Persistence of mass in a chemotaxis system with logistic source

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Abstract

This paper studies the dynamical properties of the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + ru - \mu u^2, & x \in \Omega,\, t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\, t > 0, \end{cases} \qquad (\star)$$

under homogeneous Neumann boundary conditions in bounded convex domains $\Omega \subset \mathbb{R}^n, n \geq 1$, with positive constants χ, r and μ.

Numerical simulations but also some rigorous evidence has shown that depending on the relative size of r, μ and $|\Omega|$, in comparison to the well-understood case when $\chi = 0$, this problem may exhibit quite a complex solution behavior, including unexpected effects such as asymptotic decay of the quantity u within large subdomains of Ω.

The present work indicates that any such extinction phenomenon, if occuring at all, necessarily must be of spatially local nature, whereas the population as a whole always persists. More precisely, it is shown that for any nonnegative global classical solution (u, v) of $(\star)$ with $u \not\equiv 0$ one can find $m_\star > 0$ such that

$$\int_\Omega u(x,t)dx \geq m_\star \qquad \text{for all } t > 0.$$

The proof is based on an in this context apparently novel analysis of the functional $\int_\Omega \ln u$, deriving a lower bound for this quantity along a suitable sequence of times by appropriately exploiting a differential inequality for a suitable linear combination of $\int_\Omega \ln u$, $\int_\Omega u$ and $\int_\Omega v^2$.

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1 Introduction

Chemotaxis, a biological process in which cells migrate towards higher concentrations of a chemical signal, has received great interest in biological and mathematical communities. A celebrated model for such processes, as introduced by Keller and Segel in 1970 ([9]), consists of two parabolic equations reflecting chemotactic movement through a nonlinear cross-diffusive term as their most characteristic ingredient. Since then, considerable efforts have been undertaken to comprehend possible effects of this interaction in various frameworks, with the detection of finite-time blow-up in the classical Keller-Segel system constituting the apparently most striking evidence for the strongly destabilizing action of chemotactic cross-diffusion ([8], [12], [24]). Accordingly, large bodies of the literature focus on identifying circumstances under which either global bounded solutions can be constructed, or explosions occur.

In contrast to the rich knowledge on this basic issue, understanding the qualitative properties even of bounded solutions to chemotaxis problems seems much less developed. Genuinely parabolic features such as convergence to single equilibria could up to now only be verified in some particular cases when either an appropriate entropy-dissipation structure inhibits oscillatory behavior ([3], [18]), or when some negligibility of cross-diffusion as compared to diffusion is enforced by certain smallness assumptions on the initial data or on parameters measuring the strength of chemotaxis ([22], [2], [25], [20]).

In the present work we address a dynamical property of the Keller-Segel system with logistic source given by

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + ru - \mu u^2, & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

in bounded convex domains $\Omega\subset\mathbb{R}^n, n\geq 1$, with positive constants χ, r and μ, where as usual u denotes the cell density and v represents the concentration of a signal produced by cells themselves. Here the cell kinetic term on the one hand exerts a certain growth-inhibiting influence; indeed, in the case $n\leq 2$ even arbitrarily small $\mu>0$ are sufficient to rule out any explosion by guaranteeing global existence of bounded classical solutions for all reasonably smooth initial data ([1], [15]), whereas in the case $n\geq 3$ the same conclusion holds provided that $\mu>0$ is suitably large ([21]). On the other hand, this additional logistic term apparently destroys the well-known energy structure of the corresponding free Keller-Segel system obtained in the limit case $r=\mu=0$ ([13]).

Apparently, however, the latter does not only reduce the accessibility of (1.1) to convenient mathematical tools, but beyond this it reflects a substantial change in dynamical complexity. Indeed, numerical evidence shows that even in the spatially one-dimensional setting solutions may exhibit quite a colorful behavior, ranging from simple stabilization to nonconstant equilibria over essentially periodic changes between different patterns up to apparently fully disordered behavior ([16], cf. also [14]); analytical results so far only partially capture the large variety of dynamical features in chemotaxis-growth systems, after all rigorously detecting the emergence of certain types of transient growth ([11], [26]).

A further phenomenon suggested by such simulations consists in the ability of (1.1) to enforce asymptotic smallness of the cell population density, undistinguishable from extinction, in large spatial regions (see e.g. Fig.7 (d) in [16]). Such types of solution behavior, seemingly paradoxical due to the presence of the reproduction term ru dominating e.g. the death term $-\mu u^2$ at small densities, clearly reflect a truly cross-diffusive effect in view of the evident fact that when $\chi = 0$, all solutions of the resulting decoupled problem approach the spatially homogeneous nontrivial state $\frac{r}{\mu}$ in both components.

The goal of the present work is to make sure that in fact any such extinction phenomenon must be localized in space in that the population as a whole always persists. More precisely, our main result rules out asymptotic loss of the total mass in the following quantitative sense.

Theorem 1.1 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with smooth boundary, and let $\chi > 0, r > 0$ and $\mu > 0$. Then for any choice of $m > 0$, $A > 0$ and $L > 0$ there exists $m_\star = m_\star(m, A, L, \chi, r, \mu, \Omega) > 0$ with the property that whenever (u, v) is a couple of nonnegative functions belonging to $C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$ which solve (1.1) in the classical sense with some $(u_0, v_0) \in (C^0(\bar{\Omega}))^2$ satisfying*

$$\int_\Omega u_0 \le m, \qquad \int_\Omega v_0^2 \le A \quad \textit{and} \quad \int_\Omega \ln u_0 \ge -L, \tag{1.2}$$

we have

$$\int_\Omega u(x,t)dx \ge m_\star \qquad \textit{for all } t > 0. \tag{1.3}$$

For a given individual solution of (1.1), this has an immediate consequence:

Corollary 1.2 *Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded convex domain with smooth boundary, and let $\chi > 0, r > 0$ and $\mu > 0$. Then for each nonnegative global classical solution $(u, v) \in (C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)))^2$ with $u \not\equiv 0$ we have*

$$\inf_{t>0} \int_\Omega u(\cdot, t) > 0. \tag{1.4}$$

Proof of Corollary 1.2. Since $u \not\equiv 0$, we necessarily have $u_0 \not\equiv 0$, and moreover u is actually strictly positive in $\bar{\Omega} \times (0,\infty)$ by the strong maximum principle. In particular, $\inf_{t\in(0,1)} \int_\Omega u(\cdot, t)$ is positive and $\int_\Omega \ln u(\cdot, 1) > -\infty$, so that an application of Theorem 1.1 readily yields (1.4). $\square$

Remark. i) Both Theorem 1.1 and Corollary 1.2 presuppose the existence of a global classical solution. In fact, such solutions to the initial-boundary value problem (1.1) are known to exist for all nonnegative and sufficiently smooth initial data u_0 and v_0 under the assumptions that either $n \le 2$ and $\mu > 0$ is arbitrary ([1]), or $n \ge 3$, Ω is convex and $\mu > 0$ is appropriately large ([21]). To the best of our knowledge, it is yet unclear whether for $n \ge 3$ and small values of $\mu > 0$ certain initial data may enforce finite-time blow-up of solutions (cf. [23] for a high-dimensional blow-up result in a simplified chemotaxis system with superlinear logistic-type degradation), although at least certain global weak solutions are known to exist for arbitrary $\mu > 0$ ([10]). The steady-state example $(u, v) \equiv (\frac{r}{\mu}, \frac{r}{\mu})$, however, trivially shows that some global classical solutions exist regardless of the size of $\mu > 0$. We underline that our statements above apply to any nontrivial global classical solution, even in situations when μ is so small that global existence might be expected for non-generic choices of the initial data only.

ii) By a straightforward adaptation of our arguments, the result e.g. of Corollary 1.2 can be carried over to the parabolic-elliptic counterpart of (1.1) obtained when the second PDE is replaced with

$$0 = \Delta v - v + u, \qquad x \in \Omega,\ t > 0.$$

iii) Mass persistence properties of the type considered here may also have consequences beyond the scope discussed above. Namely, in view of a positivity property of the Neumann heat semigroup ([6]), the inequality (1.3) directly implies a uniform pointwise positive lower bound for v. In some related more complex systems such an information can be used to assert boundedness or also convergence properties of solutions ([4], [6], [19]).

A main challenge in proving Theorem 1.1 will consist in creating a setup which allows for controlling both the destabilizing action of cross-diffusion and the population-diminishing effect of the quadratic death term in (1.1) by means of the regularizing influence of diffusion and the linear proliferative term ru. This will be achieved on the basis of the differential inequality

$$\frac{d}{dt}\Big\{\int_\Omega \ln u - \frac{\chi^2}{8\mu}\int_\Omega u - \frac{\chi^2}{4}\int_\Omega v^2\Big\} \ge \frac12\int_\Omega \frac{|\nabla u|^2}{u^2} + r|\Omega| - \Big(\frac{r\chi^2}{8\mu}+\mu\Big)\cdot\int_\Omega u, \tag{1.5}$$

valid on $(0,\infty)$ for any nontrivial global solution (Corollary 2.4). A careful analysis thereof will allow us to accomplish the main step in our argument in Lemma 5.1 by recursively constructing an increasing unbounded sequence of times t_k, with uniformly bounded differences $t_{k+1}-t_k$, at which $\ln u$ can be bounded from below by a fixed constant. In Section 6, this will in turn yield a sequence of times, having essentially the same properties as $(t_k)_{k\in\mathbb{N}}$, along which the integral $\int_\Omega u$ remains above some positive constant. Therefore, the outcome of Theorem 1.1 will result upon the evident observation that $\int_\Omega u$ evidently does not grow faster than exponentially (Section 6).

2 An ordinary differential inequality associated with $\int_\Omega \ln u$

We begin with three straightforward observations which on adequate combination will lead to the fundamental ODI (2.4). Let us first trace the evolution of the mass functional.

Lemma 2.1 *Any global classical solution of (1.1) satisfies*

$$\frac{d}{dt}\int_\Omega u = r\int_\Omega u - \mu\int_\Omega u^2 \qquad \textit{for all } t>0. \tag{2.1}$$

PROOF. The identity (2.1) immediately results from an integration of the first equation in (1.1) with respect to $x \in \Omega$. □

Unfortunalety, it seems far from obvious how the rightmost integral in (2.1), somewhat counteracting our goal to bound $\int_\Omega u$ from below, can be controlled. In order to gain some spatially global evidence for the intuitive idea that ru should overbalance $-\mu u^2$ when u is small, we rather divide the first equation in (1.1) by u to achieve the following.

Lemma 2.2 *If (u,v) is a global classical solution of (1.1) such that $u>0$ in $\bar\Omega\times(0,\infty)$, then*

$$\frac{d}{dt}\int_\Omega \ln u \ge \frac12\int_\Omega\frac{|\nabla u|^2}{u^2} - \frac{\chi^2}{2}\int_\Omega|\nabla v|^2 + r|\Omega| - \mu\int_\Omega u \qquad \textit{for all } t>0. \tag{2.2}$$

PROOF. We test the first equation in (1.1) by $\frac{1}{u}$ to find that

$$\begin{aligned}\frac{d}{dt}\int_\Omega \ln u &= \int_\Omega \frac{1}{u}\Delta u - \chi\int_\Omega \frac{1}{u}\nabla\cdot(u\nabla v) + r|\Omega| - \mu\int_\Omega u \\ &= \int_\Omega \frac{|\nabla u|^2}{u^2} - \chi\int_\Omega \frac{1}{u}\nabla u\cdot\nabla v + r|\Omega| - \mu\int_\Omega u \qquad \text{for all } t>0.\end{aligned}$$

Since by Young's inequality we can estimate

$$\Big| -\chi\int_\Omega \frac{1}{u}\nabla u\cdot\nabla v\Big| \le \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{2}\int_\Omega |\nabla v|^2 \qquad \text{for all } t>0,$$

this implies (2.2). $\square$

In order to control the second integral on the right of (2.2), we make use of the PDE for v through another standard testing procedure.

Lemma 2.3 *For every global classical solution of (1.1) we have*

$$\frac{d}{dt}\int_\Omega v^2 + 2\int_\Omega |\nabla v|^2 \le \frac{1}{2}\int_\Omega u^2 \qquad \textit{for all } t>0. \tag{2.3}$$

PROOF. We multiply the second equation in (1.1) by v and integrate by parts to see, using Young's inequality, that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + \int_\Omega |\nabla v|^2 + \int_\Omega v^2 = \int_\Omega uv \le \int_\Omega v^2 + \frac{1}{4}\int_\Omega u^2 \qquad \text{for all } t>0,$$

which yields (2.3). $\square$

We now combine the above lemmata to find the following differential iequality which will be applied in Lemma 5.2 and Lemma 5.4 below, and thereby forms a cornerstone for our subsequent reasoning.

Corollary 2.4 *If (u,v) is a global classical solution of (1.1) such that $u>0$ in $\bar\Omega\times(0,\infty)$, then*

$$\frac{d}{dt}\Big\{\int_\Omega \ln u - \frac{\chi^2}{8\mu}\int_\Omega u - \frac{\chi^2}{4}\int_\Omega v^2\Big\} \ge \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} + r|\Omega| - \Big(\frac{r\chi^2}{8\mu}+\mu\Big)\cdot\int_\Omega u \qquad \textit{for all } t>0. \tag{2.4}$$

PROOF. By Lemma 2.3 and Lemma 2.1,

$$\begin{aligned}\int_\Omega |\nabla v|^2 &\le -\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + \frac{1}{4}\int_\Omega u^2 \\ &= -\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 - \frac{1}{4\mu}\frac{d}{dt}\int_\Omega u + \frac{r}{4\mu}\int_\Omega u\end{aligned}$$

for all $t>0$. Hence, Lemma 2.2 entails that

$$\frac{d}{dt}\int_\Omega \ln u \ge \frac{1}{2}\int_\Omega \frac{|\nabla u|^2}{u^2} + \frac{\chi^2}{4}\frac{d}{dt}\int_\Omega v^2 + \frac{\chi^2}{8\mu}\frac{d}{dt}\int_\Omega u - \frac{r\chi^2}{8\mu}\int_\Omega u + r|\Omega| - \mu\int_\Omega u$$

for all $t>0$, as claimed. $\square$

3 Upper estimates for $\int_\Omega u$, $\int_\Omega v^2$ and $\int_t^{t+T}\int_\Omega u^2$

In order to turn (2.4) into an estimate for $\int_\Omega \ln u$ from below, it seems favorable to derive upper bounds for the integrals $\int_\Omega u$ and $\int_\Omega v^2$ appearing therein. By means of Lemma 2.1 we can easily estimate the former and at the same time also provide an upper bound for a spatio-temporal L^2 norm of u.

Lemma 3.1 *Let (u,v) be a global classical solution of (1.1) with nonnegative initial data u_0 and v_0. Then*

$$\int_\Omega u \le m^\star := \max\Big\{\int_\Omega u_0\,,\,\frac{r|\Omega|}{\mu}\Big\} \qquad \textit{for all } t>0, \tag{3.1}$$

and moreover, for any $t_0\ge 0$ and each $T>0$ the inequality

$$\int_{t_0}^{t_0+T}\int_\Omega u^2 \le \frac{(rT+1)m^\star}{\mu} \tag{3.2}$$

holds.

PROOF. Since $(\int_\Omega u)^2 \le |\Omega|\int_\Omega u^2$ by the Cauchy-Schwarz inequality, (2.1) says that $y(t) := \int_\Omega u(x,t)dx, t>0$, satisfies

$$y'(t) \le ry(t) - \frac{\mu}{|\Omega|}y^2(t) \qquad \text{for all } t>0.$$

In view of our assumption that $\int_\Omega u_0 \le m$, an ODE comparison argument thus readily entails (3.1). Now going back to (2.1) and integrating in time, we see using (3.1) and the nonnegativity of u that

$$\begin{aligned}
\mu\int_{t_0}^{t_0+T}\int_\Omega u^2 &= r\int_{t_0}^{t_0+T}\int_\Omega u - \int_\Omega u(\cdot,t_0+T) + \int_\Omega u(\cdot,t_0)\\
&\le r\int_{t_0}^{t_0+T}\int_\Omega u + \int_\Omega u(\cdot,t_0)\\
&\le rTm^\star + m^\star,
\end{aligned}$$

which yields (3.2). □

The estimate (3.2) now entails the desired upper bound for $\int_\Omega v^2$ by means of the following auxiliary lemma on boundedness of solutions to a linearly dampened ODI with an inhomogeneity appropriately controllable in an integral sense.

Lemma 3.2 *Suppose that $y\in C^0([0,\infty))\cap C^1((0,\infty))$ is nonnegative and such that there exist $a>0, f\in C^0((0,\infty))$ and $b>0$ such that $f\ge 0$ and*

$$y'(t) + ay(t) \le f(t) \qquad \textit{for a.e. } t\in(0,T) \tag{3.3}$$

as well as

$$\int_t^{t+1} f(s)ds \le b \qquad \textit{for all } t\in[0,T-1).$$

Then

$$y(t) \le \max\Big\{y(0)+b\,,\,\frac{b}{a}+2b\Big\} \qquad \textit{for all } t\in(0,T). \tag{3.4}$$

Proof. This is proved in [17, Lemma 3.4]. □

Lemma 3.3 *There exists $C_v = C_v(r, \mu, \Omega) > 0$ such that if (u, v) is a global classical solution of (1.1) with nonnegative initial data u_0 and v_0, then with $m^\star$ as in (3.1) we have*

$$\int_\Omega v^2(x,t)dx \le C_v \cdot \Big\{ \int_\Omega v_0^2 + (m^\star)^2 + 1 \Big\} \qquad \textit{for all } t > 0. \tag{3.5}$$

Proof. We first integrate the second equation in (1.1) over Ω to see that according to Lemma 3.1, with $m^\star$ given by (3.1) we have

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega u \le m^\star \qquad \text{for all } t > 0,$$

by a comparison argument implying that

$$\int_\Omega v(\cdot,t) \le c_1 := \max\Big\{ \int_\Omega v_0 \, , \, m^\star \Big\} \qquad \text{for all } t > 0. \tag{3.6}$$

Now according to the Poincaré inequality, there exists $c_2 = c_2(\Omega) > 0$ such that

$$\int_\Omega \varphi^2 \le c_2 \int_\Omega |\nabla\varphi|^2 + c_2\Big(\int_\Omega \varphi\Big)^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

so that thanks to (3.6),

$$\int_\Omega v^2 \le c_2 \int_\Omega |\nabla v|^2 + c_1^2 c_2 \qquad \text{for all } t > 0$$

and hence

$$\int_\Omega |\nabla v|^2 \ge \frac{1}{c_2}\int_\Omega v^2 - c_1^2 \qquad \text{for all } t > 0.$$

From Lemma 2.3 we therefore obtain the autonomous ODI

$$\frac{d}{dt}\int_\Omega v^2 + \frac{2}{c_2}\int_\Omega v^2 \le \frac{1}{2}\int_\Omega u^2 + 2c_1^2 \qquad \text{for all } t > 0,$$

where since

$$\int_t^{t+1} \Big\{ \frac{1}{2}\int_\Omega u^2(\cdot,s) + 2c_1^2 \Big\} ds \le c_3 := \frac{(r+1)m_\star}{2\mu} + 2c_1^2 \qquad \text{for all } t > 0,$$

Lemma 3.2 warrants that

$$\int_\Omega v^2 \le \max\Big\{ \int_\Omega v_0^2 + c_3 \, , \, \frac{c_2 c_3}{2} + 2c_3 \Big\} \qquad \text{for all } t > 0.$$

As

$$c_1^2 \le 2\Big(\int_\Omega v_0\Big)^2 + 2(m^\star)^2 \le 2|\Omega| \int_\Omega v_0^2 + 2(m^\star)^2$$

by the Cauchy-Schwarz inequality, this readily yields (3.5). □

Apart from the above application, let us also note the following consequence of (3.2) on the size of certain sets of times at which $\int_\Omega u^2$ is large. This information will be used in the proof of Lemma 5.1.

Lemma 3.4 *Let $T > 0$, $K > 0$ and $t_0 \geq 0$. Then any nonnegative global classical solution (u, v) of (1.1) satisfies*

$$\left|\left\{t \in (t_0, t_0 + T) \;\Big|\; \int_\Omega u^2(x,t)dx > K\right\}\right| \leq \frac{(rT+1)m^\star}{\mu K} \tag{3.7}$$

with $m^\star$ given by (3.1).

Proof. Abbreviating

$$S := \left\{t \in (t_0, t_0 + T) \;\Big|\; \int_\Omega u^2(x,t)dx > K\right\},$$

we have

$$\int_{t_0}^{t_0+T} \int_\Omega u^2 \geq \int_S \int_\Omega u^2(x,t)dxdt \geq |S| \cdot K.$$

Therefore, (3.7) is an immediate consequence of (3.2). $\square$

4 A lower bound for $\int_\Omega \ln \varphi$ in terms of $\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2}$ for positive $\varphi \in C^1(\bar\Omega)$

In this section we plan to prepare a lower estimate for the leftmost integral in (2.4) to be achieved in Lemma 5.1 below by deriving two statements on fairly general functions on Ω.
Let us first provide a quantitative information on the size of the set of points where a nonnegative function is conveniently large, assuming a lower and an upper bound for its norms in $L^1(\Omega)$ and $L^2(\Omega)$, respectively.

Lemma 4.1 *Let $\eta > 0$ and $K > 0$, and suppose that a nonnegative function $\psi \in L^2(\Omega)$ satisfies*

$$\int_\Omega \psi \geq \eta \qquad \text{and} \qquad \int_\Omega \psi^2 \leq K. \tag{4.1}$$

Then

$$\left|\left\{x \in \Omega \;\Big|\; \psi(x) \geq \frac{\eta}{2|\Omega|}\right\}\right| \geq \frac{\eta^2}{4K}. \tag{4.2}$$

Proof. Given any $\varepsilon > 0$, using the Cauchy-Schwarz inequality we find that

$$\begin{aligned}\int_\Omega \psi &= \int_{\{\psi<\varepsilon\}} \psi + \int_{\{\psi\geq\varepsilon\}} \psi \\ &\leq \varepsilon|\Omega| + \Big(\int_\Omega \psi^2\Big)^{\frac12} \cdot \Big|\{\psi \geq \varepsilon\}\Big|^{\frac12}.\end{aligned}$$

According to (4.1), this entails that

$$\eta \leq \varepsilon|\Omega| + \sqrt{K} \cdot \Big|\{\psi \geq \varepsilon\}\Big|^{\frac12},$$

whence specifying $\varepsilon := \frac{\eta}{2|\Omega|}$ we infer that

$$\frac{\eta}{2} \le \sqrt{K} \cdot \left| \left\{ \psi \ge \frac{\eta}{2|\Omega|} \right\} \right|^{\frac{1}{2}},$$

and that thus (4.2) holds. $\Box$

Next, in addressing the integral in question we shall essentially make use of the following variant of the Poincaré inequality, a proof of which can be found in [7, Corollary 8.1.4]. It is the only place in this paper where convexity of Ω is explicitly needed.

Lemma 4.2 *For all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that*

$$\int_\Omega \varphi^2 \le C(\varepsilon) \int_\Omega |\nabla\varphi|^2 \tag{4.3}$$

holds for all $\varphi \in W^{1,2}(\Omega)$ satisfying

$$\left| \left\{ x \in \Omega \;\middle|\; \varphi(x) = 0 \right\} \right| \ge \varepsilon.$$

In consequence, this allows us to bound $\int_\Omega \ln\varphi$ from below in terms of $\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2}$ for positive $\varphi \in C^1(\bar\Omega)$ as follows.

Lemma 4.3 *Let $\varphi \in C^1(\bar\Omega)$ be positive and such that*

$$\Big| \{ x \in \Omega \mid \varphi(x) \ge \delta \} \Big| \ge \varepsilon \tag{4.4}$$

for some $\varepsilon > 0$ and $\delta > 0$. Then

$$\int_\Omega \ln\varphi \ge |\Omega| \cdot \ln\delta - \sqrt{C(\varepsilon)|\Omega| \cdot \int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2}} \tag{4.5}$$

with $C(\varepsilon)$ taken from Lemma 4.2.

Proof. According to our assumptions,

$$\psi(x) := \left(\ln \frac{\delta}{\varphi(x)} \right)_+, \qquad x \in \bar\Omega,$$

defines a function $\psi \in W^{1,2}(\Omega)$ which, by (4.4), has the property that

$$\Big| \{ x \in \Omega \mid \psi(x) = 0 \} \Big| = \Big| \{ x \in \Omega \mid \varphi(x) \ge \delta \} \Big| \ge \varepsilon.$$

Therefore, Lemma 4.2 implies that $\int_\Omega \psi^2 \le C(\varepsilon) \int_\Omega |\nabla\psi|^2$, that is,

$$\int_\Omega \left(\ln \frac{\delta}{\varphi(x)} \right)_+^2 dx \le C(\varepsilon) \int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2},$$

so that

$$\int_\Omega \Big(\ln\frac{\delta}{\varphi(x)}\Big)_+ dx \le \sqrt{|\Omega|\cdot\int_\Omega \Big(\ln\frac{\delta}{\varphi(x)}\Big)_+^2 dx} \le \sqrt{C(\varepsilon)|\Omega|\cdot\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2}}. \tag{4.6}$$

Here combining

$$\begin{aligned}\int_\Omega \Big(\ln\frac{\delta}{\varphi(x)}\Big)_+ dx &= \int_{\{\varphi<\delta\}} \Big(\ln\delta - \ln\varphi(x)\Big)dx \\ &= |\{\varphi<\delta\}|\cdot\ln\delta - \int_{\{\varphi<\delta\}} \ln\varphi(x)dx\end{aligned}$$

with the inequality

$$\begin{aligned}\int_\Omega \ln\varphi(x)dx &= \int_{\{\varphi\ge\delta\}} \ln\varphi(x)dx + \int_{\{\varphi<\delta\}} \ln\varphi(x)dx \\ &\ge |\{\varphi\ge\delta\}|\cdot\ln\delta + \int_{\{\varphi<\delta\}} \ln\varphi(x)dx\end{aligned}$$

asserted by the monotonicity of $(0,\infty)\ni\xi\mapsto\ln\xi$, we obtain

$$\int_\Omega \Big(\ln\frac{\delta}{\varphi(x)}\Big)_+ dx \ge -\int_\Omega \ln\varphi(x)dx + |\Omega|\cdot\ln\delta.$$

Therefore, (4.5) results from (4.6). □

5 A lower bound for $\int_\Omega \ln u$

The goal of this section is to provide the following key step in the derivation of our main results.

Lemma 5.1 *Let $m>0$, $A>0$ and $L>0$. Then there exist positive constants $\overline{L}=\overline{L}(m,L,\chi,r,\mu,\Omega)$ and $T=T(m,A,L,\chi,r,\mu,\Omega)$ such that if $u_0\in C^0(\bar\Omega)$ and $v_0\in C^0(\bar\Omega)$ are nonnegative and such that*

$$\int_\Omega u_0 \le m, \qquad \int_\Omega v_0^2 \le A \quad \textit{and} \quad \int_\Omega \ln u_0 \ge -L, \tag{5.1}$$

and if (u,v) is a global classical solution of (1.1), then one can find a sequence $(t_k)_{k\in\mathbb{N}}\subset(0,\infty)$ such that $t_k\to\infty$ as $k\to\infty$ and $t_k<t_{k+1}<t_k+T$ as well as

$$\int_\Omega \ln u(\cdot,t_k) \ge -\overline{L} \qquad \textit{for all } k\in\mathbb{N}. \tag{5.2}$$

Our proof of Lemma 5.1, based on the differential inequality (2.4) and an inductive argument, will be prepared by three lemmata, the first of which exploits (2.4) in order to make sure that $\int_{t_0}^{t_0+T}\int_\Omega u$ becomes suitably large, provided that $T>0$ is chosen appropriately large in dependence on $\int_\Omega \ln u(\cdot,t_0)$. Here and below, in applying Corollary 2.4 we shall tacitly make use of the observation that under the assumption (5.1), any global classical solution (u,v) must be nontrivial and hence, due to the strong maximum principle, satisfies $u>0$ in $\bar\Omega\times(0,\infty)$.

Lemma 5.2 *Suppose that a nonnegative global classical solution (u,v) of (1.1) has the property that there exist $t_0 \ge 0$ and $L_0 \ge 0$ such that*

$$\int_\Omega \ln u(\cdot,t_0) \ge -L_0. \tag{5.3}$$

Then for any $T>0$ satisfying

$$T \ge \frac{L_0 + (\frac{\chi^2}{8\mu}+1)m^\star + \frac{\chi^2 C_v}{4}\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big)}{\frac{1}{2}r|\Omega|} \tag{5.4}$$

with $m^\star$ and C_v as given by Lemma 3.1 and Lemma 3.3, we have

$$\int_{t_0}^{t_0+T}\int_\Omega u(x,t)dxdt \ge \frac{r|\Omega|}{\frac{r\chi^2}{4\mu}+2\mu}\cdot T. \tag{5.5}$$

Proof. We drop the first term on the right of (2.4) to see upon integrating the resulting inequality that

$$\begin{aligned}
\int_\Omega \ln u(\cdot,t) - \frac{\chi^2}{8\mu}\int_\Omega u(\cdot,t) - \frac{\chi^2}{4}\int_\Omega v^2(\cdot,t) \quad &\ge \quad \int_\Omega \ln u(\cdot,t_0) - \frac{\chi^2}{8\mu}\int_\Omega u(\cdot,t_0) - \frac{\chi^2}{4}\int_\Omega v^2(\cdot,t_0) \\
&\quad + r|\Omega|(t-t_0) - \Big(\frac{r\chi^2}{8\mu}+\mu\Big)\cdot\int_{t_0}^t\int_\Omega u
\end{aligned}$$

for all $t>t_0$. Here we let $t := t_0+T$ and use (5.3), Lemma 3.1 and Lemma 3.3 to obtain

$$\begin{aligned}
\Big(\frac{r\chi^2}{8\mu}+\mu\Big)\cdot\int_{t_0}^{t_0+T}\int_\Omega u \quad &\ge \quad r|\Omega|T + \int_\Omega \ln u(\cdot,t_0) - \frac{\chi^2}{8\mu}\int_\Omega u(\cdot,t_0) - \frac{\chi^2}{4}\int_\Omega v^2(\cdot,t_0) \\
&\quad - \int_\Omega \ln u(\cdot,t_0+T) + \frac{\chi^2}{8\mu}\int_\Omega u(\cdot,t_0+T) + \frac{\chi^2}{4}\int_\Omega v^2(\cdot,t_0+T) \\
&\ge \quad r|\Omega|T - L_0 - \frac{\chi^2}{8\mu}m^\star - \frac{\chi^2}{4}C_v\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big) \\
&\quad - \int_\Omega \ln u(\cdot,t_0+T),
\end{aligned} \tag{5.6}$$

because u is nonnegative. Since $\ln\xi \le \xi$ for all $\xi>0$, we moreover have

$$-\int_\Omega \ln u(\cdot,t_0+T) \ge -\int_\Omega u(\cdot,t_0+T) \ge -m^\star,$$

whence in view of (5.4), (5.6) entails that

$$\begin{aligned}
\Big(\frac{r\chi^2}{8\mu}+\mu\Big)\cdot\int_{t_0}^{t_0+T}\int_\Omega u \quad &\ge \quad r|\Omega|T - L_0 - \Big(\frac{\chi^2}{8\mu}+1\Big)m^\star - \frac{\chi^2}{4}C_v\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big) \\
&\ge \quad \frac{r|\Omega|}{2}T
\end{aligned}$$

and therefore proves (5.5). □

This has an immediate consequence on the minimal size of the set of times at which in the above situation the integral $\int_\Omega u$ satisfies some fixed estimate from below.

Lemma 5.3 *Let (u,v) be a nonnegative global classical solution of (1.1), and suppose that $t_0 \ge 0$, $L_0 > 0$ and $T > 0$ are such that (5.3) and (5.4) hold. Then*

$$\left|\left\{t \in (t_0, t_0+T) \ \Big| \ \int_\Omega u(\cdot,t) \ge \eta\right\}\right| \ge \frac{\eta T}{m^\star} \tag{5.7}$$

is valid with

$$\eta := \min\left\{\frac{r|\Omega|}{\frac{r\chi^2}{2\mu} + 4\mu}, m^\star\right\}. \tag{5.8}$$

Proof. Writing $S := \{t \in (t_0, t_0+T) \mid \int_\Omega u(\cdot,t) \ge \eta\}$, we clearly have

$$\begin{aligned}\int_{t_0}^{t_0+T}\int_\Omega u &= \int_S\int_\Omega u + \int_{(t_0,t_0+T)\setminus S}\int_\Omega u \\ &\le m^\star|S| + \eta T.\end{aligned}$$

Thus, by means of Lemma 5.2 we can estimate

$$m^\star|S| \ge \frac{r|\Omega|}{\frac{r\chi^2}{4\mu} + 2\mu}\cdot T - \eta T,$$

which by definition of η is equivalent to (5.7). $\Box$

Upon another application of (2.4) we also obtain a quantitative information about the measure of the set of times when $\int_\Omega \frac{|\nabla u|^2}{u^2}$ is favorably large.

Lemma 5.4 *Assume that (u,v) is a nonnegative global classical solution of (1.1), and that $t_0 \ge 0$ and $L_0 > 0$ are such that (5.3) holds. Then for all $M > 0$ and $T > 0$ we have*

$$\begin{aligned}&\left|\left\{t \in (t_0, t_0+T) \ \Big| \ \int_\Omega \frac{|\nabla u(\cdot,t)|^2}{u^2(\cdot,t)} > M\right\}\right| \\ &\le \frac{(\frac{r\chi^2}{4\mu} + 2\mu)m^\star T + (\frac{\chi^2}{4\mu}+2)m^\star + \frac{\chi^2 C_v}{2}\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big) + 2L_0}{M},\end{aligned} \tag{5.9}$$

where $m^\star$ and C_v are as defined in (3.1) and (3.5), respectively.

Proof. We integrate (2.4) over $t \in (t_0, t_0+T)$ to see that

$$\begin{aligned}\frac{1}{2}\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} &\le \Big(\frac{r\chi^2}{8\mu} + \mu\Big)\cdot\int_{t_0}^{t_0+T}\int_\Omega u \\ &\quad + \int_\Omega \ln u(\cdot,t_0+T) - \frac{\chi^2}{8\mu}\int_\Omega u(\cdot,t_0+T) - \frac{\chi^2}{4}\int_\Omega v^2(\cdot,t_0+T) \\ &\quad - \int_\Omega \ln u(\cdot,t_0) + \frac{\chi^2}{8\mu}\int_\Omega u(\cdot,t_0) + \frac{\chi^2}{4}\int_\Omega v^2(\cdot,t_0),\end{aligned}$$

for we have assumed that $r>0$. Now using Lemma 3.1, Lemma 3.3 and (5.3) as well as the fact that u is nonnegative, from this we infer that

$$\frac{1}{2}\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} \le \Big(\frac{r\chi^2}{8\mu}+\mu\Big)m^\star T + m^\star + L_0 + \frac{\chi^2}{8\mu}m^\star + \frac{\chi^2}{4}C_v\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big), \tag{5.10}$$

again because $\int_\Omega \ln u(\cdot,t_0+T) \le \int_\Omega u(\cdot,t_0+T) \le m^\star$. Now since with

$$S := \Big\{ t\in (t_0,t_0+T) \ \Big| \ \int_\Omega \frac{|\nabla u(\cdot,t)|^2}{u^2(\cdot,t)} > M \Big\}$$

we have

$$\int_{t_0}^{t_0+T}\int_\Omega \frac{|\nabla u|^2}{u^2} \ge \int_S\int_\Omega \frac{|\nabla u|^2}{u^2} \ge M\cdot |S|,$$

(5.10) entails (5.9). $\square$

We can proceed to prove the main result of this section.

Proof of Lemma 5.1. We let

$$\eta := \min\Big\{ \frac{r|\Omega|}{\frac{r\chi^2}{2\mu}+4\mu}, m^\star \Big\} \tag{5.11}$$

be as in Lemma 5.3 and can then pick $K>0$ and $M>0$ fulfilling

$$\frac{rm^\star}{\mu K} < \frac{\eta}{4m^\star} \tag{5.12}$$

and

$$\frac{(\frac{r\chi^2}{4\mu}+2\mu)m^\star}{M} < \frac{\eta}{8m^\star}, \tag{5.13}$$

respectively, where $m^\star>0$ is the constant provided by Lemma 3.1. We then write

$$\delta := \frac{\eta}{2|\Omega|} \tag{5.14}$$

and

$$\varepsilon := \frac{\eta^2}{4K} \tag{5.15}$$

and let $C(\varepsilon)>0$ be as given by Lemma 4.2.
Our goal is to show that (5.2) can be achieved for an appropriate choice of $(t_k)_{k\in\mathbb{N}}\subset[0,\infty)$ satisfying $t_k\to\infty$ as $t\to\infty$ if we let

$$\overline{L} := \max\Big\{ L\,,\, -|\Omega|\cdot\ln\delta + \sqrt{C(\varepsilon)|\Omega|\cdot M} \Big\} \tag{5.16}$$

and fix any $T>0$ large enough satisfying

$$\frac{(\frac{\chi^2}{4\mu}+2)m^\star + \frac{\chi^2 C_v}{2}\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big) + 2\overline{L}}{M} < \frac{\eta T}{8m^\star} \tag{5.17}$$

and

$$\frac{m^\star}{\mu K} < \frac{\eta T}{4m^\star} \tag{5.18}$$

as well as

$$T > \frac{\overline{L} + (\frac{\chi^2}{8\mu} + 1)m^\star + \frac{\chi^2 C_v}{4}\Big(\int_\Omega v_0^2 + (m^\star)^2 + 1\Big)}{\frac{1}{2} r|\Omega|}, \tag{5.19}$$

with C_v being the constant found in Lemma 3.3.
To see this, we inductively define t_k by letting $t_1 := 0$ and claim that given $k \ge 1$ and $t_1, ..., t_k$ with the property that

$$\int_\Omega \ln u(\cdot, t_j) \ge -\overline{L} \tag{5.20}$$

for all $j \in \{1, ..., k\}$, we can find

$$t_{k+1} \in \Big(t_k + \frac{\eta T}{4m^\star}, t_k + T \Big)$$

such that (5.20) holds for $j = k + 1$.
For this purpose, we introduce the sets

$$\begin{aligned} S_1 &:= \Big\{ t \in (t_k, t_k + T) \ \Big| \ \int_\Omega u(\cdot, t) \ge \eta \Big\}, \\ S_2 &:= \Big\{ t \in (t_k, t_k + T) \ \Big| \ \int_\Omega u^2(\cdot, t) \le K \Big\} \qquad \text{and} \\ S_3 &:= \Big\{ t \in (t_k, t_k + T) \ \Big| \ \int_\Omega \frac{|\nabla u(\cdot, t)|^2}{u^2(\cdot, t)} \le M \Big\}, \end{aligned}$$

and our first objective is to make sure that

$$\Big| S_1 \cap S_2 \cap S_3 \cap \Big(t_k + \frac{\eta T}{4m^\star}, t_k + T \Big) \Big| > 0. \tag{5.21}$$

Indeed, since (5.19) and the validity of (5.20) for $j = k$ imply that (5.4) and (5.3) hold with $L_0 := \overline{L}$ and $t_0 := t_k$, we first conclude from Lemma 5.3 and our definition (5.11) of η that

$$|S_1| > \frac{\eta T}{m^\star}. \tag{5.22}$$

Next, thanks to the restrictions (5.12) and (5.18) on K and T we see from Lemma 3.4 applied to $t_0 := t_k$ that

$$\begin{aligned} |S_2| &\ge T - \frac{r m^\star}{\mu K} \cdot T - \frac{m^\star}{\mu K} \\ &> \Big(1 - \frac{\eta}{2m^\star} \Big) \cdot T, \end{aligned} \tag{5.23}$$

where we note that $(1-\frac{\eta}{2m^\star})\cdot T \ge \frac{1}{2}\cdot T>0$ due to the fact that $\eta\le m^\star$ according to (5.11). Finally, in light of (5.20), an application of Lemma 5.4 yields

$$\begin{aligned}|S_3| &= T-\bigg|\Big\{t\in(t_k,t_k+T)\ \Big|\ \int_\Omega \frac{|\nabla u(\cdot,t)|^2}{u^2(\cdot,t)}>M\Big\}\bigg| \\ &\ge T-\frac{(\frac{r\chi^2}{4\mu}+2\mu)m^\star}{M}\cdot T-\frac{(\frac{\chi^2}{4\mu}+2)m^\star+\frac{\chi^2 C_v}{2}\Big(\int_\Omega v_0^2+(m^\star)^2+1\Big)+2\overline{L}}{M},\end{aligned}$$

whence (5.13) and (5.17) assert that

$$|S_3|>\Big(1-\frac{\eta}{4m^\star}\Big)\cdot T.$$

Since (5.22) and (5.23) imply that

$$|S_1\cap S_2|=|S_1|+|S_2|-|S_1\cup S_2|>\frac{\eta T}{m^\star}+\Big(1-\frac{\eta}{2m^\star}\Big)\cdot T-T=\frac{\eta T}{2m^\star},$$

from this we obtain that

$$\begin{aligned}|(S_1\cap S_2)\cap S_3| &= |S_1\cap S_2|+|S_3|-|(S_1\cap S_2)\cup S_3| \\ &> \frac{\eta T}{2m^\star}+\Big(1-\frac{\eta}{4m^\star}\Big)\cdot T-T \\ &= \frac{\eta T}{4m^\star},\end{aligned}$$

which clearly entails (5.21).

We next claim that

$$\int_\Omega \ln u(\cdot,t)\ge -\overline{L} \qquad \text{for all } t\in S_1\cap S_2\cap S_3. \tag{5.24}$$

To verify this, given $t\in S_1\cap S_2\cap S_3$ we first use the inclusions $t\in S_1$ and $t\in S_2$ and apply Lemma 4.1 to infer that according to our definitions (5.14) and (5.15) of δ and ε we have

$$\Big|\{x\in\Omega\ |\ u(x,t)\ge\delta\}\Big|\ge\varepsilon.$$

Therefore, Lemma 4.3 asserts the inequality

$$\int_\Omega \ln u(\cdot,t)\ge|\Omega|\cdot\ln\delta-\sqrt{C(\varepsilon)|\Omega|\cdot\int_\Omega\frac{|\nabla u(\cdot,t)|^2}{u^2(\cdot,t)}},$$

so that the fact that t also belongs to S_3 allows us to conclude that

$$\int_\Omega \ln u(\cdot,t)\ge|\Omega|\cdot\ln\delta-\sqrt{C(\varepsilon)|\Omega|\cdot M}.$$

In view of the definition (5.16) of $\overline{L}$ this shows that (5.24) in fact is true, whereupon (5.20) follows for $j=k+1$ if we fix t_{k+1} to be any element of $(S_1\cap S_2\cap S_3)\cap(t_k+\frac{\eta T}{4m^\star},t_k+T)$. $\square$

6 Proof of Theorem 1.1

Combining Lemma 5.1 with Lemma 5.3 enables us to pass to the proof of our main result.

PROOF of Theorem 1.1. We first invoke Lemma 5.1 to obtain $T > 0, \overline{L} > 0$ and $(t_k)_{k\in\mathbb{N}} \subset [0, \infty)$ such that $t_k \to \infty$ as $k \to \infty$ and $t_k < t_{k+1} < t_k + T$ as well as $\int_\Omega \ln u(\cdot, t_k) \geq -\overline{L}$ for all $k \in \mathbb{N}$. Upon enlarging T if necessary we can thus achieve that the hypotheses (5.3) and (5.4) hold with $L_0 := \overline{L}$. We can therefore apply Lemma 5.3 with $t_0 := t_k$ to see that for each $k \in \mathbb{N}$ there exists $\tilde{t}_k \in (t_k, t_k + T)$ such that with η as defined in (5.8) we have

$$\int_\Omega u(\cdot, \tilde{t}_k) \geq \eta, \tag{6.1}$$

where we evidently may assume that $\tilde{t}_k < \tilde{t}_{k+1}$ for all $k \in \mathbb{N}$.
Now from Lemma 2.1 we know that

$$\frac{d}{dt}\int_\Omega u(\cdot, t) \leq r \int_\Omega u(\cdot, t) \qquad \text{for all } t > 0.$$

Along with (6.1), this shows that for any $k \in \mathbb{N}$,

$$\int_\Omega u(\cdot, t) \geq \Big(\int_\Omega u(\cdot, \tilde{t}_k)\Big) \cdot e^{-r(\tilde{t}_k - t)} \geq \eta\, e^{-r(\tilde{t}_k - t)} \qquad \text{for all } t \in [0, \tilde{t}_k), \tag{6.2}$$

and thereby directly proves that

$$\int_\Omega u(\cdot, t) \geq \eta\, e^{-r\tilde{t}_1} \qquad \text{for all } t \in [0, \tilde{t}_1). \tag{6.3}$$

As for large values of t, we note that the validity of the inequalities $t_k < t_{k-1} + T$ and $t_k < \tilde{t}_k < t_k + T$ for all $k \geq 2$ ensures that $\tilde{t}_k < \tilde{t}_{k-1} + 2T$ for all such k. Hence, (6.2) guarantees that

$$\int_\Omega u(\cdot, t) \geq \eta\, e^{-2rT} \qquad \text{for all } t \in [\tilde{t}_{k-1}, \tilde{t}_k)$$

whenever $k \geq 2$. In conjunction with (6.3) and the fact that $\tilde{t}_k > t_k \to \infty$ as $k \to \infty$, this shows that (1.3) holds if, for instance, we define $m_\star := \min\{\eta e^{-r\tilde{t}_1}, \eta e^{-2rT}\}$. □

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