Boundedness vs. blow-up in a two-species chemotaxis system with two chemicals

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Abstract

We consider a model for two species interacting through chemotaxis in such a way that each species produces a signal which directs the respective motion of the other. Specifically, we shall be concerned with nonnegative solutions of the Neumann problem, posed in bounded domains $\Omega \subset \mathbb{R}^n$ with smooth boundary, for the system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \xi \nabla \cdot (w \nabla z), & x \in \Omega,\ t > 0, \\ 0 = \Delta z - z + u, & x \in \Omega,\ t > 0, \end{cases} \qquad (\star)$$

with parameters $\chi \in \{\pm 1\}$ and $\xi \in \{\pm 1\}$, thus allowing the interaction of either attraction-repulsion, or attraction-attraction, or repulsion-repulsion type.

It is shown that

- in the attraction-repulsion case $\chi = 1$ and $\xi = -1$, if $n \le 3$ then for any nonnegative initial data $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^0(\bar{\Omega})$, there exists a unique global classical solution which is bounded;
- in the doubly repulsive case when $\chi = \xi = -1$, the same holds true;
- in the attraction-attraction case $\chi = \xi = 1$,
  - if either $n = 2$ and $\int_\Omega u_0 + \int_\Omega w_0$ lies below some threshold, or $n \ge 3$ and $\|u_0\|_{L^\infty(\Omega)}$ and $\|w_0\|_{L^\infty(\Omega)}$ are sufficiently small, then solutions exist globally and remain bounded, whereas
  - if either $n = 2$ and m is suitably large, or $n \ge 3$ and $m > 0$ is arbitrary, then there exist smooth initial data u_0 and w_0 such that $\int_\Omega u_0 + \int_\Omega w_0 = m$ and such that the corresponding solution blows up in finite time.

In particular, these results demonstrate that the circular chemotaxis mechanism underlying $(\star)$ goes along with essentially the same destabilizing features as known for the classical Keller-Segel system in the doubly attractive case, but totally suppresses any blow-up phenomenon when only one, or both, taxis directions are repulsive.

Key words: chemotaxis, attraction, repulsion, preventing blow-up
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1 Introduction

Multi-species interaction through chemotaxis. In diverse environments, the ability to adequately interact plays a vital role in biology at many levels. At a macroscopic stage, appropriately understanding a multi-species ecosystem amounts to investigating aspects of the interplay between its sub-populations. Accordingly, identifying and studying mechanisms and principles of interaction between populations has become a central objective in mathematical biology. For instance, quite a thorough theoretical comprehension has been achieved in typical situations when species cooperate and hence mutually benefit, or compete for resources through various mechanisms (see [24], or also [5] for some recent developments). In contrast to this, only few results in the mathematical literature seem to address the case when several populations interact indirectly via chemotaxis mechanisms. Such types of interplay, as found relevant in various biological applications ([15], [31], [21]), so far have been studied primarily in cases when two species produce the same signal the gradient of which directs their movement ([1], [9], [8], [37], [32]).

The purpose of the present work is to undertake a first step toward the understanding of chemotactic interaction in presence of *several* chemicals. Specifically, we shall be concerned with the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v + w - v, & x\in\Omega,\ t>0,\\ w_t = \Delta w - \xi\nabla\cdot(w\nabla z), & x\in\Omega,\ t>0,\\ 0 = \Delta z + u - z, & x\in\Omega, \end{cases} \tag{1.1}$$

which models the spatio-temporal evolution of two populations in which individuals move according to random diffusion, and in which chemotactically directed motion leads to an interaction in a circular manner: The first species, with density denoted by $u(x,t)$, adapts its motion according to a chemical substance with concentration $v(x,t)$, the latter being secreted by a second species, mathematically represented through its density $w(x,t)$. The individuals of this second population themselves orient their movement along concentration gradients of a second signal with density $z(x,t)$ which in turn is produced by the first species. The system (1.1) may be viewed as a simplified variant of a fully parabolic two-species chemotaxis model with two chemicals, involving slightly more general cross-diffusion mechanisms, as it has been proposed in [29] to describe chemotaxis-driven processes of cell sorting.

In order to capture the full variety of possible interaction types but keep the presentation clear, we allow the key parameters χ and ξ to attain either of the values $+1$ or -1, thereby addressing three prototypical situations:

1. The attraction-repulsion case. When $\chi = 1$ and $\xi = -1$, (1.1) becomes

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v + w - v, & x\in\Omega,\ t>0,\\ w_t = \Delta w + \nabla\cdot(w\nabla z), & x\in\Omega,\ t>0,\\ 0 = \Delta z + u - z, & x\in\Omega, \end{cases} \tag{1.2}$$

meaning that the first species is attracted by the signal produced by the second population, whereas the latter is repelled by the chemical secreted by the former.

2. The repulsion-repulsion case. If $\chi = \xi = -1$, (1.1) reduces to

$$\begin{cases} u_t = \Delta u + \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v + w - v, & x \in \Omega,\ t > 0, \\ w_t = \Delta w + \nabla \cdot (w \nabla z), & x \in \Omega,\ t > 0, \\ 0 = \Delta z + u - z, & x \in \Omega, \end{cases} \tag{1.3}$$

and thereby models interaction in which the orientation in both taxis processes is downward gradients of the respective signal concentration.

3. The attraction-attraction case. Likewise, both species are attracted to the signal produced by the other when we consider $\chi = \xi = 1$, that is, the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v + w - v, & x \in \Omega,\ t > 0, \\ w_t = \Delta w - \nabla \cdot (w \nabla z), & x \in \Omega,\ t > 0, \\ 0 = \Delta z + u - z, & x \in \Omega. \end{cases} \tag{1.4}$$

We shall subsequently consider these model problems (1.2), (1.3) and (1.4) along with the initial and boundary conditions

$$\begin{cases} \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

in bounded domains $\Omega \subset \mathbb{R}^n, n \geq 1$, with smooth boundary, where $\frac{\partial}{\partial \nu}$ denotes differentiation with respect to the outward normal on $\partial\Omega$.

Boundedness vs. blow-up in chemotaxis systems. The model (1.1) originates in, but essentially differs from, the classical single-species parabolic-elliptic Keller-Segel system for chemoattraction ([20]), as given by

$$\begin{cases} U_t = \Delta U - \nabla \cdot (U \nabla V), & x \in \Omega,\ t > 0, \\ 0 = \Delta V + U - V, & x \in \Omega,\ t > 0, \\ \frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ U(x,0) = U_0(x), & x \in \Omega, \end{cases} \tag{1.6}$$

which focuses on the case when the chemical is produced by the attracted cells *themselves*, known to be present in numerous biological applications ([16]). An outstanding feature of this classical Keller-Segel model is that this self-energizing chemoattractive mechanism in (1.6) has a strong tendency to destabilize homogeneous distributions. In fact, even blow-up of solutions may happen in the sense that the norm of U in $L^\infty(\Omega)$ becomes unbounded in finite time (see e.g. the review [18], and in particular [25] for the parabolic-elliptic system (1.6), as well as [14], [23] and [40] for an associated fully parabolic case).

According to the fact that in many biologically relevant situations such blow-up phenomena do not adequately reproduce experimental observations, extensive efforts attempt to develop models in which explosions in chemoattraction models are ruled out (cf. the survey [16] for an overview). Actually, various mechanisms have been identified to prevent finite-time blow-up, such as considering nonlinear variants of chemotactic sensitivity and diffusivity ([30], [4], [35]), incorporating logistic dampening ([27], [38]), or also adding a cross-diffusion term in the equation for the chemical signal ([17]).

As compared to this, models based on chemorepulsion, relevant in various biological contexts ([11], [22], [30]), apparently enjoy some global well-posedness and boundedness properties in appropriate frameworks; however, the mathematical understanding of this mechanism is yet far from complete. For instance, it is known that even in the corresponding fully parabolic chemorepulsion system associated with (1.6), solutions are global and stabilize toward a constant equilibrium when $n \le 2$, while if $n \in \{3, 4\}$ then at least certain global generalized solutions can be constructed ([6], cf. also [33] for boundedness results in a repulsion system with general nonlinear cross-diffusion). Of course, this does not exclude the possibility that some exploding solutions might occur in higher-dimensional contexts. A combination of attractive and repulsive chemotaxis was considered mathematically in [34] in the framework of a model for a single species simultaneously secreting two different signals, and it was shown there that depending on the relative strength of both taxis types, blow-up may occur or be suppressed.

Main results. The present work addresses the question whether or not solutions may become unbounded in the two-species systems (1.2), (1.3) and (1.4), thereby purposing to establish some basic information on either presence or absence of a corresponding destabilizing potential inherent to the respective taxis interaction. We shall throughout the sequel remain within the framework of classical solutions, where by a classical solution in $\Omega \times (0, T)$, $T \in (0, \infty]$, we mean a quadruple (u, v, w, z) of nonnegative functions

$$\begin{aligned}
&u \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\bar{\Omega} \times (0, T)),\\
&v \in C^{2,0}(\bar{\Omega} \times (0, T)),\\
&w \in C^0(\bar{\Omega} \times [0, T)) \cap C^{2,1}(\bar{\Omega} \times (0, T)) \qquad \text{and}\\
&z \in C^{2,0}(\bar{\Omega} \times (0, T)),
\end{aligned}$$

which solve the respective version of (1.1) along with (1.5) in the classical pointwise sense in $\Omega \times (0, T)$. Our main results will then essentially depend on the type of chemotactic interaction:

1. The attraction-repulsion case. Firstly, in presence of both types of chemotaxis we shall derive the following.

Theorem 1.1 *Let $n \le 3$ and suppose that $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^0(\bar{\Omega})$ are nonnegative. Then (1.2), (1.5) possesses a unique global classical solution which is bounded in the sense that there exists $C > 0$ satisfying*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} + \|z(\cdot, t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{1.7}$$

In particular, this means that in sharp contrast to the classical Keller-Segel system (1.6), blow-up phenomena are entirely ruled out in any situation in which chemoattraction is coupled to repulsive processes as modeled in (1.2). The above restriction $n \le 3$ may be due to technical reasons only, and we do not know if the result can be extended to higher dimensional settings; we conjecture, however, that blow-up may occur when n is appropriately large.

2. The repulsion-repulsion case. In light of the above, it will not be surprising that doubly repulsive interaction will as well never enfore any blow-up. This will actually turn out to be a by-product of our analysis concerning (1.2).

Theorem 1.2 *If $n \le 3$ and $u_0 \in C^0(\bar\Omega)$ and $w_0 \in C^0(\bar\Omega)$ are nonnegative, then (1.3), (1.5) is uniquely globally solvable, and the solution is bounded in that it satisfies (1.7) for some $C > 0$.*

3. The attraction-attraction case. Finally, in the case $\chi = \xi = 1$ we shall detect some parallels to the one-species Keller-Segel system (1.6). These are evident in the particular case when u_0 precisely coincides with w_0, in which (1.4) in fact reduces to (1.6); for general choices of the initial data, however, the apparent lack of any gradient-like structure in (1.4) will require arguments different from those used in the analysis of (1.6).
We first address the spatially two-dimensional setting in which we shall reveal a property of (1.4) which indicates the presence of a variant of the mass threshold phenomenon known for (1.6). In order to state our result in this direction, let us note that in the two-dimensional case the Gagliardo-Nirenberg inequality provides $C_{GN} > 0$ and $L > 0$ such that

$$\|\varphi\|_{L^4(\Omega)}^4 \le C_{GN}\|\nabla\varphi\|_{L^2(\Omega)}^2\|\varphi\|_{L^2(\Omega)}^2 + L\|\varphi\|_{L^2(\Omega)}^4 \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{1.8}$$

In terms of the constant C_{GN}, we can formulate a smallness condition on $\int_\Omega u_0$ and $\int_\Omega w_0$ which is sufficient to ensure boundedness of solutions.

Theorem 1.3 *Let $n = 2$.*

i) Suppose that $u_0 \in C^0(\bar\Omega)$ and $w_0 \in C^0(\bar\Omega)$ are nonnegative and satisfy

$$\max\left\{ \int_\Omega u_0, \int_\Omega w_0 \right\} < \frac{4}{C_{GN}} \tag{1.9}$$

with $C_{GN} > 0$ taken from (1.8). Then (1.4), (1.5) admits a unique global classical solution which is bounded in the sense that there exists $C > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > 0.$$

ii) For any choice of $m > 4\pi$ one can find nonnegative smooth initial data u_0 and w_0 such that

$$\min\left\{ \int_\Omega u_0, \int_\Omega w_0 \right\} = m > 4\pi, \tag{1.10}$$

and such that for some $T > 0$ the problem (1.4), (1.5) possesses a classical solution in $\Omega \times (0,T)$ which blows up in the sense that

$$\limsup_{t \nearrow T} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{1.11}$$

We have to leave open here in how far either (1.9) or (1.10) are optimal.

In the three-dimensional case, no critical mass phenomenon of the above type can be found. We shall rather derive global existence and boundedness under some smallness condition on the initial data which involves the norm in $L^\infty(\Omega)$ rather than that in $L^1(\Omega)$, and which may thus be viewed somewhat stronger than the corresponding hypothesis in Theorem 1.3 i). Moreover, blow-up of some solutions occurs for arbitrarily small masses of the initial data. Unlike in the previous cases, we can actually formulate our global well-posedness result for any $n \ge 1$, and the blow-up result can be obtained for general $n \ge 3$.

Theorem 1.4 *i) Let $n \ge 1$, and let $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first nonzero eigenvalue of the Laplacian in Ω under homogeneous Neumann boundary conditions. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ one can find $\delta(\varepsilon) > 0$ with the property that whenever $u_0 \in C^0(\bar\Omega)$ and $w_0 \in C^0(\bar\Omega)$ are nonnegative functions fulfilling*

$$\|u_0\|_{L^\infty(\Omega)} \le \delta(\varepsilon) \qquad \text{and} \qquad \|w_0\|_{L^\infty(\Omega)} \le \delta(\varepsilon), \tag{1.12}$$

the problem (1.4), (1.5) possesses a unique global classical solution which satisfies

$$\|u(\cdot,t) - \mu\|_{L^\infty(\Omega)} \le \varepsilon e^{-\lambda t} \quad \text{and} \quad \|w(\cdot,t) - \kappa\|_{L^\infty(\Omega)} \le \varepsilon e^{-\lambda t} \qquad \text{for all } t > 0, \tag{1.13}$$

where

$$\mu := \fint_\Omega u_0 \qquad \text{and} \qquad \kappa := \fint_\Omega w_0. \tag{1.14}$$

ii) Let $n \ge 3$. If Ω is star-shaped, then for any $m > 0$ one can find smooth initial data u_0 and w_0 such that

$$\min\left\{ \int_\Omega u_0, \int_\Omega w_0 \right\} = m > 0,$$

and such that for some $T > 0$ the problem (1.4), (1.5) possesses a classical solution in $\Omega \times (0,T)$ which blows up at $t = T$ in the sense specified in (1.11).

The result of Theorem 1.4 is also in good accordance with known facts for Keller-Segel systems: Whereas a similar blow-up feature of (1.6) was discussed long time ago already ([18]), a corresponding statement on global existence and diffusion-dominated large time behavior in the spirit of (1.13) was derived more recently for the fully parabolic version of (1.6) in ([39]).
We remark that in the case $n = 1$, functional analytical techniques which are well-established in the context of chemotaxis can be adapted in a straightforward manner so as to yield that as in (1.6), for all nonnegative continuous initial data the problem (1.4), (1.5) possesses a unique global classical solution which is bounded. We refrain from giving details here, and rather the reader to [28] for a demonstration of a possible procedure for the parabolic version of (1.6).

2 Preliminaries. Local existence

The local existence of solutions to (1.1) for any given nonnegative $u_0 \in C^0(\bar\Omega)$ and $w_0 \in C^0(\bar\Omega)$ can be proved by adapting approaches that are well-established in the context of parabolic-elliptic chemotaxis models (cf. [7] and [34], for instance). Actually, the following two results apply to any choice of $\chi \in \mathbb{R}$ and $\xi \in \mathbb{R}$.

Lemma 2.1 *Let $n \geq 1$, $\chi \in \mathbb{R}$ and $\xi \in \mathbb{R}$, and suppose that $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^0(\bar{\Omega})$ are nonnegative. Then there exist $T_{max} \in (0, \infty]$ and a unique quadruple (u, v, w, z) of nonnegative functions from $C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max}))$ solving (1.1), (1.5) classically in $\Omega \times (0, T_{max})$. Moreover,*

$$\text{if } T_{max} < \infty, \text{ then } \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{2.1}$$

The following simple but important mass conservation properties will frequently be used in the sequel.

Lemma 2.2 *Let $n \geq 1$, For any choice of $\chi \in \mathbb{R}$ and $\xi \in \mathbb{R}$ and each nonnegative $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^0(\bar{\Omega})$, the solution (u, v, w, z) of (1.1), (1.5) satisfies*

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|z(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \qquad \text{for all } t \in (0, T_{max}), \quad \text{and} \tag{2.2}$$

$$\|w(\cdot,t)\|_{L^1(\Omega)} = \|v(\cdot,t)\|_{L^1(\Omega)} = \|w_0\|_{L^1(\Omega)} \qquad \text{for all } t \in (0, T_{max}). \tag{2.3}$$

Proof. Integrating each equation in (1.1) with respect to $x \in \Omega$, we see that $\frac{d}{dt}\int_\Omega u \equiv 0$, $\int_\Omega w = \int_\Omega v$ as well as $\frac{d}{dt}\int_\Omega w \equiv 0$ and $\int_\Omega u = \int_\Omega z$ for $t \in (0, T_{max})$. These identities immediately yield (2.2) and (2.3). $\Box$

3 The attraction-repulsion case and the repulsion-repulsion case

In this section we shall establish the boundedness results in Theorem 1.1 and in Theorem 1.2. Throughout our presentation here, we shall assume that $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^0(\bar{\Omega})$ are nonnegative.

3.1 L^q bounds for w when $\xi = -1$

To prepare our analysis for both the attraction-repulsion and the doubly repulsive case, let us first derive two integral estimates for the components z and w of solutions to the general system (1.1), (1.5) for $\xi = -1$ irrespective of the value of χ. The following statement actually holds for arbitrary $\xi \in \mathbb{R}$. It will be used in Lemma 3.2 below and once more later in Lemma 3.5.

Lemma 3.1 *Let $n \geq 1$, $\xi \in \mathbb{R}$ and $\chi \in \mathbb{R}$. Then for all $r > 1$ fulfilling $r < \frac{n}{(n-2)_+}$, there exists $C(r) > 0$ such that the solution of (1.1), (1.5) satisfies*

$$\|v(\cdot,t)\|_{L^r(\Omega)} \leq C(r) \qquad \text{for all } t \in (0, T_{max}). \tag{3.1}$$

and

$$\|z(\cdot,t)\|_{L^r(\Omega)} \leq C(r) \qquad \text{for all } t \in (0, T_{max}). \tag{3.2}$$

Proof. By assumption, $s := \max\{1, \frac{nr}{n+r}\}$ satisfies $s \geq 1$ and $1 - \frac{n}{s} \geq -\frac{n}{r}$, so that by the Sobolev embedding theorem, $W^{1,s}(\Omega) \hookrightarrow L^r(\Omega)$ and hence

$$\|\psi\|_{L^r(\Omega)} \leq c_1 \|\psi\|_{W^{1,s}(\Omega)} \qquad \text{for all } \psi \in W^{1,s}(\Omega)$$

with some $c_1 > 0$. Moreover, according to known results on elliptic boundary-value problems with inhomogeneities in $L^1(\Omega)$ ([3]), since $s < \frac{n}{n-1}$ due to our choice of s and the fact that $(n-2)r < n$, we can find $c_2 > 0$ satisfying

$$\|\psi\|_{W^{1,s}(\Omega)} \leq c_2 \| -\Delta\psi + \psi\|_{L^1(\Omega)} + c_2 \|\psi\|_{L^1(\Omega)} \qquad \text{for all } \psi \in C^2(\bar{\Omega}) \text{ fulfilling } \frac{\partial \psi}{\partial \nu} = 0 \text{ on } \partial\Omega.$$

Since by Lemma 2.2 we know that $\int_\Omega v(\cdot,t) = \int_\Omega w(\cdot,t) = \int_\Omega w_0$ for all $t \in (0,T_{max})$, we thus infer that

$$\begin{aligned}
\|v(\cdot,t)\|_{L^r(\Omega)} &\le c_1\|v(\cdot,t)\|_{W^{1,s}(\Omega)} \\
&\le c_1c_2\|w(\cdot,t)\|_{L^1(\Omega)} + c_1c_2\|v(\cdot,t)\|_{L^1(\Omega)} \\
&\le 2c_1c_2\int_\Omega w_0 \qquad \text{for all } t \in (0,T_{max}).
\end{aligned}$$

This yields (3.1), whereas (3.2) can be derived in much the same manner using that $\int_\Omega z(\cdot,t) = \int_\Omega u(\cdot,t) = \int_\Omega u_0$ for all $t \in (0,T_{max})$. $\square$

Now when $n \le 3$ and $\xi = -1$, the above estimate for z can be used to prove boundedness of w in $L^\infty((0,T_{max});L^q(\Omega))$ for any finite $q > 1$.

Lemma 3.2 *Let $n \le 3$, $\xi = -1$ and $\chi \in \mathbb{R}$. Then for all $q \in (1,\infty)$ one can find $C(q) > 0$ with the property that for the solution of (1.1), (1.5) we have*

$$\|w(\cdot,t)\|_{L^q(\Omega)} \le C(q) \qquad \textit{for all } t \in (0,T_{max}). \tag{3.3}$$

Proof. We multiply the third equation in (1.1) by w^{q-1} and integrate by parts using the identity $\Delta z = z - u$ to find that

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega w^q + \frac{4(q-1)}{q^2}\int_\Omega |\nabla w^{\frac{q}{2}}|^2 &= -(q-1)\int_\Omega w^{q-1}\nabla w\cdot\nabla z \\
&= \frac{q-1}{q}\int_\Omega w^q\Delta z \\
&= \frac{q-1}{q}\int_\Omega w^q(z-u) \\
&\le \frac{q-1}{q}\int_\Omega w^q z \qquad \text{for all } t \in (0,T_{max}).
\end{aligned} \tag{3.4}$$

In order to estimate the integral on the right, we note that since $n \le 3$ we can fix $r > 1$ such that

$$\frac{n}{2} < r < \frac{n}{(n-2)_+}, \tag{3.5}$$

where the left inequality warrants that $r' := \frac{r}{r-1}$ satisfies

$$2r' < \frac{2n}{(n-2)_+}. \tag{3.6}$$

Now by the Hölder inequality and Lemma 3.1, the application of the latter relying on the right inequality in (3.5), we see that

$$\frac{q-1}{q}\int_\Omega w^q z \le \frac{q-1}{q}\Big(\int_\Omega w^{qr'}\Big)^{\frac{1}{r'}}\cdot\Big(\int_\Omega z^r\Big)^{\frac{1}{r}} \le c_1\Big(\int_\Omega w^{qr'}\Big)^{\frac{1}{r'}} \qquad \text{for all } t \in (0,T_{max})$$

with some $c_1 > 0$. Thanks to (3.6), we can thus invoke the Gagliardo-Nirenberg inequality to find $c_2 > 0$ fulfilling

$$\begin{aligned}
\frac{q-1}{q}\int_\Omega w^q z &\le c_1\Big(\int_\Omega (w^{\frac{q}{2}})^{2r'}\Big)^{\frac{1}{r'}} \\
&= c_1\|w^{\frac{q}{2}}\|^2_{L^{2r'}(\Omega)} \\
&\le c_2\|\nabla w^{\frac{q}{2}}\|^{2a}_{L^2(\Omega)}\cdot\|w^{\frac{q}{2}}\|^{2(1-a)}_{L^{\frac{2}{q}}(\Omega)} + c_2\|w^{\frac{q}{2}}\|^2_{L^{\frac{2}{q}}(\Omega)} \qquad \text{for all } t\in(0,T_{max})
\end{aligned}$$

with $a \in (0,1)$ given by

$$-\frac{n}{2r'} = \Big(1-\frac{n}{2}\Big)a - \frac{nq}{2}(1-a),$$

that is with

$$a = \frac{\frac{nq}{2}-\frac{n}{2r'}}{1-\frac{n}{2}+\frac{nq}{2}}.$$

Since

$$\|w^{\frac{q}{2}}\|^{\frac{2}{q}}_{L^{\frac{2}{q}}(\Omega)} = \int_\Omega w_0 \qquad \text{for all } t\in(0,T_{max}) \tag{3.7}$$

due to Lemma 2.2, we may apply Young's inequality to obtain $c_3 > 0$ and $c_4 > 0$ such that

$$\begin{aligned}
\frac{q-1}{q}\int_\Omega w^q z &\le c_3\|\nabla w^{\frac{q}{2}}\|^{2a}_{L^2(\Omega)} + c_3 \\
&\le \frac{2(q-1)}{q^2}\int_\Omega |\nabla w^{\frac{q}{2}}|^2 + c_4 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{3.8}$$

Now by the Poincaré inequality and (3.7) we moreover find $c_5 > 0$ such that

$$\int_\Omega w^q \le c_5\int_\Omega |\nabla w^{\frac{q}{2}}|^2 + c_5 \qquad \text{for all } t\in(0,T_{max}),$$

which combined with (3.4) and (3.8) shows that

$$\frac{d}{dt}\int_\Omega w^q + c_6\int_\Omega w^q \le c_7 \qquad \text{for all } t\in(0,T_{max})$$

with $c_6 := \frac{2(q-1)}{qc_5}$ and $c_7 := qc_4 + \frac{2(q-1)}{q}$. Upon an ODE comparison, we thus conclude that

$$\int_\Omega w^q(\cdot,t) \le \max\Big\{\int_\Omega w_0^q\ ,\ \frac{c_7}{c_6}\Big\} \qquad \text{for all } t\in(0,T_{max}),$$

which proves (3.3). $\square$

3.2 Boundedness in the attraction-repulsion system. Proof of Theorem 1.1

We now concentrate on the system (1.2), for which the estimate in Lemma 3.2 can be turned into the following.

Lemma 3.3 *Let $n \le 3$ and $p \in (1,\infty)$. Then there exists $C(p) > 0$ such that the solution of (1.2), (1.5) satisfies*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le C(p) \qquad \textit{for all } t \in (0,T_{max}). \tag{3.9}$$

Proof. By straightforward computation involving three integrations by parts and the first two equations in (1.2), similar to the proof of Lemma 3.2 we obtain

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 &= (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v \\ &= -\frac{p-1}{p}\int_\Omega u^p(v-w) \\ &\le \frac{p-1}{p}\int_\Omega u^p w \qquad \text{for all } t\in(0,T_{max}). \end{aligned} \tag{3.10}$$

Here we pick $r > 1$ such that $r > \frac{n}{2}$ and use the Hölder inequality and Lemma 3.2 to estimate

$$\frac{p-1}{p}\int_\Omega u^p w \le \frac{p-1}{p}\Big(\int_\Omega u^{pr'}\Big)^{\frac{1}{r'}}\cdot\Big(\int_\Omega w^r\Big)^{\frac{1}{r}} \le c_1\Big(\int_\Omega u^{pr'}\Big)^{\frac{1}{r'}} \qquad \text{for all } t\in(0,T_{max})$$

with some $c_1 > 0$, where $r' := \frac{r}{r-1}$. Since $2r' < \frac{2n}{(n-2)_+}$, the Gagliardo-Nirenberg inequality provides $c_2 > 0$ fulfilling

$$\begin{aligned} c_1\Big(\int_\Omega u^{pr'}\Big)^{\frac{1}{r'}} &= c_1\|u^{\frac{p}{2}}\|^2_{L^{2r'}(\Omega)} \\ &\le c_2\|\nabla u^{\frac{p}{2}}\|^{2b}_{L^2(\Omega)}\cdot\|u^{\frac{p}{2}}\|^{2(1-b)}_{L^{\frac{2}{p}}(\Omega)} + c_2\|u^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)} \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{3.11}$$

where

$$-\frac{n}{2r'} = \Big(1-\frac{n}{2}\Big)b - \frac{np}{2}(1-b), \qquad \text{so that} \qquad b = \frac{\frac{np}{2}-\frac{n}{2r'}}{1-\frac{n}{2}+\frac{np}{2}} \in (0,1).$$

Now Lemma 2.2 says that

$$\|u^{\frac{p}{2}}\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega u_0 \qquad \text{for all } t\in(0,T_{max}), \tag{3.12}$$

whence by Young's inequality we obtain from (3.11) that

$$c_1\Big(\int_\Omega u^{pr'}\Big)^{\frac{1}{r'}} \le \frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_3 \qquad \text{for all } t\in(0,T_{max})$$

with come $c_3 > 0$. As once more by the Poincaré inequality and (3.12) we can find $c_4 > 0$ satisfying

$$\int_\Omega u^p \le c_4\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_4 \qquad \text{for all } t\in(0,T_{max}),$$

from (3.10) we thus gain the ODI

$$\frac{d}{dt}\int_\Omega u^p + c_5 \int_\Omega u^p \le c_6 \qquad \text{for all } t \in (0, T_{max}),$$

where $c_5 := \frac{2(p-1)}{pc_4}$ and $c_6 := pc_3 + \frac{2(p-1)}{p}$. Therefore, by comparison we obtain

$$\int_\Omega u^p(\cdot,t) \le \max\Big\{ \int_\Omega u_0^p\,,\, \frac{c_6}{c_5} \Big\} \qquad \text{for all } t \in (0, T_{max}),$$

which completes the proof. $\Box$

Now combining Lemma 3.2 and Lemma 3.3 with standard elliptic theory and an iterative parabolic regularity argument, we obtain boundedness of both u and w.

Lemma 3.4 *Let $n \le 3$. Then there exists $C > 0$ such that the solution of (1.2), (1.5) has the property that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \quad (3.13)$$

PROOF. From Lemma 3.2, Lemma 3.3 and standard elliptic regularity theory ([10]) we obtain bounds for both v and z in all spaces $L^\infty((0,T_{max}); W^{2,s}(\Omega))$ for any $s \in (1,\infty)$, whence in particular

$$|\nabla v(x,t)| + |\nabla z(x,t)| \le c_1 \qquad \text{for all } (x,t) \in \Omega \times (0, T_{max})$$

with some $c_1 > 0$. In conjunction with Lemma 3.2 and Lemma 3.3 this ensures that Lemma A.1 in [35] becomes applicable so as to assert via a Moser-type iteration that (3.13) holds. $\Box$

The proof of Theorem 1.1 is now obvious.

PROOF of Theorem 1.1. The statement is an evident consequence of Lemma 3.4 and the extensibility criterion provided by Lemma 2.1. $\Box$

3.3 Boundedness in the fully repulsive system. Proof of Theorem 1.2

The system (1.3) can now be easily analyzed using the above preparations.

Lemma 3.5 *Let $n \le 3$. Then for all $q \in (1,\infty)$ there exists $C(q) > 0$ such that the solution of (1.3), (1.5) satisfies*

$$\|u(\cdot,t)\|_{L^q(\Omega)} \le C(q) \qquad \textit{for all } t \in (0, T_{max}). \quad (3.14)$$

PROOF. Since in (1.1) we have $\chi = \xi = -1$ now, in the proof of Lemma 3.2 we may replace w by u and z by v. Then relying on (3.2) rather than on (3.1), we may copy the proof of Lemma 3.2 word by word to end up with (3.14). $\Box$

PROOF of Theorem 1.2. Since Lemma 3.2 still applies, we may combine its outcome with that of Lemma 3.5 to first conclude, as in Lemma 3.4 again on the basis of elliptic regularity theory, that for some $c_1 > 0$ we have

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max}).$$

Once more by a Moser-type iteration, this implies the existence of $c_2 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le c_2 \qquad \text{for all } t \in (0, T_{max}).$$

Thereupon, the claim becomes immediate from (2.1). $\square$

4 Boundedness vs. blow-up in the attraction-attraction system

This section will provide sufficient conditions for solutions to exist globally, and will report on some occurrences of explosions. While the latter can be obtained by a simple reduction to the Keller-Segel system (1.6), the former will require some additional efforts.

4.1 Small-mass solutions in the case $n = 2$. Proof of Theorem 1.3 i)

For convenience, let us recall the interpolation inequality referred to in Theorem 1.3, namely

$$\|\varphi\|^4_{L^4(\Omega)} \le C_{GN}\|\nabla\varphi\|^2_{L^2(\Omega)}\|\varphi\|^2_{L^2(\Omega)} + L\|\varphi\|^4_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{4.1}$$

The role of the constant C_{GN} therein becomes clear in the following argument.

Lemma 4.1 *Let $n = 2$, and let C_{GN} and L denote positive constants such that the interpolation inequality (4.1) holds. Then for all nonnegative $u_0 \in C^0(\bar\Omega)$ and $w_0 \in C^0(\bar\Omega)$ satisfying*

$$\max\left\{\int_\Omega u_0, \int_\Omega w_0\right\} < \frac{4}{C_{GN}}, \tag{4.2}$$

there exists $C > 0$ such that the solution of (1.4), (1.5) has the property that

$$\int_\Omega u(x,t)\ln u(x,t)dx \le C \quad \textit{and} \quad \int_\Omega w(x,t)\ln w(x,t)dx \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.3}$$

Proof. We test the first equation in (1.4) by $\ln u$ and integrate by parts using the identity $\Delta v = v - w$ to obtain that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u\ln u + \int_\Omega \frac{|\nabla u|^2}{u} &= \int_\Omega \nabla u\cdot\nabla v \\
&= -\int_\Omega u\Delta v \\
&= -\int_\Omega u(v-w) \\
&\le \int_\Omega uw \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{4.4}$$

because $v \ge 0$ and $u \ge 0$. In the same way, we find that

$$\frac{d}{dt}\int_\Omega w\ln w + \int_\Omega \frac{|\nabla w|^2}{w} \le \int_\Omega wu \qquad \text{for all } t \in (0, T_{max}).$$

Adding this to (4.4) and invoking Young's inequality yields

$$\frac{d}{dt}\left\{\int_\Omega u\ln u+\int_\Omega w\ln w\right\}+\int_\Omega\frac{|\nabla u|^2}{u}+\int_\Omega\frac{|\nabla w|^2}{w}\le\int_\Omega u^2+\int_\Omega w^2 \qquad \text{for all } t\in(0,T_{max}). \tag{4.5}$$

Since by means of (4.1) we can estimate

$$\begin{aligned}\int_\Omega u^2=\|u^{\frac12}\|^4_{L^4(\Omega)} &\le C_{GN}\|\nabla u^{\frac12}\|^2_{L^2(\Omega)}\|u^{\frac12}\|^2_{L^2(\Omega)}+L\|u^{\frac12}\|^4_{L^2(\Omega)}\\ &\le \frac{C_{GN}}{4}\left(\int_\Omega\frac{|\nabla u|^2}{u}\right)\cdot\left(\int_\Omega u\right)+L\left(\int_\Omega u\right)^2 \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

and, similarly,

$$\int_\Omega w^2\le\frac{C_{GN}}{4}\left(\int_\Omega\frac{|\nabla w|^2}{w}\right)\cdot\left(\int_\Omega w\right)+L\left(\int_\Omega w\right)^2 \qquad \text{for all } t\in(0,T_{max}),$$

we thus see writing $m_1:=\int_\Omega u_0\equiv\int_\Omega u(\cdot,t)$ and $m_2:=\int_\Omega w_0\equiv\int_\Omega w(\cdot,t)$ that (4.5) entails

$$\frac{d}{dt}\left\{\int_\Omega u\ln u+\int_\Omega w\ln w\right\}+\frac{4}{m_1C_{GN}}\left(\int_\Omega u^2-Lm_1^2\right)+\frac{4}{m_2C_{GN}}\left(\int_\Omega w^2-Lm_2^2\right)\le\int_\Omega u^2+\int_\Omega w^2$$

for all $t\in(0,T_{max})$. Using that $\sigma:=\min\left\{\frac{4}{m_1C_{GN}}-1,\frac{4}{m_2C_{GN}}-1\right\}$ is positive thanks to our assumption (4.2), we therefore find that

$$\frac{d}{dt}\left\{\int_\Omega u\ln u+\int_\Omega w\ln w\right\}+\sigma\left\{\int_\Omega u^2+\int_\Omega w^2\right\}\le c_1 \qquad \text{for all } t\in(0,T_{max})$$

with $c_1:=\frac{4L(m_1+m_2)}{C_{GN}}$. In view of the elementary inequality $y\ln y\le y^2$, valid for all $y>0$, we thus obtain

$$\frac{d}{dt}\left\{\int_\Omega u\ln u+\int_\Omega w\ln w\right\}+\sigma\left\{\int_\Omega u\ln u+\int_\Omega w\ln w\right\}\le c_1 \qquad \text{for all } t\in(0,T_{max}),$$

which upon integration shows that

$$\int_\Omega u\ln u+\int_\Omega w\ln w\le\max\left\{\int_\Omega u_0\ln u_0+\int_\Omega w_0\ln w_0,\frac{c_1}{\sigma}\right\} \qquad \text{for all } t\in(0,T_{max}).$$

Since $y\ln y\ge-\frac1e$ for all $y>0$ and hence $-\int_\Omega w\ln w\le\frac{|\Omega|}{e}$ for all $t\in(0,T_{max})$, this proves the lemma. $\square$

Now in order to turn the above result into an estimate for u and w with respect to the norm in some space $L^p(\Omega)$, we shall follow a standard procedure in the context of two-dimensional Keller-Segel-type systems ([26]) and rely on the non-homogeneous interpolation inequality

$$\|\varphi\|^3_{L^3(\Omega)}\le\varepsilon\|\nabla\varphi\|^2_{L^2(\Omega)}\|\varphi\ln|\varphi|\|^2_{L^1(\Omega)}+C_\varepsilon\|\varphi\|^3_{L^1(\Omega)}+C_\varepsilon \qquad \text{for all } \varphi\in W^{1,2}(\Omega), \tag{4.6}$$

which for any $\varepsilon>0$ is known to hold with some $C_\varepsilon>0$ (see [2], for instance).

Lemma 4.2 *Let $n = 2$, and suppose that $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^0(\bar{\Omega})$ are nonnegative and satisfy the smallness condition (4.2) with $C_{GN} > 0$ as in (4.1). Then there exists $C > 0$ such that the solution of (1.4), (1.5) fulfills*

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.7}$$

Proof. We multiply the first equation in (1.4) by $2u$ and integrate by parts using the identity $\Delta v = v - w$ to obtain that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u^2 + 2\int_\Omega |\nabla u|^2 &= 2\int_\Omega u\nabla u\cdot\nabla v \\
&= -\int_\Omega u^2\Delta v \\
&= -\int_\Omega u^2(v-w) \\
&\le \int_\Omega u^2 w \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{4.8}$$

because $v \ge 0$, whence using Young's inequality we infer that

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 + 2\int_\Omega |\nabla u|^2 \le \int_\Omega u^3 + \int_\Omega w^3 + c_1$$

holds for all $t \in (0, T_{max})$ with a certain $c_1 > 0$. Similarly, we can find $c_2 > 0$ fulfilling

$$\frac{d}{dt}\int_\Omega w^2 + \int_\Omega w^2 + 2\int_\Omega |\nabla w|^2 \le \int_\Omega u^3 + \int_\Omega w^3 + c_2 \qquad \text{for all } t \in (0, T_{max}).$$

Combining the latter two inequalities yields

$$\frac{d}{dt}\Big\{\int_\Omega u^2 + \int_\Omega w^2\Big\} + \int_\Omega u^2 + \int_\Omega w^2 + 2\int_\Omega |\nabla u|^2 + 2\int_\Omega |\nabla w|^2 \le \int_\Omega u^3 + \int_\Omega w^3 + c_3 \qquad \text{for all } t \in (0, T_{max}) \tag{4.9}$$

with $c_3 := c_1 + c_2 > 0$. Now according to Lemma 4.1 we can find $c_4 > 0$ fulfilling

$$\|u(\cdot,t)\ln u(\cdot,t)\|_{L^1(\Omega)} \le c_4 \quad \text{and} \quad \|w(\cdot,t)\ln w(\cdot,t)\|_{L^1(\Omega)} \le c_4 \qquad \text{for all } t \in (0, T_{max}),$$

whence an application of the interpolation inequality (4.6) to $\varepsilon := \frac{2}{c_4}$ yields $c_5 > 0$ such that

$$\begin{aligned}
\|u\|^3_{L^3(\Omega)} &\le \varepsilon\|\nabla u\|^2_{L^2(\Omega)}\cdot\|u\ln u\|_{L^1(\Omega)} + c_5\|u\|^3_{L^1(\Omega)} + c_5 \qquad \text{for all } t \in (0, T_{max}) \quad \text{and} \\
\|w\|^3_{L^3(\Omega)} &\le \varepsilon\|\nabla w\|^2_{L^2(\Omega)}\cdot\|w\ln w\|_{L^1(\Omega)} + c_5\|u\|^3_{L^1(\Omega)} + c_5 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

In view of (2.2) and (2.3), this entails that

$$\int_\Omega u^3 + \int_\Omega w^3 \le 2\int_\Omega |\nabla u|^2 + 2\int_\Omega |\nabla w|^2 + c_6 \qquad \text{for all } t \in (0, T_{max})$$

is valid with some $c_6 > 0$. Inserted into (4.9) this shows that

$$\frac{d}{dt}\Big\{\int_\Omega u^2 + \int_\Omega w^2\Big\} + \int_\Omega u^2 + \int_\Omega w^2 \le c_2 + c_6 \qquad \text{for all } t \in (0, T_{max})$$

and thereby proves (4.7) on integration. $\Box$

Proof of Theorem 1.3 i). Thanks to the mass constraint under consideration, Lemma 4.2 becomes applicable to provide $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^2(\Omega)} + \|w(\cdot,t)\|_{L^2(\Omega)} \le C \qquad \text{for all } t \in (0, T_{max}).$$

Now in the spatially two-dimensional setting addressed here, it is known ([36, Lemma 4.1]) that this is sufficient to ensure via two Moser-type iterations that both u and w are bounded in $\Omega \times (0, T_{max})$. Hence, $T_{max} = \infty$ according to (2.1), and using that $\|v(\cdot,t)\|_{L^\infty(\Omega)} \le \|w(\cdot,t)\|_{L^\infty(\Omega)}$ and $\|z(\cdot,t)\|_{L^\infty(\Omega)} \le \|u(\cdot,t)\|_{L^\infty(\Omega)}$ for all $t > 0$, we obtain boundedness of (u, v, w, z) in $\Omega \times (0, \infty)$. $\Box$

4.2 Boundedness under a smallness condition in $L^\infty(\Omega)$. Proof of Theorem 1.4 i)

For initial data which are suitably small when measured in $L^\infty(\Omega)$, a fixed-point-type argument yields global existence, boundedness and even stabilization of solutions toward homogeneous steady states. The reasoning pursued here is inspired by a corresponding procedure applied to the parabolic counterpart of the Keller-Segel system (1.6) in ([39]).

Proof of Theorem 1.4 i). Since $\lambda < \lambda_1$, we can pick $\eta \in (0,1)$ sufficiently small such that

$$\tilde{\lambda} := (1-\eta)\lambda_1 - \eta^2 > \lambda. \tag{4.10}$$

Moreover, we fix any $\beta \in (0, \frac{1}{2})$ and can then choose $p \in (1, \infty)$ large enough such that

$$2\beta - \frac{n}{p} > 0. \tag{4.11}$$

Then the realization of the operator $A := -\Delta + \eta$ in $L^p(\Omega)$ under homogeneous Neumann boundary conditions is sectorial with the fractional power A^β having its domain $D(A^\beta)$ continuously embedded into $L^\infty(\Omega)$ ([13]) thanks to (4.11); in particular, there exists $c_1 > 0$ satisfying

$$\|\varphi\|_{L^\infty(\Omega)} \le c_1 \|A^\beta \varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in C^2(\bar{\Omega}) \text{ such that } \tfrac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{4.12}$$

Furthermore, if we let $(e^{-tA})_{t\ge 0}$ denote the corresponding semigroup, then ([10]) we can find $c_2 > 0$ such that for all $t > 0$,

$$\|A^\beta e^{-tA}\varphi\|_{L^p(\Omega)} \le c_2 t^{-\beta} \|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \varphi \in L^p(\Omega). \tag{4.13}$$

Next, standard smoothing and decay estimates for the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ on Ω (see e.g. [39, Lemma 1.3] for versions appropriate for our purpose) provide positive constants c_3, c_4 and c_5 such that for all $t > 0$ we have

$$\|e^{t\Delta}\varphi\|_{L^\infty(\Omega)} \le c_3 e^{-\lambda_1 t} \|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \varphi \in L^\infty(\Omega) \text{ such that } \textstyle\int_\Omega \varphi = 0 \tag{4.14}$$

and

$$\begin{aligned}\|e^{t\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} &\le c_4(1+t^{-\frac12})e^{-\lambda_1 t}\|\varphi\|_{L^p(\Omega)}\\ &\le c_5(1+t^{-\frac12})e^{-\lambda_1 t}\|\varphi\|_{L^\infty(\Omega)},\end{aligned} \tag{4.15}$$

where in the latter inequality we have used the boundedness of Ω.
By means of standard elliptic regularity theory ([10]), we can furthermore pick $c_6 > 0$ such that

$$\|\nabla\varphi\|_{L^\infty(\Omega)} \le c_6\|-\Delta\varphi+\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \varphi\in C^2(\bar\Omega) \text{ such that } \tfrac{\partial\varphi}{\partial\nu}=0 \text{ on } \partial\Omega. \tag{4.16}$$

Finally, since $\tilde\lambda > \lambda$ and $\beta+\frac12 < 1$, in a straightforward manner (see [39, Lemma 1.2]) one can derive the existence of $c_7 > 0$ fulfilling

$$\int_0^t \Big\{1+(t-s)^{-\beta-\frac12}\Big\}e^{-\tilde\lambda(t-s)}e^{-\lambda s}ds \le c_7 e^{-\lambda t} \qquad \text{for all } t>0. \tag{4.17}$$

Let us now set

$$c_8 := 4c_1c_2c_5c_6c_7\eta^{-\beta}(1-\eta)^{-\frac12} \tag{4.18}$$

and define

$$\varepsilon_0 := \frac{1}{4c_8}. \tag{4.19}$$

Then given any $\varepsilon\in(0,\varepsilon_0)$, we claim that the desired conclusion is valid if we let

$$\delta\equiv\delta(\varepsilon) := \min\Big\{\frac{\varepsilon}{4}, \frac{\varepsilon}{8c_3}\Big\}. \tag{4.20}$$

To see this, for nonnegative functions $u_0\in C^0(\bar\Omega)$ and $w_0\in C^0(\bar\Omega)$ fulfilling (1.12), in accordance with Lemma 2.1 we let (u,v,w,z) denote the associated solution of (1.4), (1.5), defined up to its maximal existence time $T_{max}\in(0,\infty]$ satisfying (2.1). We then let μ and κ be as given by (1.14) and introduce

$$T := \sup\Big\{T_0\in(0,T_{max}) \ \Big|\ \|u(\cdot,t)-\mu\|_{L^\infty(\Omega)} \le \varepsilon e^{-\lambda t} \text{ and } \|w(\cdot,t)-\kappa\|_{L^\infty(\Omega)} \le \varepsilon e^{-\lambda t} \text{ for all } t\in(0,T_0)\Big\}. \tag{4.21}$$

Here we note that by continuity of both u and w, the set on the right is not empty, and hence $T\in(0,\infty]$ is well-defined, because (4.20) asserts that

$$\|u(\cdot,0)-\mu\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)}+\mu \le 2\|u_0\|_{L^\infty(\Omega)} \le 2\delta \le \frac{\varepsilon}{2}$$

and, similarly, $\|w(\cdot,0)\|_{L^\infty(\Omega)} \le \frac{\varepsilon}{2}$.
In order to show that actually $T=T_{max}$, let us assume on the contrary that $T<T_{max}$, whence in particular $T<\infty$. Again by continuity of u and w, we then know that

$$\text{either } \|u(\cdot,T)-\mu\|_{L^\infty(\Omega)} = \varepsilon e^{-\lambda T} \quad \text{or} \quad \|w(\cdot,T)-\kappa\|_{L^\infty(\Omega)} = \varepsilon e^{-\lambda T}. \tag{4.22}$$

To derive a contradiction from this, we first observe that by the second equation in (1.4) we have

$$-\Delta(v-\kappa)+(v-\kappa) = w-\kappa \qquad \text{in } \Omega\times(0,T_{max}),$$

so that (4.16) and (4.21) yield

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} &= \|\nabla(v(\cdot,t)-\kappa)\|_{L^\infty(\Omega)} \\
&\le c_6\|-\Delta(v(\cdot,t)-\kappa)+(v(\cdot,t)-\kappa)\|_{L^\infty(\Omega)} \\
&= c_6\|w(\cdot,t)-\kappa\|_{L^\infty(\Omega)} \\
&\le c_6\varepsilon e^{-\lambda t} \qquad \text{for all } t\in(0,T).
\end{aligned} \tag{4.23}$$

We next estimate u by using the variation-of-constants formula associated with the first equation in (1.4) according to

$$\|u(\cdot,t)-\mu\|_{L^\infty(\Omega)} \le \|e^{t\Delta}(u_0-\mu)\|_{L^\infty(\Omega)} + \int_0^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \quad \text{for all } t\in(0,T), \tag{4.24}$$

where by (4.14) and (1.12),

$$\begin{aligned}
\|e^{t\Delta}(u_0-\mu)\|_{L^\infty(\Omega)} &\le c_3 e^{-\lambda_1 t}\|u_0-\mu\|_{L^\infty(\Omega)} \\
&\le 2c_3\delta e^{-\lambda_1 t} \\
&\le 2c_3\delta e^{-\lambda t} \qquad \text{for all } t\in(0,T),
\end{aligned} \tag{4.25}$$

because $\lambda<\lambda_1$. In the integral on the right of (4.24), we first apply (4.12) and then decompose the semigroup part rewriting $\Delta=-A+\eta$ to see that whenever $0<s<t<T$,

$$\begin{aligned}
&\Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} \\
&\le c_1\Big\|A^\beta e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)} \\
&\le c_1 e^{\eta^2(t-s)}\Big\|A^\beta e^{-\eta(t-s)A}e^{(1-\eta)(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)} \\
&\le c_1c_2\eta^{-\beta}(t-s)^{-\beta}e^{\eta^2(t-s)}\Big\|e^{(1-\eta)(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^p(\Omega)} \\
&\le c_1c_2\eta^{-\beta}(t-s)^{-\beta}e^{\eta^2(t-s)}\,\times \\
&\quad \times c_5\Big\{1+[(1-\eta)(t-s)]^{-\frac12}\Big\}e^{-\lambda_1(1-\eta)(t-s)}\Big\|u(\cdot,s)\nabla v(\cdot,s)\Big\|_{L^\infty(\Omega)}
\end{aligned} \tag{4.26}$$

because of (4.13) and (4.15). Here we observe that

$$\begin{aligned}
(t-s)^{-\beta}\cdot\Big\{1+[(1-\eta)(t-s)]^{-\frac12}\Big\} &\le (1-\eta)^{-\frac12}\cdot\Big\{(t-s)^{-\beta}+(t-s)^{-\beta-\frac12}\Big\} \\
&\le 2(1-\eta)^{-\frac12}\cdot\Big\{1+(t-s)^{-\beta-\frac12}\Big\} \qquad \text{for } 0<s<t<\infty,
\end{aligned}$$

and that (4.21), (1.12) and (4.23) warrant that

$$\begin{aligned}
\Big\|u(\cdot,s)\nabla v(\cdot,s)\Big\|_{L^\infty(\Omega)} &\le \|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla v\|_{L^\infty(\Omega)} \\
&\le \Big(\|u(\cdot,s)-\mu\|_{L^\infty(\Omega)}+\mu\Big)\cdot\|\nabla v\|_{L^\infty(\Omega)} \\
&\le c_6\Big(\varepsilon e^{-\lambda s}+\mu\Big)\cdot\varepsilon e^{-\lambda s} \\
&\le 2c_6\varepsilon^2 e^{-\lambda s} \qquad \text{for all } s\in(0,T),
\end{aligned}$$

because $\delta \le \frac{\varepsilon}{4} < \varepsilon$ by (4.20). Therefore, from (4.24)-(4.26) we obtain the inequality

$$\begin{aligned} \|u(\cdot,t)-\mu\|_{L^\infty(\Omega)} &\le 2c_3\delta e^{-\lambda t} \\ &\quad +4c_1c_2c_5c_6\eta^{-\beta}(1-\eta)^{-\frac{1}{2}}\varepsilon^2 \int_0^t \Big\{1+(t-s)^{-\beta-\frac{1}{2}}\Big\} e^{[\eta^2-\lambda_1(1-\eta)](t-s)} e^{-\lambda s} ds \end{aligned}$$

for all $t \in (0,T)$. As $\eta^2 - \lambda_1(1-\eta) = -\tilde{\lambda}$ and $\tilde{\lambda} > \lambda$ by (4.10), invoking (4.17) and recalling the definition (4.19) of ε_0 we find that

$$\|u(\cdot,t)-\mu\|_{L^\infty(\Omega)} \le (2c_3\delta + c_8\varepsilon^2)e^{-\lambda t} \qquad \text{for all } t \in (0,T). \tag{4.27}$$

Now due to (4.20), (4.19) and the fact that $\varepsilon < \varepsilon_0$, we have

$$2c_3\delta + c_8\varepsilon^2 \le \frac{1}{4}\varepsilon + c_8\varepsilon^2 = \Big(\frac{1}{4}+c_8\varepsilon\Big)\varepsilon \le \Big(\frac{1}{4}+c_8\varepsilon_0\Big)\varepsilon \le \frac{\varepsilon}{2},$$

so that, once more by continuity, (4.27) in particular implies that

$$\|u(\cdot,T)-\mu\|_{L^\infty(\Omega)} \le \frac{\varepsilon}{2}e^{-\lambda T}.$$

Since in precisely the same manner one can verify that also

$$\|w(\cdot,T)-\kappa\|_{L^\infty(\Omega)} \le \frac{\varepsilon}{2}e^{-\lambda T},$$

this shows that (4.22) cannot be valid, and that hence we indeed must have $T = T_{max}$. By definition of T and the extensibility criterion (2.1), however, this entails that actually $T_{max} = \infty$, and that thus u and w satisfy (1.13). □

4.3 Blow-up. Proof of Theorem 1.3 ii) and Theorem 1.4 ii)

Let us finally observe how some large-data blow-up solutions to (1.4), (1.5) can easily be constructed using the following well-known blow-up result for the Keller-Segel system (1.6).

Proposition 4.3 ([25], [18]) *i) Let $n = 2$. Then for each $x_0 \in \Omega$ there exists $C > 0$ such that if $U_0 \in C^0(\bar{\Omega})$ is nonnegative and such that $\int_\Omega U_0 > 4\pi$ and $\int_\Omega |x-x_0|^2 U_0(x)dx \le C$, then the corresponding solution of (1.6) blows up in finite time in the sense that for some $T > 0$ we have*

$$\limsup_{t \nearrow T} \|U(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{4.28}$$

ii) Let $n \ge 3$, and assume that Ω is star-shaped with respect to some $x_0 \in \Omega$. Then one can pick $C > 0$ with the property that whenever $U_0 \in C^0(\bar{\Omega})$ is nonnegative and satisfies $\int_\Omega |x-x_0|^2 U_0(x)dx \le C$, the solution of (1.6) emanating from U_0 blows up after a finite time in the sense that (4.28) is valid with some $T > 0$.

In particular, this immediately implies the blow-up statements in both Theorem 1.3) and Theorem 1.4:

Proof of Theorem 1.3 ii). Given $m > 4\pi$, we take U_0 as provided by Proposition 4.3 i) and let (U, V) denote the corresponding solution of (1.6). We choose $u_0 := U_0$ and $w_0 := U_0$ as initial data in (1.4), (1.5) and then clearly obtain that (1.10) is satisfied. Since by uniqueness we know that $(u, v, w, z) := (U, V, U, V)$ is the solution of (1.4), (1.5), the fact that U satisfies (4.28) completes the proof. □

Proof of Theorem 1.4 ii). In quite the same manner as above, one can derive the statement in question directly from Proposition 4.3 ii). □

5 Conclusion

We have considered a prototypical model for two-species chemotaxis processes involving two signals. Even this particular system exhibits remarkable aspects of such types of interaction. Firstly, in presence of either exclusively attractive, or entirely repulsive mechanisms, the picture is in perfrect accordance with well-known facts on the corresponding one-species chemotaxis systems: Whereas the former interplay may give rise to exploding solutions, the latter enforces solutions to be global and remain bounded. Secondly, however, we have seen that a hybrid variant thereof, that is, an attraction-repulsion-type of interaction as described by (1.2), by no means interpolates between these antipodes, but rather fully belongs to the latter in that it solely produces bounded solutions.

Aiming at providing a first step toward a comprehensive understanding of the system dynamics, we have not addressed any question related to the large time behavior of bounded solutions. A natural next issue consists of studying the structure of the corresponding sets of equilibria, as well as their respective attractivity properties. This might provide relevant information concerning the role of multi-species chemotaxis mechanisms e.g. in processes of cell sorting ([9]).

For technical purposes we have concentrated on the apparently simplest reasonable mathematical framework for the study of the addressed type of interaction. The analysis in Section 3 can easily be extended so as to cover the case of arbitrary $\xi < 0$ and $\chi \in \mathbb{R}$; likewise, appropriate counterparts of the global existence results in Theorem 1.3 i) and Theorem 1.4 i) can be derived for any choice of $\chi > 0$ and $\xi > 0$. In the case when χ and ξ are both positive but different from each other, a reduction to one-species cases as in Theorem 1.3 ii) and Theorem 1.4 ii) seems no longer available, so that proving the occurrence of blow-up will require different approaches, e.g. by going back to the analysis of moment functionals such as in the one-species Keller-Segel system ([25]). Such an independent blow-up proof might be of interest of its own, because by providing explosion criteria more general than ours it might reveal blow-up as an essentially generic feature of the doubly attractive system.

We have also concentrated on the parabolic-elliptic setting in (1.1) only, thereby ignoring all technical challenges which may occur in the analysis of the fully parabolic counterpart; indeed, even in the corresponding one-species case a full understanding of the dampening effect of repulsion seems to be lacking when also the signal evolution is of parabolic type ([6]). Similarly, our approach is appearently inadequate to cover also the physically irrelevant case $n \geq 4$ in the two-species atttactive-repulsive model (1.1). It would be mathematically interesting to see whether or not blow-up may occur in a sufficiently high-dimensional framework.

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