Boundedness and decay enforced by quadratic degradation in a three-dimensional chemotaxis-fluid system

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Abstract

The coupled chemotaxis-fluid system

$$\begin{cases} n_t + u \cdot \nabla n &=& \Delta n - \nabla \cdot (n \nabla c) + rn - \mu n^2, \\ c_t + u \cdot \nabla c &=& \Delta c - c + n, \\ u_t + \nabla P &=& \Delta u + n \nabla \phi + g(x,t), \\ \nabla \cdot u &=& 0, \end{cases}$$

is considered under no-flux boundary conditions for n and c and no-slip boundary conditions for u in three-dimensional bounded domains with smooth boundary, where $r \geq 0$ and $\mu > 0$ are given constants and $\phi \in W^{1,\infty}(\Omega)$ and $g \in C^1(\bar{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty))$ are prescribed parameter functions.
It is shown that under the explicit condition $\mu \geq 23$ and suitable regularity assumptions on the initial data, the corresponding initial-boundary problem possesses a global classical solution which is bounded.
Apart from this, it is proved that if $r = 0$ then both $n(\cdot,t)$ and $c(\cdot,t)$ decay to zero with respect to the norm in $L^\infty(\Omega)$ as $t \to \infty$, and that if moreover $\int_0^\infty \int_\Omega |g|^2 < \infty$ then also $u(\cdot,t) \to 0$ in $L^\infty(\Omega)$ as $t \to \infty$.

Key words: chemotaxis, logistic source, Stokes, boundedness
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1 Introduction

Chemotaxis, the directed movement toward higher concentrations of a chemical signal, is known to be a mechanism of great significance for pattern formation in numerous biological contexts ([15], [9]). An outstanding role in the theoretical understanding of such processes is played by the classical Keller-Segel model ([12]) which despite its simple structure has turned out to yield quite an appropriate theoretical description of collective behavior of bacterial populations under the influence of a chemoattractant produced by the cells themselves ([9], [2]).

In various situations, however, the migration of bacteria is furthermore substantially affected by changes in their environment. For instance, experimental observations by Goldstein et al. ([25]) report pattern generation and spontaneous emergence of turbulence in populations of aerobic bacteria suspended in sessile drops of water. Another important example for a significant interaction of chemotactic movement with a surrounding environment is the phenomenon of broadcast spawning in which chemotaxis plays a crucial role for successful coral fertilization in enabling an effective mixing within a flowing fluid ([3], [16]).

It is the goal of the present work to study basic mathematical features of a simple model for chemotaxis-fluid interaction in cases when the evolution of the chemoattractant, as in the original Keller-Segel system, is essentially governed by production through cells. More precisely, we shall consider the Keller-Segel-Stokes system

$$\begin{cases} n_t + u\cdot\nabla n &= \Delta n - \nabla\cdot(n\nabla c) + rn - \mu n^2, & x\in\Omega,\ t>0,\\ c_t + u\cdot\nabla c &= \Delta c + n - c, & x\in\Omega,\ t>0,\\ u_t + \nabla P &= \Delta u + n\nabla\phi + g(x,t), & x\in\Omega,\ t>0,\\ \nabla\cdot u &= 0, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

for the unknown (n, c, u, P), where $r \geq 0$ and $\mu > 0$ are given parameters, and $\phi = \phi(x)$ and $g = g(x,t)$ are prescribed functions. Here n denotes the bacteria density, c represents the signal concentration, and u and P stand for the fluid velocity and the associated pressure. In (1.1) it is assumed that cell kinetics follows a logistic-type law determined by parameters r and μ, where allowing for the borderline case $r = 0$ we explicitly include cases when cell proliferation can be neglected. The model (1.1) moreover presupposes that in addition to the driving action of cells through buoyant forces within the gravitational field with potential ϕ, the motion of the fluid might be controlled by a given external force g. In particular, the system (1.1) thereby includes the outcomes of the modeling approaches for biomixing recently presented by Espejo and Suzuki ([5]) on the basis of an original work by Kiselev and Ryzhik ([13], [14]).

As far as we know, there are only two results which deal with chemotaxis-fluid interaction in the presence of a signal production mechanism. Espejo and Suzuki ([5]) proved global existence of certain weak solutions for the particular two-dimensional version of (1.1) obtained by letting $r = 0$ and $g = 0$. Very recently, the global existence and large time behavior of classical solutions to a corresponding more complex Keller-Segel-Navier-Stokes system has been investigated in more detail in this two-dimensional context ([22]).

As compared to this, the three-dimensional Keller-Segel-Stokes system (1.1) seems much less understood; to the best of our knowledge, not even a mere existence result seems available. The purpose of the present work is to firstly establish a result on global existence of classical solutions under reasonably mild assumptions, and to secondly provide some information on the large time behavior of solutions at least in cases when the dynamics in (1.1) is fairly simple. To state our results precisely, we specify the precise problem context by considering (1.1) along with the boundary conditions

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{for } x \in \partial\Omega \text{ and } t > 0, \tag{1.2}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

We shall assume throughout this paper that the initial data satisfy

$$\begin{cases} n_0 \in C^0(\bar{\Omega}), & n_0 > 0 \quad \text{in } \bar{\Omega}, \\ c_0 \in W^{1,\infty}(\Omega), & c_0 \geq 0 \quad \text{in } \bar{\Omega} \quad \text{and} \\ u_0 \in W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \text{ fulfilling } \nabla \cdot u_0 = 0, \end{cases} \tag{1.4}$$

and that the potential function ϕ and the forcing term g in (1.1) are suitably regular in fulfilling

$$\phi \in W^{1,\infty}(\Omega) \tag{1.5}$$

as well as

$$g \in C^1(\bar{\Omega} \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)). \tag{1.6}$$

Within this framework, our main result on global existence and boundedness of classical solutions to (1.1) is the following.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary, and suppose that $r \geq 0$ and that ϕ and g comply with (1.5) and (1.6). Then whenever $\mu \geq 23$, for any choice of n_0, c_0 and u_0 fulfilling (1.4), the problem (1.1)-(1.3) possesses a global classical solution (n, c, u, P) for which n, c and u are bounded in $\Omega \times (0,\infty)$ in the sense that there exists $C > 0$ fulfilling*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0. \tag{1.7}$$

Here we emphasize that Theorem 1.1 detects an explicit parameter condition $\mu \geq 23$ under which the suitable quadratic degradation of bacteria is sufficient to exclude any blow-up phenomenon, as known to occur when $\mu = 0$ even in the case of the corresponding Keller-Segel system obtained on letting $u \equiv 0$ ([29]). Apart from this, Theorem 1.1 moreover somewhat improves the knowledge on the latter chemotaxis-only system also in the case $\mu > 0$, for which, namely, up to now a corresponding boundedness result has been achieved only for suitably large values of μ beyond a certain number not explicitly known ([26]).

The limit case $r = 0$ becomes relevant when either the considered time scales are much smaller than those of cell proliferation, or when cells are a priori unable to reproduce themselves, such as e.g. in the model for broadcast spawning phenomena discussed in [13] and [5]. In such situations, the solution constructed in Theorem 1.1 enjoys the following decay properties.

Theorem 1.2 *Assume that $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, that ϕ, g and (n_0, c_0, u_0) satisfy (1.5), (1.6) and (1.4).*
i) If $\mu \geq 23$ and $r = 0$, then the global classical solution (n, c, u, P) of (1.1)-(1.3) constructed in Theorem 1.1) has the property that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \textit{and} \quad \|c(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{1.8}$$

ii) If $\mu \geq 23$ and $r = 0$ as well as

$$\int_0^\infty \int_\Omega |g(x,t)|^2 dx dt < \infty,$$

then the above solution moreover satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{1.9}$$

We do not pursue here in detail down to which critical value $\mu_0 \geq 0$ the number $\mu = 23$ can possibly decreased. The goal of this work is rather to provide *some* explicit condition on μ sufficient for the properties detected above. Accordingly, the interesting open problem of determining μ_0 will be left for future research, possibly within a more general framework allowing also further crucial model parameters such as the chemotactic sensitivity to vary.

Before going into details, let us briefly mention that there exist some works addressing qualitative properties of coupled chemotaxis-fluid systems in which the signal is consumed by bacteria, rather than produced by cells (cf. [4], [28], [30] and [31], for instance). Since due to the dampening effect of signal absorption, the structure of these systems is somewhat different from that of (1.1) with signal production, the mathematical approaches already developed for these chemotaxis-fluid systems can apparently not be applied to (1.1).

A key role in our approach will be played by an analysis of the coupled functional

$$y(t) := 6\int_\Omega n^2(\cdot,t) + \int_\Omega |\nabla c(\cdot,t)|^4 + \int_\Omega n(\cdot,t)|\nabla c(\cdot,t)|^2, \qquad t > 0.$$

Indeed, on the basis of some elementary integral estimates essentially gained due to the pure existence of the quadratic dampening term $-\mu n^2$ in (1.1) (see Section 2), this functional can be shown to enjoy an entropy-like property, provided that $\mu \geq 23$ (Lemma 3.5). Suitable bootstrap arguments thereafter provide higher regularity properties which shall firstly yield the global existence and boundedness result from Theorem 1.1, and which will secondly allow for proving Theorem 1.2 by turning some basic decay information (Lemma 5.1 and Lemma 5.2) into the uniform convergence properties (1.8) and (1.9) in the respective situations.

We close this introduction with a remark that the full space problem corresponding to Theorem 1.2 remains untouched here, because for analyzing the asymptotic behavior it seems that different mathematical techniques are required.

2 Preliminaries

In the sequel, we let $A = -\mathcal{P}\Delta$ denote the realization of the Stokes operator in $L^2_\sigma(\Omega)$ with domain $D(A) := W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$, where $\mathcal{P}$ denotes the Helmholtz projection mapping $L^2(\Omega)$ onto its subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla\cdot\varphi = 0 \text{ in } \mathcal{D}'(\Omega)\}$ of $L^2(\Omega)$ of all solenoidal vector fields. For $\alpha \in (0,1)$, we let A^α denote the corresponding closed fractional power of A ([19]).

By a straightforward adaptation of the reasoning in [28, Lemma 2.1], one can derive the following basic statement on local solvability and extensibility of solutions to (1.1)-(1.3).

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary and let $\mu > 0$. Suppose that (1.5) and (1.6) hold, and that n_0, c_0 and u_0 satisfy (1.4). Then there exist $T_{max} \in (0,\infty]$ and a classical solution (n,c,u,P) of (1.1)-(1.3) in $\Omega\times(0,T_{max})$ such that $n>0$ and $c>0$ in $\bar\Omega\times(0,T_{max})$, that*

$$\begin{cases} n \in C^0(\bar\Omega\times[0,T_{max}))\cap C^{2,1}(\bar\Omega\times(0,T_{max})), \\ c \in C^0(\bar\Omega\times[0,T_{max}))\cap C^{2,1}(\bar\Omega\times(0,T_{max})) \quad \textit{and} \\ u \in C^0(\bar\Omega\times[0,T_{max}))\cap C^{2,1}(\bar\Omega\times(0,T_{max})), \end{cases}$$

and such that

$$\begin{aligned} &\textit{either } T_{max}=\infty, \textit{ or} \\ &\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \to \infty \textit{ for all } \alpha\in(\tfrac34,1) \quad \textit{as } t\nearrow T_{max}. \end{aligned} \tag{2.1}$$

The following preliminary boundedness property immediately results from integration in the first equation in (1.1).

Lemma 2.2 *There exist $m>0$ and $C>0$ such that the solution of (1.1)-(1.3) satisfies*

$$\int_\Omega n(\cdot,t) \le m \qquad \textit{for all } t\in(0,T_{max}) \tag{2.2}$$

and

$$\int_t^{t+\tau}\int_\Omega n^2 \le C \qquad \textit{for all } t\in(0,T_{max}-\tau), \tag{2.3}$$

where

$$\tau := \min\Big\{1\,,\,\frac12 T_{max}\Big\}. \tag{2.4}$$

Proof. From integration of the first equation in (1.1) we obtain

$$\frac{d}{dt}\int_\Omega n = r\int_\Omega n - \mu\int_\Omega n^2 \qquad \text{for all } t\in(0,T_{max}) \tag{2.5}$$

and hence

$$\frac{d}{dt}\int_\Omega n \le r\int_\Omega n - \frac{\mu}{|\Omega|}\Big(\int_\Omega n\Big)^2 \qquad \text{for all } t\in(0,T_{max}),$$

because $\int_\Omega n^2 \ge \frac{1}{|\Omega|}(\int_\Omega n)^2$ for all $t\in(0,T_{max})$ by the Cauchy-Schwarz inequality. This implies (2.2), whereafter (2.3) can be obtained upon a time integration of (2.5). □

In our subsequent analysis we shall make use of the following auxiliary statement on a boundedness property in an ODI. For its elementary proof we refer to [20, Lemma 3.4] where the particular case $\tau = 1$ is detailed.

Lemma 2.3 *Let $T > 0$, $\tau \in (0, T)$, $a > 0$ and $b > 0$, and suppose that $y : [0, T) \to [0, \infty)$ is absolutely continuous and such that*

$$y'(t) + ay(t) \le h(t) \qquad \text{for a.e. } t \in (0, T)$$

with some nonnegative function $h \in L^1_{loc}([0, T))$ satisfying

$$\int_t^{t+\tau} h(s)ds \le b \qquad \text{for all } t \in [0, T - \tau).$$

Then

$$y(t) \le \max\left\{ y(0) + b \, , \frac{b}{a\tau} + 2b \right\} \qquad \text{for all } t \in (0, T).$$

As a consequence of Lemma 2.2, let us establish two fundamental estimates associated with u.

Lemma 2.4 *There exists $C > 0$ such that*

$$\int_\Omega |\nabla u|^2 \le C \qquad \text{for all } t \in (0, T_{max}) \tag{2.6}$$

and

$$\int_t^{t+\tau} \int_\Omega |Au|^2 \le C \qquad \text{for all } t \in (0, T_{max} - \tau), \tag{2.7}$$

where $\tau = \min\{1 \, , \, \frac{1}{2} T_{max}\}$ is as in (2.4).

Proof. We apply the Helmholtz projection $\mathcal{P}$ to both sides of the third equation in (1.1) and test the resulting identity by Au. Since A is self-adjoint and $\mathcal{P}$ is an orthogonal projector, by means of Young's inequality we can thereby estimate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac{1}{2}} u|^2 + \int_\Omega |Au|^2 &= \int_\Omega Au \cdot \mathcal{P}[n\nabla\phi + g] \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \frac{1}{2}\int_\Omega |\mathcal{P}[n\nabla\phi + g]|^2 \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \int_\Omega |n\nabla\phi|^2 + \int_\Omega |g|^2 \\
&\le \frac{1}{2}\int_\Omega |Au|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)} \int_\Omega n^2 + \int_\Omega |g|^2 \quad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{2.8}$$

Using that $\int_\Omega |A^{\frac{1}{2}} u|^2 = \int_\Omega |\nabla u|^2$ ([19, p. 133, Lemma a)]) and that with some $C_1 > 0$ we have

$$\int_\Omega |\nabla u|^2 \le C_1 \int_\Omega |Au|^2$$

for all $t \in (0, T_{max})$ thanks to the fact that $\|\cdot\|_{W^{2,2}(\Omega)}$ and $\|A(\cdot)\|_{L^2(\Omega)}$ are equivalent on $D(A)$ ([19, p. 129, Theorem e)]), we see that for $y(t) := \int_\Omega |\nabla u(\cdot, t)|^2$, $t \in [0, T_{max})$ and $h(t) := 2\|\nabla\phi\|^2_{L^\infty(\Omega)} \int_\Omega n^2(\cdot, t) + 2\int_\Omega |g(\cdot, t)|^2$, $t \in (0, T_{max})$, (2.8) implies the inequality

$$y'(t) + \frac{1}{2C_1} y(t) + \frac{1}{2}\int_\Omega |Au|^2 \le h(t) \qquad \text{for all } t \in (0, T_{max}). \tag{2.9}$$

As from Lemma 2.2 and our boundedness assumptions on g it is clear that there exists $C_2 > 0$ such that $\int_t^{t+\tau} h(s)ds \le C_2$ for all $t \in (0, T_{max}-\tau)$, thanks to Lemma 2.3 this firstly entails (2.6), whereafter (2.7) follows by integrating (2.9) in time. $\square$

By straightforward interpolation, the latter implies the following estimate on u which will play a crucial role in obtaining a bound for $\int_\Omega n^2$ and $\int_\Omega |\nabla c|^4$ in Lemma 3.5.

Lemma 2.5 *There exists $C > 0$ such that*

$$\int_t^{t+\tau} \int_\Omega |u|^{10} \le C \qquad \textit{for all } t \in (0, T_{max} - \tau), \tag{2.10}$$

where $\tau = \min\{1, \frac{1}{2}T_{max}\}$ is as in (2.4).

PROOF. We interpolate using the Gagliardo-Nirenberg inequality and the well-known fact that $\|A(\cdot)\|_{L^2(\Omega)}$ defines a norm equivalent to $\|\cdot\|_{W^{2,2}(\Omega)}$ on $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ ([19, p. 129, Theorem e)]) to find $C_1 > 0$ fulfilling

$$\int_t^{t+\tau} \|u(\cdot,s)\|_{L^{10}(\Omega)}^{10} ds \le C_1 \int_t^{t+\tau} \|Au(\cdot,s)\|_{L^2(\Omega)}^2 \|u(\cdot,s)\|_{L^6(\Omega)}^8 ds \qquad \text{for all } t \in (0, T_{max} - \tau).$$

Since the embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ warrants the existence of $C_2 > 0$ such that $\|u(\cdot,s)\|_{L^6(\Omega)} \le C_2 \|\nabla u(\cdot,s)\|_{L^2(\Omega)}$ for all $s \in (0, T_{max})$, (2.10) therefore is a consequence of Lemma 2.4. $\square$

The above estimates on n and u also entail some basic regularity properties of c.

Lemma 2.6 *One can find $C > 0$ such that*

$$\int_\Omega c \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{2.11}$$

and

$$\int_\Omega |\nabla c|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{2.12}$$

PROOF. We first integrate the second equation in (1.1) over Ω and recall (2.2) to obtain

$$\frac{d}{dt}\int_\Omega c + \int_\Omega c = \int_\Omega n \le m \qquad \text{for all } t \in (0, T_{max}),$$

from which (2.11) immediately follows.

We next multiply the second equation in (1.1) by $-\Delta c$, integrate by parts and use Young's inequality and the Hölder inequality to see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 + \int_\Omega |\nabla c|^2 &= -\int_\Omega n\Delta c + \int_\Omega (u \cdot \nabla c)\Delta c \\ &\le \frac{1}{2}\int_\Omega |\Delta c|^2 + \int_\Omega n^2 + \int_\Omega |u \cdot \nabla c|^2 \\ &\le \frac{1}{2}\int_\Omega |\Delta c|^2 + \int_\Omega n^2 + \|u\|_{L^{10}(\Omega)}^2 \|\nabla c\|_{L^{\frac{5}{2}}(\Omega)}^2 \end{aligned} \tag{2.13}$$

for all $t \in (0, T_{max})$. Here the Gagliardo-Nirenberg inequality and standard elliptic regularity theory combined with (2.11) provide $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned}\int_\Omega |\nabla c|^{\frac{5}{2}} &\leq C_1 \|\Delta c\|^2_{L^2(\Omega)} \|c\|^{\frac{1}{2}}_{L^1(\Omega)} + C_1 \|c\|^{\frac{5}{2}}_{L^1(\Omega)} \\ &\leq C_2 \Big\{ \int_\Omega |\Delta c|^2 + 1 \Big\} \qquad \text{for all } t \in (0, T_{max}),\end{aligned}$$

whence again by Young's inequality we find $C_3 > 0$ fulfilling

$$\begin{aligned}\|u\|^2_{L^{10}(\Omega)} \|\nabla c\|^2_{L^{\frac{5}{2}}(\Omega)} &\leq \|u\|^2_{L^{10}(\Omega)} \cdot C_2^{\frac{4}{5}} \Big\{ \int_\Omega |\Delta c|^2 + 1 \Big\}^{\frac{4}{5}} \\ &\leq \frac{1}{2} \int_\Omega |\Delta c|^2 + C_3 \int_\Omega |u|^{10} + C_3 \qquad \text{for all } t \in (0, T_{max}).\end{aligned}$$

Therefore, (2.13) implies that for the functions given by $y(t) := \int_\Omega |\nabla c(\cdot, t)|^2$, $t \in [0, T_{max})$, and $h(t) := 2\int_\Omega n^2(\cdot, t) + 2C_3 \int_\Omega |u(\cdot, t)|^{10} + 2C_3$, $t \in (0, T_{max})$, we have

$$y'(t) + 2y(t) \leq h(t) \qquad \text{for all } t \in (0, T_{max}).$$

Since Lemma 2.2 and Lemma 2.5 provide $C_4 > 0$ such that $\int_t^{t+\tau} h(s)ds \leq C_4$ for all $t \in (0, T_{max} - \tau)$ with $\tau = \min\{1, \frac{1}{2}T_{max}\}$, in conjunction with Lemma 2.3 this establishes (2.12). □

3 Estimating $\int_\Omega n^2$ and $\int_\Omega |\nabla c|^4$

This section contains the main step of our analysis, to be achieved in Lemma 3.5, by establishing an estimate for a certain linear combination of the functionals $\int_\Omega n^2$, $\int_\Omega |\nabla c|^4$ and $\int_\Omega n|\nabla c|^2$. As a starting point, let derive differential inequalities for the two uncoupled of these functionals.

Lemma 3.1 *The solution of (1.1) from Lemma 2.1 satisfies*

$$\frac{d}{dt}\int_\Omega n^2 + \int_\Omega |\nabla n|^2 \leq \int_\Omega n^2 |\nabla c|^2 + 2r \int_\Omega n^2 - 2\mu \int_\Omega n^3 \qquad \textit{for all } t \in (0, T_{max}). \tag{3.1}$$

PROOF. Since $\nabla \cdot u \equiv 0$, testing the first equation in (1.1) against n yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega n^2 = -\int_\Omega |\nabla n|^2 + \int_\Omega n \nabla n \cdot \nabla c + r\int_\Omega n^2 - \mu \int_\Omega n^3$$

for all $t \in (0, T_{max})$, which implies (3.1) due to the fact that by Young's inequality,

$$\int_\Omega n \nabla n \cdot \nabla c \leq \frac{1}{2}\int_\Omega |\nabla n|^2 + \frac{1}{2}\int_\Omega n^2 |\nabla c|^2$$

for all $t \in (0, T_{max})$. □

Lemma 3.2 *We have*

$$\frac{d}{dt}\int_\Omega |\nabla c|^4 + \int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 \le 14\int_\Omega n^2|\nabla c|^2 + 14\int_\Omega |u|^2|\nabla c|^4 + 2\int_{\partial\Omega} |\nabla c|^2 \frac{\partial|\nabla c|^2}{\partial\nu} \qquad \textit{for all } t\in(0,T_{max}). \tag{3.2}$$

PROOF. Using that $\nabla c\cdot\nabla\Delta c = \frac{1}{2}\Delta|\nabla c|^2 - |D^2c|^2$, by a straightforward computation using the second equation in (1.1) and several integrations by parts we find that

$$\begin{aligned}
\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla c|^4 &= \int_\Omega |\nabla c|^2\nabla c\cdot\nabla(\Delta c - c + n - u\cdot\nabla c)\\
&= \frac{1}{2}\int_\Omega |\nabla c|^2\Delta|\nabla c|^2 - \int_\Omega |\nabla c|^2|D^2c|^2 - \int_\Omega |\nabla c|^4\\
&\quad - \int_\Omega n\nabla\cdot(|\nabla c|^2\nabla c) + \int_\Omega (u\cdot\nabla c)\nabla\cdot(|\nabla c|^2\nabla c)\\
&= -\frac{1}{2}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla c|^2\frac{\partial|\nabla c|^2}{\partial\nu} - \int_\Omega |\nabla c|^2|D^2c|^2 - \int_\Omega |\nabla c|^4\\
&\quad - \int_\Omega n|\nabla c|^2\Delta c - \int_\Omega n\nabla c\cdot\nabla|\nabla c|^2\\
&\quad + \int_\Omega (u\cdot\nabla c)|\nabla c|^2\Delta c + \int_\Omega (u\cdot\nabla c)\nabla c\cdot\nabla|\nabla c|^2 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{3.3}$$

Here since $|\Delta c| \le \sqrt{3}|D^2c|$, by Young's inequality we can estimate

$$\begin{aligned}
-\int_\Omega n|\nabla c|^2\Delta c &\le \sqrt{3}\int_\Omega n|\nabla c|^2|D^2c|\\
&\le \frac{1}{2}\int_\Omega |\nabla c|^2|D^2c|^2 + \frac{3}{2}\int_\Omega n^2|\nabla c|^2
\end{aligned}$$

and, similarly,

$$\begin{aligned}
\int_\Omega (u\cdot\nabla c)|\nabla c|^2\Delta c &\le \frac{1}{2}\int_\Omega |\nabla c|^2|D^2c|^2 + \frac{3}{2}\int_\Omega |u\cdot\nabla c|^2|\nabla c|^2\\
&\le \frac{1}{2}\int_\Omega |\nabla c|^2|D^2c|^2 + \frac{3}{2}\int_\Omega |u|^2|\nabla c|^4
\end{aligned}$$

for all $t\in(0,T_{max})$. As moreover

$$-\int_\Omega n\nabla c\cdot\nabla|\nabla c|^2 \le \frac{1}{8}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + 2\int_\Omega n^2|\nabla c|^2$$

and

$$\begin{aligned}
\int_\Omega (u\cdot\nabla c)\nabla c\cdot\nabla|\nabla c|^2 &\le \frac{1}{8}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + 2\int_\Omega |u\cdot\nabla c|^2|\nabla c|^2\\
&\le \frac{1}{8}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + 2\int_\Omega |u|^2|\nabla c|^4
\end{aligned}$$

for all $t \in (0, T_{max})$, from (3.3) we infer that

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla c|^4 &\le -\frac{1}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{2}\int_{\partial\Omega} |\nabla c|^2 \frac{\partial|\nabla c|^2}{\partial\nu} - \int_\Omega |\nabla c|^4 \\ &\quad + \frac{7}{2}\int_\Omega n^2|\nabla c|^2 + \frac{7}{2}\int_\Omega |u|^2|\nabla c|^4 \qquad \text{for all } t \in (0, T_{max}),\end{aligned}$$

which on dropping a nonpositive term on its right-hand side yields (3.2). □

In order to cope with the first integrals on the right-hand sides of (3.1) and (3.2), we shall additionally make use of a differential inequality related to $\int_\Omega n|\nabla c|^2$.

Lemma 3.3 *For all $t \in (0, T_{max})$, we have*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega n|\nabla c|^2 + (\mu - 3)\int_\Omega n^2|\nabla c|^2 \\ &\quad \le 5\int_\Omega |\nabla n|^2 + \frac{3}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + (r-2)\int_\Omega n|\nabla c|^2 \\ &\qquad + \frac{5}{4}\int_\Omega |u|^2|\nabla c|^4 + \frac{9}{16}\int_\Omega |u|^4|\nabla c|^2 \\ &\qquad + \int_{\partial\Omega} n\frac{\partial|\nabla c|^2}{\partial\nu}. \end{aligned} \tag{3.4}$$

PROOF. We differentiate the integral on the left of (3.4) using the first two equations in (1.1) to see upon integrating by parts and employing the identity $\nabla c \cdot \nabla\Delta c = \frac{1}{2}\Delta|\nabla c|^2 - |D^2 c|^2$ that

$$\begin{aligned}\frac{d}{dt}\int_\Omega n|\nabla c|^2 &= \int_\Omega |\nabla c|^2\Big\{\Delta n - \nabla\cdot(n\nabla c) + rn - \mu n^2 - u\cdot\nabla n\Big\} \\ &\quad + 2\int_\Omega n\nabla c\cdot\nabla\Big\{\Delta c - c + n - u\cdot\nabla c\Big\} \\ &= \int_\Omega |\nabla c|^2\Delta n - \int_\Omega |\nabla c|^2\nabla\cdot(n\nabla c) + r\int_\Omega n|\nabla c|^2 - \mu\int_\Omega n^2|\nabla c|^2 - \int_\Omega (u\cdot\nabla n)|\nabla c|^2 \\ &\quad + \int_\Omega n\Delta|\nabla c|^2 - 2\int_\Omega n|D^2c|^2 - 2\int_\Omega n|\nabla c|^2 + 2\int_\Omega n\nabla n\cdot\nabla c - 2\int_\Omega n\nabla c\cdot\nabla(u\cdot\nabla c) \\ &= -2\int_\Omega \nabla n\cdot\nabla|\nabla c|^2 + \int_\Omega n\nabla c\cdot\nabla|\nabla c|^2 + (r-2)\int_\Omega n|\nabla c|^2 \\ &\quad - \mu\int_\Omega n^2|\nabla c|^2 - \int_\Omega (u\cdot\nabla n)|\nabla c|^2 \\ &\quad + \int_{\partial\Omega} n\frac{\partial|\nabla c|^2}{\partial\nu} - 2\int_\Omega n|D^2c|^2 + 2\int_\Omega n\nabla n\cdot\nabla c \\ &\quad + 2\int_\Omega (u\cdot\nabla c)\nabla n\cdot\nabla c + 2\int_\Omega (u\cdot\nabla c)n\Delta c \qquad \text{for all } t \in (0, T_{max}).\end{aligned} \tag{3.5}$$

Here by Young's inequality,

$$-2\int_\Omega \nabla n\cdot\nabla|\nabla c|^2 \le 2\int_\Omega |\nabla n|^2 + \frac{1}{2}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 \tag{3.6}$$

and

$$\int_\Omega n\nabla c\cdot\nabla|\nabla c|^2 \le \int_\Omega n^2|\nabla c|^2 + \frac{1}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 \tag{3.7}$$

and

$$-\int_\Omega (u\cdot\nabla n)|\nabla c|^2 \le \int_\Omega |\nabla n|^2 + \frac{1}{4}\int_\Omega |u|^2|\nabla c|^4 \tag{3.8}$$

as well as

$$2\int_\Omega n\nabla n\cdot\nabla c \le \int_\Omega |\nabla n|^2 + \int_\Omega n^2|\nabla c|^2 \tag{3.9}$$

and

$$2\int_\Omega (u\cdot\nabla c)\nabla n\cdot\nabla c \le \int_\Omega |\nabla n|^2 + \int_\Omega |u|^2|\nabla c|^4 \tag{3.10}$$

for all $t\in(0,T_{max})$. Since finally two more applications of Young's inequality along with the pointwise estimate $|\Delta c|\le\sqrt{3}|D^2c|$ show that

$$\begin{aligned}
2\int_\Omega (u\cdot\nabla c)n\Delta c &\le 2\sqrt{3}\int_\Omega |u\cdot\nabla c|n|D^2c| \\
&\le 2\int_\Omega n|D^2c|^2 + \frac{3}{2}\int_\Omega n|u|^2|\nabla c|^2 \\
&\le 2\int_\Omega n|D^2c|^2 + \int_\Omega n^2|\nabla c|^2 + \frac{9}{16}\int_\Omega |u|^4|\nabla c|^2 \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

in view of (3.6)-(3.10) the identity (3.5) readily implies (3.4). □

Properly combining Lemmata 3.1-3.3, we arrive at the following.

Corollary 3.4 *We have*

$$\begin{aligned}
\frac{d}{dt}\Big\{6\int_\Omega n^2 + \int_\Omega |\nabla c|^4 + \int_\Omega n|\nabla c|^2\Big\} &+ \int_\Omega |\nabla n|^2 + \frac{1}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 \\
&+(\mu-23)\int_\Omega n^2|\nabla c|^2 + 12\mu\int_\Omega n^3 \\
&\le 12r\int_\Omega n^2 + (r-2)\int_\Omega n|\nabla c|^2 + \frac{61}{4}\int_\Omega |u|^2|\nabla c|^4 + \frac{9}{16}\int_\Omega |u|^4|\nabla c|^2 \\
&+2\int_{\partial\Omega} |\nabla c|^2\frac{\partial|\nabla c|^2}{\partial\nu} + \int_{\partial\Omega} n\frac{\partial|\nabla c|^2}{\partial\nu}
\end{aligned} \tag{3.11}$$

for all $t\in(0,T_{max})$.

Proof. We only need to take an evident linear combination of the inequalities provided by Lemma 3.1, Lemma 3.2 and Lemma 3.3. □

Here it turns out that if μ is suitably large, then in (3.11), all integrals on the right can adequately be estimated in terms of the respective dissipated quantities on the left, in consequence implying the following main result of this section.

Lemma 3.5 *Suppose that $\mu \geq 23$. Then there exists $C > 0$ such that*

$$\int_\Omega n^2(\cdot,t) \leq C \qquad \textit{for all } t \in (0,T_{max}) \tag{3.12}$$

and

$$\int_\Omega |\nabla c(\cdot,t)|^4 \leq C \qquad \textit{for all } t \in (0,T_{max}) \tag{3.13}$$

PROOF. Since $\mu \geq 23$, Corollary 3.4 implies that

$$y(t) := 6\int_\Omega n^2(\cdot,t) + \int_\Omega |\nabla c(\cdot,t)|^4 + \int_\Omega n(\cdot,t)|\nabla c(\cdot,t)|^2, \qquad t \in [0,T_{max}),$$

satisfies

$$\begin{aligned} y'(t) + y(t) &+ \int_\Omega |\nabla n|^2 + \frac{1}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + 12\mu\int_\Omega n^3 \\ &\leq (12r+6)\int_\Omega n^2 + \int_\Omega |\nabla c|^4 + (r-1)\int_\Omega n|\nabla c|^2 + \frac{61}{4}\int_\Omega |u|^2|\nabla c|^4 + \frac{9}{16}\int_\Omega |u|^4|\nabla c|^2 \\ &\quad + 2\int_{\partial\Omega} |\nabla c|^2 \frac{\partial|\nabla c|^2}{\partial\nu} + \int_{\partial\Omega} n\frac{\partial|\nabla c|^2}{\partial\nu} \end{aligned} \tag{3.14}$$

In order to take full advantage of the dissipated quantities appearing on the left-hand side herein, we first invoke the Gagliardo-Nirenberg inequality which provides $C_1 > 0$ such that

$$\Big\||\nabla c|^2\Big\|_{L^{\frac{8}{3}}(\Omega)}^{\frac{8}{3}} \leq C_1\Big\|\nabla|\nabla c|^2\Big\|_{L^2(\Omega)}^2 \Big\||\nabla c|^2\Big\|_{L^1(\Omega)}^{\frac{2}{3}} + C_1\Big\||\nabla c|^2\Big\|_{L^1(\Omega)}^{\frac{8}{3}} \qquad \text{for all } t \in (0,T_{max}).$$

Since from Lemma 2.6 we know that

$$\int_\Omega |\nabla c|^2 \leq C_2 \qquad \text{for all } t \in (0,T_{max}) \tag{3.15}$$

with some $C_2 \geq 1$, this shows that

$$\int_\Omega |\nabla c|^{\frac{16}{3}} \leq C_1 C_2^{\frac{2}{3}} \int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + C_1 C_2^{\frac{8}{3}}$$

and hence

$$\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 \geq C_3\int_\Omega |\nabla c|^{\frac{16}{3}} - 1 \qquad \text{for all } t \in (0,T_{max}), \tag{3.16}$$

where $C_3 := C_1^{-1} C_2^{-\frac{8}{3}}$. Using Young's inequality, on the right-hand side of (3.14) we can therefore estimate

$$(r-1)\int_\Omega n|\nabla c|^2 \leq \frac{(r-1)_+^2}{4}\int_\Omega n^2 + \int_\Omega |\nabla c|^4$$

and hence, with some $C_4 > 0$,

$$
\begin{aligned}
(r-1)\int_\Omega n|\nabla c|^2 + \int_\Omega |\nabla c|^4 + (12r+6)\int_\Omega n^2 &\le \Big\{\frac{(r-1)_+^2}{4} + 12r+6\Big\}\int_\Omega n^2 + 2\int_\Omega |\nabla c|^4 \\
&\le 12\mu\int_\Omega n^3 + \frac{C_3}{8}\int_\Omega |\nabla c|^{\frac{16}{3}} + C_4 \\
&\le 12\mu\int_\Omega n^3 + \frac{1}{8}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{8} + C_4 \qquad (3.17)
\end{aligned}
$$

for all $t \in (0, T_{max})$, and similarly, again by means of Young's inequality and (3.16) we obtain positive constants C_5 and C_6 fulfilling

$$
\begin{aligned}
\frac{61}{4}\int_\Omega |u|^2|\nabla c|^4 &\le \frac{C_3}{16}\int_\Omega |\nabla c|^{\frac{16}{3}} + C_5\int_\Omega |u|^8 \\
&\le \frac{1}{16}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{16} + C_5\int_\Omega |u|^{10} + C_5|\Omega| \qquad (3.18)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{9}{16}\int_\Omega |u|^4|\nabla c|^2 &\le \frac{C_3}{32}\int_\Omega |\nabla c|^{\frac{16}{3}} + C_6\int_\Omega |u|^{\frac{32}{5}} \\
&\le \frac{1}{32}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \frac{1}{32} + C_6\int_\Omega |u|^{10} + C_6|\Omega| \qquad (3.19)
\end{aligned}
$$

for all $t \in (0, T_{max})$. In estimating the boundary integrals in (3.14), we make use of the one-sided pointwise inequality

$$
\frac{\partial|\nabla c|^2}{\partial\nu} \le C_7|\nabla c|^2 \qquad \text{for all } x \in \partial\Omega \text{ and } t \in (0, T_{max}),
$$

valid with some $C_7 > 0$ due to the fact that $\frac{\partial c}{\partial\nu} = 0$ on $\partial\Omega$ ([17]). We moreover recall the boundary trace embedding $W^{\frac{1}{2},2}(\Omega) \hookrightarrow L^2(\partial\Omega)$ in finding $C_8 > 0$ such that

$$
\|\varphi\|_{L^2(\partial\Omega)} \le C_8\|\varphi\|_{W^{\frac{1}{2},2}(\Omega)} \qquad \text{for all } \varphi \in W^{\frac{1}{2},2}(\Omega),
$$

which by Ehrling's lemma, since $W^{1,2}(\Omega) \hookrightarrow\hookrightarrow W^{\frac{1}{2},2}(\Omega) \hookrightarrow L^1(\Omega)$, entails that for each $\delta > 0$ one can pick $C_9(\delta) > 0$ such that

$$
\int_{\partial\Omega}\varphi^2 \le \delta\int_\Omega |\nabla\varphi|^2 + C_9(\delta)\Big(\int_\Omega |\varphi|\Big)^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega).
$$

Applying this first to $\delta = \delta_1 := \frac{1}{96C_7}$ and then to $\delta = \delta_2 := \frac{4}{C_7}$, thanks to Young's inequality, (3.15) and (2.2) this enables us to estimate the two rightmost summands in (3.14) according to

$$
2\int_{\partial\Omega}|\nabla c|^2\frac{\partial|\nabla c|^2}{\partial\nu} + \int_{\partial\Omega} n\frac{\partial|\nabla c|^2}{\partial\nu} \le 2C_7\int_{\partial\Omega}|\nabla c|^4 + C_7\int_{\partial\Omega} n|\nabla c|^2
$$

$$
\begin{aligned}
&\leq 3C_7 \int_{\partial\Omega} |\nabla c|^4 + \frac{C_7}{4} \int_{\partial\Omega} n^2 \\
&\leq \frac{1}{32} \int_\Omega \Big|\nabla |\nabla c|^2\Big|^2 + 3C_7C_9(\delta_1)\Big(\int_\Omega |\nabla c|^2\Big)^2 \\
&\quad + \int_\Omega |\nabla n|^2 + \frac{C_7C_9(\delta_2)}{4}\Big(\int_\Omega n\Big)^2 \\
&\leq \frac{1}{32} \int_\Omega \Big|\nabla |\nabla c|^2\Big|^2 + \int_\Omega |\nabla n|^2 + C_{10} \qquad (3.20)
\end{aligned}
$$

for all $t \in (0, T_{max})$, where $C_{10} := 3C_2^2 C_7 C_9(\delta_1) + \frac{C_7C_9(\delta_2)m^2}{4}$.
In summary, from (3.14) and (3.17)-(3.20) we infer that

$$
\begin{aligned}
y'(t) + y(t) &+ \frac{1}{4}\int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 + \int_\Omega |\nabla n|^2 + 12\mu \int_\Omega n^3 \\
&\leq \Big\{\frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{32}\Big\} \int_\Omega \Big|\nabla|\nabla c|^2\Big|^2 \\
&\quad + \int_\Omega |\nabla n|^2 + 12\mu \int_\Omega n^3 \\
&\quad + (C_5 + C_6)\int_\Omega |u|^{10} + C_{11} \qquad \text{for all } t \in (0, T_{max})
\end{aligned}
$$

and hence

$$
y'(t) + y(t) \leq (C_5 + C_6)\int_\Omega |u|^{10} + C_{11} \qquad \text{for all } t \in (0, T_{max})
$$

with $C_{11} := (\frac{1}{8} + C_4) + (\frac{1}{16} + C_5|\Omega|) + (\frac{1}{32} + C_6|\Omega|) + C_{10}$. As a consequence of Lemma 2.3 and the spatio-temporal L^{10} bound for u asserted by Lemma 2.5, we thus conclude that there exists $C_{12} > 0$ such that

$$
y(t) \equiv 6\int_\Omega n^2 + \int_\Omega |\nabla c|^4 + \int_\Omega n|\nabla c|^2 \leq C_{12} \qquad \text{for all } t \in (0, T_{max}),
$$

which implies both (3.12) and (3.13). □

4 Higher regularity properties. Global existence

In order to prove Theorem 1.1, in view of Lemma 2.1 we need to further establish higher regularity estimates. We first use the L^2 estimate for n from Lemma 3.5 in order to control the solution component u.

Lemma 4.1 *Suppose that $\mu \geq 23$. Then for all $\alpha \in (0,1)$ one can find $C(\alpha) > 0$ such that*

$$
\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \leq C(\alpha) \qquad \text{for all } t \in (0, T_{max}). \tag{4.1}
$$

In particular, there exist $\theta \in (0,1)$ and $C > 0$ satisfying

$$
\|u(\cdot,t)\|_{C^\theta(\bar\Omega)} \leq C \qquad \text{for all } t \in (0, T_{max}). \tag{4.2}
$$

Proof. On the basis of the variation-of-constants formula for the projected version of the third equation in (1.1), that is of the identity $u_t + Au = \mathcal{P}[n\nabla\phi + g]$, according to standard smoothing propeties of the Stokes semigroup we see that there exist $C_1 > 0$ and $\lambda > 0$ such that

$$\begin{aligned}
\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} &= \left\| A^\alpha e^{-tA}u_0 + \int_0^t A^\alpha e^{-(t-s)A}\mathcal{P}[n(\cdot,s)\nabla\phi + g(\cdot,s)]ds \right\|_{L^2(\Omega)} \\
&\le \|A^\alpha u_0\|_{L^2(\Omega)} + C_1 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \Big\| \mathcal{P}[n(\cdot,s)\nabla\phi + g(\cdot,s)] \Big\|_{L^2(\Omega)} ds \\
&\le \|A^\alpha u_0\|_{L^2(\Omega)} + C_1 \int_0^t (t-s)^{-\alpha} e^{-\lambda(t-s)} \Big\{ \|\nabla\phi\|_{L^\infty(\Omega)} \|n(\cdot,s)\|_{L^2(\Omega)} + \|g(\cdot,s)\|_{L^2(\Omega)} \Big\} ds
\end{aligned}$$

for all $t \in (0, T_{max})$, again because $\mathcal{P}$ acts as an orthogonal projection. Since from Lemma 3.5 and our overall assumptions on boundedness of $\nabla\phi$ and g we know that there exists $C_2 > 0$ fulfilling

$$\|\nabla\phi\|_{L^\infty(\Omega)} \|n(\cdot,s)\|_{L^2(\Omega)} + \|g(\cdot,s)\|_{L^2(\Omega)} \le C_2 \qquad \text{for all } t \in (0, T_{max}),$$

this shows that

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le \|A^\alpha u_0\|_{L^2(\Omega)} + C_1 C_2 \int_0^t \sigma^{-\alpha} e^{-\lambda\sigma} d\sigma \qquad \text{for all } t \in (0, T_{max}).$$

As the inequality $\alpha < 1$ warrants that $\int_0^\infty \sigma^{-\alpha} e^{-\lambda\sigma} d\sigma$ converges, and that thanks to (1.4) also $\|A^\alpha u_0\|_{L^2(\Omega)}$ is finite, we thereby obtain (4.1), which in turn entails (4.2) upon choosing any $\alpha \in (\frac{3}{4}, 1)$ and recalling a well-known embedding property of $D(A^\alpha)$ into spaces of Hölder continuous functions ([6], [7]). □

In conjunction with the estimate for ∇c in $L^4(\Omega)$ provided by Lemma 3.5, the latter entails boundedness of n.

Lemma 4.2 *If $\mu \ge 23$, then there exists $C > 0$ such that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.3}$$

Proof. Given $T \in (0, T_{max})$, in order to prepare an estimation of the finite number $M(T) := \sup_{t\in(0,T)} \|n(\cdot,t)\|_{L^\infty(\Omega)}$ we write $h := \nabla c + u$ and then obtain from Lemma 3.5 and Lemma 4.1 that there exists $C_1 > 0$ such that

$$\|h(\cdot,t)\|_{L^4(\Omega)} \le C_1 \qquad \text{for all } t \in (0, T_{max}). \tag{4.4}$$

Since $n_t = \Delta n - \nabla\cdot(nh) + rn - \mu n^2$ in $\Omega \times (0, T_{max})$ due to the fact that $\nabla \cdot u \equiv 0$, by means of an associate variation-of-constants formula we can represent $n(\cdot,t)$ for each $t \in (0,T)$ according to

$$\begin{aligned}
n(\cdot,t) &= e^{(t-t_0)\Delta} n(\cdot,t_0) - \int_{t_0}^t e^{(t-s)\Delta} \nabla\cdot(n(\cdot,s)h(\cdot,s))ds \\
&\quad + \int_{t_0}^t e^{(t-s)\Delta}(rn(\cdot,s) - \mu n^2(\cdot,s))ds \\
&=: n_1(\cdot,t) + n_2(\cdot,t) + n_3(\cdot,t),
\end{aligned} \tag{4.5}$$

where $t_0 := (t-1)_+$. Here by the maximum principle we can estimate

$$\|n_1(\cdot,t)\|_{L^\infty(\Omega)} \le \|n_0\|_{L^\infty(\Omega)} \qquad \text{if } t \in (0,1], \tag{4.6}$$

whereas if $t > 1$ then standard $L^p - L^q$ estimates for the Neumann heat semigroup provide $C_2 > 0$ such that

$$\|n_1(\cdot,t)\|_{L^\infty(\Omega)} \le C_2 (t-t_0)^{-\frac{3}{2}} \|n(\cdot,t_0)\|_{L^1(\Omega)} = C_2 \|n(\cdot,t_0)\|_{L^1(\Omega)} \le C_2 m \tag{4.7}$$

holds because of (2.2).
Next, since $r\xi - \mu\xi^2 \le C_3 := \frac{r^2}{4\mu}$ for all $\xi \in \mathbb{R}$, again by the maximum principle we have

$$n_3(\cdot,t) \le \int_{t_0}^t e^{(t-s)\Delta} C_3 ds = C_3(t-t_0) \le C_3. \tag{4.8}$$

Finally, to estimate n_2 we fix an arbitrary $p \in (3,4)$ and then once more invoke known smoothing properties of $(e^{\sigma\Delta})_{\sigma\ge 0}$ ([27, Lemma 1.3 (iv)]) and the Hölder inequality to find $C_4 > 0$ such that

$$\begin{aligned}
\|n_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_4 \int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|n(\cdot,s)h(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le C_4 \int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|n(\cdot,s)\|_{L^{\frac{4p}{4-p}}(\Omega)} \|h(\cdot,s)\|_{L^4(\Omega)} ds \\
&\le C_4 \int_{t_0}^t (t-s)^{-\frac{1}{2}-\frac{3}{2p}} \|n(\cdot,s)\|^a_{L^\infty(\Omega)} \|n(\cdot,s)\|^{1-a}_{L^1(\Omega)} \|h(\cdot,s)\|_{L^4(\Omega)} ds,
\end{aligned}$$

where $a := \frac{5p-4}{4p} \in (0,1)$. In view of (2.2), (4.4) and the definition of $M(T)$, this entails that

$$\|n_2(\cdot,t)\|_{L^\infty(\Omega)} \le C_1 C_4 m^{1-a} \int_0^1 \sigma^{-\frac{1}{2}-\frac{3}{2p}} d\sigma \cdot M^a(T),$$

so that since $\frac{1}{2} + \frac{3}{2p} < 1$ according to our restriction $p > 3$, in combination with (4.5)-(4.7) we obtain $C_5 > 0$ such that

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} = \sup_{x\in\Omega} n(x,t) \le \sup_{x\in\Omega} n_1(x,t) + \sup_{x\in\Omega} n_2(x,t) + \sup_{x\in\Omega} n_3(x,t) \le C_5 + C_5 M^a(T)$$

for all $t \in (0,T)$. Therefore,

$$M(T) \le C_5 + C_5 M^a(T) \qquad \text{for all } T \in (0, T_{max})$$

and hence

$$M(T) \le \max\left\{1\,,\,(2C_5)^{\frac{1}{1-a}}\right\} \qquad \text{for all } T \in (0, T_{max}).$$

This proves (4.3). □

This information now readily implies boundedness of ∇c in $L^q(\Omega)$ for arbitrary finite q.

Lemma 4.3 *Assume that $\mu \geq 23$. Then for each $q > 1$ one can find $C(q) > 0$ such that*

$$\|c(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C(q) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.9}$$

PROOF. With the regularity properties from Lemma 4.1 and Lemma 4.2 at hand, one can readily derive this by means of standard parabolic regularity arguments applied to the second equation in (1.1) (cf. e.g. [22, Lemma 3.12] for details in a related situation). □

Our main result on global existence and boundedness thereby becomes a straightforward consequence of Lemma 2.1.

PROOF of Theorem 1.1. In view of the extensibility criterion in Lemma 2.1, the estimates gathered in Lemma 4.2, Lemma 4.3 and Lemma 4.1 assert that $T_{max} = \infty$, and that hence the global boundedness properties in (1.7) hold. □

5 Decay. Proof of Theorem 1.2

In the case $r = 0$ addressed in Theorem 1.2, the absence of any cell reproduction term in (1.1) implies the following basic decay properties of n and c.

Lemma 5.1 *Suppose that $\mu \geq 23$ and $r = 0$. Then*

$$\int_0^\infty \int_\Omega n^2(x,t)dxdt < \infty, \tag{5.1}$$

and there exists $C > 0$ such that

$$\int_\Omega n(x,t)dx \leq \frac{C}{t+1} \qquad \textit{for all } t > 0 \tag{5.2}$$

as well as

$$\int_\Omega c(x,t)dx \leq \frac{C}{t+1} \qquad \textit{for all } t > 0. \tag{5.3}$$

PROOF. As in Lemma 2.2 we simply integrate in space to obtain under the assumption $r = 0$ that

$$\frac{d}{dt}\int_\Omega n = -\mu \int_\Omega n^2 \leq -\frac{\mu}{|\Omega|}\left(\int_\Omega n\right)^2 \qquad \text{for all } t > 0,$$

which readily implies both (5.1) and (5.2).
Thereupon, (5.3) follows upon an ODE comparison of the functions y and $\overline{y}$ which for suitably large $C_1 > 0$ are given by $y(t) := \int_\Omega c(x,t)dx$ and $\overline{y}(t) := \frac{C_1}{t+2}$, $t \geq 0$, noting that from the second equation in (1.1) and (5.2) we obtain $C_2 > 0$ fulfilling

$$y'(t) \leq -y(t) + \frac{C_2}{t+1} \qquad \text{for all } t > 0$$

whereas a straightforward computation yields

$$\overline{y}'(t) + \overline{y}(t) - \frac{C_2}{t+1} \geq 0 \qquad \text{for all } t > 0$$

(cf. [22, Lemma 4.2] for details). $\square$

Under an additional assumption on the temporal decay of g, Lemma 5.1 also entails asymptotic extinction of u at least with respect to the norm in $L^2(\Omega)$.

Lemma 5.2 *Let $\mu \ge 23$ and $r = 0$, and assume that*

$$\int_0^\infty \int_\Omega |g(x,t)|^2 dxdt < \infty. \tag{5.4}$$

Then

$$\int_\Omega |u(x,t)|^2 dx \to 0 \qquad \text{as } t \to \infty. \tag{5.5}$$

PROOF. Using u as a test function for the third equation in (1.1), employing the Hölder inequality we see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 &= \int_\Omega n\nabla\phi \cdot u + \int_\Omega g \cdot u \\ &\le \Big\{ \|\nabla\phi\|_{L^\infty(\Omega)} \|n\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \Big\} \cdot \|u\|_{L^2(\Omega)} \qquad \text{for all } t > 0. \end{aligned}$$

Since the Poincaré inequality provides $C_1 > 0$ such that $\int_\Omega |u|^2 \le C_1 \int_\Omega |\nabla u|^2$ for all $t > 0$, we thereby readily obtain $C_2 > 0$ and $C_3 > 0$ such that $y(t) := \int_\Omega |u^2(\cdot,t)|^2$, $t \ge 0$, satisfies

$$y'(t) + C_2 y(t) \le h(t) := C_3 \cdot \Big\{ \int_\Omega n^2(\cdot,t) + \int_\Omega |g(\cdot,t)|^2 \Big\} \qquad \text{for all } t > 0$$

and hence, by an ODE comparison,

$$\begin{aligned} y(t) &\le e^{-C_2 t} y(0) + \int_0^t e^{-C_2(t-s)} h(s) ds \\ &\le e^{-C_2 t} y(0) + e^{-\frac{C_2 t}{2}} \int_0^{\frac{t}{2}} h(s) ds + \int_{\frac{t}{2}}^t h(s) ds \qquad \text{for all } t > 0. \end{aligned}$$

As $\int_0^\infty \int_\Omega h(t) dt < \infty$ according to Lemma 5.1 and our assumption on g, this entails (5.5). $\square$

In turning the basic decay information on n from Lemma 5.1 into the uniform convergence property asserted in Theorem 1.2, we shall make use of the following Hölder estimate implied by the regularity properties collected in the previous section.

Lemma 5.3 *Let $\mu \ge 23$. Then there exist $\theta \in (0,1)$ and $C > 0$ such that*

$$\|n\|_{C^{\theta,\frac{\theta}{2}}(\bar\Omega \times [t,t+1])} \le C \qquad \text{for all } t > 1. \tag{5.6}$$

PROOF. Writing the first equation of (1.1) in the form

$$n_t = \nabla \cdot \Big(\nabla n - h_1(x,t) \Big) + h_2(x,t), \qquad x \in \Omega,\ t > 0, \tag{5.7}$$

with

$$h_1(x,t) := n(x,t)\nabla c(x,t) + n(x,t)u(x,t)$$

and

$$h_2(x,t) := rn(x,t) - \mu n^2(x,t)$$

for $x \in \Omega$ and $t > 0$, we can estimate

$$(\nabla n - h_1) \cdot \nabla n \geq \frac{1}{2}|\nabla n|^2 - \frac{1}{2}|h_1|^2$$

and

$$|\nabla n - h_1| \leq |\nabla n| + |h_1|$$

in $\Omega \times (0,\infty)$. As Lemma 4.1, Lemma 4.2 and Lemma 4.3 imply that both h_1 and h_2 are bounded in $L^\infty((0,\infty); L^q(\Omega))$ for any $q \in (1,\infty)$, and that n is a bounded solution of (5.7), a known result on parabolic Hölder regularity ([18, Theorem 1.3]) immediately asserts (5.6). □

By means of standard arguments, we can finally verify the claimed statements on decay of solutions in the case $r = 0$.

PROOF of Theorem 1.2. i) To verify the first statement in (1.8), supposing on the contrary that this be false we could find $C_1 > 0$ and $(t_j)_{j\in\mathbb{N}} \subset (1,\infty)$ such that $t_j \to \infty$ as $j \to \infty$ and

$$\|n(\cdot,t_j)\|_{L^\infty(\Omega)} \geq C_1 \qquad \text{for all } j \in \mathbb{N}. \tag{5.8}$$

On the other hand, invoking Lemma 5.3, by means of the Arzelà-Ascoli theorem we see that $(n(\cdot,t))_{t>1}$ is relatively compact in $C^0(\bar{\Omega})$, and thus on extracting a subsequence we may assume that

$$n(\cdot,t_j) \to n_\infty \quad \text{in } L^\infty(\Omega) \qquad \text{as } j \to \infty$$

with some nonnegative $n_\infty \in C^0(\bar{\Omega})$. However, the decay property (5.2) implies that

$$n(\cdot,t) \to 0 \quad \text{in } L^1(\Omega) \qquad \text{as } t \to \infty.$$

Therefore, combining the above two observations we see that necessarily

$$n_\infty \equiv 0,$$

which contradicts (5.8) and thereby proves the first claim in (1.8). The claimed stabilization property of c can be derived along the same lines, relying on an application of Lemma 4.3 to any $q > 3$, and on (5.3).

ii) Likewise, under the assumption that $g \in L^2(\Omega \times (0,\infty))$ the claimed uniform decay of u results from Lemma 5.2, because also $(u(\cdot,t))_{t>1}$ is relatively compact in $C^0(\bar{\Omega})$ according to Lemma 4.1. □

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