Large-data global generalized solutions in a chemotaxis system with tensor-valued sensitivities

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Abstract

The chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot \Big(uS(x,u,v) \cdot \nabla v\Big), \qquad x \in \Omega,\ t>0, \\ v_t = \Delta v - uf(v), \qquad x \in \Omega,\ t>0, \end{cases} \qquad (\star)$$

for the density $u = u(x,t)$ of a cell population and the concentration $v = v(x,t)$ of an attractive chemical consumed by the former, is considered under no-flux boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \ge 1$, with smooth boundary, where $f \in C^1([0,\infty);[0,\infty))$ and $S \in C^2(\bar{\Omega} \times [0,\infty)^2;\mathbb{R}^{n\times n})$ are given functions such that $f(0)=0$.
In contrast to related Keller-Segel-type problems with scalar sensitivities, in presence of such matrix-valued S the system $(\star)$ in general apparently does not possess any useful gradient-like structure. Accordingly, its analysis needs to be based on new types of a priori bounds.
Using a spatio-temporal L^2 estimate for $\nabla \ln(u+1)$ as a starting point, we derive a series of compactness properties of solutions to suitably regularized versions of $(\star)$. Motivated by these, we develop a generalized solution concept which requires solutions to satisfy very mild regularity hypotheses only, especially for the component u; in particular, the chemotactic flux $uS(x,u,v)\cdot\nabla v$ needs not be integrable in this context.
On the basis of the above compactness properties, it is finally shown that within this framework, under a mild growth assumption on S and for all sufficiently regular nonnegative initial data, the corresponding initial-boundary value problem for $(\star)$ possesses at least one global generalized solution. This extends known results which in the case of such general matrix-valued S provide statements on global existence only in the two-dimensional setting and under the additional restriction that $\|v_0\|_{L^\infty(\Omega)}$ be small.

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1 Introduction

Chemotaxis with tensor-valued sensitivities. This work is concerned with solutions of the parabolic initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot \Big(uS(x,u,v) \cdot \nabla v \Big), & x \in \Omega, \ t > 0, \\ v_t = \Delta v - uf(v), & x \in \Omega, \ t > 0, \\ \nabla u \cdot \nu = u(S(x,u,v) \cdot \nabla v) \cdot \nu, \quad \nabla v \cdot \nu = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $n \geq 1$ and ν denotes the outward normal vector field on $\partial\Omega$, and where $f : [0,\infty) \to \mathbb{R}$ and the matrix-valued function $S : \Omega \times [0,\infty)^2 \to \mathbb{R}^{n\times n}$ are supposed to be given parameter functions.

Systems of this type arise in mathematical biology as models for the evolution of cell populations, in which individuals, besides moving randomly, are able to partially adapt their motion to gradients of a chemical signal substance. This mechanism, also known as *chemotaxis*, in prototypical situations is such that the preferred direction of motion is either toward increasing signal concentrations, or away from the latter ([7]). A simple model for these processes of *chemoattractive* and *chemorepulsive* movement was proposed by Keller and Segel in 1970 ([9]), at its core containing an equation of the form

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v) \tag{1.2}$$

for the evolution of the population density $u = u(x,t)$ in response to the gradient of the chemical concentration $v = v(x,t)$, where the constant $\chi \in \mathbb{R}$ is positive in the attractive and negative in the repulsive case. Such Keller-Segel-type systems, obtained upon complementing (1.2) or variants thereof by appropriate equations for the chemical in the respective situations, have widely been used as models in quite diverse particular biological contexts, including spontaneous aggregation phenomena in populations of *Dictyostelium discoideum* ([9]), tumor cell invasion ([3]), and also self-organization during embryonic development ([15]).

In contrast to this, more recent modeling approaches ([25], [14], [4]) suggest to allow for more general mechanisms of chemotactic migration in certain situations, including directions not necessarily parallel to the gradient of the signal. Corresponding models then require the so-called chemotactic sensitivity, in (1.2) represented by the constant scalar χ, to be a general matrix-valued function such as in (1.1). For instance, a concise derivation of a macroscopic model for the behavior of swimming bacteria near the surface of their surrounding fluid, as presented in [25], in the corresponding parabolic limit leads to a description of the cell population density by the first equation in (1.1), with sensitivity tensors of the form

$$S(x,u,v) = \chi \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \beta \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.3}$$

in their nondiagonal parts inter alia reflecting that when cells swim e.g. parallel to a surface, larger viscous forces are exerted on those parts of the cells which are closer to the surface. Here in the simplest conceivable setting χ and β are assumed to be positive constants, but they may as well vary with x such as e.g. in cases when rotational flux components are neglected far from boundary regions, and moreover possibly depend on the the variables u and v if further mechanisms are accounted for such as saturation effects at large cell or signal densities ([24], [7]).

Boundedness vs. blow-up. Guided by this example, in this work we will concentrate on the case when besides such a general type of chemotactic motion, the coupling between the quantities u and v is governed by signal consumption through cells; that is, we shall assume that cells absorb the chemical in question upon contact, as reflected in the particular form of the second equation in (1.1). A fundamental mathematical question is then whether and in which sense the resulting initial-boundary value problem (1.1) can be solved globally in time. Since in view of the choice of boundary conditions, the system (1.1) formally preserves the total mass of cells in the sense that $\int_\Omega u(\cdot,t) \equiv \int_\Omega u_0$, addressing this question essentially amounts to either ruling out or showing the occurrence of finite-time mass accumulation. Indeed, in the setting of the minimal version of the full original Keller-Segel system,

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.4}$$

in which the signal is thus *produced* by the cells, such a singularity formation, mathematically represented as finite-time blow-up of the solution component u with respect to the norm in $L^\infty(\Omega)$, may occur in certain situations: For appropriate initial data, explosions of this type have been detected when either $\Omega \subset \mathbb{R}^2$ is a disk and the total mass of cells is supercritical in the sense that $\int_\Omega u_0 > 8\pi$ ([6], [11]), or when Ω is a ball in $\mathbb{R}^n$, $n \geq 3$, and $\int_\Omega u_0$ is an arbitrary prescribed number ([22]). Here the criticality of the spatially two-dimensional setting is underlined by a complementing result which asserts that in this case the condition $\int_\Omega u_0 < 4\pi$ is sufficient to ensure global existence of bounded solutions, thereby ruling out any blow-up phenomenon; in the radial case, this condition can even be relaxed to the essentially optimal inequality $\int_\Omega u_0 < 8\pi$ ([12]). In the one-dimemsional version of (1.4), all solutions are global and bounded ([13]), whereas in the three- and higher-dimensional case alternative smallness assumptions on the initial data, involving norms of u_0 in $L^p(\Omega)$ for $p \geq \frac{n}{2}$, warrant global boundedness ([20], [1]).

On the other hand, signal consumption as in (1.1) is known to inhibit this tendency toward blow-up, at least to a certain extent, when coupled to the mechanisms in (1.2): For instance, the corresponding Neumann initial-boundary value problem for the prototypical system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \end{cases} \tag{1.5}$$

possesses global classical solutions in smoothly bounded convex domains $\Omega \subset \mathbb{R}^2$ for all reasonably regular initial data, and moreover all these solutions approach the constant equilibrium given by $u \equiv \frac{1}{|\Omega|} \int_\Omega u_0$ and $v \equiv 0$ in the large time limit ([16]). For the three-dimensional analogue, at least certain generalized global solutions can be constructed. These eventually become bounded and smooth and stabilize in the aforementioned manner, but it is unknown whether they may develop singularities at an intermediate stage ([16]). Related systems involving nonlinear cell diffusion, essentially modeled by a porous medium-type operator Δu^m or non-degenerate variants thereof, have recently been studied in [2] and [18], where it has been shown that global bounded solutions can be constructed if the enhancement of diffusion at high densities is sufficiently large in the sense that $m > 2 - \frac{2}{n}$.
Some of these global existence and boundedness properties of (1.5) can even be found in a more complex model for swimming aerobic bacteria which, in addition to the mechanisms reflected in (1.5),

includes the interaction of cells and chemoattractant with the surrounding fluid ([5], [21], [23]).

The mathematical challenge: Deriving boundedness despite loss of energy structure. From a point of view of mathematical analysis, passing from (1.5) to (1.1) by allowing for more complex cross-diffusion mechanisms in (1.1) appears to bring about a significant structural change: For (1.5), the integral

$$\int_\Omega \Big\{ u \ln u + 2|\nabla \sqrt{v}|^2 \Big\}$$

plays the role of an energy functional in that it decreases along trajectories ([16], cf. also [5]). A corresponding gradient-like structure, along with all its consequences for the a priori knowledge on the regularity of solutions, apparently cannot be expected for general matrix-valued sensitivities S in (1.1). It is thus not clear how far the blow-up preventing effect of signal absorption in (1.5) extends to the general system (1.1). As far as we know, the only available result in this direction asserts global existence of bounded solutions to (1.1) in bounded convex planar domains, even in the classical sense, under mild assumptions on S and f (essentially coinciding with (1.6), (1.7) and (1.8) below), but only under the restrictive additional assumption that $\|v_0\|_{L^\infty(\Omega)}$ be small enough ([10]). Without such a smallness condition, the recent paper [2] proves global existence of bounded weak solutions to the related system obtained from (1.1) upon replacing Δu by the porous medium-type nonlinear diffusion term Δu^m with arbitrary $m > 1$.

Main results. The purpose of the present paper is to establish a result on global existence for (1.1) under fairly general assumptions on f and S. More precisely, throughout our analysis we will assume that

$$f \in C^1([0,\infty)) \quad \text{is nonnegative with} \quad f(0) = 0, \tag{1.6}$$

and that $S = (S_{ij})_{i,j\in\{1,...,n\}}$ is a chemotactic sensitivity tensor with

$$S_{ij} \in C^2(\bar\Omega \times [0,\infty) \times [0,\infty)) \qquad \text{for } i,j \in \{1,...,n\}. \tag{1.7}$$

Moreover, we suppose that with some nondecreasing function S_0 on $[0,\infty)$, S satisfies the growth hypothesis

$$|S(x,u,v)| \le S_0(v) \qquad \text{for all } (x,u,v) \in \bar\Omega \times [0,\infty) \times [0,\infty). \tag{1.8}$$

Our main result then says that within this framework, for all suitably smooth initial data the problem (1.1) is globally solvable in an appropriate generalized sense. In particular, unlike in [10] we do neither need to impose any smallness asumption on the initial data here, nor do we require any restriction on the spatial dimension.

Theorem 1.1 *Suppose that $n \ge 1$ and that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, and let f and S satisfy (1.6), (1.7) and (1.8). Then for any choice of nonnegative functions $u_0 \in C^0(\bar\Omega)$ and $v_0 \in W^{1,\infty}(\Omega)$, the problem (1.1) posseses at least one global generalized solution (u,v) in the sense of Definition 2.2. This solution can be obtained as the limit a.e. in $\Omega \times (0,\infty)$ of a sequence $((u_\varepsilon, v_\varepsilon))_{\varepsilon=\varepsilon_j \searrow 0}$ of smooth classical solutions to the regularized problems (3.1) below.*

Key steps in our analysis. In order to highlight the main ideas underlying our approach, and to outline the structure of this work, let us note that unlike the case when $n = 2$ and $\|v_0\|_{L^\infty(\Omega)}$ is assumed to be small enough, a priori estimates for the solution component u in some reflexive Lebesgue space seem hard to obtain. As seen in [10] for convex planar domains, such an additional smallness assumption indeed allows for the derivation of bounds for both u and $|\nabla v|^2$ in $L^\infty((0,T);L^p(\Omega))$ with arbitrarily large $p>1$ through an essentially straightforward approach using suitable differential inequalities for $\int_\Omega u^p + \int_\Omega |\nabla v|^{2p}$. Instead, our analysis needs to be based on alternative a priori information on solutions $(u_\varepsilon, v_\varepsilon)$ to adequately regularized versions of (1.1) (see (3.1) below). Here beyond the immediate boundedness properties associated with the conservation of mass functional $\int_\Omega u_\varepsilon(\cdot,t)$ and the nonincrease of $\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}$ (Lemma 3.2), of fundamental importance to our approach will be the key estimate

$$\int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le C \tag{1.9}$$

with some $C>0$ independent of the regularization parameter $\varepsilon\in(0,1)$ (Lemma 4.1). Due to the strong dampening at large values of u_ε of the weight function $\frac{1}{(u_\varepsilon+1)^2}$ therein, however, we do not expect (1.9) to initiate an appropriate bootstrap process yielding substantial further regularity properties which would allow for passing to the limit $\varepsilon\searrow 0$ suitably so as to obtain a limit object solving (1.1) in one of the standard weak formulations. We shall accordingly introduce a generalized solution concept, to be specified in Definitions 2.1, 2.2 and 2.3, which at its core refers to the transformed quantity $\ln(u+1)$ rather than to u itself.

Indeed, viewing (1.9) as an inequality for $\nabla\ln(u_\varepsilon+1)$ and establishing an appropriate estimate for $\partial_t \ln(u_\varepsilon+1)$, we will thereby infer in Corollary 4.3 that

$$(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } L^2_{loc}(\bar\Omega\times[0,\infty)) \text{ with respect to the strong topology.} \tag{1.10}$$

Furthermore, (1.9) will be essential in deriving in Lemma 6.2 that

$$(u_\varepsilon f(v_\varepsilon))_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } L^1_{loc}(\bar\Omega\times[0,\infty)) \text{ with respect to the weak topology.} \tag{1.11}$$

This will on the one hand allow for passing to the limit along suitable subsequences so as to obtain a limit object (u,v), for which v solves the second equation in (1.1) in the natural weak sense (Section 7). On the other hand, (1.11) will enable us to refine straightforward compactness properties of $(v_\varepsilon)_{\varepsilon\in(0,1)}$, as expressed in Section 5 and in Section 7, so as to obtain in Section 8 that

$$(\nabla v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is relatively compact in } L^2_{loc}(\bar\Omega\times[0,\infty)) \text{ with respect to the strong topology.} \tag{1.12}$$

In the natural weak version of the first equation in (3.1) associated with $\ln(u_\varepsilon+1)$ (see (9.1)), these compactness properties (1.10) and (1.12) will form a main ingredient in taking $\varepsilon\searrow 0$ termwise, with one exception being an integral containing $\frac{|\nabla u|^2}{(u+1)^2}$, for which it seems that only a one-sided control can be obtained by using lower semicontinuity of norms with respect to weak convergence.

Therefore, in our solution concept we shall require $\ln(u+1)$ to satisfy the respective integral *inequality* only, thus generalizing a *supersolution* property of u with regard to the first equation in (1.1) (Definition 2.2). As seen in Lemma 2.1, the role of a complementing *subsolution*-like property can be played by the simple nonincrease of mass (cf. (2.6)), and within this framework the above limit (u,v) indeed is a generalized solution of (1.1) (see Section 9).

2 A generalized solution concept

To begin with, let us first specify our solution concept. As far as the second component v is concerned, a generalization of the respective sub-problem of (1.1) is rather straightforward, because there the only nonlinear part $uf(v)$ is of lowest differentiability order.

Definition 2.1 *Let $u \in L^1_{loc}(\bar\Omega \times [0,\infty))$, let f satisfy (1.6), and assume that $v_0 \in W^{1,2}(\Omega)$. Then a nonnegative function*

$$v \in L^\infty_{loc}(\bar\Omega \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega))$$

is said to be a global weak solution *of*

$$\begin{cases} v_t = \Delta v - uf(v), \qquad x \in \Omega, \ t>0, \\ \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega, \ t>0, \\ v(x,0) = v_0(x), \qquad x \in \Omega, \end{cases} \tag{2.1}$$

if for all $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ having compact support in $\bar\Omega \times [0,\infty)$ with $\varphi_t \in L^2(\Omega \times (0,\infty))$, the identity

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega uf(v)\varphi \tag{2.2}$$

holds.

The most important part of our solution concept refers to the cross-diffusive equation in (1.1).

Definition 2.2 *Assume that S complies with (1.7) and (1.8), and that $u_0 \in L^\infty(\Omega)$ is nonnegative. Moreover, let $\phi \in C^2([0,\infty))$ be such that $\phi' > 0$ on $(0,\infty)$, and suppose that $v \in L^\infty_{loc}(\bar\Omega \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega))$ is nonnegative. Then a nonnegative function $u : \Omega \times (0,\infty) \to \mathbb{R}$ will be called a* global very weak ϕ-supersolution *of the problem*

$$\begin{cases} u_t = \Delta u - \nabla \cdot \Big(uS(x,u,v)\cdot\nabla v\Big), \qquad x \in \Omega, t>0, \\ \Big(\nabla u - u(S(x,u,v)\cdot\nabla v)\Big)\cdot\nu = 0, \qquad x \in \partial\Omega, \ t>0, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{2.3}$$

if

$$\begin{aligned} &\phi(u) \text{ and } \phi''(u)|\nabla u|^2 \text{ belong to } L^1_{loc}(\bar\Omega \times [0,\infty)), \\ &u\phi''(u)\nabla u \text{ and } u\phi'(u) \text{ belong to } L^2_{loc}(\bar\Omega \times [0,\infty)), \end{aligned} \tag{2.4}$$

and if for each nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, the inequality

$$\begin{aligned} -\int_0^\infty \int_\Omega \phi(u)\varphi_t - \int_\Omega \phi(u_0)\varphi(\cdot,0) \quad \ge \quad & \int_0^\infty \int_\Omega \phi(u)\Delta\varphi - \int_0^\infty \int_\Omega \phi''(u)|\nabla u|^2\varphi \\ &+ \int_0^\infty \int_\Omega u\phi''(u)\nabla u \cdot \Big(S(x,u,v)\cdot\nabla v\Big)\varphi \\ &+ \int_0^\infty \int_\Omega u\phi'(u)\Big(S(x,u,v)\cdot\nabla v\Big)\cdot\nabla\varphi \end{aligned} \tag{2.5}$$

is satisfied.

Remark. i) It can easily be checked using (1.8) that the required regularity properties of v along with (2.4) ensure that all integrals in (2.5) are well-defined.
ii) In our final existence argument given in Lemma 9.2, we shall eventually choose $\phi(s) := \ln(s+1)$ for $s \geq 0$.

It is evident that in order to become meaningful, the above supersolution property has to be complemented by an additional condition which rules out that the component u has its time derivative exceeding the one dictated by the first equation in (1.1). We shall see that in a generalized sense, for this it is already sufficient to require that only the total mass $\int_\Omega u(\cdot,t)$ be bounded *from above* by $\int_\Omega u_0$:

Definition 2.3 *A couple (u,v) of nonnegative functions u and v defined in $\Omega \times (0,\infty)$ and satisfying*

$$u \in L^\infty((0,\infty); L^1(\Omega))$$

with

$$\int_\Omega u(\cdot,t) \leq \int_\Omega u_0 \qquad \textit{for a.e. } t > 0 \tag{2.6}$$

as well as

$$v \in L^\infty_{loc}(\bar{\Omega} \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega))$$

will be named a global generalized solution *of (1.1) if v is a global weak solution of (2.1) in the sense of Definition 2.1, and if for some $\phi \in C^2([0,\infty))$ with $\phi' > 0$ on $[0,\infty)$, u is a global very weak ϕ-supersolution of (2.3) in the sense of Definition 2.2.*

Indeed, this concept is fully compatible with that of classical solutions in the following sense:

Lemma 2.1 *Suppose that u and v are nonnegative functions from $C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty))$. Then if (u,v) is a global generalized solution of (1.1), it follows that (u,v) also is a classical solution of (1.1) in $\Omega \times (0,\infty)$.*

Proof. Since it is clear upon a standard reasoning that v is a classical solution of (2.1), we only need to prove that u is a classical solution of (2.3). To this end, we first fix a sequence $(\zeta_j)_{j\in\mathbb{N}} \subset C_0^\infty([0,\infty))$ such that $0 \leq \zeta_j \leq 1 = \zeta(0)$, $\zeta_j' \leq 0$ and $\operatorname{supp} \zeta_j \subset [0, \frac{1}{j}]$ for $j \in \mathbb{N}$, and given any nonnegative $\psi \in C_0^\infty(\Omega)$ we choose $\varphi(x,t) := \zeta_j(t)\psi(x)$, $(x,t) \in \bar{\Omega} \times [0,\infty)$, in (2.5). Then thanks to (2.4), the dominated convergence theorem and the fact that ζ_j' approaches the Dirac measure $-\delta(t)$, in the limit $j \to \infty$ we obtain

$$\int_\Omega \phi(u(\cdot,0))\psi - \int_\Omega \phi(u_0)\psi \geq 0$$

for any such ψ. This implies that $\phi(u(\cdot,0)) \geq \phi(u_0)$ in Ω and hence $u(\cdot,0) \geq u_0$ in Ω, because $\phi' > 0$ on $[0,\infty)$. Therefore, (2.6) and the continuity of u at $t=0$ warrant that actually $u(\cdot,0) = u_0$ in Ω. Secondly, choosing arbitrary nonnegative $\varphi \in C_0^\infty(\Omega \times (0,\infty))$ in (2.5), by a similar density argument we see that

$$\frac{\partial}{\partial t}\phi(u) \geq \Delta\phi(u) - \phi''(u)|\nabla u|^2 - \phi'(u)\nabla \cdot \Big(uS(x,u,v)\cdot\nabla v\Big) \qquad \text{in } \Omega \times (0,\infty)$$

holds in the classical sense. This is equivalent to

$$\phi'(u)u_t \geq \phi'(u)\Delta u - \phi'(u)\nabla \cdot \Big(uS(x,u,v)\cdot\nabla v\Big) \qquad \text{in } \Omega\times(0,\infty),$$

and using that $\phi' > 0$ on $[0,\infty)$ we conclude that u is a classical supersolution of the first equation in (1.1), that is,

$$u_t \geq \Delta u - \nabla \cdot \Big(uS(x,u,v)\cdot\nabla v\Big) \qquad \text{in } \Omega\times(0,\infty). \tag{2.7}$$

Finally, choosing arbitrary nonnegative $\varphi \in C_0^\infty(\bar\Omega\times(0,\infty))$ supported near $\partial\Omega\times[0,\infty)$ and such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega$, in a standard manner we moreover obtain from (2.5) that

$$\frac{\partial u}{\partial\nu} \geq u\Big(S(x,u,v)\cdot\nabla v\Big)\cdot\nu \qquad \text{on } \partial\Omega\times(0,\infty). \tag{2.8}$$

Now if u was not a classical solution of (2.3) then, by (2.7), (2.8) and a continuity argument, for some open subset $G_1\subset\Omega$ and some open interval $J_1\subset(0,\infty)$ we would have

$$u_t > \Delta u - \nabla \cdot \Big(uS(x,u,v)\cdot\nabla v\Big) \qquad \text{in } G_1\times J_1, \tag{2.9}$$

or there would exist a relatively open set $G_2\subset\partial\Omega$ and an open interval $J_2\subset(0,\infty)$ fulfilling

$$\frac{\partial u}{\partial\nu} > u\Big(S(x,u,v)\cdot\nabla v\Big) \qquad \text{in } G_2\times J_2. \tag{2.10}$$

In the former case, (2.9) together with (2.7) and (2.8) would imply that for all $t\in J_1$,

$$\begin{aligned}\int_\Omega u(\cdot,t) - \int_\Omega u_0 = \int_0^t\int_\Omega u_t \quad &> \quad \int_0^t\int_\Omega \Big(\Delta u - \nabla\cdot\Big(uS(x,u,v)\cdot\nabla v\Big)\Big) \\ &= \quad \int_0^t\int_{\partial\Omega}\Big(\frac{\partial u}{\partial\nu} - \Big(uS(x,u,v)\cdot\nabla v\Big)\cdot\nu\Big) \\ &\geq \quad 0,\end{aligned}$$

meaning that $\int_\Omega u(\cdot,t) > \int_\Omega u_0$ for all $t\in J_1$ and thereby contradicting the second assumption (2.6) on u. Along with a similar argument in the case when (2.10) holds, this completes the proof. $\Box$

3 Global solutions of regularized problems

In order to introduce an appropriate regularization of (1.1), let us fix families $(\rho_\varepsilon)_{\varepsilon\in(0,1)}$ and $(\chi_\varepsilon)_{\varepsilon\in(0,1)}$ of functions

$$\rho_\varepsilon \in C_0^\infty(\Omega) \quad \text{such that} \quad 0\leq\rho_\varepsilon\leq 1 \text{ in } \Omega \quad \text{and} \quad \rho_\varepsilon \nearrow 1 \text{ in } \Omega \text{ as } \varepsilon\searrow 0$$

and

$$\chi_\varepsilon \in C_0^\infty([0,\infty)) \quad \text{such that} \quad 0\leq\chi_\varepsilon\leq 1 \text{ in } [0,\infty) \quad \text{and} \quad \chi_\varepsilon \nearrow 1 \text{ in } [0,\infty) \text{ as } \varepsilon\searrow 0,$$

define

$$S_\varepsilon(x,u,v) := \rho_\varepsilon(x)\cdot\chi_\varepsilon(u)\cdot S(x,u,v), \qquad x\in\bar\Omega,\ u\ge 0,\ v\ge 0,$$

and consider the problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla\cdot\Big(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big), \qquad x\in\Omega,\ t>0,\\ v_{\varepsilon t} = \Delta v_\varepsilon - u_\varepsilon f(v_\varepsilon), \qquad x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0 \qquad x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad v_\varepsilon(x,0)=v_0(x), \qquad x\in\Omega, \end{cases} \tag{3.1}$$

for $\varepsilon\in(0,1)$. These are indeed globally solvable in the classical sense:

Lemma 3.1 *For all $\varepsilon\in(0,1)$, there exists a pair $(u_\varepsilon,v_\varepsilon)\in(C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\bar\Omega\times(0,\infty))$ of nonnegative functions which solve (3.1) classically in $\Omega\times(0,\infty)$.*

Proof. Local existence of a smooth solution can be seen by a well-established contraction mapping argument in the space $C^0(\bar\Omega\times[0,T])\times L^\infty((0,T);W^{1,q}(\Omega))$ for arbitrary fixed $q>\max\{2,n\}$ and suitably small $T>0$ (see [19], for instance). Since $S_\varepsilon(x,u,v)\equiv 0$ for all sufficiently large u, standard estimation techniques yield extensibility of this local solution for all times (cf. e.g. [8]). □

The following basic properties of solutions to (3.1) are immediate.

Lemma 3.2 *The solution of (3.1) satisfies*

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \textit{for all } t>0 \tag{3.2}$$

as well as

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t>0. \tag{3.3}$$

In particular, with S_0 as defined in (1.8) we have the pointwise estimate

$$|S_\varepsilon(x,u_\varepsilon,v_\varepsilon)| \le S_1 := S_0(\|v_0\|_{L^\infty(\Omega)}) \qquad \textit{in } \Omega\times(0,\infty). \tag{3.4}$$

Proof. The identity (3.2) directly results upon integration of the first equation in (3.1) with respect to $x\in\Omega$. The estimate (3.3) is a straightforward consequence of the maximum principle applied to the second equation in (3.1), because we already know that $u_\varepsilon\ge 0$, and because f was assumed to be nonnegative throughout. □

Two more testing procedures easily yield further information:

Lemma 3.3 *The solution of (3.1) has the properties*

$$\int_0^\infty\int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{2}\int_\Omega v_0^2 \tag{3.5}$$

and

$$\int_0^\infty\int_\Omega u_\varepsilon f(v_\varepsilon) \le \int_\Omega v_0. \tag{3.6}$$

PROOF. Multiplying the second equation in (3.1) by v_ε and integrating by parts over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^2 = -\int_\Omega u_\varepsilon v_\varepsilon f(v_\varepsilon) \qquad \text{for all } t>0.$$

Since here by nonnegativity of f, u_ε and v_ε the right-hand side is nonpositive, integrating in time yields (3.5).
Likewise, testing the second equation in (3.1) against a nontrivial constant shows that

$$\frac{d}{dt}\int_\Omega v_\varepsilon = -\int_\Omega u_\varepsilon f(v_\varepsilon) \qquad \text{for all } t>0,$$

from which (3.6) results upon a time integration. □

4 Estimates for $\ln(u_\varepsilon+1)$

We proceed to derive further estimates for u_ε. The first of these provides an integral bound for the gradient of $\ln(u_\varepsilon+1)$.

Lemma 4.1 *For each $\varepsilon\in(0,1)$, the solution of (3.1) satisfies*

$$\int_0^\infty \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le K_1 := 2\int_\Omega u_0 + \frac{S_1^2}{2}\cdot\int_\Omega v_0^2, \tag{4.1}$$

with the number S_1 being as defined in (3.4).

PROOF. We multiply the first equation in (3.1) by $\frac{1}{u_\varepsilon+1}$ and integrate by parts over Ω, which results in the identity

$$\begin{aligned}\frac{d}{dt}\int_\Omega \ln(u_\varepsilon+1) &= -\int_\Omega \nabla u_\varepsilon\cdot\nabla\frac{1}{u_\varepsilon+1} + \int_\Omega \nabla\frac{1}{u_\varepsilon+1}\cdot\Big(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\\ &= \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} - \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big) \qquad \text{for all } t>0. \end{aligned}\tag{4.2}$$

By Young's inequality and (3.4),

$$\begin{aligned}\left|-\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\right| &\le \frac{1}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\\ &\quad +\frac{1}{2}\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}\cdot|S_\varepsilon(x,u_\varepsilon,v_\varepsilon)|^2\cdot|\nabla v_\varepsilon|^2\\ &\le \frac{1}{2}\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \frac{S_1^2}{2}\cdot\int_\Omega|\nabla v_\varepsilon|^2.\end{aligned}$$

Therefore, an integration of (4.2) with respect to the time variable yields

$$\int_\Omega \ln(u_\varepsilon(\cdot,t)+1) - \int_\Omega \ln(u_0+1) \;\ge\; \frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} - \frac{S_1^2}{2}\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \qquad \text{for all } t>0$$

and thus, since $0 \le \ln(\xi+1) \le \xi$ for all $\xi \ge 0$,

$$\begin{aligned}
\frac{1}{2}\int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} &\le \int_\Omega u_\varepsilon(\cdot,t) + \frac{S_1^2}{2}\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \\
&= \int_\Omega u_0 + \frac{S_1^2}{2}\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \qquad \text{for all } t>0
\end{aligned}$$

thanks to (3.2). As $\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 \le \frac{1}{2}\int_\Omega v_0^2$ by Lemma 3.3, this establishes (4.1). $\square$

In order to prepare pointwise convergence a.e. in $\Omega\times(0,\infty)$ for u_ε along a suitable sequence of numbers $\varepsilon=\varepsilon_j \searrow 0$, we next aim at deriving a strong compactness property of $(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$. This is prepared by the following.

Lemma 4.2 *Let $m\in\mathbb{N}$ be such that $m>\frac{n}{2}$. Then there exists $K_2>0$ with the property that for each $\varepsilon\in(0,1)$, the solution of (3.1) satisfies*

$$\int_0^T \Big\|\partial_t \ln(u_\varepsilon(\cdot,t)+1)\Big\|_{(W_0^{m,2}(\Omega))^\star} dt \le K_2\cdot(1+T) \qquad \textit{for all } T>0. \tag{4.3}$$

Proof. For fixed $t>0$ and arbitrary $\psi\in W_0^{m,2}(\Omega)$, using the first equation in (3.1) and integrating by parts we obtain

$$\begin{aligned}
\int_\Omega \partial_t \ln(u_\varepsilon(x,t)+1)\cdot\psi(x)dx &= \int_\Omega \frac{u_{\varepsilon t}}{u_\varepsilon+1}\cdot\psi \\
&= \int_\Omega \frac{1}{u_\varepsilon+1}\Delta u_\varepsilon\cdot\psi - \int_\Omega \frac{1}{u_\varepsilon+1}\nabla\cdot\Big(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\psi \\
&= -\int_\Omega \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi + \int_\Omega \frac{1}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\psi \\
&\quad + \int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\cdot\nabla\psi \\
&\quad - \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\psi.
\end{aligned}\tag{4.4}$$

Here, by the Cauchy-Schwarz inequality we have

$$\Big|-\int_\Omega \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi\Big| \le \Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)^{\frac{1}{2}}\cdot\|\nabla\psi\|_{L^2(\Omega)},$$

and by the same token we see that

$$\Big|\int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\cdot\nabla\psi\Big| \le S_1\cdot\Big(\int_\Omega |\nabla v_\varepsilon|^2\Big)^{\frac{1}{2}}\cdot\|\nabla\psi\|_{L^2(\Omega)}$$

and

$$\Big|-\int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\psi\Big| \le S_1\cdot\Big(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\Big)^{\frac{1}{2}}\cdot\Big(\int_\Omega |\nabla v_\varepsilon|^2\Big)^{\frac{1}{2}}\cdot\|\psi\|_{L^\infty(\Omega)}.$$

Since clearly

$$\left|\int_\Omega \frac{1}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\psi\right| \le \left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\right)\cdot\|\psi\|_{L^\infty(\Omega)},$$

(4.4) therefore yields

$$\begin{aligned}\left|\int_\Omega \partial_t \ln(u_\varepsilon(x,t)+1)\cdot\psi(x)dx\right| \;\le\; & \Bigg\{\left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\right)^{\frac12} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \\ & + S_1\cdot\left(\int_\Omega |\nabla v_\varepsilon|^2\right)^{\frac12} + S_1\cdot\left(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\right)^{\frac12}\cdot\left(\int_\Omega|\nabla v_\varepsilon|^2\right)^{\frac12}\Bigg\}\times \\ & \times\Big(\|\nabla\psi\|_{L^2(\Omega)}+\|\psi\|_{L^\infty(\Omega)}\Big) \qquad \text{for all } \psi\in W_0^{m,2}(\Omega).\end{aligned}$$

As our condition $m>\frac{n}{2}$ ensures that the space $W_0^{m,2}(\Omega)$ is continuously embedded into $L^\infty(\Omega)$, by Young's inequality this implies that with some $c_1>0$,

$$\left|\int_\Omega \partial_t\ln(u_\varepsilon(x,t)+1)\cdot\psi(x)dx\right| \le c_1\cdot\left\{1+\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\int_\Omega|\nabla v_\varepsilon|^2\right\}\cdot\|\psi\|_{W_0^{m,2}(\Omega)}$$

for all $\psi\in W_0^{n,2}(\Omega)$, meaning that

$$\Big\|\partial_t\ln(u_\varepsilon(\cdot,t)+1)\Big\|_{(W_0^{m,2}(\Omega))^\star} \le c_1\cdot\left\{1+\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+\int_\Omega|\nabla v_\varepsilon|^2\right\} \qquad \text{for all } t>0.$$

Since according to Lemma 4.1 and (3.5) we have $\int_0^\infty\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\le K_1$ and $\int_0^\infty\int_\Omega|\nabla v_\varepsilon|^2\le\frac12\int_\Omega v_0^2$, an integration over $(0,T)$ easily yields (4.3) with an evident choice of K_2. □

Now a straightforward application of (a variant of) the Aubin-Lions lemma can be used to establish the following compactness properties of $(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$.

Corollary 4.3 *Let $T>0$. Then $(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$ is relatively compact in $L^2((0,T);W^{1,2}(\Omega))$ with respect to the weak topology, and relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology.*

Proof. As $(\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$ is bounded in $L^2((0,T);W^{1,2}(\Omega))$ according to Lemma 4.1 and (3.2), the first statement is immediate. Using that moreover $(\partial_t\ln(u_\varepsilon+1))_{\varepsilon\in(0,1)}$ is bounded in $L^1((0,T);(W_0^{n,2}(\Omega))^\star)$ by Lemma 4.2, since $(W_0^{n,2}(\Omega))^\star$ is a Hilbert space we may invoke a version of the Aubin-Lions lemma ([17, Theorem 2.3]) to obtain the claimed strong precompactness property. □

5 Compactness properties of $(v_\varepsilon)_{\varepsilon\in(0,1)}$

By a simplified variant of the argument of the previous section, we can readily derive the following.

Lemma 5.1 *Let $T>0$. Then $(v_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact in $L^2(\Omega\times(0,T))$ with respect to the strong topology.*

PROOF. We let $m \in \mathbb{N}$ be such that $m > \frac{n}{2}$, and take an arbitrary $\psi \in W_0^{m,2}(\Omega)$. Then from the second equation in (3.1) and the Cauchy-Schwarz inequality, for each fixed $t \in (0,T)$ we obtain

$$\begin{aligned}\left|\int_\Omega v_{\varepsilon t}(x,t)\psi(x)dx\right| &= \left|-\int_\Omega \nabla v_\varepsilon \cdot \nabla\psi - \int_\Omega u_\varepsilon f(v_\varepsilon)\psi\right| \\ &\le \left(\int_\Omega |\nabla v_\varepsilon|^2\right)^{\frac{1}{2}} \cdot \|\nabla\psi\|_{L^2(\Omega)} + \left(\int_\Omega u_\varepsilon f(v_\varepsilon)\right) \cdot \|\psi\|_{L^\infty(\Omega)}.\end{aligned}$$

Again since $W_0^{m,2}(\Omega) \hookrightarrow L^\infty(\Omega)$, we thus find that

$$\int_0^T \|v_{\varepsilon t}(\cdot,t)\|_{(W_0^{m,2}(\Omega))^\star} dt \le c_1 \int_0^T \left\{1 + \int_\Omega |\nabla v_\varepsilon|^2 + \int_\Omega u_\varepsilon f(v_\varepsilon)\right\} dt$$

with some $c_1 > 0$, and hence in light of Lemma 3.3 we conclude that

$$\int_0^T \|v_{\varepsilon t}(\cdot,t)\|_{(W_0^{m,2}(\Omega))^\star} dt \le c_1 T + \frac{c_1}{2}\int_\Omega v_0^2 + c_1 \int_\Omega v_0.$$

Therefore, the Aubin-Lions lemma in [17, Theorem 2.3] along with the boundedness of $(v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2((0,T);W^{1,2}(\Omega))$, as asserted by (3.3) and (3.5), yields the claim. $\square$

6 Precompactness of $(u_\varepsilon f(v_\varepsilon))_{\varepsilon\in(0,1)}$

In passing to the limit in the taxis term in (3.1), we will also need strong precompactness of $(\nabla v_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{loc}(\bar\Omega \times [0,\infty))$, rather than the corresponding weak compactness property implied by (3.5). This will finally be achieved in Lemma 8.2 below, but prepared by a series of steps, the first of which can be interpreted as providing some superlinear integrability property of the inhomogeneity $h_\varepsilon := u_\varepsilon f(v_\varepsilon)$ in the semilinear heat equation $v_{\varepsilon t} = \Delta v_\varepsilon - h_\varepsilon$.

Lemma 6.1 *For each $\varepsilon \in (0,1)$ we have the inequality*

$$\int_0^\infty \int_\Omega u_\varepsilon \ln(u_\varepsilon+1) f(v_\varepsilon) \le K_3, \tag{6.1}$$

where

$$K_3 := \int_\Omega v_0 \ln(u_0+1) + (\|v_0\|_{L^\infty(\Omega)} + 2) \cdot K_1 + \left(\frac{1}{2} + \frac{S_1}{2} + \frac{1}{8}\|v_0\|^2_{L^\infty(\Omega)} S_1^2\right) \cdot \int_\Omega v_0^2$$

with S_1 and K_1 taken from (3.4) and (4.1), respectively.

PROOF. Using the first and second equation in (3.1), we track the time evolution of $\int_\Omega v_\varepsilon \ln(u_\varepsilon+1)$ by computing

$$\begin{aligned}\frac{d}{dt}\int_\Omega v_\varepsilon \ln(u_\varepsilon+1) &= \int_\Omega v_{\varepsilon t}\ln(u_\varepsilon+1) + \int_\Omega \frac{v_\varepsilon}{u_\varepsilon+1} u_{\varepsilon t} \\ &= \int_\Omega \Delta v_\varepsilon \cdot \ln(u_\varepsilon+1) - \int_\Omega u_\varepsilon \ln(u_\varepsilon+1) f(v_\varepsilon) \\ &\quad + \int_\Omega \frac{v_\varepsilon}{u_\varepsilon+1}\Delta u_\varepsilon - \int_\Omega \frac{v_\varepsilon}{u_\varepsilon+1}\nabla\cdot\Big(u_\varepsilon S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big) \qquad (6.2)\end{aligned}$$

for all $t > 0$. Integrating by parts, we find that

$$\int_\Omega \Delta v_\varepsilon \cdot \ln(u_\varepsilon + 1) = -\int_\Omega \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon$$

and

$$\begin{aligned}\int_\Omega \frac{v_\varepsilon}{u_\varepsilon + 1} \Delta u_\varepsilon &= -\int_\Omega \nabla\Big(\frac{v_\varepsilon}{u_\varepsilon + 1}\Big) \cdot \nabla u_\varepsilon \\ &= -\int_\Omega \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int_\Omega \frac{v_\varepsilon}{(u_\varepsilon + 1)^2} |\nabla u_\varepsilon|^2\end{aligned}$$

as well as

$$\begin{aligned}-\int_\Omega \frac{v_\varepsilon}{u_\varepsilon + 1} \nabla \cdot \Big(u_\varepsilon S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon\Big) &= \int_\Omega u_\varepsilon \nabla\Big(\frac{v_\varepsilon}{u_\varepsilon + 1}\Big) \cdot \Big(S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon\Big) \\ &= \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1} \nabla v_\varepsilon \cdot \Big(S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon\Big) \\ &\quad - \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon + 1)^2} \nabla u_\varepsilon \cdot \Big(S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon\Big)\end{aligned}$$

for $t > 0$. Upon a time integration, (6.2) therefore becomes

$$\begin{aligned}\int_0^t \int_\Omega u_\varepsilon \ln(u_\varepsilon + 1) f(v_\varepsilon) + \int_\Omega v_\varepsilon(\cdot, t) \ln(u_\varepsilon(\cdot, t) + 1) &= \int_\Omega v_0 \ln(u_0 + 1) \\ &\quad -2\int_0^t \int_\Omega \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon + \int_0^t \int_\Omega \frac{v_\varepsilon}{(u_\varepsilon + 1)^2} |\nabla u_\varepsilon|^2 \\ &\quad + \int_0^t \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1} \nabla v_\varepsilon \cdot \Big(S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon\Big) \\ &\quad - \int_0^t \int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon + 1)^2} \nabla u_\varepsilon \cdot \Big(S_\varepsilon(x, u_\varepsilon, v_\varepsilon) \cdot \nabla v_\varepsilon\Big) \qquad (6.3)\end{aligned}$$

for all $t > 0$. Here we use Young's inequality, Lemma 4.1 and (3.5) in estimating

$$\begin{aligned}-2\int_0^t \int_\Omega \frac{1}{u_\varepsilon + 1} \nabla u_\varepsilon \cdot \nabla v_\varepsilon &\le \int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} + \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 \\ &\le K_1 + \frac{1}{2} \int_\Omega v_0^2 \qquad \text{for all } t > 0,\end{aligned}$$

whereas Lemma 4.1 combined with (3.3) shows that

$$\int_0^t \int_\Omega \frac{v_\varepsilon}{(u_\varepsilon + 1)^2} |\nabla u_\varepsilon|^2 \le \|v_0\|_{L^\infty(\Omega)} \cdot \int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon + 1)^2} \le \|v_0\|_{L^\infty(\Omega)} \cdot K_1 \qquad \text{for all } t > 0.$$

Moreover, by means of (3.4) and (3.5) we find that

$$\begin{aligned}\int_0^t \int_\Omega \frac{u_\varepsilon}{u_\varepsilon + 1} \nabla v_\varepsilon \cdot \Big(S_\varepsilon(x, u_\varepsilon, v_\varepsilon \cdot \nabla v_\varepsilon\Big) &\le S_1 \cdot \int_0^t \int_\Omega |\nabla v_\varepsilon|^2 \\ &\le S_1 \cdot \frac{1}{2} \int_\Omega v_0^2 \qquad \text{for all } t > 0,\end{aligned}$$

and similarly (4.1), (3.3) and (3.5) in view of Young's inequality yield

$$
\begin{aligned}
-\int_0^t\int_\Omega \frac{u_\varepsilon v_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big) &\le \int_0^t\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \\
&\quad +\frac14\|v_0\|_{L^\infty(\Omega)}^2\cdot S_1^2\cdot\int_0^t\int_\Omega \frac{u_\varepsilon^2}{(u_\varepsilon+1)^2}|\nabla v_\varepsilon|^2 \\
&\le K_1+\frac14\|v_0\|_{L^\infty(\Omega)}^2\cdot S_1^2\cdot\frac12\int_\Omega v_0^2
\end{aligned}
$$

for all $t>0$. Since $\int_\Omega v_\varepsilon \ln(u_\varepsilon+1)$ is nonnegative, (6.3) therefore implies (6.1). $\square$

Along with the Pettis theorem, the above lemma yields the following.

Lemma 6.2 *For each $T>0$, the family $(u_\varepsilon f(v_\varepsilon))_{\varepsilon\in(0,1)}$ is relatively compact in $L^1(\Omega\times(0,T))$ with respect to the weak topology.*

PROOF. Let $w_\varepsilon := u_\varepsilon f(v_\varepsilon)$, $\varepsilon\in(0,1)$. Then since $f(v_\varepsilon)\le c_1 := \|f\|_{L^\infty((0,\|v_0\|_{L^\infty(\Omega)}))}$ according to (3.3), using Lemma 3.3 and Lemma 6.1 and writing $c_2:=\max\{1,c_1\}$ and $m:=\int_\Omega u_0$ we find that

$$
\begin{aligned}
\int_0^T\int_\Omega w_\varepsilon\ln(w_\varepsilon+1) &\le \int_0^T\int_\Omega u_\varepsilon f(v_\varepsilon)\cdot\ln\left(c_1u_\varepsilon+1\right) \\
&\le \int_0^T\int_\Omega u_\varepsilon f(v_\varepsilon)\cdot\ln\Big(c_2(u_\varepsilon+1)\Big) \\
&= \ln c_2\cdot\int_0^T\int_\Omega u_\varepsilon f(v_\varepsilon)+\int_0^T\int_\Omega u_\varepsilon\ln(u_\varepsilon+1)f(v_\varepsilon) \\
&\le \ln c_2\cdot\int_\Omega v_0+K_3.
\end{aligned}
$$

In view of Pettis' theorem, this equi-integrability property already guarantees that $(w_\varepsilon)_{\varepsilon\in(0,1)}$ is relatively compact with respect to the weak topology in $L^1(\Omega\times(0,T))$. $\square$

7 Passing to the limit. Solution properties of v

We can now perform a first subsequence extraction procedure, resulting in a limit object (u,v) the second component of which can already be shown to be a weak solution of its respective equation in (1.1).

Lemma 7.1 *There exists a sequence $(\varepsilon_j)_{j\in\mathbb{N}}$ of numbers $\varepsilon_j\in(0,1)$ such that $\varepsilon_j\searrow 0$ as $j\to\infty$ and*

$$
\begin{aligned}
& u_\varepsilon\to u \qquad \textit{a.e. in } \Omega\times(0,\infty), && (7.1)\\
& \ln(u_\varepsilon+1)\rightharpoonup \ln(u+1) \qquad \textit{in } L^2_{loc}([0,\infty);W^{1,2}(\Omega)), && (7.2)\\
& v_\varepsilon\to v \qquad \textit{a.e. in } \Omega\times(0,\infty), && (7.3)\\
& v_\varepsilon\to v \qquad \textit{in } L^2_{loc}(\bar\Omega\times[0,\infty)), && (7.4)\\
& v_\varepsilon\overset{\star}{\rightharpoonup} v \qquad \textit{in } L^\infty(\Omega\times(0,\infty)), && (7.5)\\
& \nabla v_\varepsilon\rightharpoonup\nabla v \qquad \textit{in } L^2(\Omega\times(0,\infty)) \qquad \textit{and} && (7.6)\\
& u_\varepsilon f(v_\varepsilon)\to uf(v) \qquad \textit{in } L^1_{loc}(\bar\Omega\times[0,\infty)) && (7.7)
\end{aligned}
$$

as $\varepsilon = \varepsilon_j \searrow 0$ with certain nonnegative functions u and v defined in $\Omega \times (0,\infty)$. Moreover, v is a weak solution of (2.1) in the sense of Definition 2.1.

Proof. According to (3.3), (3.5) and Lemma 5.1, (7.3)-(7.6) can be achieved through a straightforward extraction process. Similarly, Corollary 4.3 and Lemma 4.1 imply that (7.1) and (7.2) hold along a further subsequence. In particular, by continuity of f this entails that

$$u_\varepsilon f(v_\varepsilon) \to uf(v) \qquad \text{a.e. in } \Omega \times (0,\infty) \tag{7.8}$$

as $\varepsilon = \varepsilon_j \searrow 0$, which combined with Lemma 6.2 and Egorov's theorem ensures that upon another extraction we may assume that

$$u_\varepsilon f(v_\varepsilon) \rightharpoonup uf(v) \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty))$$

as $\varepsilon = \varepsilon_j \searrow 0$. In light of Lemma 10.3 below, again using (7.8) we conclude that even (7.7) holds. Now the verification of the claimed solution property of v is quite standard: Given φ with the properties listed in Definition 2.1, testing the second equation in (3.1) against φ yields

$$\int_0^\infty \int_\Omega v_\varepsilon \varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega u_\varepsilon f(v_\varepsilon)\cdot \varphi \tag{7.9}$$

for all $\varepsilon \in (0,1)$. Since φ has compact support in $\bar\Omega \times [0,\infty)$, the properties $\varphi_t \in L^2(\Omega\times(0,\infty)), \nabla\varphi \in L^2(\Omega \times (0,\infty))$ and $\varphi \in L^\infty(\Omega\times(0,\infty))$ in conjunction with (7.4), (7.6) and (7.7), respectively, imply that the identity (2.2) results from (7.9) upon taking $\varepsilon = \varepsilon_j \searrow 0$ in each integral separately. $\Box$

8 Strong precompactness of $(\nabla v_\varepsilon)_{\varepsilon\in(0,1)}$

Let us next fully concentrate on the problem of asserting strong precompactness of $(\nabla v_\varepsilon)_{\varepsilon\in(0,1)}$. Having ∇v as a candidate for the desired limit at hand now, and knowing that by the weak convergence statement in (7.6) we have $\int_0^T\int_\Omega |\nabla v|^2 \le \liminf_{\varepsilon=\varepsilon_j\searrow 0} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2$ for $T>0$, in order to show that actually $\nabla v_\varepsilon \to \nabla v$ in $L^2(\Omega\times(0,T))$ it is sufficient to make sure that $\int_0^T\int_\Omega|\nabla v|^2$ satisfies a corresponding estimate from below. This will be a consequence of the following lemma which is concerned with the standard entropy identity

$$\frac{1}{2}\frac{d}{dt}\int_\Omega v^2 + \int_\Omega |\nabla v|^2 = -\int_\Omega uvf(v), \qquad t>0,$$

that clearly holds for smooth solutions of (2.1), but which seems not to be extensible in a straightforward way to arbitrary weak solutions v of (2.1) with the function u on the right-hand side only belonging to the non-reflexive space $L^\infty((0,T);L^1(\Omega))$. After all, upon a suitable choice of test functions in (2.2) it is possible to derive a corresponding inequality which will be sufficient for our purpose.

Lemma 8.1 *There exists a null set $N \subset (0,\infty)$ such that the limit functions u and v gained in Lemma 7.1 satisfy the inequality*

$$\frac{1}{2}\int_\Omega v^2(\cdot,T) - \frac{1}{2}\int_\Omega v_0^2 + \int_0^T\int_\Omega |\nabla v|^2 \ge -\int_0^T\int_\Omega uvf(v) \qquad \textit{for all } T \in (0,\infty)\setminus N. \tag{8.1}$$

PROOF. Since $v \in L^\infty(\Omega \times (0,\infty))$, $z(t) := \int_\Omega v^2(x,t)dx,\ t>0$, defines a function $z \in L^1_{loc}([0,\infty))$. Therefore there exists a null set $N \subset (0,\infty)$ such that each $T \in (0,\infty) \setminus N$ is a Lebesgue point of z; in particular,

$$\frac{1}{\delta}\int_T^{T+\delta}\int_\Omega v^2(x,t)dxdt \to \int_\Omega v^2(x,T)dx \quad \text{for all } T \in (0,\infty)\setminus N \qquad \text{as } \delta \searrow 0. \tag{8.2}$$

To see that (8.1) holds with this choice of N, given any $T \in (0,\infty)\setminus N$ and $\delta \in (0,1)$ we let

$$\zeta_\delta(t) := \begin{cases} 1, & t \in [0,T], \\ 1 - \frac{t-T}{\delta}, & t \in (T, T+\delta), \\ 0, & t \geq T, \end{cases}$$

and define

$$\tilde{v}_k(x,t) := \begin{cases} v(x,t), & (x,t) \in \Omega \times (0,\infty), \\ v_{0k}(x), & (x,t) \in \Omega \times (-1,0], \end{cases}$$

for $k \in \mathbb{N}$, where $(v_{0k})_{k\in\mathbb{N}} \subset C^1(\bar{\Omega})$ is such that $v_{0k} \to v_0$ in $L^2(\Omega)$. Then for $\delta \in (0,1), k \in \mathbb{N}$ and $h \in (0,1)$ we introduce

$$\varphi(x,t) := \varphi_{\delta,k,h}(x,t) := \zeta_\delta(t) \cdot (A_h\tilde{v}_k)(x,t), \qquad (x,t) \in \Omega \times (0,\infty),$$

where the temporal average $A_h\tilde{v}_k$ is defined as

$$(A_h\tilde{v}_k)(x,t) := \frac{1}{h}\int_{t-h}^t \tilde{v}_k(x,s)ds, \qquad (x,t) \in \Omega \times (0,\infty).$$

Since $v \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ by Lemma 7.1, it can easily be checked that also φ belongs to $L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$, and that in addition φ is supported in $\bar{\Omega} \times [0, T+1]$ with

$$\varphi_t(x,t) = \zeta_\delta'(t) \cdot (A_h\tilde{v}_k)(x,t) + \zeta_\delta(t) \cdot \frac{1}{h}\Big(\tilde{v}_k(x,t) - \tilde{v}_k(x,t-h)\Big), (x,t) \in \Omega \times (0,\infty),$$

implying that $\varphi_t \in L^2(\Omega \times (0,\infty))$. We may therefore insert φ into (2.2) to obtain

$$\begin{aligned} I_1(\delta,k,h) + I_2(\delta,k,h) \quad &:= \int_0^\infty \int_\Omega \zeta_\delta(t) \nabla v(x,t) \cdot \nabla(A_h\tilde{v}_k)(x,t)dxdt \\ &\quad + \int_0^\infty \int_\Omega \zeta_\delta(t) u(x,t) f(v(x,t)) \cdot (A_h\tilde{v}_k)(x,t)dxdt \\ &= \int_0^\infty \int_\Omega \zeta_\delta'(t) v(x,t) \cdot (A_h\tilde{v}_k)(x,t)dxdt \\ &\quad + \int_0^\infty \int_\Omega \zeta_\delta(t) v(x,t) \cdot \frac{1}{h}\Big(\tilde{v}_k(x,t) - \tilde{v}_k(x,t-h)\Big)dxdt \\ &\quad + \int_\Omega v_0(x) v_{0k}(x)dx \\ &=: I_3(\delta,k,h) + I_4(\delta,k,h) + I_5(\delta,k,h), \end{aligned} \tag{8.3}$$

where we have used that

$$\varphi(x,0) = \zeta_\delta(0) \cdot \frac{1}{h} \int_{-h}^0 \tilde{v}_k(x,s)ds = v_{0k}(x), \qquad x \in \Omega,$$

by definition of ζ_δ and $\tilde{v}_k$. Now since $v_{0k} \in C^1(\bar{\Omega})$, it follows that $\nabla \tilde{v}_k \in L^2(\Omega \times (-1, T+1))$, so that Lemma 10.2 a) below applies to yield

$$\nabla(A_h \tilde{v}_k) = A_h(\nabla \tilde{v}_k) \rightharpoonup \nabla \tilde{v}_k = \nabla v \quad \text{in } L^2(\Omega \times (0, T+1)) \qquad \text{as } h \searrow 0,$$

so that

$$I_1(\delta,k,h) \to \int_0^\infty \int_\Omega \zeta_\delta(t) \cdot |\nabla v|^2(x,t) dxdt \qquad \text{as } h \searrow 0. \tag{8.4}$$

Similarly, the inclusion $\tilde{v}_k \in L^\infty(\Omega \times (-1, T+1))$ along with Lemma 10.2 b) ensures that

$$A_h \tilde{v}_k \overset{\star}{\rightharpoonup} \tilde{v}_k = v \quad \text{in } L^\infty(\Omega \times (0, T+1)) \qquad \text{as } h \searrow 0, \tag{8.5}$$

and that hence

$$I_2(\delta,k,h) \to \int_0^\infty \int_\Omega \zeta_\delta(t) u(x,t) v(x,t) f(v(x,t)) dxdt \qquad \text{as } h \searrow 0, \tag{8.6}$$

because $uf(v) \in L^1((\Omega \times (0, T+1))$. By (8.5) we clearly also see that

$$I_3(\delta,k,h) \to \int_0^\infty \int_\Omega \zeta'_\delta(t) v^2(x,t) dxdt \qquad \text{as } h \searrow 0. \tag{8.7}$$

In order to analyze the corresponding limit behavior of $I_4(\delta,k,h)$, we split this integral according to

$$I_4(\delta,k,h) = \frac{1}{h} \int_0^\infty \int_\Omega \zeta_\delta(t) \tilde{v}_k^2(x,t) dxdt - \frac{1}{h} \int_0^\infty \int_\Omega \zeta_\delta(t) \tilde{v}_k(v,t) \tilde{v}_k(x,t-h) dxdt$$

and estimate the second term on the right by means of Young's inequality to see that

$$\begin{aligned} \frac{1}{h} \int_0^\infty \int_\Omega \zeta_\delta(t) \tilde{v}_k(v,t) \tilde{v}_k(x,t-h) dxdt \quad &\le \quad \frac{1}{2h} \int_0^\infty \int_\Omega \zeta_\delta(t) \tilde{v}_k^2(x,t) dxdt \\ &\quad + \frac{1}{2h} \int_0^\infty \int_\Omega \zeta_\delta(t) \tilde{v}_k^2(x,t-h) dxdt. \end{aligned}$$

Thus, upon substituting $s = t - h$, we find that

$$\begin{aligned} I_4(\delta,k,h) \quad &\ge \quad \frac{1}{2h} \int_0^\infty \int_\Omega \zeta_\delta(t) v^2(x,t) dxdt - \frac{1}{2h} \int_0^\infty \int_\Omega \zeta_\delta(t) \tilde{v}_k^2(x,t-h) dxdt \\ &= \quad \frac{1}{2h} \int_0^\infty \int_\Omega \zeta_\delta(t) v^2(x,t) dxdt - \frac{1}{2h} \int_0^\infty \int_\Omega \zeta_\delta(s+h) v^2(x,s) dxds \\ &\quad - \frac{1}{2h} \int_0^h \int_\Omega \zeta_\delta(t) v_{0k}^2(x) dxdt \\ &= \quad -\frac{1}{2} \int_0^\infty \int_\Omega \frac{\zeta_\delta(t+h) - \zeta_\delta(t)}{h} \cdot v^2(x,t) dxdt - \frac{1}{2h} \int_0^h \int_\Omega \zeta_\delta(t) v_{0k}^2(x) dxdt, \end{aligned}$$

again because $\tilde{v}_k(\cdot,t) = v_{0k}$ for $t \in (-1,0)$. Here since ζ_δ is continuous with $\zeta_\delta(0) = 0$, we have

$$-\frac{1}{2h}\int_0^h \int_\Omega \zeta_\delta(t) v_{0k}^2(x)dxdt \to -\frac{1}{2}\int_\Omega v_{0k}^2(x)dx \qquad \text{as } h \searrow 0,$$

whereas by the dominated convergence theorem we conclude that

$$-\frac{1}{2}\int_0^\infty \int_\Omega \frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h} \cdot v^2(x,t)dxdt \to -\frac{1}{2}\int_0^\infty \int_\Omega \zeta_\delta'(t)v^2(x,t)dxdt \qquad \text{as } h \searrow 0,$$

whence altogether we infer that

$$\liminf_{h\searrow 0} I_4(\delta,k,h) \geq -\frac{1}{2}\int_0^\infty \int_\Omega \zeta_\delta'(t)v^2(x,t)dxdt - \frac{1}{2}\int_\Omega v_{0k}^2(x)dx.$$

Therefore, taking $h \searrow 0$ and recalling (8.4), (8.6) and (8.7), from (8.3) we obtain the inequality

$$\begin{aligned}\int_0^\infty \int_\Omega \zeta_\delta(t)|\nabla v(x,t)|^2 dxdt \quad &+ \quad \int_0^\infty \int_\Omega \zeta_\delta(t)u(x,t)v(x,t)f(v(x,t))dxdt \\ &\geq \frac{1}{2}\int_0^\infty \int_\Omega \zeta_\delta'(t)v^2(x,t)dxdt \\ &\quad -\frac{1}{2}\int_\Omega v_{0k}^2(x)dx + \int_\Omega v_0(x)v_{0k}(x)dx\end{aligned}$$

for all $k \in \mathbb{N}$, in the limit $k \to \infty$ implying that

$$\begin{aligned}\int_0^\infty \int_\Omega \zeta_\delta(t)|\nabla v(x,t)|^2 dxdt \quad &+ \quad \int_0^\infty \int_\Omega \zeta_\delta(t)u(x,t)v(x,t)f(v(x,t))dxdt \\ &\geq \frac{1}{2}\int_0^\infty \int_\Omega \zeta_\delta'(t)v^2(x,t)dxdt + \frac{1}{2}\int_\Omega v_0^2(x)dx.\end{aligned} \tag{8.8}$$

Now by definition of ζ_δ, the first term on the right satisfies

$$\begin{aligned}\frac{1}{2}\int_0^\infty \int_\Omega \zeta_\delta'(t)v^2(x,t)dxdt &= -\frac{1}{2\delta}\int_T^{T+\delta} \int_\Omega v^2(x,t)dxdt \\ &\to -\frac{1}{2}\int_\Omega v^2(x,T)dx \qquad \text{as } \delta \searrow 0\end{aligned}$$

according to the Lebesgue point property of T. Applying the monotone convergence theorem to both integrals on the left of (8.8), we thereupon readily arrive at (8.1). $\Box$

We can now establish the desired strong convergence result. Besides on the above inequality (8.1), its derivation essentially relies on the strong convergence statement in (7.7).

Lemma 8.2 *Let $(\varepsilon_j)_{j\in\mathbb{N}}$ be as provided by Lemma 7.1. Then there exists a subsequence, again denoted by $(\varepsilon_j)_{j\in\mathbb{N}}$, such that for each $T > 0$ we have*

$$\nabla v_\varepsilon \to \nabla v \quad \text{in } L^2(\Omega\times(0,T)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{8.9}$$

Proof. Since we know from (7.4) that $v_\varepsilon \to v$ in $L^2_{loc}(\bar\Omega \times [0,\infty))$, upon passing to a subsequence if necessary we may assume that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$\int_\Omega v_\varepsilon^2(\cdot,T) \to \int_\Omega v^2(\cdot,T) \qquad \text{for all } T \in (0,\infty) \setminus N_1$$

with some null set $N_1 \subset (0,\infty)$. Taking $N \subset (0,\infty)$ as in Lemma 8.1, we then evidently only need to verify (8.9) for all $T \in (0,\infty) \setminus (N \cup N_1)$. Given any such T, we apply (7.7) to see that as $\varepsilon = \varepsilon_j \searrow 0$,

$$u_\varepsilon f(v_\varepsilon) \to u f(v) \qquad \text{in } L^1(\Omega \times (0,T)),$$

which thanks to the fact that

$$v_\varepsilon \overset{\star}{\rightharpoonup} v \qquad \text{in } L^\infty(\Omega \times (0,T))$$

by (7.5) implies that

$$\int_0^T \int_\Omega u_\varepsilon v_\varepsilon f(v_\varepsilon) \to \int_0^T \int_\Omega u v f(v).$$

Therefore Lemma 8.1 says that due to our choice of T,

$$\begin{aligned}
\int_0^T \int_\Omega |\nabla v|^2 &\ge -\frac{1}{2}\int_\Omega v^2(\cdot,T) + \frac{1}{2}\int_\Omega v_0^2 - \int_0^T \int_\Omega uvf(v) \\
&= \lim_{\varepsilon=\varepsilon_j \searrow 0} \left\{ -\frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,T) + \frac{1}{2}\int_\Omega v_0^2 - \int_0^T \int_\Omega u_\varepsilon v_\varepsilon f(v_\varepsilon) \right\}.
\end{aligned}$$

Since testing the second equation in (3.1) by v_ε yields

$$-\frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,T) + \frac{1}{2}\int_\Omega v_0^2 - \int_0^T \int_\Omega u_\varepsilon v_\varepsilon f(v_\varepsilon) = \int_0^T \int_\Omega |\nabla v_\varepsilon|^2,$$

this entails that

$$\int_0^T \int_\Omega |\nabla v|^2 \ge \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2. \tag{8.10}$$

On the other hand, by lower semicontinuity of the norm in $L^2(\Omega \times (0,T))$ with respect to weak convergence,

$$\int_0^T \int_\Omega |\nabla v|^2 \le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^T \int_\Omega |\nabla v_\varepsilon|^2, \tag{8.11}$$

whence (8.9) results from (8.10) and (8.11) together with (7.6) upon a well-known argument. □

9 Solution properties of u. Proof of Theorem 1.1

We are now in the position to show that also the limit u from Lemma 7.1 solves its associated subsystem in (1.1) in the sense specified in Definition 2.3. We first establish the mass inequality (2.6).

Lemma 9.1 *The function u gained in Lemma 7.1 satisfies $u \in L^\infty((0,\infty); L^1(\Omega))$ and*

$$\int_\Omega u(\cdot,t) \le \int_\Omega u_0 \qquad \textit{for a.e. } t>0.$$

PROOF. Since according to (3.2) we have $\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0$ for all $t>0$ and each $\varepsilon \in (0,1)$, both statements are consequences from (7.1) and Fatou's lemma. $\square$

The derivation of a corresponding ϕ-supersolution property of u in the spirit of Definition 2.2 is more delicate and crucially involves Lemma 8.2.

Lemma 9.2 *Let u and v be as constructed in Lemma 7.1. Then u is a global very weak ϕ-supersolution of (2.3) with*

$$\phi(s) := \ln(s+1), \qquad s \ge 0,$$

in the sense of Definition 2.2.

PROOF. Using (7.2), we first see that $\phi(u)$ and $u\phi'(u) = \frac{u}{u+1}$ belong to $L^1_{loc}(\bar\Omega \times [0,\infty))$ and to $L^2_{loc}(\bar\Omega \times [0,\infty))$, respectively. Moreover, (7.2) guarantees that

$$\phi''(u)|\nabla u|^2 = -\frac{|\nabla u|^2}{(u+1)^2} = -|\nabla \ln(u+1)|^2 \in L^1_{loc}(\bar\Omega \times [0,\infty)),$$

and that since

$$|u\phi''(u)\nabla u| = \frac{u|\nabla u|}{(u+1)^2} \le |\nabla \ln(u+1)|,$$

we also have $u\phi''(u)\nabla u \in L^2_{loc}(\bar\Omega \times [0,\infty))$.
Now given any nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ with $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$, multiplying the first equation in (3.1) by $\phi'(u_\varepsilon)\cdot\varphi = \frac{1}{u_\varepsilon+1}\cdot\varphi$ and integrating by parts we derive the identity

$$\begin{aligned}
\int_0^\infty \int_\Omega \frac{1}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\varphi &= -\int_0^\infty \int_\Omega \ln(u_\varepsilon+1)\varphi_t - \int_\Omega \ln(u_0+1)\varphi(\cdot,0) \\
&\quad - \int_0^\infty \int_\Omega \ln(u_\varepsilon+1)\Delta\varphi \\
&\quad + \int_0^\infty \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2}\nabla u_\varepsilon \cdot \Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\cdot\varphi \\
&\quad - \int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1}\Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon)\cdot\nabla v_\varepsilon\Big)\cdot\nabla\varphi \qquad (9.1)
\end{aligned}$$

for all $\varepsilon \in (0,1)$. Here, thanks to (7.2) we have

$$-\int_0^\infty \int_\Omega \ln(u_\varepsilon+1)\varphi_t \to -\int_0^\infty \int_\Omega \ln(u+1)\varphi_t \qquad (9.2)$$

and

$$-\int_0^\infty \int_\Omega \ln(u_\varepsilon+1)\Delta\varphi \to -\int_0^\infty \int_\Omega \ln(u+1)\Delta\varphi \qquad (9.3)$$

as $\varepsilon = \varepsilon_j \searrow 0$. Furthermore, by definition of S_ε the statements (7.1) and (7.3) imply that as $\varepsilon = \varepsilon_j \searrow 0$ we have

$$\frac{u_\varepsilon}{u_\varepsilon+1} S_\varepsilon(x,u_\varepsilon,v_\varepsilon) \to \frac{u}{u+1} S(x,u,v) \qquad \text{a.e. in } \Omega \times (0,\infty).$$

In light of the uniform majorization

$$\left| \frac{u_\varepsilon}{u_\varepsilon+1} S_\varepsilon(x,u_\varepsilon,v_\varepsilon) \right| \le S_1 \quad \text{in } \Omega \times (0,\infty) \qquad \text{for all } \varepsilon \in (0,1),$$

as warranted by Lemma 3.2, and the fact that due to Lemma 8.2 we have

$$\nabla v_\varepsilon \to \nabla v \qquad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)),$$

according to Lemma 10.4 this implies the strong convergence property

$$\frac{u_\varepsilon}{u_\varepsilon+1} \Big(S_\varepsilon(\cdot,u_\varepsilon,v_\varepsilon) \cdot \nabla v_\varepsilon \Big) \to \frac{u}{u+1} \Big(S(\cdot,u,v) \cdot \nabla v \Big) \qquad \text{in } L^2_{loc}(\bar\Omega \times [0,\infty)) \tag{9.4}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Together with (7.2) this entails that

$$\begin{aligned}
\int_0^\infty \int_\Omega \frac{u_\varepsilon}{(u_\varepsilon+1)^2} \nabla u_\varepsilon \cdot \Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon) \cdot \nabla v_\varepsilon \Big) \cdot \varphi &= \int_0^\infty \int_\Omega \nabla \ln(u_\varepsilon+1) \cdot \Big(\frac{u_\varepsilon}{u_\varepsilon+1} S_\varepsilon(x,u_\varepsilon,v_\varepsilon) \cdot \nabla v_\varepsilon \Big) \cdot \varphi \\
&\to \int_0^\infty \int_\Omega \nabla \ln(u+1) \cdot \Big(\frac{u}{u+1} S(x,u,v) \cdot \nabla v \Big) \cdot \varphi \\
&= \int_0^\infty \int_\Omega \frac{u}{(u+1)^2} \nabla u \cdot \Big(S(x,u,v) \cdot \nabla v \Big) \cdot \varphi
\end{aligned} \tag{9.5}$$

as $\varepsilon = \varepsilon_j \searrow 0$. Moreover, (9.4) clearly also guarantees that

$$-\int_0^\infty \int_\Omega \frac{u_\varepsilon}{u_\varepsilon+1} \Big(S_\varepsilon(x,u_\varepsilon,v_\varepsilon) \cdot \nabla v_\varepsilon \Big) \cdot \nabla\varphi \to -\int_0^\infty \int_\Omega \frac{u}{u+1} \Big(S(x,u,v) \cdot \nabla v \Big) \cdot \nabla \varphi \tag{9.6}$$

as $\varepsilon = \varepsilon_j \searrow 0$.

Collecting (9.2), (9.3), (9.5) and (9.6), by a lower semicontinuity argument we thus infer from (9.1) and the nonegativity of φ that

$$\begin{aligned}
\int_0^\infty \int_\Omega \frac{1}{(u+1)^2} |\nabla u|^2 \varphi &\le \liminf_{\varepsilon=\varepsilon_j \searrow 0} \int_0^\infty \int_\Omega \frac{1}{(u_\varepsilon+1)^2} |\nabla u_\varepsilon|^2 \varphi \\
&= -\int_0^\infty \int_\Omega \ln(u+1)\varphi_t - \int_\Omega \ln(u_0+1)\varphi(\cdot,0) \\
&\quad - \int_0^\infty \int_\Omega \ln(u+1)\Delta\varphi \\
&\quad + \int_0^\infty \int_\Omega \frac{u}{(u+1)^2} \nabla u \cdot \Big(S(x,u,v) \cdot \nabla v \Big) \cdot \varphi \\
&\quad - \int_0^\infty \int_\Omega \frac{u}{u+1} \Big(S(x,u,v) \cdot \nabla v \Big) \cdot \nabla\varphi
\end{aligned}$$

for any such test function φ, meaning that u indeed is a global very weak ϕ-supersolution of (2.3). □

Thereby our main result on global existence has actually been established already:

Proof of Theorem 1.1. We only need to combine Lemma 9.1 and Lemma 9.2 with Lemma 7.1. □

10 Appendix

Let us briefly collect some basic facts on approximation properties of the Steklov averages defined by

$$(A_h w)(x,t) := \frac{1}{h}\int_{t-h}^{t} w(x,s)ds, \qquad x \in \Omega,\ t \in (0,T),\ h \in (0,1),$$

of a given function $w \in L^1(\Omega \times (-1,T)), T > 0$.

Lemma 10.1 *As $h \searrow 0$, we have $A_h w \to w$ a.e. in $\Omega \times (0,T)$.*

Proof. Since $w \in L^1(\Omega \times (-1,T))$, there exists a null set $N \subset \Omega$ such that $(-1,T) \ni t \mapsto w(x,t)$ belongs to $L^1((-1,T))$ for all $x \in \Omega \setminus N$. Then for fixed $x \in \Omega \setminus N$, by a known result one can pick a null set $N(x) \subset (0,T)$ such that each $t \in (0,T) \setminus N(x)$ is a Lebesgue point of $(0,T) \ni \tilde{t} \mapsto w(x,\tilde{t})$. This means that with

$$N_\star := \Big\{(x,t) \in \Omega \times (0,T) \ \Big| \ x \in N \text{ or } (x \in \Omega \setminus N \text{ and } t \in N(x))\Big\},$$

we have $(A_h w)(x,t) \to w(x,t)$ as $h \searrow 0$ for any $(x,t) \in (\Omega \times (0,T)) \setminus N_\star$. But since

$$\begin{aligned}\int_\Omega \int_0^T \chi_{N_\star}(x,t)dtdx &= \int_N \int_0^T dtdx + \int_{\Omega\setminus N}\int_{N(x)} dtdx \\ = |N| \cdot T + \int_{\Omega\setminus N} |N(x)|dx = 0, &\end{aligned}$$

the Tonelli theorem ensures that $N_\star$ is a null set in $\Omega \times (0,T)$. $\square$

Lemma 10.2 *a) If $w \in L^p(\Omega \times (-1,T))$ for some $p \in (1,\infty)$, then $A_h w \rightharpoonup w$ in $L^p(\Omega \times (0,T))$ as $h \searrow 0$.*
b) If $w \in L^\infty(\Omega \times (-1,T))$, then $A_h w \overset{\star}{\rightharpoonup} w$ in $L^\infty(\Omega \times (0,T))$ as $h \searrow 0$.

Proof. In view of a standard argument involving appropriate extraction of subsequences, Lemma 10.1 and Egorov's theorem, it is sufficient to assert boundedness of $(A_h w)_{h\in(0,1)}$ in the respective spaces. In the case in a) this follows on applying the Hölder inequality and Fubini's theorem in estimating

$$\begin{aligned}\|A_h w\|^p_{L^p(\Omega\times(0,T))} &= \frac{1}{h^p}\int_\Omega\int_0^T \left|\int_{t-h}^t w(x,s)ds\right|^p dtdx \\ &\le \frac{1}{h^p}\cdot h^{p-1}\int_\Omega\int_0^T\int_{t-h}^t |w(x,s)|^p dsdtdx \\ &= \frac{1}{h}\int_\Omega\int_0^T\int_s^{s+h} |w(x,s)|^p dtdsdx \\ &= \|w\|^p_{L^p(\Omega\times(-h,T-h))} \le \|w\|^p_{L^p(\Omega\times(-1,T))} \qquad \text{for all } h \in (0,1).\end{aligned}$$

In the situation in b) it is immediate that $\|A_h w\|_{L^\infty(\Omega\times(0,T))} \le \|w\|_{L^\infty(\Omega\times(-1,T))}$ for all $h \in (0,1)$. $\square$

The derivation of the following criterion for strong convergence in L^1 is quite straightforward. Since we could not find a precise reference in the literature, we include a short proof.

Lemma 10.3 *Let $N \geq 1$ and $M \subset \mathbb{R}^N$ be measurable, and suppose that $(w_j)_{j\in\mathbb{N}} \subset L^1(M)$ is such that $w_j \geq 0$ a.e. in M for all $j \in \mathbb{N}$ and*

$$w_j \rightharpoonup w \quad \textit{in } L^1(M) \qquad \textit{and} \qquad w_j \to w \quad \textit{a.e. in } M \qquad \textit{as } j \to \infty \tag{10.1}$$

with some $w \in L^1(M)$. Then

$$w_j \to w \quad \textit{in } L^1(M) \qquad \textit{as } j \to \infty \tag{10.2}$$

PROOF. Since $(w_j)_{j\in\mathbb{N}}$ is necessarily bounded in $L^1(M)$, the sequence $(\sqrt{w_j})_{j\in\mathbb{N}}$ is bounded in the Hilbert space $L^2(M)$ and hence relatively compact in this space with respect to the weak topology. In view of the second assumption in (10.1) and Egorov's theorem we thus have $\sqrt{w_j} \rightharpoonup \sqrt{w}$ in $L^2(M)$ as $j \to \infty$.
Using constant test functions in the weak convergence statement in (10.1), we moreover obtain that

$$\int_M (\sqrt{w_j})^2 = \int_M w_j \cdot 1 \to \int_M w \cdot 1 = \int_M (\sqrt{w})^2 \qquad \text{as } j \to \infty,$$

which combined with the former yields $\sqrt{w_j} \to \sqrt{w}$ in $L^2(M)$ as $j \to \infty$. By means of the Cauchy-Schwarz inequality, this in turn implies that

$$\begin{aligned} \|w_j - w\|_{L^1(M)} &= \int_M \Big|(\sqrt{w_j})^2 - (\sqrt{w})^2\Big| = \int_M |\sqrt{w_j} - \sqrt{w}| \cdot |\sqrt{w_j} + \sqrt{w}| \\ &\leq \Big\|\sqrt{w_j} - \sqrt{w}\Big\|_{L^2(M)} \cdot \Big(\|\sqrt{w_j}\|_{L^2(M)} + \|\sqrt{w}\|_{L^2(M)}\Big) \\ &\to 0 \qquad \text{as } j \to \infty, \end{aligned}$$

as claimed. $\square$

We finally note a useful consequence of the dominated convergence theorem.

Lemma 10.4 *Let $N \geq 1$ and $M \subset \mathbb{R}^N$ be measurable, and suppose that $(w_j)_{j\in\mathbb{N}} \subset L^\infty(M)$ and $(z_j)_{j\in\mathbb{N}} \subset L^2(M)$ are such that*

$$|w_j| \leq C \quad \textit{in } M \qquad \textit{for all } j \in \mathbb{N} \tag{10.3}$$

as well as

$$w_j \to w \qquad \textit{a.e. in } M \tag{10.4}$$

and

$$z_j \to z \qquad \textit{in } L^2(M) \tag{10.5}$$

as $j \to \infty$ for some $C > 0$, $w \in L^\infty(M)$ and $z \in L^2(M)$. Then

$$w_j z_j \to wz \quad \textit{in } L^2(M) \qquad \textit{as } j \to \infty. \tag{10.6}$$

PROOF. We directly estimate

$$\int_N |w_j z_j - wz|^2 \leq 2\int_M (w_j - w)^2 z^2 + 2\int_M w_j^2 (z_j - z)^2, \tag{10.7}$$

where by (10.3)-(10.5) and the dominated convergence theorem we have

$$2\int_M (w_j - w)^2 z^2 \to 0 \qquad \text{as } j \to \infty.$$

Since thanks to (10.3) and (10.5) we know that also

$$2\int_M w_j^2 (z_j - z)^2 \le 2C^2 \|z_j - z\|^2_{L^2(M)} \to 0 \qquad \text{as } j \to \infty,$$

(10.7) implies (10.6). □

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