# Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics


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**Abstract**

The fully parabolic two-species chemotaxis system

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1-u-a_1 v), \qquad x\in\Omega,\ t>0, \\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1-v-a_2 u), \qquad x\in\Omega,\ t>0, \\ w_t = d_3\Delta w - \gamma w + \alpha u + \beta v, \qquad x\in\Omega,\ t>0, \end{cases} \qquad (\star)$$

is considered in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary.

It is shown that if $n\le 2$ and all parameters in $(\star)$ are merely positive, then for all appropriately regular nonnegative initial data $u_0, v_0$ and $w_0$ the corresponding Neumann initial-boundary value problem possesses a unique global bounded solution.

Moreover, by means of the construction of suitable energy functionals it is proved that whenever $n\ge 1$,

- if $a_1<1$ and $a_2<1$ and both $\mu_1$ and $\mu_2$ are sufficiently large, then any global bounded solution emanating from adequately regular initial data fulfilling $u_0\not\equiv 0\not\equiv v_0$ satisfies

$$(u,v,w)(\cdot,t)\to(u_\star,v_\star,w_\star) \quad \text{uniformly in } \Omega \qquad \text{as } t\to\infty,$$

where $(u_\star,v_\star,w_\star)$ denotes the unique positive spatially homogeneous equilibrium of $(\star)$, and that

- if $a_1\ge 1$ and $a_2<1$ and $\mu_2$ is large enough, then all global bounded solution with reasonably smooth initial data satisfying $v_0\not\equiv 0$ have the property that

$$(u,v,w)(\cdot,t)\to\Big(0,1,\frac{\alpha}{\gamma}\Big) \quad \text{uniformly in } \Omega \qquad \text{as } t\to\infty.$$

The respective rates of convergence are shown to be at least exponential when $a_1\ne 1$, and algebraic if $a_1=1$.



**Key words:** multi-species chemotaxis; competition, logistic source; stability; domain of attraction
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# 1 Introduction

We consider the initial-boundary value problem

$$\begin{cases} u_t = d_1\Delta u - \chi_1\nabla\cdot(u\nabla w) + \mu_1 u(1-u-a_1 v), & x\in\Omega,\ t>0,\\ v_t = d_2\Delta v - \chi_2\nabla\cdot(v\nabla w) + \mu_2 v(1-v-a_2 u), & x\in\Omega,\ t>0,\\ w_t = d_3\Delta w - \gamma w + \alpha u + \beta v, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x),\quad v(x,0)=v_0(x),\quad w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, where $d_1,d_2,d_3,\chi_1,\chi_2,\mu_1,\mu_2,\alpha,\beta$ and $\gamma$ are positive constants, and where $u_0, v_0$ and $w_0$ are given nonnegative functions.

The problem (1.1) arises in mathematical biology as a model for the spatio-temporal evolution of two populations which proliferate and compete according to a Lotka-Volterra-type kinetics, and in which individuals are moreover able to move according to both random diffusion and chemotaxis toward a signal jointly produced by themselves. In this setting, $u = u(x,t)$ and $v = v(x,t)$ represent the respective densities of the two populations and $w = w(x,t)$ denotes the concentration of the chemical (see e.g. [32] for a discussion of chemotaxis-competition models and [9] for a broader survey).

The existing literature on two-species chemotaxis systems mainly concentrates on simplified models obtained on describing the evolution of the signal by an elliptic rather than a parabolic equation. Without kinetic terms, that is, when $\mu_1=\mu_2=0$, the resulting system then inherits some important properties from the original Keller-Segel model for single-species chemotaxis; in particular, the striking phenomenon of finite-time blow-up, known to occur in both parabolic-elliptic and fully parabolic versions of the latter ([25], [40], [23]), has also been detected in parabolic-parabolic-elliptic two-species systems ([1], [2], [3], [4], [6]).

On the other hand, well-known results from the analysis of one-species chemotaxis systems indicate that such explosions might be ruled out by absorption terms as in (1.1), essentially quadratic with regard to the respective unknown: Indeed, in the special case when $v\equiv 0$ in (1.1), all solutions emanating from reasonably regular initial data are global in time and remain bounded whenever $n\le 2$ and $\mu_1>0$ is arbitrary ([28]), or $n\ge 3$ and $\mu_1$ is sufficiently large ([38]), with the corresponding condition on the size of $\mu_1$ being even explicit in the parabolic-elliptic counterpart ([34]). If $n=3$ and $\mu_1$ is merely positive, then at least certain global weak solutions can be constructed ([20]). Numerical experiments suggest that despite possible absence of blow-up, such chemotaxis-growth systems may exhibit quite rich dynamics, including even chaotic behavior ([10], [17]). This is partially supported by results on structured steady states ([17]) and by recent analytical findings on transient growth phenomena ([41], [19]). We mention that also subquadratic degradation may allow for global solvability in certain modified variants of the systems, provided that its effect is suitably relative to the aggregative mechanisms of cross-diffusion and signal production ([26], [27], [36]), but counterexamples show that not any superlinear absorption is sufficient to rule out blow-up ([39]).

As for two-species models with logistic-type growth restrictions as in (1.1), a comprehensive theory of global solvability is apparently lacking, and even much less is known about qualitative behavior of bounded solutions. A fully parabolic variant of (1.1) is studied in [42], where global bounded solutions are constructed under the assumptions that the chemotactic sensitivities, allowed to depend on the signal concentration $w$ in the considered model, decay sufficiently fast as $w \to \infty$. Beyond questions of global existence, the large time behavior in the parabolic-elliptic counterpart of (1.1) has been addressed in [35] for the case of weak competition when both $a_1 < 1$ and $a_2 < 1$, and in [32] for the case $a_1 > 1$ and $a_2 < 1$. In this simplified situation, namely, the system actually reduces to merely two parabolic equations with spatially nonlocal terms, and thereby becomes accessible to certain elaborate comparison techniques. Based on this powerful tool, the results in [35] and [32] show that under appropriate conditions on the size of $\mu_1$ and $\mu_2$ as related to the chemotactic sensitivities $\chi_1$ and $\chi_2$, essentially requiring suitable smallness assumptions on the latter, all reasonable solutions of the PDE system are global and bounded, and that their large time behavior is determined by the asymptotics in the ODE system

$$\begin{cases} u_t = \mu_1 u(1-u-a_1 v), & t>0, \\ v_t = \mu_2 v(1-v-a_2 u), & t>0, \end{cases} \tag{1.2}$$

associated with (1.1). In particular, this means that in the large time limit, $(u,v)$ approaches the constant vector $(\frac{1-a_1}{1-a_1a_2}, \frac{1-a_2}{1-a_1a_2})$ when $a_1 < 1$ and $a_2 < 1$ ([35]), whereas $(u,v) \to (0,1)$ as $t \to \infty$ if $a_1 > 1$ and $a_2 < 1$ ([32]). Strong use of an elliptic simplification in the signal evolution is also made in the analysis of a two-species chemotaxis model involving two chemicals in [33]).

**Main results.** In the currently considered fully parabolic system (1.1), such a reduction is apparently impossible. The goal of this work is to develop an approach which despite this complification provides insight into the dynamical properties of (1.1). In order to achieve this, let us complete the problem setting by requiring that the initial data in (1.1) satisfy

$$\begin{cases} u_0 \in C^0(\bar\Omega) \quad \text{with } u_0 \ge 0 \text{ in } \Omega, \\ v_0 \in C^0(\bar\Omega) \quad \text{with } u_0 \ge 0 \text{ in } \Omega, \qquad \text{and that} \\ w_0 \in W^{1,q}(\bar\Omega) \quad \text{for some } q > \max\{2,n\} \quad \text{with } w \ge 0 \text{ in } \Omega. \end{cases} \tag{1.3}$$

A fundamental and necessary first step then consists in establishing a satisfactory existence theory, which will be accomplished by means of suitable a priori estimates to be derived in Section 2. Our result in this direction does not require any condition on the size of $\mu_1$ and $\mu_2$, and hence in view of the mentioned boundedness results for one-species systems the restriction $n \le 2$ herein appears to be natural and optimal, extending a corresponding statement on global existence in [13] to the two-dimensional case.

**Theorem 1.1** *Let $n \le 2$, and let $d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2, \alpha, \beta$ and $\gamma$ be arbitrary positive constants. Then for any choice of functions $u_0, v_0$ and $w_0$ satisfying (1.3) for some $q > \max\{2,n\}$, the problem (1.1) possesses a globally defined classical solution $(u,v,w)$ which is unique within the class of functions fulfilling*

$$\begin{aligned} & u \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ & v \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \qquad \textit{and} \\ & w \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,q}(\Omega)). \end{aligned}$$

*Moreover, this solution is bounded in $\bar\Omega \times (0,\infty)$.*

Next focusing on the asymptotic behavior of solutions, we will first consider the situation when both competitive kinetic terms in (1.1) are weak in the sense that $a_1 \in (0,1)$ and $a_2 \in (0,1)$. In this case, there exists precisely one positive spatially homogeneous steady state of (1.1), that is, a triple $(u_\star, v_\star, w_\star)$ of positive numbers solving the linear algebraic system

$$\begin{cases} 1 - u_\star - a_1 v_\star = 0, \\ 1 - v_\star - a_2 u_\star = 0, \\ -\gamma w_\star + \alpha u_\star + \beta v_\star = 0; \end{cases} \tag{1.4}$$

in fact, this equilibrium is explicitly given by

$$u_\star := \frac{1-a_1}{1-a_1 a_2}, \qquad v_\star := \frac{1-a_2}{1-a_1 a_2} \qquad \text{and} \qquad w_\star := \frac{\alpha(1-a_1)+\beta(1-a_2)}{\gamma(1-a_1 a_2)}, \tag{1.5}$$

and it is known ([24]) to be attractive to all positive solutions of the corresponding ODE system

$$\begin{cases} u_t = \mu_1 u(1-u-a_1 v), & t > 0, \\ v_t = \mu_2 v(1-v-a_2 u), & t > 0, \\ w_t = -\gamma w + \alpha u + \beta v, & t > 0. \end{cases} \tag{1.6}$$

This property continues to hold when diffusion is involved, that is, in the Neumann problem for the parabolic system

$$\begin{cases} u_t = d_1 \Delta u + \mu_1 u(1-u-a_1 v), & x \in \Omega,\ t > 0, \\ v_t = d_2 \Delta v + \mu_2 v(1-v-a_2 u), & x \in \Omega,\ t > 0 \\ w_t = d_3 \Delta w - \gamma w + \alpha u + \beta v, & x \in \Omega, t > 0, \end{cases} \tag{1.7}$$

with positive coefficients $d_1, d_2$ and $d_3$, in which the first two equations are actually decoupled from the third. Then, namely, the system possesses a Lyapunov functional of a form introduced in [8] and [11], and in consequence allows for the conclusion that all nontrivial nonnegative solutions satisfy $(u(\cdot,t), v(\cdot,t), w(\cdot,t)) \to (u_\star, v_\star, w_\star)$ with respect to the norm in $L^\infty(\Omega)$ as $t \to \infty$ (see e.g. [12] for more details on the construction of Lyapunov functionals, and [5, 29, 43] for an independent proof based on comparison arguments).

Section 3.1 will reveal that this global attractivity property of $(u_\star, v_\star, w_\star)$ is actually inherited by the chemotaxis system (1.1) despite the considerably more complex cross-diffusive coupling, provided that the overall effect of the Lotka-Volterra kinetics, as measured by the size of the coefficients $\mu_1$ and $\mu_2$, is sufficiently strong. In that case, namely, we shall see that the system still admits an energy-type inequality (cf. (3.1) and (3.4)), and an analysis thereof will show that any nontrivial solution approaches $(u_\star, v_\star, w_\star)$ exponentially fast with respect to the topology in $L^\infty(\Omega)$:

**Theorem 1.2** *Let $a_1 \in (0,1)$ and $a_2 \in (0,1)$, and suppose that the positive numbers $d_1, d_2, d_3, \chi_1, \chi_2$, $\mu_1, \mu_2, \alpha, \beta$ and $\gamma$ satisfy the relations*

$$\mu_1 > \frac{d_2 \chi_1^2 u_\star}{\frac{4a_1\gamma(1-a_1a_2)d_1d_2d_3}{(a_1\alpha^2+a_2\beta^2-2a_1a_2\alpha\beta)} - \frac{d_1 a_1 \chi_2^2 v_\star}{4\mu_2 a_2}} \tag{1.8}$$

*and*

$$\mu_2 > \frac{\chi_2^2 v_\star (a_1\alpha^2 + a_2\beta^2 - 2a_1a_2\alpha\beta)}{16 d_2 d_3 a_2 \gamma (1 - a_1 a_2)}. \tag{1.9}$$

*Then whenever $n \geq 1$ and $(u, v, w)$ is a global bounded classical solution of (1.1) with initial data $(u_0, v_0, w_0)$ which satisfy (1.3) and $u_0 \not\equiv 0 \not\equiv v_0$, one can find $\lambda > 0$ and $C > 0$ such that*

$$\|u(\cdot,t) - u_\star\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v_\star\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \qquad \text{for all } t > 0, \tag{1.10}$$

*where $(u_\star, v_\star, w_\star)$ is given by (1.5).*

In the strongly asymmetric case when $a_1 \geq 1$ but still $a_2 < 1$, an appropriate largeness assumption on the kinetic influence once more warrants the existence of a Lyapunov functional (see (3.11) and (3.14), again implying that the behavior in (1.1) will essentially be determined by the asymptotics in (1.6); as known to occur in the latter ([24]), we shall see in Section 3.2 that also in this case any nontrivial choice of $v_0$ will imply that the second species eventually outcompetes the first, the corresponding convergence rate being again at least exponential when $a_1 > 1$, but only, and necessarily (cf. the remark following Lemma 3.7), algebraic in the borderline case $a_1 = 1$.

**Theorem 1.3** *Let $d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, \alpha, \beta$ and $\gamma$ be positive constants.*
*i) Let $a_1 > 1$ and $a_2 \in (0, 1)$, and suppose that for some $a_1' \in (1, a_1]$ such that $a_1' a_2 < 1$ we have*

$$\mu_2 > \frac{\chi_2^2 v_\star (a_1'\alpha^2 + a_2\beta^2 - 2a_1'a_2\alpha\beta)}{16 d_2 d_3 a_2 \gamma (1 - a_1' a_2)}. \tag{1.11}$$

*Then for any global bounded classical solution $(u, v, w)$ of (1.1) emanating from initial data satisfying (1.3) as well as $v_0 \not\equiv 0$, one can fix $\lambda > 0$ and $C > 0$ fulfilling*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \left\|w(\cdot,t) - \frac{\alpha}{\gamma}\right\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t} \qquad \text{for all } t > 0. \tag{1.12}$$

*ii) Suppose that $a_1 = 1$ and $a_2 \in (0, 1)$, and that (1.11) holds with $a_1' = 1$. Then if (1.3) holds with $v_0 \not\equiv 0$, any global bounded classical solution $(u, v, w)$ of (1.1) satisfies*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \left\|w(\cdot,t) - \frac{\alpha}{\gamma}\right\|_{L^\infty(\Omega)} \leq C(t+1)^{-\kappa} \qquad \text{for all } t > 0 \tag{1.13}$$

*with some $\kappa > 0$ and $C > 0$.*

**Remark.** i) In Theorem 1.3 one may always choose $a_1' := \frac{1+\min\{a_1, \frac{1}{a_2}\}}{2}$, for instance, and thereby reduce (1.11) to a requirement on $\mu_2$ only.
ii) We do not expect the conditions (1.8), (1.9) and (1.11) to be optimal. However, the results in [13] indicate that even in the spatially one-dimensional case, spatially inhomogeneous positive steady states may exist when the sensitivities $\chi_1$ and $\chi_2$ are appropriately large.

This paper does not address the situation when both competitive effects in (1.1) are strong in the sense that $a_1 > 1$ and $a_2 > 1$, in which we expect the dynamics to be rather complicated. In fact, even in the associated ODE system (1.6) the solution behavior is then more involved due to the presence of a separatrix $h : (0,\infty) \to (0,\infty)$ with the property that if $v(0) < h(u(0))$ then $(u(t), v(t)) \to (1,0)$ as $t \to \infty$, whereas whenever $v(0) > h(u(0))$, we have $(u(t), v(t)) \to (0,1)$ as $t \to \infty$. The picture even becomes significantly more sophisticated in the corresponding system with diffusion, obtained on dropping the third equation in (1.7): For this, namely, it is easy to show that e.g. in the topology of $L^\infty(\Omega)$, both semitrivial steady states $(1,0)$ and $(0,1)$ are locally stable and the trivial solution $(0,0)$ as well as the unique positive constant equilibrium $(\frac{a_1-1}{a_1a_2-1}, \frac{a_2-1}{a_1a_2-1})$ are unstable. Moreover, it is known that if $\Omega$ is convex, then besides $(1,0)$ and $(0,1)$ there are no further stable steady states ([16]). In nonconvex domains $\Omega$, however, there may exist other stable nonconstant equilibria ([21], [22], [15]). Correspondingly, the knowledge on the large time behavior, especially in general domains, is comparatively rudimentary already in the system (1.7) without cross-diffusion (see [14] and the references therein for some results). Describing stability properties of spatially homogeneous equilibria for the full system (1.1) in the case when $a_1 > 1$ and $a_2 > 1$ accordingly remains a challenging open topic; even establishing the mere existence of non-constant steady states seems far from trivial.

# 2 Global existence for $n \le 2$. Uniform regularity of bounded solutions

## 2.1 Preliminaries

To begin with, let us state a result on local existence and uniqueness of classical solutions.

**Lemma 2.1** *Let $n \ge 1$, let $d_1, d_2, d_3, \chi_1, \chi_2, \mu_1, \mu_2, a_1, a_2, \alpha, \beta$ and $\gamma$ be positive, and let $q > \max\{2, n\}$. Then for each nonnegative $u_0 \in C^0(\bar\Omega)$, $v_0 \in C^0(\bar\Omega)$ and $w_0 \in W^{1,q}(\Omega)$, there exists $T_{max} \in (0,\infty]$ and a uniquely determined triple $(u,v,w)$ of functions*

$$\begin{aligned}
&u \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})),\\
&v \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \qquad \textit{and}\\
&w \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,q}(\Omega)),
\end{aligned}$$

*which solves (1.1) classically in $\Omega \times (0,T_{max})$, and which is such that*

$$\textit{if } T_{max} < \infty \textit{ then } \limsup_{t \nearrow T_{max}} \Big( \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{2.1}$$

Proof. This can be seen using well-established methods in the local existence theory for chemotaxis problems (cf. e.g. [38]). □

The following basic boundedness properties are immediate but important consequences of the presence of logistic-type dampening in the first two equations in (1.1).

**Lemma 2.2** *Let $n \ge 1$. Then the solution of (1.1) satisfies*

$$\int_\Omega u(x,t)dx \le m_1 := \max\Big\{ \int_\Omega u_0, |\Omega| \Big\} \qquad \textit{for all } t \in (0,T_{max}) \tag{2.2}$$

*and*

$$\int_\Omega v(x,t)dx \le m_2 := \max\Big\{ \int_\Omega v_0, |\Omega| \Big\} \qquad \textit{for all } t \in (0,T_{max}) \tag{2.3}$$

*as well as*

$$\int_t^{t+1}\int_\Omega u^2(x,s)dxds \le K_1 := m_1 + \frac{m_1}{\mu_1} \qquad \textit{for all } t \in [0, T_{max}-1) \tag{2.4}$$

*and*

$$\int_t^{t+1}\int_\Omega v^2(x,s)dxds \le K_2 := m_2 + \frac{m_2}{\mu_2} \qquad \textit{for all } t \in [0, T_{max}-1). \tag{2.5}$$

PROOF. We integrate the first equation in (1.1) over $x \in \Omega$ and use the Cauchy-Schwarz inequality to estimate

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &= \mu_1\int_\Omega u - \mu_1\int_\Omega u^2 \\ &\le \mu_1\int_\Omega u - \frac{\mu_1}{|\Omega|}\Big(\int_\Omega u\Big)^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned} \tag{2.6}$$

The latter inequality implies (2.2) by a stratightforward ODE comparison argument, whereupon a time integration in the first identity in (2.6) yields

$$\begin{aligned} \int_\Omega u(x,t+1)dx + \mu_1\int_t^{t+1}\int_\Omega u^2(x,s)dxds &= \int_\Omega u(x,t)dx + \int_t^{t+1}\int_\Omega u(x,s)dxds \\ &\le m_1 + \mu_1 m_1 \qquad \text{for all } t \in [0, T_{max}-1) \end{aligned}$$

and hence proves (2.4). The inequalities (2.3) and (2.5) can be derived similarly. □

## 2.2 An $L^2$ bound for $(u, v)$ in the case $n = 2$

In order to prepare the derivation of further a priori estimates from the properties asserted by from Lemma 2.2, let us cite the following auxiliary statement from [31, Lemma 3.4].

**Lemma 2.3** *Let $T > 0$, and suppose that $y$ is a nonnegative absolutely continuous function on $[0, T)$ satisfying*

$$y'(t) + ay(t) \le f(t) \qquad \textit{for a.e. } t \in (0, T) \tag{2.7}$$

*with some $a > 0$ and a nonnegative function $f \in L^1_{loc}([0, T))$ for which there exists $b > 0$ such that*

$$\int_t^{t+1} f(s)ds \le b \qquad \textit{for all } t \in [0, T-1).$$

*Then*

$$y(t) \le \max\left\{y(0) + b\,, \frac{b}{a} + 2b\right\} \qquad \textit{for all } t \in (0, T). \tag{2.8}$$

Using this, we can apply a standard testing procedure to the third equation in (1.1) to derive the following consequence of Lemma 2.2 on the regularity of $w$.

**Lemma 2.4** *Let $n \ge 1$. Then there exists $C > 0$ such that the solution of (1.1) satisfies*

$$\int_t^{t+1}\int_\Omega |\Delta w(x,s)|^2 dxds \le C \qquad \textit{for all } t \in [0, T_{max}-1). \tag{2.9}$$

PROOF. Testing the third equation in (1.1) by $-\Delta w$ and using Young's inequality, we see that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w|^2 + d_3\int_\Omega |\Delta w|^2 + \gamma\int_\Omega |\nabla w|^2 &= -\alpha\int_\Omega u\Delta w - \beta\int_\Omega v\Delta w \\ &\le \frac{d_3}{2}\int_\Omega |\Delta w|^2 + \frac{\alpha^2}{d_3}\int_\Omega u^2 + \frac{\beta^2}{d_3}\int_\Omega v^2\end{aligned}$$

for all $t \in (0, T_{max})$. Thus, $y(t) := \int_\Omega |\nabla w(x,t)|^2 dx,\ t \in [0, T_{max})$, satisfies

$$y'(t) + 2\gamma y(t) + d_3\int_\Omega |\Delta w(x,t)|^2 dx \le f(t) \qquad \text{for all } t \in (0, T_{max}), \tag{2.10}$$

where

$$f(t) := \frac{2\alpha^2}{d_3}\int_\Omega u^2(x,t)dx + \frac{2\beta^2}{d_3}\int_\Omega v^2(x,t)dx \qquad \text{for } t \in (0, T_{max}),$$

whence from Lemma 2.2 we know that

$$\int_t^{t+1} f(s)ds \le c_1 := \frac{2\alpha^2 K_1 + 2\beta^2 K_2}{d_3} \qquad \text{for all } t \in (0, T_{max}-1).$$

Accordingly, Lemma 2.3 ensures that

$$y(t) = \int_\Omega |\nabla w(x,t)|^2 dx \le c_2 := \max\Big\{\int_\Omega |\nabla w_0(x)|^2 + c_1, \frac{c_1}{2\gamma} + 2c_1\Big\} \qquad \text{for all } t \in (0, T_{max}).$$

Thereupon, an integration of (2.10) over $(t, t+1)$ yields

$$\begin{aligned}y(t+1) + 2\gamma\int_t^{t+1} y(s)ds + d_3\int_t^{t+1}\int_\Omega |\nabla w(x,s)|^2 dxds &\le y(t) + \int_t^{t+1} f(s)ds \\ &\le c_2 + c_1 \qquad \text{for all } t \in [0, T_{max}-1),\end{aligned}$$

which in view of the nonnegativity of $y$ implies (2.9). $\Box$

Now in the particular case $n = 2$, by means of two more testing procedures and an interpolation argument the latter bound can be turned into estimates for $u$ and $v$ in $L^\infty((0, T_{max}); L^2(\Omega))$.

**Lemma 2.5** *Let $n = 2$. Then there exists $C > 0$ such that for the solution of (1.1) we have*

$$\int_\Omega u^2(x,t)dx \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{2.11}$$

*and*

$$\int_\Omega v^2(x,t)dx \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{2.12}$$

PROOF. We multiply the first equation in (1.1) by $u$ and integrate by parts over $\Omega$ to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 + d_1\int_\Omega |\nabla u|^2 = \chi_1\int_\Omega u\nabla u\cdot\nabla w + \mu_1\int_\Omega u^2(1-u-a_1v) \qquad \text{for all } t\in(0,T_{max}), \quad (2.13)$$

where using the pointwise inequality $\xi^2(1-\xi)\le\frac{4}{27}$, valid for all $\xi\ge 0$, we see that

$$\mu_1\int_\Omega u^2(1-u-a_1v)\le\frac{4\mu_1|\Omega|}{27} \qquad \text{for all } t\in(0,T_{max}). \quad (2.14)$$

Once more integrating by parts, by the Cauchy-Schwarz inequality we moreover find that

$$\chi_1\int_\Omega u\nabla u\cdot\nabla w = -\frac{\chi_1}{2}\int_\Omega u^2\Delta w\le\frac{\chi_1}{2}\|u\|^2_{L^4(\Omega)}\|\Delta w\|_{L^2(\Omega)} \qquad \text{for all } t\in(0,T_{max}). \quad (2.15)$$

Here we invoke the Gagliardo-Nirenberg inequality and recall (2.2) to find $c_1>0$ such that

$$\begin{aligned}\frac{\chi_1}{2}\|u\|^2_{L^4(\Omega)} &\le c_1\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}+c_1\|u\|^2_{L^1(\Omega)}\\ &\le c_1\|\nabla u\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)}+c_1m_1^2 \qquad \text{for all } t\in(0,T_{max}).\end{aligned}$$

By Young's inequality, (2.15) thus implies that

$$\chi_1\int_\Omega u\nabla u\cdot\nabla w\le d_1\int_\Omega|\nabla u|^2+c_2\Big(\int_\Omega u^2\Big)\cdot\Big(\int_\Omega|\Delta w|^2\Big)+c_2\Big(\int_\Omega|\Delta w|^2+1\Big) \qquad \text{for all } t\in(0,T_{max})$$

with some $c_2>0$. In conjunction with (2.14) and (2.13), this warrants the existence of $c_3>0$ fulfilling

$$\frac{d}{dt}\int_\Omega u^2\le 2c_2\Big(\int_\Omega u^2\Big)\cdot\Big(\int_\Omega|\Delta w|^2\Big)+c_3\Big(\int_\Omega|\Delta w|^2+1\Big) \qquad \text{for all } t\in[0,T_{max}). \quad (2.16)$$

In order to integrate this appropriately, we first note that by Lemma 2.4 we can find $c_4>0$ such that

$$\int_{t_0}^{t_0+1}\int_\Omega|\Delta w(x,s)|^2dxds\le c_4 \qquad \text{for all } t_0\in(0,T_{max}-1). \quad (2.17)$$

We next fix $t\in(0,T_{max})$ and then obtain from Lemma 2.2 that there exist $t_0\in[0,T_{max})$ such that $t-1\le t_0\le t$ and

$$\int_\Omega u^2(x,t_0)dx\le c_5:=\max\Big\{K_1,\int_\Omega u_0^2(x)dx\Big\}. \quad (2.18)$$

Now an integration of (2.16) over $(t_0,t)$ shows that

$$\begin{aligned}\int_\Omega u^2(x,t)dx &\le \Big(\int_\Omega u^2(x,t_0)dx\Big)\cdot e^{2c_2\int_{t_0}^t\int_\Omega|\Delta w(x,s)|^2dxds}\\ &\quad +c_3\int_{t_0}^t e^{2c_2\int_s^t\int_\Omega|\Delta w(x,\sigma)|^2dxd\sigma}\cdot\Big(\int_\Omega|\Delta w(x,s)|^2dx+1\Big)ds,\end{aligned}$$

which in light of (2.17) and (2.18) implies that

$$\begin{aligned}\int_\Omega u^2(x,t)dx &\le c_5\cdot e^{2c_2c_4}+c_3\cdot\int_{t_0}^t e^{2c_2c_4}\cdot\Big(\int_\Omega|\Delta w(x,s)|^2dx+21\Big)ds\\ &\le c_5\cdot e^{2c_2c_4}+c_3\cdot e^{2c_2c_4}\cdot(c_4+1),\end{aligned}$$

because $t\le t_0+1$. The proof of (2.12) can be carried out in much the same manner. $\square$

## 2.3 Smoothness implied by $L^p$ bounds. Proof of Theorem 1.1

The following lemma, containing a general statement on extensibility and regularity of solutions known to be bounded in $L^\infty((0,T_{max});L^p(\Omega))$ for some $p>\frac{n}{2}$, will be used to prove global existence and boundedness in the case $n\le 2$, but beyond this the higher order information (2.21) therein will moreover allow us to deal with arbitrary global bounded solutions for general $n\ge 1$ in the sequel.

**Lemma 2.6** *Let $n\ge 1$, and suppose that there exists $p\ge 1$ such that $p>\frac{n}{2}$ and*

$$\sup_{t\in(0,T_{max})}\Big(\|u(\cdot,t)\|_{L^p(\Omega)}+\|v(\cdot,t)\|_{L^p(\Omega)}\Big)<\infty. \tag{2.19}$$

*Then $T_{max}=\infty$ and*

$$\sup_{t>0}\Big(\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w(\cdot,t)\|_{L^\infty(\Omega)}\Big)<\infty. \tag{2.20}$$

*Moreover, there exist $\theta\in(0,1)$ and $C>0$ such that*

$$\|u\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])}+\|v\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])}+\|w\|_{C^{2+\theta,1+\frac{\theta}{2}}(\bar\Omega\times[t,t+1])}\le C\qquad \textit{for all } t\ge 1. \tag{2.21}$$

Proof. Without loss of generality we may assume that $p\le n$. Then since $p>\frac{n}{2}$, we have $\frac{np}{n-p}>n$, so that we can fix $r>n$ such that $r<\frac{np}{n-p}$ and $r<q$, and then choose $\theta>1$ such that

$$2\le r\theta<\frac{np}{n-p}\qquad\text{and}\qquad r\theta\le q. \tag{2.22}$$

Now for each $T\in(0,T_{max})$, the number

$$M(T):=\sup_{t\in(0,T)}\Big(\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{L^\infty(\Omega)}\Big)$$

evidently is finite. In order to estimate $M(T)$ adequately, we fix an arbitrary $t\in(0,T)$, let $t_0:=(t-1)_+$ and use the variation-of-constants formula and the order preserving property of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ in $\Omega$, as asserted by the parabolic comparison principle, to derive the inequality

$$\begin{aligned}\|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{d_1(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty(\Omega)}+\chi_1\int_{t_0}^t\Big\|e^{d_1(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla w(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds\\ &\quad+\mu_1\int_{t_0}^t\Big\|e^{d_1(t-s)\Delta}u(\cdot,s)\Big(1-u(\cdot,s)-a_1v(\cdot,s)\Big)_+\Big\|_{L^\infty(\Omega)}ds.\end{aligned} \tag{2.23}$$

Here if $t\le 1$ and hence $t_0=0$, we once more use the comparison principle to see that

$$\|e^{d_1(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty(\Omega)}\le\|u_0\|_{L^\infty(\Omega)}, \tag{2.24}$$

whereas in the case $t>1$ we invoke (2.2) and known smoothing properties of $(e^{\tau\Delta})_{\tau\ge0}$ ([30], [37]) to find $c_1>0$, as all constants $c_2,c_3,...$ appearing below independent of $t$ and $T$, such that

$$\begin{aligned}\|e^{d_1(t-t_0)\Delta}u(\cdot,t_0)\|_{L^\infty(\Omega)} &\le c_1(t-t_0)^{-\frac{n}{2}}\|u(\cdot,t_0)\|_{L^1(\Omega)}\\ &\le c_1m_1,\end{aligned} \tag{2.25}$$

because then we have $t - t_0 = 1$.
Next, once again by the maximum principle, the validity of the pointwise one-sided inequality

$$u(1-u-a_1v) \le \frac{1}{4} \qquad \text{in } \Omega \times (0,T_{max})$$

implies that

$$\begin{aligned} \mu_1 \int_{t_0}^t \Big\| e^{d_1(t-s)\Delta} u(\cdot,s)\Big(1-u(\cdot,s)-a_1v(\cdot,s)\Big)_+ \Big\|_{L^\infty(\Omega)} ds \\ &\le \mu_1 \int_{t_0}^t \Big\| u(\cdot,s)\Big(1-u(\cdot,s)-a_1v(\cdot,s)\Big)_+ \Big\|_{L^\infty(\Omega)} ds \\ &\le \frac{\mu_1}{4}, \end{aligned} \tag{2.26}$$

again since $t \le t_0 + 1$.
Finally, in treating the second integral on the right of (2.23) we recall ([7, Lemma 3.3]) that there exists $c_2 > 0$ fulfilling

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_2 \tau^{-\frac{1}{2}-\frac{n}{2r}} \|\varphi\|_{L^r(\Omega)} \qquad \begin{array}{l} \text{for all } \tau > 0 \text{ and each } \varphi \in C^1(\bar\Omega;\mathbb{R}^n) \\ \text{such that } \varphi\cdot\nu = 0 \text{ on } \partial\Omega, \end{array}$$

to estimate the term in question according to

$$\begin{aligned} \chi_1 \int_{t_0}^t \Big\| e^{d_1(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla w(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \\ &\le c_2\chi_1 \int_{t_0}^t \Big(d_1(t-s)\Big)^{-\frac{1}{2}-\frac{n}{2r}} \cdot \|u(\cdot,s)\nabla w(\cdot,s)\|_{L^r(\Omega)} ds. \end{aligned} \tag{2.27}$$

Here, twice applying the Hölder inequality and writing $\theta' := \frac{\theta}{\theta-1}$, from Lemma 2.2 we obtain that

$$\begin{aligned} \|u(\cdot,s)\nabla w(\cdot,s)\|_{L^r(\Omega)} &\le \|u(\cdot,s)\|_{L^{r\theta'}(\Omega)} \cdot \|\nabla w(\cdot,s)\|_{L^{r\theta}(\Omega)} \\ &\le \|u(\cdot,s)\|^{\delta}_{L^\infty(\Omega)} \cdot \|u(\cdot,s)\|^{1-\delta}_{L^1(\Omega)} \cdot \|\nabla w(\cdot,s)\|_{L^{r\theta}(\Omega)} \\ &\le M^\delta(T)\cdot m_1^{1-\delta} \cdot \|\nabla w(\cdot,s)\|_{L^{r\theta}(\Omega)} \qquad \text{for all } s \in (t_0,t) \end{aligned} \tag{2.28}$$

with $\delta := 1 - \frac{1}{r\theta'} \in (0,1)$. Now by (1.1), $\nabla w$ can be represented according to

$$\nabla w(\cdot,s) = e^{-\gamma s}\nabla e^{d_3 s\Delta} w_0 + \int_0^s e^{-\gamma(s-\sigma)} e^{d_3(s-\sigma)\Delta}\Big(\alpha u(\cdot,\sigma) + \beta v(\cdot,\sigma)\Big) d\sigma \qquad \text{for all } s \in (0,T_{max}),$$

so that in light of our assumption (2.19), again employing standard $L^p$-$L^q$ estimates for $(e^{\tau\Delta})_{\tau\ge 0}$ ([30], [37, Lemma 1.3]) we can find positive constants $c_3$ and $c_4$ such that

$$\begin{aligned}
\|\nabla w(\cdot,s)\|_{L^{r\theta}(\Omega)} &\le e^{-\gamma s}\|\nabla e^{d_3 s\Delta}w_0\|_{L^{r\theta}(\Omega)} + \int_0^s e^{-\gamma(s-\sigma)}\Big\|\nabla e^{d_3(s-\sigma)\Delta}\Big(\alpha u(\cdot,\sigma)+\beta v(\cdot,\sigma)\Big)\Big\|_{L^{r\theta}(\Omega)} d\sigma \\
&\le c_3 e^{-\gamma s}\|\nabla w_0\|_{L^{r\theta}(\Omega)} \\
&\quad + c_3\int_0^s e^{-\gamma(s-\sigma)}\cdot\Big(1+(s-\sigma)^{-\frac12-\frac n2(\frac1p-\frac1{r\theta})}\Big)\cdot\Big(\|u(\cdot,\sigma)\|_{L^p(\Omega)}+\|v(\cdot,\sigma)\|_{L^p(\Omega)}\Big)d\sigma \\
&\le c_3\|\nabla w_0\|_{L^{r\theta}(\Omega)} + c_4\int_0^s e^{-\gamma(s-\sigma)}\cdot\Big(1+(s-\sigma)^{-\frac12-\frac n2(\frac1p-\frac1{r\theta})}\Big)d\sigma \\
&\le c_5 \qquad \text{for all } s\in(0,T_{max}),
\end{aligned}$$

where we have used that $r\theta\ge 2$ in applying [37, Lemma 1.3 (iii)], and where $c_5 := c_3\|\nabla w_0\|_{L^{r\theta}(\Omega)} + c_4\int_0^\infty e^{-\gamma\xi}(1+\xi^{-\frac12-\frac n2(\frac1p-\frac1{r\theta})})d\xi$ is finite thanks to the fact that $r\theta\le q$ and $r\theta<\frac{np}{n-p}$ by (2.22). As a consequence, (2.28) yields

$$\|u(\cdot,s)\nabla w(\cdot,s)\|_{L^r(\Omega)} \le c_5 m_1^{1-\delta}M^\delta(T) \qquad \text{for all } s\in(t_0,t),$$

so that since $\frac12+\frac{n}{2r}<1$ due to the fact that $r>n$, (2.27) shows that

$$\chi_1\int_{t_0}^t \Big\|e^{d_1(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla w(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \le c_6 M^\delta(T)$$

with $c_6 := c_2c_5\chi_1 m_1^{1-\delta}\cdot\int_0^1(d_1\xi)^{-\frac12-\frac{n}{2r}}d\xi<\infty$. Combining this with (2.24), (2.25) and (2.26), from (2.23) we infer that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \max\{\|u_0\|_{L^\infty(\Omega)}, c_1m_1\} + c_6M^\delta(T) \qquad \text{for all } t\in(0,T).$$

Along with an analogous estimate for $v$, this implies that there exist $c_7>0$ and $c_8>0$ such that

$$M(T)\le c_7 + c_8M^\delta(T) \qquad \text{for all } T\in(0,T_{max}),$$

which by an elementary argument entails that

$$M(T)\le\max\left\{\Big(\frac{c_7}{c_8}\Big)^{\frac1\delta}\, , \,(2c_8)^{\frac{1}{1-\delta}}\right\} \qquad \text{for all } T\in(0,T_{max})$$

and thereby establishes (2.20), because $c_7, c_8$ and $\delta$ are independent of $T\in(0,T_{max})$. By means of (2.20), the additional properties in (2.21) can be derived through a straightforward reasoning involving standard parabolic regularity theory ([18]). □

Combining Lemma 2.6 with the estimates gained above, we readily arrive at our main result on global solvability and boundedness in the case $n\le 2$.

PROOF of Theorem 1.1. In the case $n=2$, the statement results from Lemma 2.1 upon applying Lemma 2.6 to $p:=2$ and employing Lemma 2.5. When $n=1$, we instead use Lemma 2.6 with $p:=1$ and then only need to recall (2.2) and (2.3). □

# 3 Stabilization

The goal of this section will be to establish the convergence properties stated in Theorem 1.2 and Theorem 1.3. The key idea of our approach is to use an energy functional, the form of which is inspired by [11].
Let us first provide the following tool from elementary analysis.

**Lemma 3.1** *Suppose that $f$ : $(1,\infty)$ is a uniformly continuous nonnegative function such that $\int_1^\infty f(t)dt < \infty$. Then $f(t) \to 0$ as $t \to \infty$.*

PROOF. We only need to combine the assumed integrability property with the uniform continuity of $f$ to see that $\limsup_{t\to\infty} f(t)$ cannot be positive. □

## 3.1 Convergence in the co-existence case when $a_1 < 1$ and $a_2 < 1$

Let us first concentrate on the case when both competition parameters are small in the sense that $a_1 < 1$ and $a_2 < 1$. The key step toward the corresponding convergence statement in Theorem 1.2 will consist in the construction of the energy functional $E_1$ in the following lemma. Its definition (3.1) involves a positive parameter $\delta$ which under the assumptions (1.8) and (1.9) can be adjusted in such a way that indeed $E_1$ decreases along trajectories.

**Lemma 3.2** *Let $a_1 \in (0,1)$, $a_2 \in (0,1)$ and $(u_\star, v_\star, w_\star)$ be as in (1.5), and assume that (1.8) and (1.9) hold. Assume that $(u_0, v_0, w_0)$ satisfies (1.3), and that $(u,v,w)$ is a global bounded classical solution of (1.1). Then there exist $\delta > 0$ and $\varepsilon > 0$ such that the functions $E_1$ and $F_1$ defined by*

$$E_1(t) := \int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star}\Big) + \frac{\mu_1 a_1}{\mu_2 a_2} \int_\Omega \Big(v(\cdot,t) - v_\star - v_\star \ln \frac{v(\cdot,t)}{v_\star}\Big) + \frac{\delta}{2} \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2, \qquad t > 0, \tag{3.1}$$

*and*

$$\begin{aligned} F_1(t) \quad := \quad & \int_\Omega \Big|\frac{\nabla u(\cdot,t)}{u(\cdot,t)}\Big|^2 + \int_\Omega \Big|\frac{\nabla v(\cdot,t)}{v(\cdot,t)}\Big|^2 dx + \int_\Omega |\nabla w(\cdot,t)|^2 \\ & + \int_\Omega \Big(u(\cdot,t) - u_\star\Big)^2 + \int_\Omega \Big(v(\cdot,t) - v_\star\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2, \qquad t > 0, \end{aligned} \tag{3.2}$$

*satisfy*

$$E_1(t) \ge 0 \qquad \textit{for all } t > 0 \tag{3.3}$$

*as well as*

$$\frac{d}{dt} E_1(t) \le -\varepsilon F_1(t) \qquad \textit{for all } t > 0. \tag{3.4}$$

PROOF. Let us first note that by straightforward computation it can be checked that as a consequence of (2.11) and (2.12) we have

$$\frac{1}{d_1 d_2 d_3}\Big(\frac{d_1 \chi_2^2 v_\star \mu_1 a_1}{4\mu_2 a_2} + \frac{d_2 \chi_1^2 u_\star}{4}\Big) < \frac{4 a_1 \gamma (1 - a_1 a_2)\mu_1}{(a_1\alpha^2 + a_2\beta^2 - 2a_1 a_2 \alpha\beta)},$$

which enables us to fix some $\delta > 0$ which simultaneously fulfils

$$\delta < \frac{4a_1\gamma(1-a_1a_2)\mu_1}{(a_1\alpha^2 + a_2\beta^2 - 2a_1a_2\alpha\beta)} \tag{3.5}$$

and

$$\delta > \frac{1}{d_1d_2d_3}\Big(\frac{d_1\chi_2^2 v_\star \mu_1 a_1}{4\mu_2 a_2} + \frac{d_2\chi_1^2 u_\star}{4}\Big). \tag{3.6}$$

With this value of $\delta$ fixed henceforth, we let $E_1$ be as defined in (3.1) and decompose $E_1$ according to

$$E_1(t) = A_1(t) + \frac{\mu_1 a_1}{\mu_2 a_2} B_1(t) + C_1(t), \qquad t > 0,$$

where

$$\begin{aligned} A_1(t) &:= \int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star}\Big) \\ B_1(t) &:= \int_\Omega \Big(v(\cdot,t) - v_\star - v_\star \ln \frac{v(\cdot,t)}{v_\star}\Big) \qquad \text{and} \\ C_1(t) &:= \frac{\delta}{2}\int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 \end{aligned}$$

for $t > 0$.
To prove the nonnegativity of $E_1$, we let $H(\bar{u}) := \bar{u} - u_\star \ln \bar{u}$ for $\bar{u} > 0$ and use Taylor's formula to see that for all $x \in \Omega$ and each $t > 0$ we can find $\tau = \tau(x,t) \in (0,1)$ such that

$$\begin{aligned} H(u(x,t)) - H(u_\star) &= H'(u_\star)\cdot\Big(u(x,t) - u_\star\Big) + \frac{1}{2}H''\Big(\tau u(x,t) + (1-\tau)u_\star\Big)\cdot\Big(u(x,t) - u_\star\Big)^2 \\ &= \frac{u_\star}{2\Big(\tau u(x,t) + (1-\tau)u_\star\Big)^2}\Big(u(x,t) - u_\star\Big)^2 \\ &\geq 0. \end{aligned}$$

From this we immediately obtain that $A_1(t) = \int_\Omega \Big(H(u(\cdot,t)) - H(u_\star)\Big) \geq 0$, and and by a similar argument it follows that also $B_1(t) \geq 0$ for all $t \geq 0$. Since clearly also $C_1$ is nonnegative by positivity of $\delta$, this implies (3.3).

In order to prove that (3.5) and (3.6) ensure the validity of (3.4) for some $\varepsilon > 0$, we first use (1.1) in computing

$$\begin{aligned} \frac{d}{dt}A_1(t) &= \int_\Omega \Big(u_t - \frac{u_\star}{u}u_t\Big) \\ &= \int_\Omega \mu_1(u-u_\star)(1-u-a_1v) - d_1u_\star\int_\Omega \frac{|\nabla u|^2}{u} + \chi_1 u_\star \int_\Omega \frac{\nabla u}{u}\cdot\nabla w \\ &= -\mu_1\int_\Omega (u-u_\star)^2 - \mu_1 a_1\int_\Omega (u-u_\star)(v-v_\star) - d_1u_\star\int_\Omega \frac{|\nabla u|^2}{u} + \chi_1 u_\star\int_\Omega \frac{\nabla u}{u}\cdot\nabla w \end{aligned}$$

and

$$\frac{d}{dt}B_1(t) \;\;=\;\; -\mu_2\int_\Omega (v-v_\star)^2 - \mu_2 a_2\int_\Omega (u-u_\star)(v-v_\star) - d_2 v_\star\int_\Omega \frac{|\nabla v|^2}{v} + \chi_2 v_\star\int_\Omega \frac{\nabla v}{v}\cdot\nabla w$$

as well as

$$\frac{d}{dt}C_1(t) = -\delta d_3\int_\Omega |\nabla w|^2 - \gamma\delta\int_\Omega (w-w_\star)^2 + \alpha\delta\int_\Omega (w-w_\star)(u-u_\star) + \beta\delta\int_\Omega (w-w_\star)(v-v_\star)$$

for all $t>0$. Combining these three equalities, we obtain the identity

$$\frac{d}{dt}E_1(t) = -\int_\Omega X\cdot(\mathbb{P}\cdot X) - \int_\Omega Y\cdot(\mathbb{S}\cdot Y) \qquad \text{for all } t>0, \tag{3.7}$$

with the vector functions $X$ and $Y$ defined through

$$X(x,t) := \Big(u(x,t)-u_\star, v(x,t)-v_\star, w(x,t)-w_\star\Big) \quad\text{and}\quad Y(x,t) := \left(\frac{|\nabla u(x,t)|}{u(x,t)}, \frac{|\nabla v(x,t)|}{v(x,t)}, |\nabla w(x,t)|\right)$$

for $x\in\Omega$ and $t>0$, and the constant matrices $\mathbb{P}$ and $\mathbb{S}$ given by

$$\mathbb{P} := \begin{pmatrix} \mu_1 & \mu_1 a_1 & -\frac{\alpha\delta}{2} \\ \mu_1 a_1 & \frac{\mu_1 a_1}{a_2} & -\frac{\beta\delta}{2} \\ -\frac{\alpha\delta}{2} & -\frac{\beta\delta}{2} & \gamma\delta \end{pmatrix}$$

and

$$\mathbb{S} := \begin{pmatrix} d_1 u_\star & 0 & \frac{\chi_1 u_\star}{2} \\ 0 & \frac{d_2 v_\star \mu_1 a_1}{\mu_2 a_2} & \frac{\chi_2 v_\star \mu_1 a_1}{2\mu_2 a_2} \\ \frac{\chi_1 u_\star}{2} & \frac{\chi_2 v_\star \mu_1 a_1}{2\mu_2 a_2} & d_3\delta \end{pmatrix}.$$

Now the key step in establishing (3.4) consists in proving that both $\mathbb{P}$ and $\mathbb{S}$ are positive definite. Once this has been shown, namely, it will follow that for some $\varepsilon>0$ we have

$$X(x,t)\cdot(\mathbb{P}\cdot X(x,t)) \ge \varepsilon|X(x,t)|^2 \quad\text{and}\quad Y(x,t)\cdot(\mathbb{S}\cdot Y(x,t)) \ge \varepsilon|Y(x,t)|^2 \quad \text{for all } x\in\Omega \text{ and } t>0, \tag{3.8}$$

whereupon (3.4) will become an evident consequence of (3.7).
Thus concentrating on the desired definiteness properties, we first compute the first two principal minors $M_1 := |\mu_1|$ and $M_2 := \begin{vmatrix} \mu_1 & \mu_1 a_1 \\ \mu_1 a_1 & \mu_1\frac{a_1}{a_2} \end{vmatrix}$ of $\mathbb{P}$ to obtain that

$$M_1 = \mu_1 > 0 \qquad\text{and}\qquad M_2 = (1-a_1a_2)\mu_1^2\cdot\frac{a_1}{a_2} > 0,$$

because $a_1 \in (0,1)$ and $a_2 \in (0,1)$. Since moreover $M_3 := |\mathbb{P}|$ satisfies

$$\begin{aligned}
\begin{vmatrix} \mu_1 & \mu_1 a_1 & -\frac{\alpha\delta}{2} \\ \mu_1 a_1 & \mu_1 \frac{a_1}{a_2} & -\frac{\beta\delta}{2} \\ -\frac{\alpha\delta}{2} & -\frac{\beta\delta}{2} & \gamma\delta \end{vmatrix}
&= \mu_1 \begin{vmatrix} \mu_1 \frac{a_1}{a_2} & -\frac{\beta\delta}{2} \\ -\frac{\beta\delta}{2} & \gamma\delta \end{vmatrix} - \mu_1 a_1 \begin{vmatrix} \mu_1 a_1 & -\frac{\beta\delta}{2} \\ -\frac{\alpha\delta}{2} & \gamma\delta \end{vmatrix} + \frac{\alpha\delta}{2} \begin{vmatrix} \mu_1 a_1 & \mu_1 \frac{a_1}{a_2} \\ -\frac{\alpha\delta}{2} & -\frac{\beta\delta}{2} \end{vmatrix} \\
&= \mu_1 \left( \frac{\mu_1 a_1 \gamma\delta}{a_2} - \frac{\beta^2\delta^2}{4} \right) - \mu_1 a_1 \left( \mu_1 a_1 \gamma\delta - \frac{\alpha\beta\delta^2}{4} \right) + \frac{\alpha\delta}{2} \left( \frac{\mu_1 a_1 \beta\delta}{2} - \frac{\mu_1 a_1 \alpha\delta}{2a_2} \right) \\
&= \mu_1 \delta \left( \mu_1 \gamma \frac{a_1}{a_2} + \frac{a_1 \alpha\beta\delta}{2} - \frac{\beta^2\delta}{4} - \frac{a_1 \alpha^2 \delta}{4a_2} - \mu_1 a_1^2 \gamma \right) \\
&> 0
\end{aligned}$$

thanks to (3.5), Sylvester's criterion guarantees that indeed $\mathbb{P}$ is positive definite.
Likewise, for the matrix $\mathbb{S}$ we use the observations that

$$d_1 u_\star > 0 \qquad \text{and} \qquad \begin{vmatrix} d_1 u_\star & 0 \\ 0 & \frac{d_2 v_\star \mu_1 a_1}{\mu_2 a_2} \end{vmatrix} = \frac{d_1 d_2 u_\star v_\star \mu_1 a_1}{\mu_2 a_2} > 0,$$

and that our restriction (3.6) on $\delta$ ensures that

$$\begin{aligned}
\begin{vmatrix} d_1 u_\star & 0 & \frac{\chi_1 u_\star}{2} \\ 0 & \frac{\mu_1 a_1 d_2 v_\star}{\mu_2 a_2} & \frac{\mu_1 a_1 \chi_2 v_\star}{2\mu_2 a_2} \\ \frac{\chi_1 u_\star}{2} & \frac{\mu_1 a_1 \chi_2 v_\star}{2\mu_2 a_2} & d_3\delta \end{vmatrix}
&= d_1 u_\star \begin{vmatrix} \frac{\mu_1 a_1 d_2 v_\star}{\mu_2 a_2} & \frac{\mu_1 a_1 \chi_2 v_\star}{2\mu_2 a_2} \\ \frac{\mu_1 a_1 \chi_2 v_\star}{2\mu_2 a_2} & d_3\delta \end{vmatrix} + \frac{\chi_1 u_\star}{2} \begin{vmatrix} 0 & \frac{\mu_1 a_1 d_2 v_\star}{\mu_2 a_2} \\ \frac{\chi_1 u_\star}{2} & \frac{\mu_1 a_1 \chi_2 v_\star}{2\mu_2 a_2} \end{vmatrix} \\
&= \frac{\mu_1 a_1 u_\star v_\star}{\mu_2 a_2} \left( d_1 d_2 d_3 \delta - \frac{d_1 v_\star \mu_1 a_1 \chi_2^2}{4\mu_2 a_2} - \frac{d_2 \mu_1 u_\star \chi_1^2}{4} \right) \\
&> 0,
\end{aligned}$$

to conclude again by the Sylvester criterion that also $\mathbb{S}$ is positive definite. The proof is thereby complete. □

A first use of the energy inequality (3.4) in conjunction with Lemma 3.1 and the global regularity properties asserted by Lemma 2.6 now yields stabilization as claimed in Theorem 1.2, but yet without any information on the rate of convergence. That this is actually exponential will be proved by means of a second application of (3.14) in Lemma 3.7 below.

**Lemma 3.3** *Let $a_1 \in (0,1)$ and $a_2 \in (0,1)$ and let $\mu_1$ and $\mu_2$ be such that (1.8) and (1.9) are valid. Suppose that $u_0, v_0$ and $w_0$ are such that (1.3) holds as well as $u_0 \not\equiv 0 \not\equiv v_0$, and that $(u,v,w)$ is a global bounded classical solution of (1.1). Then*

$$\|u(\cdot,t) - u_\star\|_{L^\infty(\Omega)} + \|v(\cdot,t) - v_\star\|_{L^\infty(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty, \tag{3.9}$$

*where $u_\star, v_\star$ and $w_\star$ are as given by (1.5).*

PROOF. We let $f(t) := \int_\Omega (u(\cdot,t)-u_\star)^2 + \int_\Omega (v(\cdot,t)-v_\star)^2 + \int_\Omega (w(\cdot,t)-w_\star)^2$ for $t \ge 0$ and choose $\delta$ as in Lemma 3.2. Then with $E_1$ and $F_1$ as given by (3.1) and (3.2), we clearly have $f(t) \le F_1(t)$ for all $t > 0$, so that (3.4) implies the inequality

$$\frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \le -\varepsilon f(t) \qquad \text{for all } t > 0.$$

Since $E_1(t)$ is nonnegative by Lemma 3.3, it follows that

$$\int_1^\infty f(t)dt \le \frac{1}{\varepsilon}(E_1(1) - E_1(t)) \le \frac{E_1(1)}{\varepsilon} < \infty.$$

Since from Lemma 2.6 we know that $u, v$ and $w$ are Hölder continuous in $\bar\Omega \times [t, t+1]$, uniformly with respect to $t > 1$, we infer that $f(t)$ is uniformly continuous in $(1, \infty)$, whence an application of Lemma 3.1 shows that

$$\int_\Omega \Big(u(\cdot,t) - u_\star\Big)^2 + \int_\Omega \Big(v(\cdot,t) - v_\star\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 = f(t) \to 0 \qquad \text{as } t \to \infty. \tag{3.10}$$

In order to turn this into a respective convergence statement with respect to the norm in $L^\infty(\Omega)$, we invoke the Gagliardo-Nirenberg inequality to find $c_1 > 0$ fulfilling

$$\|\varphi\|_{L^\infty(\Omega)} \le c_1 \|\varphi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\varphi\|_{L^2(\Omega)}^{\frac{2}{n+2}} \qquad \text{for all } \varphi \in W^{1,\infty}(\Omega).$$

Applying this to $u(\cdot,t) - u_\star$ for $t > 0$ and using that $(u(\cdot,t))_{t>1}$ is bounded in $W^{1,\infty}(\Omega)$ according to Lemma 2.6, we conclude from (3.10) that indeed $u(\cdot,t) \to u_\star$ in $L^\infty(\Omega)$ as $t \to \infty$. Repeating this argument for $v$ and $w$ yields (3.9). $\Box$

### 3.2 Convergence in the extinction case when $a_1 \ge 1 > a_2$

In the case when still $a_2 < 1$ but the competitive effect of $v$ on $u$ is strong in that $a_2 \ge 1$, we will see that $u$ will become extinct asymptotically whenever $v_0 \not\equiv 0$ and $\mu_2$ is large fulfilling (1.11). Our proof of this follows a strategy similar to that in Section 3.1, a slight difference consisting in an adaptation of the Lyapunov functional to the present setting in which $u$ no longer approaches a positive equilibrium but rather decays to zero.

**Lemma 3.4** *Assume that $a_1 \ge 1$ and $a_2 \in (0,1)$, and that (1.11) holds. Let $(u_0, v_0, w_0)$ satisfy (1.3), and suppose that $(u, v, w)$ is a global bounded classical solution of (1.1). Then there exist $\delta > 0$ and $\varepsilon > 0$ such that if we let*

$$E_2(t) := \int_\Omega u(\cdot,t) + \frac{\mu_1 a_1'}{\mu_2 a_2} \int_\Omega \Big(v(\cdot,t) - 1 - v_\star \ln \frac{v(\cdot,t)}{v_\star}\Big) + \frac{\delta}{2}\int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2, \qquad t > 0, \tag{3.11}$$

*and*

$$F_2(t) := \int_\Omega |\nabla w|^2 + \int_\Omega \Big|\frac{\nabla v(\cdot,t)}{v(\cdot,t)}\Big|^2 + \int_\Omega |\nabla w(\cdot,t)|^2 + \int_\Omega u^2(\cdot,t) + \int_\Omega \Big(v(\cdot,t) - 1\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2, \qquad t > 0, \tag{3.12}$$

*then*

$$E_2(t) \geq 0 \qquad \textit{for all } t > 0 \tag{3.13}$$

*and*

$$\frac{d}{dt}E_2(t) \leq -\varepsilon F_2(t) - \mu_1(a_1' - 1)\int_\Omega u(\cdot,t) \qquad \textit{for all } t > 0. \tag{3.14}$$

PROOF. We fix $\delta > 0$ such that

$$\delta < \frac{4a_1'\gamma(1-a_1'a_2)\mu_1}{(a_1'\alpha^2 + a_2\beta^2 - 2a_1'a_2\alpha\beta)}, \tag{3.15}$$

and that

$$\delta > \frac{\chi_2^2 v_\star \mu_1 a_1'}{4d_2 d_3 \mu_2 a_2}, \tag{3.16}$$

and then define $E_2$ and $F_2$ as in (3.11) and (3.12). Then copying the repsective arguments from the proof of Lemma 3.2, we easily obtain nonnegativity of $E_2$, and on the basis of Sylvester's criterion we can verify in a straightforward manner that

$$\mathbb{P}' := \begin{pmatrix} \mu_1 & \mu_1 a_1' & -\frac{\alpha\delta}{2} \\ \mu_1 a_1' & \frac{\mu_1 a_1'}{a_2} & -\frac{\beta\delta}{2} \\ -\frac{\alpha\delta}{2} & -\frac{\beta\delta}{2} & \gamma\delta \end{pmatrix}$$

is positive definite thanks to (3.15), and that (3.16) warrants that also

$$\mathbb{S}' := \begin{pmatrix} \frac{d_2 v_\star \mu_1 a_1'}{\mu_2 a_2} & \frac{\chi_2 v_\star \mu_1 a_1'}{2\mu_2 a_2} \\ \frac{\chi_2 v_\star \mu_1 a_1'}{2\mu_2 a_2} & d_3\delta \end{pmatrix}.$$

is positive definite.
We now rewrite $E_2$ according to

$$E_2(t) = A_2(t) + \frac{\mu_1 a_1'}{\mu_2 a_2}B_2(t) + C_2(t),$$

with

$$\begin{aligned} A_2(t) &:= \int_\Omega \Big(u(\cdot,t) - u_\star\Big), \\ B_2(t) &:= \int_\Omega \Big(v(\cdot,t) - v_\star - v_\star \ln\frac{v(\cdot,t)}{v_\star}\Big) \qquad \text{and} \\ C_2(t) &:= \frac{\delta}{2}\int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 \end{aligned}$$

for $t > 0$, and use (1.1) in computing

$$
\begin{aligned}
\frac{d}{dt} A_2(t) &= \int_\Omega \mu_1 u(1-u-a_1 v) \\
&\le \int_\Omega \mu_1 u(1-u-a_1' v) \\
&= -\mu_1(a_1'-1)\int_\Omega u - \mu_1 \int_\Omega u^2 - \mu_1 a_1' \int_\Omega u(v-1) \\
&= -\mu_1(a_1'-1)\int_\Omega u - \mu_1 \int_\Omega (u-u_\star)^2 - \mu_1 a_1' \int_\Omega (u-u_\star)(v-v_\star)
\end{aligned}
$$

and

$$
\frac{d}{dt} B_2(t) = -\mu_2 \int_\Omega (v-v_\star)^2 - \mu_2 a_2 \int_\Omega (u-u_\star)(v-v_\star) - d_2 v_\star \int_\Omega \frac{|\nabla v|^2}{v} + \chi_2 v_\star \int_\Omega \frac{\nabla v}{v} \cdot \nabla w
$$

as well as

$$
\frac{d}{dt} C_2(t) = -\delta d_3 \int_\Omega |\nabla w|^2 - \gamma\delta \int_\Omega (w-w_\star)^2 + \alpha\delta \int_\Omega (w-w_\star)(u-u_\star) + \beta\delta \int_\Omega (w-w_\star)(v-v_\star)
$$

for $t > 0$. We thus obtain that

$$
\frac{d}{dt} E_2(t) \le -\int_\Omega X \cdot (\mathbb{P}' \cdot X) - \int_\Omega Z \cdot (\mathbb{S}' \cdot Z) - \mu_1(a_1'-1)\int_\Omega u \qquad \text{for all } t > 0, \tag{3.17}
$$

where

$$
X(x,t) := \Big(u(x,t)-u_\star, v(x,t)-v_\star, w(x,t)-w_\star\Big) \quad \text{and} \quad Z(x,t) := \left(\frac{|\nabla v(x,t)|}{v(x,t)}, |\nabla w(x,t)|\right)
$$

for $x \in \Omega$ and $t > 0$. As $\mathbb{P}'$ and $\mathbb{S}'$ are positive definite, it can be seen as in Lemma 3.2 that (3.17) implies (3.14). $\square$

By a reasoning almost identical to that in Lemma 3.3, we thereby immediately obtain the following qualitative convergence result.

**Lemma 3.5** *Let $a_1 \ge 1$ and $a_2 \in (0,1)$, and let $\mu_2$ be such that (1.11) holds. Then for any choice of $u_0, v_0$ and $w_0$ which satisfy (1.3) and are such that $v_0 \not\equiv 0$ and that (1.1) possesses a global bounded classical solution $(u,v,w)$, we have*

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)-1\|_{L^\infty(\Omega)} + \Big\|w(\cdot,t) - \frac{\alpha}{\gamma}\Big\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{3.18}
$$

### 3.3 Convergence rates

In order to complete the proofs of Theorem 1.2 and Theorem 1.3, it remains to describe the rates of convergence in (1.10), (1.12) and (1.13). Let us prepare our argument therefor by the following.

**Lemma 3.6** *Let $(u_\star, v_\star, w_\star) \in \mathbb{R}^3$ be any solution of (1.4), and suppose that $(u, v, w)$ is a global bounded classical solution of (1.1) emanating from initial data fulfilling (1.3). Moreover, assume that there exist two decreasing functions $h_1$ and $h_2$ on $(0, \infty)$ with the properties that*

$$\|u(\cdot,t) - u_\star\|_{L^{2n}(\Omega)} + \|v(\cdot,t) - v_\star\|_{L^{2n}(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^{2n}(\Omega)} \le h_1(t) \qquad \text{for all } t > 0 \tag{3.19}$$

*and*

$$\Big(\int_{t-1}^t \int_\Omega |\nabla w|^2\Big)^{\frac{1}{2n+2}} \le h_2(t) \qquad \text{for all } t > 1. \tag{3.20}$$

*Then there exists $C > 0$ such that*

$$\|u(\cdot,s) - u_\star\|_{L^\infty(\Omega)} + \|v(\cdot,s) - v_\star\|_{L^\infty(\Omega)} + \|w(\cdot,s) - w_\star\|_{L^\infty(\Omega)} \le C \cdot \Big(h_1(t-1) + h_2(t)\Big) \qquad \text{for all } t > 2. \tag{3.21}$$

Proof. By the variation-of-constants formula associated with the first equation in (1.1), for each $t > 2$ we can estimate $u - u_\star$ according to

$$\begin{aligned}
\|u(\cdot,t) - u_\star\|_{L^\infty(\Omega)} &\le \Big\|e^{d_1\Delta}(u(\cdot,t_0) - u_\star)\Big\|_{L^\infty(\Omega)} + \chi_1 \int_{t-1}^t \Big\|e^{d_1(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla w(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \\
&\quad + \mu_1 \int_{t-1}^t \Big\|e^{d_1(t-s)\Delta} u(\cdot,s)\Big(1 - u(\cdot,s) - a_1 v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)} ds \\
&:= I_1 + I_2 + I_3,
\end{aligned} \tag{3.22}$$

where known smoothing properties of the heat semigroup yield $c_1 > 0$ such that

$$I_1 \le c_1\Big(t - (t-1)\Big)^{-\frac{1}{4}} \|u(\cdot,t_0) - u_\star\|_{L^{2n}(\Omega)} \le c_1 h_1(t-1) \tag{3.23}$$

because of (3.19). Likewise, invoking $L^p - L^q$ estimates for $(e^{\tau\Delta})_{\tau\ge0}$ and applying the Hölder inequality along with (3.20) provide positive constants $c_2$ and $c_3$ fulfilling

$$\begin{aligned}
I_2 &\le c_2 \int_{t-1}^t (t-s)^{-\frac{1}{2} - \frac{n}{2}\frac{1}{2n+2}} \Big\|u(\cdot,s)\nabla w(\cdot,s)\Big\|_{L^{2n+2}(\Omega)} ds \\
&\le c_2 \Big(\int_{t-1}^t (t-s)^{-\frac{3n+2}{4n+2}} ds\Big)^{\frac{2n+1}{2n+2}} \Big(\int_{t-1}^t \|u(\cdot,s)\nabla w(\cdot,s)\|_{L^{2n+2}(\Omega)}^{2n+2} ds\Big)^{\frac{1}{2n+2}} \\
&\le c_3 \|u\|_{L^\infty(\Omega\times(0,\infty))} \|\nabla w\|_{L^\infty(\Omega\times(1,\infty))}^{\frac{n}{n+1}} \Big(\int_{t-1}^t \int_\Omega |\nabla w(x,s)|^2 dx ds\Big)^{\frac{1}{2n+2}} \\
&\le c_4 h_2(t),
\end{aligned} \tag{3.24}$$

where $c_4 := c_3 \|u\|_{L^\infty(\Omega\times(0,\infty))} \|\nabla w\|_{L^\infty(\Omega\times(1,\infty))}^{\frac{n}{n+1}}$ is finite thanks to the boundedness properties asserted by Theorem 1.1 and Lemma 2.6.

To estimate $I_3$, we first note that as a consequence of the equilibrium property of $(u_\star, v_\star, w_\star)$ implied by (1.4) we have the pointwise identity

$$u(1 - u + a_1 v) = \begin{cases} (u - u_\star)(1 - u + a_1 v) & \text{if } a_1 \ge 1, \\ u \cdot \Big((u_\star - u) + a_1(v_\star - v)\Big) & \text{if } a_1 < 1, \end{cases}$$

which in view of (3.19) and the downward monotonicity of $h_1$ readily ensures the existence of $c_5 > 0$ such that

$$\Big\|u(\cdot,s)\Big(1-u(\cdot,s)+a_1v(\cdot,s)\Big)\Big\|_{L^{2n}(\Omega)} \le c_5h_1(t-1) \qquad \text{for all } s\in(t-1,t).$$

Hence, $L^p - L^q$ estimates entail that for some $c_6 > 0$ we have

$$\begin{aligned} I_3 &\le c_6\int_{t-1}^t (t-s)^{-\frac{n}{2}\cdot\frac{1}{2n}}\Big\|u(\cdot,s)\Big(1-u(\cdot,s)+a_1v(\cdot,s)\Big)\Big\|_{L^{2n}(\Omega)}ds \\ &\le c_6c_5h_1(t-1)\int_{t-1}^t (t-s)^{-\frac{1}{4}}ds \\ &= c_7h_1(t-1) \end{aligned} \tag{3.25}$$

with $c_7 := \frac{4}{3}c_5c_6$. On substituting (3.23), (3.24) and (3.25) into (3.22), we infer that there exists $c_8 > 0$ such that

$$\|u(\cdot,t)-u_\star\|_{L^\infty(\Omega)} \le c_8\cdot\Big(h_1(t-1)+h_2(t)\Big) \qquad \text{for all } t>2.$$

Now an analogous estimate for $\|v-v_\star\|_{L^\infty(\Omega)}$ can be obtained in precisely the same manner, whereas a similar inequality for $\|w-w_\star\|_{L^\infty(\Omega)}$ can be derived using simplified variant of the above arguments, based on the representation

$$w(\cdot,t)-w_\star = e^{(d_3\Delta-\gamma I)}(w(\cdot,t-1)-w_\star)+\int_{t-1}^t e^{(t-s)(d_3\Delta-\gamma I)}\Big(\alpha(u(\cdot,s)-u_\star)+\beta(v(\cdot,s)-v_\star)\Big)ds,$$

which is valid for all $t > 1$ again due to the fact that $(u_\star, v_\star, w_\star)$ is a steady state of (1.1). □

Building on the uniform convergence properties asserted by Lemma 3.3 and Lemma 3.5, with Lemma 3.6 at hand we can now perform a refined analysis of the energy inequalities (3.4) and (3.14) to establish the desired quantitative statements on stabilization.

**Lemma 3.7** *i) Assume that $a_1 \in (0,1)$ and $a_2 \in (0,1)$, and that (1.8) and (1.9) hold. Then for each global bounded classical solution $(u,v,w)$ of (1.1) evolving from initial data fulfilling (1.3), there exist $C > 0$ and $\lambda > 0$ such that*

$$\|u(\cdot,t)-u_\star\|_{L^\infty(\Omega)}+\|v(\cdot,t)-v_\star\|_{L^\infty(\Omega)}+\|w(\cdot,t)-w_\star\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \qquad \textit{for all } t>0, \tag{3.26}$$

*where $(u_\star, v_\star, w_\star)$ is given by (1.5).*
*ii) Let $a_1 \ge 1$ and $a_2 \in (0,1)$, and suppose that $\mu_2$ has the property (1.11).*
*ii.i) If $a_1 > 1$ and $(u,v,w)$ is a global bounded classical solution of (1.1) with initial data complying with (1.3), then for some $C > 0$ and $\lambda > 0$ we have*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)-1\|_{L^\infty(\Omega)}+\Big\|w(\cdot,t)-\frac{\alpha}{\gamma}\Big\|_{L^\infty(\Omega)} \le Ce^{-\lambda t} \qquad \textit{for all } t>0. \tag{3.27}$$

*ii.ii) In the case $a_1 = 1$, for any choice of $(u_0, v_0, w_0)$ satisfying (1.3) and admitting a global bounded classical solution $(u,v,w)$ of (1.1), one can find $C > 0$ and $\kappa > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)-1\|_{L^\infty(\Omega)}+\Big\|w(\cdot,t)-\frac{\alpha}{\gamma}\Big\|_{L^\infty(\Omega)} \le C(1+t)^{-\kappa} \qquad \textit{for all } t>0. \tag{3.28}$$

PROOF. i) We again use the function $H$ given by $H(\bar u) := \bar u - u_\star \ln \bar u$ for $\bar u > 0$, which according to L'Hôpital's rule has the property that

$$\lim_{\bar u\to u_\star} \frac{H(\bar u) - H(u_\star)}{(\bar u - u_\star)^2} = \lim_{\bar u\to u_\star} \frac{H'(\bar u)}{2(\bar u - u_\star)} = \frac{1}{2u_\star}. \tag{3.29}$$

Since we already know from Lemma 3.3 that $\|u(\cdot,t) - u_\star\|_{L^\infty} \to 0$ as $t\to\infty$, we can thus choose $t_0 > 0$ such that

$$\begin{aligned} \int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star}\Big) &= \int_\Omega \Big(H(u(\cdot,t)) - H(u_\star)\Big) \\ &\le \frac{1}{u_\star}\int_\Omega \Big(u(\cdot,t) - u_\star\Big)^2 \qquad \text{for all } t > t_0 \end{aligned} \tag{3.30}$$

and

$$\int_\Omega \Big(u(\cdot,t) - u_\star - u_\star \ln \frac{u(\cdot,t)}{u_\star}\Big) \ge \frac{1}{4u_\star}\int_\Omega \Big(u(\cdot,t) - u_\star\Big)^2 \qquad \text{for all } t > t_0. \tag{3.31}$$

By a similar argument, upon enlarging $t_0$ if necessary we can clearly achieve that also

$$\frac{1}{4v_\star}\int_\Omega \Big(v(\cdot,t) - v_\star\Big)^2 \le \int_\Omega \Big(v(\cdot,t) - v_\star - v_\star \ln \frac{v(\cdot,t)}{v_\star}\Big) \le \frac{1}{v_\star}\int_\Omega \Big(v(\cdot,t) - v_\star\Big)^2 \qquad \text{for all } t > t_0. \tag{3.32}$$

In view of the definitions (3.1) and (3.2) of $E_1$ and $F_1$, (3.30) and the right inequality in (3.32) imply that for some $c_1 > 0$ we have $E_1(t) \le c_1 F_1(t)$ for all $t > t_0$. Substituting this into the energy inequality (3.4), we obtain

$$\frac{d}{dt}E_1(t) \le -\varepsilon F_1(t) \le -\varepsilon c_1 E_1(t) \qquad \text{for all } t > t_0, \tag{3.33}$$

which on integration shows that there exist $c_2 > 0$ and $l > 0$ fulfilling

$$E_1(t) \le c_2 e^{-lt} \qquad \text{for all } t > 0.$$

Now thanks to (3.31) and the left inequality in (3.32), this entails the existence of positive constants $c_3$, $c_4$ and $c_5$ satisfying

$$\int_\Omega \Big(u(\cdot,t) - u_\star\Big)^2 + \int_\Omega \Big(v(\cdot,t) - v_\star\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 \le c_3 E_1(t) \le c_4 e^{-lt} \qquad \text{for all } t > t_0 \tag{3.34}$$

and

$$\int_{t-1}^t \int_\Omega |\nabla w|^2 \le \int_{t-1}^t F_1(s)ds \le -\frac{1}{\varepsilon}\int_{t-1}^t \frac{d}{ds}E_1(s)ds \le \frac{1}{\varepsilon}E_1(t-1) \le c_5 e^{-lt} \qquad \text{for all } t > t_0 + 1. \tag{3.35}$$

Now since the Hölder inequality ensures that $\|\varphi\|_{L^{2n}(\Omega)} \le \|\varphi\|_{L^\infty(\Omega)}^{\frac{n-1}{n}} \|\varphi\|_{L^2(\Omega)}^{\frac{1}{n}}$ for all $\varphi \in L^\infty(\Omega)$, and since $(u,v,w)$ is bounded in $\Omega\times(0,\infty)$ by Theorem 1.1, (3.34) warrants that

$$\|u(\cdot,t) - u_\star\|_{L^{2n}(\Omega)} + \|v(\cdot,t) - v_\star\|_{L^{2n}(\Omega)} + \|w(\cdot,t) - w_\star\|_{L^{2n}(\Omega)} \le c_6 e^{-\frac{l}{2n}t} \qquad \text{for all } t > t_0$$

with some $c_6 > 0$. Together with (3.35), this allows for an application of Lemma 3.6 which upon evident choices of $h_1$ and $h_2$ shows (3.26) with $\lambda := \frac{1}{2n+2}$.

ii.i) This part can be proved by a discussion similar to that in i).

ii.ii) In the critical case $a_1 = 1$, we note that the definitions (3.11) and (3.12) of the functionals $E_2$ and $F_2$ actually reduce to the identities

$$E_2(t) = \int_\Omega u(\cdot,t) + \frac{\mu_1 a_1'}{\mu_2 a_2}\int_\Omega \Big(v(\cdot,t) - 1 - \ln v(\cdot,t)\Big) + \frac{\delta}{2}\int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2, \qquad t > 0,$$

and

$$F_2(t) = \int_\Omega \Big|\frac{\nabla v(\cdot,t)}{v(\cdot,t)}\Big|^2 + \int_\Omega \big|\nabla w(\cdot,t)\big|^2 + \int_\Omega u^2(\cdot,t) + \int_\Omega \Big(v(\cdot,t) - 1\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2, \qquad t > 0,$$

with $w_\star = \frac{\alpha}{\gamma}$, and that Lemma 3.4 says that there exists $\varepsilon > 0$ such that

$$\frac{d}{dt}E_2(t) \le -\varepsilon F_2(t) \qquad \text{for all } t > 0. \tag{3.36}$$

Here we use an evident analogue of the right inequality in (3.32) and then apply the Cauchy-Schwarz inequality to see that since $(u, v, w)$ is bounded in $\Omega \times (0,\infty)$, there exist $t_1 > 0$, $c_7 > 0, c_8 > 0$ and $c_9 > 0$ such that

$$\begin{aligned} E_2(t) &\le \int_\Omega u(\cdot,t) + c_7\int_\Omega \Big(v(\cdot,t) - 1\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 \\ &\le c_8\Big(\int_\Omega u^2(\cdot,t)\Big)^{\frac12} + c_8\Big(\int_\Omega \Big(v(\cdot,t) - 1\Big)^2\Big)^{\frac12} + c_8\Big(\int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2\Big)^{\frac12} \\ &\le c_9 F_2^{\frac12}(t) \qquad \text{for all } t > t_1. \end{aligned}$$

Thus, (3.36) implies that for some $c_{10} > 0$ we have

$$\frac{d}{dt}E_2(t) \le -c_{10}E_2^2(t) \qquad \text{for all } t > t_1,$$

and that we can hence find $c_{11} > 0$ satisfying

$$E_2(t) \le \frac{c_{11}}{t+1} \qquad \text{for all } t > 0. \tag{3.37}$$

Thanks to an adapted version of the left inequality in (3.32), this guarantees the existence of $t_2 > t_1 + 1$ and $c_{12} > 0$ such that

$$\int_\Omega u(\cdot,t) + \int_\Omega \Big(v(\cdot,t) - 1\Big)^2 + \int_\Omega \Big(w(\cdot,t) - w_\star\Big)^2 \le c_{12}E_2(t) \le \frac{c_{12}c_{11}}{t+1} \qquad \text{for all } t > t_2.$$

and

$$\int_{t-1}^t \int_\Omega |\nabla w|^2 \le \int_{t-1}^t F_2(s)ds \le -\frac{1}{\varepsilon}\int_{t-1}^t \frac{d}{ds}E_2(s)ds \le CE_2(t-1) \le \frac{C}{t+1} \qquad \text{for all } t > t_2.$$

Another application of Lemma 3.6 thereupon yields (3.28). $\square$

**Remark.** In the critical case $a_1 = 1$, exponential decay as in (3.27) cannot be expected in general. Indeed, if all the initial data $u_0, v_0, w_0$ are positive constants with $v_0 \le 1$, then clearly also $u(\cdot,t), v(\cdot,t)$ and $w(\cdot,t)$ are spatially homogeneous for all $t > 0$, whence the PDE system in (1.1) actually reduces to the ODE system (1.6), that is, to

$$\begin{cases} u_t = \mu_1 u(1 - u - a_1 v), & x \in \Omega,\ t > 0, \\ v_t = \mu_2 v(1 - v - a_2 u), & x \in \Omega,\ t > 0, \\ w_t = -\gamma w + \alpha u + \beta v, & x \in \Omega,\ t > 0, \end{cases}$$

Here the second equation implies that $v_t \le \mu_2 v(1 - v)$ for all $x \in \Omega$ and $t > 0$, which in view of our restriction $v_0 \le 1$ shows that $v \le 1$ in $\Omega \times (0, \infty)$. Since $a_1 \le 1$, the first equation herein now yields the inequality $u_t \ge -\mu_1 u^2$ in $\Omega \times (0, \infty)$, which upon integration entails that with some $c > 0$ we have

$$u(x,t) \ge \frac{c}{t+1} \qquad \text{for all } x \in \Omega \text{ and } t > 0,$$

in particular meaning that in fact the largest possible constant $\kappa$ in Lemma 3.7 ii.ii) is $\kappa = 1$.

Finally, our main results on equilibration immediately result from Lemma 3.7.

Proof of Theorem 1.2. We only need to apply Lemma 3.7 i). $\square$

Proof of Theorem 1.3. Both statements are precisely asserted by Lemma 3.7 ii.i) and ii.ii). $\square$

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