|^2 \\
&\le \frac{1}{2}\int_\Omega |\nabla u|^2|D^2u|^2 + \frac{a^2n}{2}\int_\Omega |\nabla u|^6 + \frac{a^2n}{2}\int_\Omega |\nabla v|^6
\end{aligned}$$

and

$$\begin{aligned}
\int_\Omega u|\nabla u|^2\Delta u\Delta w &\le c_1\sqrt{n}\int_\Omega |\nabla u|^2|D^2u||\Delta w| \\
&\le \frac{1}{4}\int_\Omega |\nabla u|^2|D^2u|^2 + c_1^2 n\int_\Omega |\nabla u|^2|\Delta w|^2 \\
&\le \frac{1}{4}\int_\Omega |\nabla u|^2|D^2u|^2 + c_1^2 n\int_\Omega |\nabla u|^6 + c_1^2 n\int_\Omega |\Delta w|^3
\end{aligned}$$

and

$$\begin{aligned}
-\frac{1}{2}\int_\Omega |\nabla u|^2\nabla v\cdot\nabla|\nabla u|^2 &= -a\int_\Omega |\nabla u|^2\nabla v\cdot(D^2u\cdot\nabla u) \\
&\le a\int_\Omega |\nabla u|^3|\nabla v||D^2u| \\
&\le \frac{1}{8}\int_\Omega |\nabla u|^2|D^2u|^2 + 2a^2\int_\Omega |\nabla u|^4|\nabla v|^2 \\
&\le \frac{1}{8}\int_\Omega |\nabla u|^2|D^2u|^2 + 2a^2\int_\Omega |\nabla u|^6 + 2a^2\int_\Omega |\nabla v|^6
\end{aligned}$$

as well as

$$\begin{aligned}
-2a\int_\Omega (\nabla u\cdot\nabla v)\nabla u\cdot\nabla|\nabla u|^2 &\le 4a\int_\Omega |\nabla u|^3|\nabla v||D^2u| \\
&\le \frac{1}{16}\int_\Omega |\nabla u|^2|D^2u|^2 + 64a^2\int_\Omega |\nabla u|^6 + 64a^2\int_\Omega |\nabla v|^6
\end{aligned}$$

and

$$b\int_\Omega |\nabla u|^4\Delta u \le b\sqrt{n}\int_\Omega |\nabla u|^4|D^2u| \le \frac{1}{32}\int_\Omega |\nabla u|^2|D^2u|^2 + 8b^2n\int_\Omega |\nabla u|^6$$

and

$$-\frac{b}{2}\int_\Omega |\nabla u|^2\nabla u\cdot\nabla|\nabla u|^2 \le b\int_\Omega |\nabla u|^4|D^2u| \le \frac{1}{64}\int_\Omega |\nabla u|^2|D^2u|^2 + 16b^2\int_\Omega |\nabla u|^6$$

and

$$\begin{aligned}
\int_\Omega u\Big(\nabla u\cdot\nabla|\nabla u|^2\Big)\Delta w &\le 2c_1\int_\Omega |\nabla u|^2|D^2u||\Delta w| \\
&\le \frac{1}{128}\int_\Omega |\nabla u|^2|D^2u|^2 + 256c_1^2\int_\Omega |\nabla u|^2|\Delta w|^2 \\
&\le \frac{1}{128}\int_\Omega |\nabla u|^2|D^2u|^2 + 256c_1^2\int_\Omega |\nabla u|^6 + 256c_1^2\int_\Omega |\Delta w|^3.
\end{aligned}$$

According to (5.22), we can therefore find $c_2 > 0$ such that

$$I_2 \le \frac{127}{128}\int_\Omega |\nabla u|^2|D^2u|^2 + c_2 \cdot \left(\int_\Omega |\nabla u|^6 + \int_\Omega |\nabla v|^6 \right) + c_2 \int_\Omega |\Delta w|^3 \qquad \text{for all } t \in (0, T_{max}), \tag{5.23}$$

and to finally estimate the three rightmost terms in (5.12) we once more use Young's inequality to see that

$$\begin{aligned} I_3 + I_4 + I_5 &\le (\mu+1)\int_\Omega |\nabla u|^4 - \mu\alpha_2 \int_\Omega |\nabla u|^2 \nabla u \cdot (u\nabla v + v\nabla u) \\ &\le (\mu+1)\int_\Omega |\nabla u|^4 - \mu\alpha_2 \int_\Omega u|\nabla u|^2 \nabla u \cdot \nabla v \\ &\le (\mu+1) \cdot \left(\int_\Omega |\nabla u|^6 + |\Omega| \right) \\ &\quad + \mu\alpha_2 c_1 \cdot \left(\int_\Omega |\nabla u|^6 + \int_\Omega |\nabla v|^6 + 1 \right) \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{5.24}$$

Combining (5.14), (5.23) and (5.24) with (5.12) thus shows that with some $c_3 > 0$ we have

$$\begin{aligned} \frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla u|^4 + \frac{1}{128}\int_\Omega |\nabla u|^2|D^2u|^2 &\le c_3 \int_\Omega |\nabla u|^6 + c_3 \int_\Omega |\nabla v|^6 \\ &\quad + c_3 \int_\Omega |\Delta w|^3 + c_3 \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which together with a similar procedure applied to the second equation in (1.2) provides $c_4 > 0$ such that

$$\begin{aligned} &\frac{1}{4}\frac{d}{dt}\left(\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla v|^4 \right) + \left(\int_\Omega |\nabla u|^4 + \int_\Omega |\nabla v|^4 \right) + \frac{1}{128}\left(\int_\Omega |\nabla u|^2|D^2u|^2 + \int_\Omega |\nabla v|^2|D^2v|^2 \right) \\ &\qquad \le c_4 \int_\Omega |\nabla u|^6 + c_4 \int_\Omega |\nabla v|^6 + c_4 \int_\Omega |\Delta w|^3 + c_4 \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{5.25}$$

Now since $(u(\cdot,t))_{t\in[0,T_{max})}$ and $(v(\cdot,t))_{t\in[0,T_{max})}$ are bounded in $L^\infty(\Omega)$ and equicontinuous in $\bar\Omega$ by Lemma 4.12 and Lemma 4.13, we may apply Lemma 5.1 to gain $c_5 > 0$ fulfilling

$$c_4 \int_\Omega |\nabla u|^6 \le \frac{1}{128}\int_\Omega |\nabla u|^2|D^2u|^2 + c_5 \qquad \text{for all } t \in (0, T_{max})$$

and

$$c_4 \int_\Omega |\nabla v|^6 \le \frac{1}{128}\int_\Omega |\nabla v|^2|D^2v|^2 + c_5 \qquad \text{for all } t \in (0, T_{max}),$$

whence (5.25) entails that $y(t) := \int_\Omega |\nabla u(\cdot,t)|^4 + \int_\Omega |\nabla v(\cdot,t)|^4,\ t \in [0, T_{max})$, satisfies

$$\frac{1}{4}y'(t) + y(t) \le h(t) \qquad \text{for all } t \in (0, T_{max}),$$

where by Lemma 4.12 and Lemma 4.2, $h(t) := c_4 \int_\Omega |\Delta w(\cdot,t)|^3 + c_4 + c_5,\ t \in (0, T_{max})$, has the property that

$$\int_t^{t+\tau} h(\sigma) d\sigma \le c_6 \qquad \text{for all } t \in (0, \widehat{T}_{max})$$

with some $c_6 > 0$. Therefore, Lemma 3.3 says that y is bounded in $[0, T_{max})$, which is equivalent to (5.11). □

Now our final statement on global existence becomes an immediate consequence.

PROOF of Theorem 1.1. Since $n \le 3 < 4$, we only need to combine the outcomes of Lemma 5.2 and Lemma 4.12 with the local existence and extensibility statement in Lemma 2.1. □

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