Stabilization in a chemotaxis model for tumor invasion

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Abstract

This paper deals with the chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + wz, & x \in \Omega,\ t > 0, \\ w_t = -wz, & x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + u, & x \in \Omega,\ t > 0, \end{cases}$$

in a smoothly bounded domain $\Omega \subset \mathbb{R}^n$, $n \le 3$, that has recently been proposed as a model for tumor invasion in which the role of an active extracellular matrix is accounted for.

It is shown that for any choice of nonnegative and suitably regular initial data (u_0, v_0, w_0, z_0), a corresponding initial-boundary value problem of Neumann type possesses a global solution which is bounded. Moreover, it is proved that whenever $u_0 \not\equiv 0$, these solutions approach a certain spatially homogeneous equilibrium in the sense that as $t \to \infty$,

$$u(x,t) \to \overline{u_0}, \qquad v(x,t) \to \overline{v_0} + \overline{w_0}, \qquad w(x,t) \to 0 \qquad \text{and} \qquad z(x,t) \to \overline{u_0},$$

uniformly with respect to $x \in \Omega$, where $\overline{u_0} := \frac{1}{|\Omega|} \int_\Omega u_0$, $\overline{v_0} := \frac{1}{|\Omega|} \int_\Omega v_0$ and $\overline{w_0} := \frac{1}{|\Omega|} \int_\Omega w_0$.

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1 Introduction

This paper is concerned with the chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v + wz, \\ w_t = -wz, \\ z_t = \Delta z - z + u, \end{cases} \tag{1.1}$$

which has been proposed in [5] as a modification of the tumor invasion model originally introduced by Chaplain and Anderson in [2]. A particular focus of the model (1.1) consists in accounting for a chemotactic attraction induced by a so-called *active extracellular matrix*, ECM*, which is produced by a biological reaction between the extracellular matrix, ECM, and a matrix-degrading enzyme, MDE. Accordingly, besides the densities u, w and z of tumor cells, ECM and MDE, a fourth relevant quantity becomes the concentration of ECM*, which is represented by the function v in (1.1).

In the past two decades, a large variety of mathematical models describing tumor invasion phenomena has been developed by focusing on different aspects. Besides models purely based on reaction-diffusion equations ([6]), most of these models at their core assume taxis mechanisms which are of *haptotaxis* type, meaning that the respective attractant is non-diffusible (see e.g. [2] and [1] or also the discussion in [5]). Analytical results on such haptotaxis systems, essentially containing certain memory-type evolution problems as subsystems, are yet quite fragmentary, so far mainly concentrating on issues such as global existence and boundedness ([13], [14], [19], [20], [22], [27]); more detailed answers have been given only in certain special cases ([4], [8], [9], [12]). After all, certain global existence results can be achieved for such haptotaxis systems even when expanded to more realistic models ([3]) by including additional mechanisms ([21], [23], [24], [25], [26]).

As compared to this, cross-diffusion in (1.1) is of *chemotaxis* type in that it is directed toward the diffusible ECM*, the latter being produced by the static ECM in conjunction with the chemical MDE. From a mathematical point of view, one might expect this additional influence of diffusion to entail certain improved regularity properties of solutions. On the other hand, the literature shows that also such chemotactic cross-diffusion may have a strong destabilizing effect: For instance, in the Keller-Segel system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla z), \\ z_t = \Delta z - z + u, \end{cases} \tag{1.2}$$

widely considered as a prototypical model for chemoattractive processes, it is known that solutions are global and remain bounded if either $n = 1$ ([17]), or $n \geq 2$ and the initial data are suitably small ([16], [28]), but that some large-data solutions become unbounded even within finite time in the cases $n = 2$ ([7], [15]) and $n \geq 3$ ([29]), where n denotes the space dimension.

Main results. As opposed to (1.1), in (1.2) the substance secreted by the cells is immediately directing chemoattraction, whereas in (1.1) this chemical only has an indirect taxis effect by stimulating the signal production. It is the purpose of the present paper to clarify how far this indirect chemotactic feedback may enhance the regularity and boundedness properties of solutions. Indeed, we shall see that any type of blow-up is thereby entirely suppressed in the physically relevant case $n \leq 3$, and that furthermore basically all solutions approach a spatially homogeneous equilibrium in the large time limit.

In order to precisely formulate our results in this direction, let us specify the full problem setting by considering the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \qquad x \in \Omega,\ t > 0, \\ v_t = \Delta v + wz, \qquad x \in \Omega,\ t > 0, \\ w_t = -wz, \qquad x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + u, \qquad x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad w(x,0) = w_0(x), \quad z(x,0) = z_0(x), \qquad x \in \Omega, \end{cases} \tag{1.3}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where throughout this paper we shall assume that

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) \quad \text{is nonnegative,} \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{is nonnegative,} \\ w_0 \in C^2(\bar{\Omega}) \quad \text{is nonnegative} \qquad \text{and} \\ z_0 \in C^0(\bar{\Omega}) \quad \text{is nonnegative.} \end{cases} \tag{1.4}$$

The first of our main results asserts that under this condition, (1.3) admits for global existence of a bounded classical solution when $n \le 3$. We underline that the following statement on this does not require any smallness condition on the initial data, such as necessary for global boundedness in the Keller-Segel system.

Theorem 1.1 *Let $n \le 3$, and suppose that (1.4) holds. Then there exists a uniquely determined quadruple (u, v, w, z) of nonnegative functions*

$$\begin{aligned} &u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ &v \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\infty}(\Omega)), \\ &w \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{0,1}(\bar{\Omega} \times (0,\infty)) \qquad \textit{and} \\ &z \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \end{aligned}$$

which solve (1.3) classically in $\Omega \times (0,\infty)$. Moreover the solution (u, v, w, z) of (1.3) is bounded in the sense that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0.$$

Moreover, whenever u_0 is nontrivial, the above solution approaches a certain spatially homogeneous steady state:

Theorem 1.2 *Let $n \le 3$. Assume that u_0, v_0, w_0 and z_0 comply with (1.4), and that $u_0 \not\equiv 0$. Then the solution (u, v, w, z) of (1.3) satisfies*

$$\begin{aligned} &\|u(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0, \\ &\|v(\cdot,t) - (\overline{v_0} + \overline{w_0})\|_{L^\infty(\Omega)} \to 0, \\ &\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{and} \\ &\|z(\cdot,t) - \overline{u_0}\|_{L^\infty(\Omega)} \to 0 \end{aligned}$$

as $t \to \infty$, where the constants $\overline{u_0}$, $\overline{v_0}$ and $\overline{w_0}$ are given by

$$\overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0, \qquad \overline{v_0} := \frac{1}{|\Omega|}\int_\Omega v_0, \qquad \textit{and} \qquad \overline{w_0} := \frac{1}{|\Omega|}\int_\Omega w_0. \tag{1.5}$$

In consequence, the indirect mechanism of signal production in (1.3) is apparently insufficient to generate any significant instability of homogeneous distributions: In fact, the results from Theorem 1.1 and Theorem 1.2 indicate that at least when $n \le 3$, the cross-diffusive term in the first equation in (1.3) is substantially overbalanced by diffusion, and that hence the overall behavior of the model, with respect to both global solvability and asymptotic behavior, is essentially the same as that of the correspondingly modified system obtained on fully disregarding this taxis mechanism.

This paper is organized as follows. After collecting some preliminary facts including local existence in Section 2, we shall establish Theorem 1.1 in Section 3 by deriving suitable a priori estimates through a two-step bootstrap argument which eventually yields a bound for the crucial component u with respect to the norm in $L^\infty(\Omega)$ (Lemma 3.5). The large time behavior will be addressed in Section 4, as a starting point using the integrability property

$$\int_0^\infty \int_\Omega w(x,t) z(x,t) dx dt < \infty$$

(Lemma 4.1). Thanks to global regularity estimates implied by the boundedness of solutions (Lemma 3.6), this will entail convergence of v to some nonnegative constant L in $W^{1,\infty}(\Omega)$ (Lemma 4.3). This in turn warrants stabilization of u (Lemma 4.4) and then of z (Lemma 4.5) in the sense claimed by Theorem 1.2, where the latter property along with the assumption $\overline{u_0} > 0$ enforces decay of w (Lemma 4.6) and thereupon allows for determining L (Lemma 4.7), thus completing the proof of Theorem 1.2.

2 Preliminaries. Local existence and basic estimates

The following statement on local existence and uniqueness is contained in [5, Theorem 3.1].

Lemma 2.1 *Let $n \ge 1$, and assume that u_0, v_0, w_0 and z_0 satisfy (1.4). Then there exist $T_{max} \in (0, \infty]$ and a unique classical solution (u, v, w, z) of (1.3) in $\Omega \times (0, T_{max})$ which is such that*

$$\begin{aligned}
& 0 \le u \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})), \\
& 0 \le v \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})) \cap L^\infty_{loc}([0, T_{max}); W^{1,\infty}(\Omega)), \\
& 0 \le w \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{0,1}(\bar\Omega \times (0, T_{max})) \qquad \textit{and} \\
& 0 \le z \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})),
\end{aligned}$$

and such that

$$\textit{if } T_{max} < \infty \quad \textit{then} \quad \lim_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \Big) = \infty. \tag{2.1}$$

Throughout the sequel, we suppose that (u_0, v_0, w_0, z_0) is given such that (1.4) holds, and let (u, v, w, z) and $T_{max} \in (0, \infty]$ denote the corresponding solution of (1.3) and its maximal existence time as specified in Lemma 2.1.

The following statement on conservation of the total mass $\int_\Omega u$ of cells is obvious but essential to our analysis.

Lemma 2.2 *The first solution component u satisfies*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx \qquad \textit{for all } t \in (0, T_{max}). \tag{2.2}$$

Proof. This can immediately be seen upon integrating the first equation in (1.3) over $\Omega \times (0,t)$ for $t \in (0, T_{max})$. $\Box$

Likewise, it is evident from (1.3) that w is nonincreasing with time. We shall frequently use the following implication thereof.

Lemma 2.3 *The third solution component w fulfills*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \le \|w_0\|_{L^\infty(\Omega)} \qquad \textit{for all } t \in (0, T_{max}).$$

Proof. Since both w and z are nonnegative, this is obvious from the third equation in (1.3). $\Box$

The particular structure of the nonlinearities in the second and third equations in (1.3) moreover enables us to derive boundedness of v with respect to the norm in $L^1(\Omega)$.

Lemma 2.4 *The second solution component has the property that*

$$\int_\Omega v(x,t)dx \le \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx \qquad \textit{for all } t \in (0, T_{max}). \tag{2.3}$$

Proof. We add the third to the second equation in (1.3) and integrate with respect to $x \in \Omega$ to obtain

$$\frac{d}{dt}\int_\Omega (v+w) = \int_\Omega \Delta v = 0 \qquad \text{for all } t \in (0, T_{max}),$$

because $\frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$. Thus,

$$\int_\Omega v(x,t)dx + \int_\Omega w(x,t)dx = \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx \qquad \text{for all } t \in (0, T_{max}), \tag{2.4}$$

from which (2.3) follows by nonnegativity of w. $\Box$

3 Boundedness. Proof of Theorem 1.1

Throughout our subsequent analysis, we shall frequently make use of well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ in Ω. Unless stated otherwise, corresponding proofs can be carried out in a standard manner such as reported in [18] mainly for the neighboring case of Dirichlet boundary conditions; a precise demonstration for the particular case of Neumann boundary data can be found e.g. in [28, Lemma 1.3].

Let us first use these regularization properties to derive the following estimate for the solution component z under an appropriate boundedness assumption on u.

Lemma 3.1 *Let $p \geq 1$ and*

$$\begin{cases} q \in [1, \frac{np}{n-2p}) & \text{if } p \leq \frac{n}{2}, \\ q \in [1, \infty] & \text{if } p > \frac{n}{2}. \end{cases} \tag{3.1}$$

Then for all $M > 0$ there exists $C_z(p,q,M) > 0$ such that if for some $T \in (0, T_{max})$ we have

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq M \qquad \text{for all } t \in (0,T), \tag{3.2}$$

then

$$\|z(\cdot,t)\|_{L^q(\Omega)} \leq C_z(p,q,M) \qquad \text{for all } t \in (0,T). \tag{3.3}$$

Proof. In view of the Hölder inequality, we may clearly assume that $q > p$. Then according to standard L^p-L^q estimates for $(e^{t\Delta})_{t\geq 0}$, we can find $c_1 > 0$ such that

$$\|e^{\tau\Delta}\varphi\|_{L^q(\Omega)} \leq c_1 \Big(1 + \tau^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Big) \cdot \|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in L^p(\Omega),$$

and using the maximum principle for the heat equation, we easily obtain $c_2 > 0$ fulfilling

$$\|e^{\tau\Delta}\varphi\|_{L^q(\Omega)} \leq c_2 \|\varphi\|_{L^\infty(\Omega)} \qquad \text{for all } \tau > 0 \text{ and arbitrary } \varphi \in L^\infty(\Omega).$$

Therefore, from the variation-of-constants representation of z,

$$z(\cdot,t) = e^{t(\Delta-1)}z_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds \qquad \text{for all } t \in (0,T),$$

we infer that the assumption (3.2) entails the inequality

$$\begin{aligned} \|z(\cdot,t)\|_{L^q(\Omega)} &\leq e^{-t}\|e^{t\Delta}z_0\|_{L^q(\Omega)} + \int_0^t e^{-(t-s)}\|e^{(t-s)\Delta}u(\cdot,s)\|_{L^q(\Omega)}ds \\ &\leq c_2 e^{-t} \cdot \|z_0\|_{L^\infty(\Omega)} + c_1 M \int_0^t e^{-(t-s)} \cdot \Big(1 + (t-s)^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\Big)ds \qquad \text{for all } t \in (0,T). \end{aligned}$$

Since (3.1) ensures that $c_3 := \int_0^\infty (1 + \sigma^{-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}) \cdot e^{-\sigma}d\sigma$ is finite, this implies that

$$\|z(\cdot,t)\|_{L^q(\Omega)} \leq c_2\|z_0\|_{L^\infty(\Omega)} + c_1c_3M \qquad \text{for all } t \in (0,T)$$

and thereby proves (3.3). $\square$

Next, a boundedness property of z of the above form entails a certain regularity for ∇v.

Lemma 3.2 *Let $q \geq 1$ and*

$$\begin{cases} r \in [1, \frac{nq}{n-q}) & \text{if } q \leq n, \\ r \in [1, \infty] & \text{if } q > n. \end{cases} \tag{3.4}$$

Then for all $M > 0$ there exists $C_v(q,r,M) > 0$ with the property that if $t \in (0,T_{max})$ is such that

$$\|z(\cdot,t)\|_{L^q(\Omega)} \leq M \qquad \text{for all } t \in (0,T), \tag{3.5}$$

then

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \leq C_v(q,r,M) \qquad \text{for all } t \in (0,T).$$

PROOF. Again in view of the Hölder inequality, we need to consider the case $r \ge q$ only, in which according to known regularization properties of $(e^{t\Delta})_{t\ge 0}$, as contained in [28, Lemma 1.3], for all $s \in [1,q]$ we can find $c_1(s) > 0$ such that

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^r(\Omega)} \le c_1(s)\Big(1+\tau^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{s}-\frac{1}{r})}\Big)\|\varphi\|_{L^s(\Omega)} \qquad \text{for all } \tau > 0 \text{ and each } \varphi \in L^s(\Omega), \tag{3.6}$$

and moreover there exists $c_2 > 0$ satisfying

$$\|\nabla e^{\tau\Delta}\varphi\|_{L^r(\Omega)} \le c_2\|\varphi\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } \tau > 0 \text{ and any } \varphi \in W^{1,\infty}(\Omega). \tag{3.7}$$

We now fix a nonnegative integer k and represent $v(\cdot,t)$ according to

$$v(\cdot,t) = e^{(t-k)\Delta}v(\cdot,k) + \int_k^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds \qquad \text{for all } t \in (k,\infty)\cap(0,T). \tag{3.8}$$

Here if $k \ge 1$, we may apply (3.6) to $s := 1$ and use Lemma 2.4 to estimate

$$\begin{aligned}
\|\nabla e^{(t-k)\Delta}v(\cdot,k)\|_{L^r(\Omega)} &\le c_1(1)\Big(1+(t-k)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\Big)\|v(\cdot,k)\|_{L^1(\Omega)} \\
&\le c_1(1)c_3\Big(1+(t-k)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{r})}\Big) \\
&\le 2c_1(1)c_3 \qquad \text{for all } t \in [k+1,\infty)\cap(0,T)
\end{aligned} \tag{3.9}$$

with $c_3 := \int_\Omega v_0 + \int_\Omega w_0$. In the case $k=0$, we instead employ (3.7) to obtain

$$\|\nabla e^{(t-k)\Delta}v(\cdot,k)\|_{L^r(\Omega)} = \|\nabla e^{t\Delta}v_0\|_{L^r(\Omega)} \le c_2\|v_0\|_{W^{1,\infty}(\Omega)} \qquad \text{for all } t > 0. \tag{3.10}$$

In the second summand on the right of (3.8) we use (3.6) with $s := q$ to see that

$$\begin{aligned}
\Big\|\nabla \int_k^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds\Big\|_{L^r(\Omega)} &\le c_1(q)\int_k^t \Big(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\Big)\|w(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)}ds \\
&\qquad \text{for all } t \in (k,\infty)\cap(0,T),
\end{aligned} \tag{3.11}$$

where thanks to our hypothesis (3.5) and Lemma 2.3 we know that

$$\|w(\cdot,s)z(\cdot,s)\|_{L^q(\Omega)} \le \|w(\cdot,s)\|_{L^\infty(\Omega)}\|z(\cdot,s)\|_{L^q(\Omega)} \le c_4M \qquad \text{for all } s \in (0,T)$$

with $c_4 := \|w_0\|_{L^\infty(\Omega)}$. Therefore, (3.11) entails that

$$\begin{aligned}
\Big\|\nabla \int_k^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds\Big\|_{L^r(\Omega)} &\le c_1(q)c_4M\int_k^t \Big(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\Big)ds \\
&\le c_1(q)c_4M\cdot c_5 \qquad \text{for all } t \in (k,k+2)\cap(0,T),
\end{aligned} \tag{3.12}$$

where the assumption (3.4) on r warrants that

$$c_5 := \int_0^2 \Big(1+\sigma^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{r})}\Big)d\sigma$$

is finite. Hence, in the case $t \in (0,2) \cap (0,T)$ we infer from (3.8), (3.10) and (3.12) that

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le c_2\|v_0\|_{W^{1,\infty}(\Omega)} + c_1(q)c_4c_5M,$$

whereas whenever $t \in (0,T)$ is such that $t \ge 2$, we can pick an integer $k \ge 1$ such that $t \in [k+1,k+2)$ and thereupon obtain from (3.8), (3.9) and (3.12) that

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le c_1(1)c_3 + c_1(q)c_4c_5M.$$

The proof is thus complete. $\Box$

In the proof of Lemma 3.4 below we shall need the following extension of a regularization estimate for $(e^{t\Delta})_{t\ge0}$, as contained in [28, Lemma 1.3], to the case of the space $L^\infty(\Omega)$.

Lemma 3.3 *Let $p \in (1,\infty]$. Then there exists $C>0$ such that for all $\varphi \in C^1(\bar\Omega;\mathbb{R}^n)$ fulfilling $\varphi\cdot\nu = 0$ on $\partial\Omega$ we have*

$$\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le Ct^{-\frac12-\frac{n}{2p}}\|\varphi\|_{L^p(\Omega)} \qquad \textit{for all } t>0. \tag{3.13}$$

Proof. According to known smoothing properties of $(e^{t\Delta})_{t\ge0}$, there exists $c_1>0$ such that for all $\psi \in C_0^\infty(\Omega)$ we have

$$\|\nabla e^{t\Delta}\psi\|_{L^{p'}(\Omega)} \le c_1 t^{-\frac12-\frac{n}{2}(1-\frac{1}{p'})}\|\psi\|_{L^1(\Omega)} \qquad \text{for all } t>0,$$

where $p' \in [1,\infty)$ is such that $\frac1p + \frac1{p'} = 1$. By the duality characterization of the norm in $L^\infty(\Omega) \simeq (L^1(\Omega))^\star$, and by density of $C_0^\infty(\Omega)$ in $L^1(\Omega)$, we thus obtain on integrating by parts and using the self-adjointness of $e^{t\Delta}$ in $L^2(\Omega)$ that

$$\begin{aligned}
\|e^{t\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} &= \sup_{\substack{\psi\in C_0^\infty(\Omega)\\ \|\psi\|_{L^1(\Omega)}\le1}} \left|\int_\Omega (e^{t\Delta}\nabla\cdot\varphi)\cdot\psi\right| \\
&= \sup_{\substack{\psi\in C_0^\infty(\Omega)\\ \|\psi\|_{L^1(\Omega)}\le1}} \left|\int_\Omega \varphi\cdot\nabla e^{t\Delta}\psi\right| \\
&\le \|\varphi\|_{L^p(\Omega)}\cdot \sup_{\substack{\psi\in C_0^\infty(\Omega)\\ \|\psi\|_{L^1(\Omega)}\le1}} \|\nabla e^{t\Delta}\psi\|_{L^{p'}(\Omega)} \\
&\le \|\varphi\|_{L^p(\Omega)}\cdot c_1 t^{-\frac12-\frac{n}{2}(1-\frac{1}{p'})} \qquad \text{for all } t>0.
\end{aligned}$$

Since $1-\frac1{p'} = \frac1p$, this proves (3.13). $\Box$

We can now prepare a closure of our regularity reasoning by deriving an estimate for u from a supposedly present appropriate boundedness property of ∇v.

Lemma 3.4 *Suppose that $r>n$. Then for all $M>0$ there exists $C_u(r,M)>0$ such that if*

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le M \qquad \textit{for all } t\in(0,T) \tag{3.14}$$

with some $T\in(0,T_{max})$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_u(r,M) \qquad \textit{for all } t\in(0,T).$$

PROOF. Since $r > n$, we can fix a number θ such that

$$\theta > \frac{r}{r+1} \tag{3.15}$$

and

$$n < \theta < r. \tag{3.16}$$

Then according to Lemma 3.3 there exists $c_1 > 0$ fulfilling

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_1\tau^{-\frac{1}{2}-\frac{n}{2\theta}}\|\varphi\|_{L^\theta(\Omega)} \qquad \text{for all } \varphi\in C^1(\bar\Omega;\mathbb{R}^n) \text{ such that } \varphi\cdot\nu=0 \text{ on } \partial\Omega. \tag{3.17}$$

Moreover, standard L^p-L^q estimates yield $c_2>0$ satisfying

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \le c_2\tau^{-\frac{n}{2}}\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \tau>0 \text{ and each } \varphi\in L^1(\Omega) \text{ such that } \int_\Omega\varphi=0. \tag{3.18}$$

Now proceeding in a way similar to that in the proof of Lemma 3.2, for a given integer $k\ge 0$ we use a variation-of-constants representation of u to estimate

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\Omega)} &= \left\|e^{(t-k)\Delta}u(\cdot,k) - \int_k^t e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)ds\right\|_{L^\infty(\Omega)} \\
&\le \|e^{(t-k)\Delta}u(\cdot,k)\|_{L^\infty(\Omega)} + \int_k^t \left\|e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s)\right\|_{L^\infty(\Omega)}ds
\end{aligned} \tag{3.19}$$

for all $t>k$. Here when $k=0$, by the maximum principle we obtain

$$\|e^{(t-k)\Delta}u(\cdot,k)\|_{L^\infty(\Omega)} = \|e^{t\Delta}u_0\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \qquad \text{for all } t>0, \tag{3.20}$$

while in the case $k\ge 1$ we use (3.18) and recall (2.2) to see that

$$\begin{aligned}
\|e^{(t-k)\Delta}u(\cdot,k)\|_{L^\infty(\Omega)} &\le \|e^{(t-k)\Delta}(u(\cdot,k)-\overline{u_0})\|_{L^\infty(\Omega)} + \overline{u_0} \\
&\le c_2(t-k)^{-\frac{n}{2}}\|u(\cdot,k)-\overline{u_0}\|_{L^1(\Omega)} + \overline{u_0} \\
&\le 2c_2(t-k)^{-\frac{n}{2}}\|u_0\|_{L^1(\Omega)} + \overline{u_0} \\
&\le 2c_2\|u_0\|_{L^1(\Omega)} + \overline{u_0} \qquad \text{for all } t\ge k+1,
\end{aligned} \tag{3.21}$$

due to the relation $e^{t\Delta}\overline{u_0}\equiv\overline{u_0}$ for all $t>0$. In the rightmost integral in (3.19), we invoke (3.17) to find that

$$\int_k^t \left\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\right\|_{L^\infty(\Omega)}ds \le c_1\int_k^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^\theta(\Omega)}ds \qquad \text{for all } t\in(k,\infty)\cap(0,T), \tag{3.22}$$

where an application of the Hölder inequality combined with our hypothesis (3.14) shows that

$$\begin{aligned}
\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^\theta(\Omega)} &\le \|\nabla v(\cdot,s)\|_{L^r(\Omega)}\cdot\|u(\cdot,s)\|_{L^{\frac{r\theta}{r-\theta}}(\Omega)} \\
&\le M\cdot\|u(\cdot,s)\|_{L^{\frac{r\theta}{r-\theta}}(\Omega)} \qquad \text{for all } s\in(0,T).
\end{aligned} \tag{3.23}$$

Since the property (3.15) ensures that $\frac{r\theta}{r-\theta} > 1$ and that hence $\kappa := \frac{r-\theta}{r\theta} \in (0,1)$, we may once again use the Hölder inequality and (2.2) to estimate

$$\begin{aligned}
\|u(\cdot,s)\|_{L^{\frac{r\theta}{r-\theta}}(\Omega)} &\le \|u(\cdot,s)\|_{L^1(\Omega)}^{\kappa} \cdot \|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\kappa} \\
&= \|u_0\|_{L^1(\Omega)}^{\kappa} \cdot \|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\kappa} \qquad \text{for all } s \in (0,T),
\end{aligned}$$

so that (3.22) and (3.23) imply that

$$\int_k^t \Big\| e^{(t-s)\Delta} \nabla \cdot \Big(u(\cdot,s) \nabla v(\cdot,s) \Big) \Big\|_{L^\infty(\Omega)} ds \le c_1 M \cdot \|u_0\|_{L^1(\Omega)}^{\kappa} \cdot \int_k^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}} \|u(\cdot,s)\|_{L^\infty(\Omega)}^{1-\kappa} ds$$
$$\text{for all } t \in (k,\infty) \cap (0,T). \qquad (3.24)$$

Thus, writing

$$K \equiv K(T) := \sup_{t \in (0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)},$$

from (3.19), (3.20) and (3.24) we obtain that if $t \in (0,2) \cap (0,T)$ then

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|u_0\|_{L^\infty(\Omega)} + c_1 M \|u_0\|_{L^1(\Omega)}^{\kappa} \cdot K^{1-\kappa} \cdot \int_0^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}} ds \\
&\le \|u_0\|_{L^\infty(\Omega)} + c_1 c_3 M \|u_0\|_{L^1(\Omega)}^{\kappa} \cdot K^{1-\kappa} \qquad (3.25)
\end{aligned}$$

holds with $c_3 := \int_0^2 \sigma^{-\frac{1}{2}-\frac{n}{2\theta}} d\sigma$ being finite due to the left inequality in (3.16). On the other hand, if $t \in (0,T)$ is such that $t \ge 2$ then for some integer $k \ge 1$ we have $t \in [k+1, k+2)$ and hence infer from (3.19), (3.21) and (3.24) that

$$\begin{aligned}
\|u(\cdot,t)\|_{L^\infty(\Omega)} &\le 2c_2 \|u_0\|_{L^1(\Omega)} + \overline{u_0} + c_1 M \|u_0\|_{L^1(\Omega)}^{\kappa} \cdot K^{1-\kappa} \cdot \int_k^t (t-s)^{-\frac{1}{2}-\frac{n}{2\theta}} ds \\
&\le 2c_2 \|u_0\|_{L^1(\Omega)} + \overline{u_0} + c_1 c_3 M \|u_0\|_{L^1(\Omega)}^{\kappa} \cdot K^{1-\kappa}. \qquad (3.26)
\end{aligned}$$

Combining (3.25) with (3.26) thus shows that if we let $c_4 := \max\{\|u_0\|_{L^\infty(\Omega)}, 2c_2\|u_0\|_{L^1(\Omega)} + \overline{u_0}\}$ and $c_5 := c_1 c_3 \|u_0\|_{L^1(\Omega)}^{\kappa}$, then

$$K \le c_4 + c_5 M K^{1-\kappa},$$

from which upon an elementary argument we conclude that

$$K \le \max\left\{ (2c_5 M)^{\frac{1}{\kappa}} \, , \, \Big(\frac{c_4}{c_5 M}\Big)^{\frac{1}{1-\kappa}} \right\},$$

as desired. $\square$

Combining Lemma 3.1, Lemma 3.2 and Lemma 3.4 and using the mass conservation property (2.2) as a starting point, we can now prove that u in fact must be bounded when $n \le 3$.

Lemma 3.5 *Suppose that $n \le 3$. Then there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{3.27}$$

Proof. Since $n \le 3$, we have $\frac{n}{2} < \frac{n}{(n-2)_+}$, so that it is possible to find $q \in [1, n]$ satisfying

$$\frac{n}{2} < q < \frac{n}{(n-2)_+}. \tag{3.28}$$

Here the left inequality warrants that $\frac{nq}{n-q} > n$, whence we can pick a number r fulfilling

$$n < r < \frac{nq}{n-q}. \tag{3.29}$$

We now write $M_1 := \|u_0\|_{L^1(\Omega)}$, let

$$M_2 := C_z(1, q, M_1)$$

be as provided by Lemma 3.1 and

$$M_3 := C_v(q, r, M_2)$$

be as given by Lemma 3.2, and claim that then for any choice of $T \in (0, T_{max})$ we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_u(r, M_3) \qquad \text{for all } t \in (0, T) \tag{3.30}$$

with $C_u(r, M_3)$ taken from Lemma 3.4. Indeed, for any such T, thanks to the right inequality in (3.28) we may apply Lemma 3.1 which in view of (2.2) and our definitions of M_1 and M_2 shows that

$$\|z(\cdot,t)\|_{L^q(\Omega)} \le M_2 \qquad \text{for all } t \in (0, T).$$

Due to the right inequality in (3.29), we thus obtain from Lemma 3.2 that

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le M_3 \qquad \text{for all } t \in (0, T),$$

whereupon Lemma 3.4 implies (3.30), because $r > n$ by (3.29). Since $T \in (0, T_{max})$ was arbitrary, this directly yields (3.27). $\Box$

In light of the extensibility statement in Lemma 2.1, the above readily shows that the local solution actually exists globally in time and has some further boundedness properties.

Lemma 3.6 *Let $n \le 3$. Then the solution (u, v, w, z) of (1.3) is global in time; that is, $T_{max} = \infty$. Moreover, there exist $\alpha \in (0,1)$ and $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} + \|z(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0 \tag{3.31}$$

as well as

$$\|u\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} + \|v\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} + \|z\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])} \le C \qquad \textit{for all } t \ge 1. \tag{3.32}$$

PROOF. From Lemma 3.5 we know that $\sup_{t\in(0,T_{max})} \|u(\cdot,t)\|_{L^\infty(\Omega)}$ is finite, whence applying Lemma 3.1 to some conveniently large $p \ge 1$ and then Lemma 3.2 to suitably large $q \ge 1$ we infer that also $\sup_{t\in(0,T_{max})} \|z(\cdot,t)\|_{L^\infty(\Omega)}$ and $\sup_{t\in(0,T_{max})} \|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}$ are finite. In conjunction with Lemma 2.3 and the extensibility criterion (2.1) in Lemma 2.1, this shows that $T_{max} = \infty$ and, by independence of the obtained estimate with respect to $t \in (0,T_{max}) = (0,\infty)$, establishes (3.31). Thereupon, straightforward bootstrap arguments involving standard interior parabolic regularity theory ([11]) readily yield (3.32). $\Box$

Now the proof of our main result on global well-posedness and boundedness is obvious.

PROOF of Theorem 1.1. We only need to combine Lemma 2.1 with Lemma 3.6. $\Box$

4 Large time behavior. Proof of Theorem 1.2

The core of our proof of the stabilization result in Theorem 1.2 consists in the following observation.

Lemma 4.1 *The solution of (1.3) has the property that*

$$\int_0^\infty \int_\Omega w(x,t)z(x,t)dxdt < \infty. \tag{4.1}$$

PROOF. For arbitrary $t > 0$, integrating the third equation in (1.3) over $\Omega \times (0,t)$ we obtain

$$\int_0^t \int_\Omega w(x,s)z(x,s)dxds = \int_\Omega w_0(x)dx - \int_\Omega w(x,t)dx.$$

Since w is nonnegative, this implies (4.1). $\Box$

When combined with appropriate compactness properties such as e.g. implied by Lemma 3.6, the above integrability statement can step by step be turned into the convergence results from Theorem 1.2. We first derive a weak version of the claimed stabilization property of v.

Lemma 4.2 *There exists a constant $L \ge 0$ such that*

$$\|v(\cdot,t) - L\|_{L^1(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{4.2}$$

PROOF. According to Lemma 3.6 and e.g. the Arzelà-Ascoli theorem, we can find $(t_k)_{k\in\mathbb{N}} \subset (1,\infty)$ and a nonnegative function $v_\infty \in C^0(\bar\Omega)$ such that $t_k \to \infty$ and

$$v(\cdot,t_k) \to v_\infty \qquad \text{in } L^1(\Omega) \tag{4.3}$$

as $k \to \infty$. To show that we actually have

$$v(\cdot,t) \to \overline{v_\infty} := \frac{1}{|\Omega|}\int_\Omega v_\infty \quad \text{in } L^1(\Omega) \qquad \text{as } k \to \infty, \tag{4.4}$$

we let $\varepsilon > 0$ be given. Then in view of (4.3) and Lemma 4.1 we can fix $k \in \mathbb{N}$ large enough such that

$$\|v(\cdot,t_k) - v_\infty\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \tag{4.5}$$

and

$$\int_{t_k}^\infty \int_\Omega w(x,t)z(x,t)\,dxdt < \frac{\varepsilon}{3}. \tag{4.6}$$

Moreover, using the well-known fact that for any $\varphi \in L^1(\Omega)$ we have $e^{\tau\Delta}\varphi \to \frac{1}{|\Omega|}\int_\Omega \varphi$ in $L^1(\Omega)$ as $\tau \to \infty$, we can choose some suitably large $\tau_0 > 0$ fulfilling

$$\|e^{\tau\Delta}v_\infty - \overline{v_\infty}\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } \tau > \tau_0. \tag{4.7}$$

Then by means of the variation-of-constants representation of v we see that

$$\begin{aligned} v(\cdot,t) - \overline{v_\infty} &= e^{(t-t_k)\Delta}\Big(v(\cdot,t_k) - v_\infty\Big) + \Big(e^{(t-t_k)\Delta}v_\infty - \overline{v_\infty}\Big) \\ &\quad + \int_{t_k}^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds \qquad \text{for all } t > t_k, \end{aligned} \tag{4.8}$$

where from (4.7) we obtain

$$\|e^{(t-t_k)\Delta}v_\infty - \overline{v_\infty}\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t > t_k + \tau_0. \tag{4.9}$$

Next, since $e^{\tau\Delta}$ acts as a contraction on $L^1(\Omega)$, we can use (4.5) to estimate

$$\Big\|e^{(t-t_k)\Delta}\Big(v(\cdot,t_k) - v_\infty\Big)\Big\|_{L^1(\Omega)} \le \|v(\cdot,t_k) - v_\infty\|_{L^1(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t > t_k, \tag{4.10}$$

and invoke (4.6) to infer that

$$\begin{aligned} \Big\| \int_{t_k}^t e^{(t-s)\Delta}w(\cdot,s)z(\cdot,s)ds\Big\|_{L^1(\Omega)} &\le \int_{t_k}^t \|w(\cdot,s)z(\cdot,s)\|_{L^1(\Omega)}ds \\ &\le \int_{t_k}^\infty \int_\Omega w(x,s)z(x,s)\,dxds \\ &< \frac{\varepsilon}{3}. \end{aligned} \tag{4.11}$$

Collecting (4.8)-(4.11) shows that

$$\|v(\cdot,t) - \overline{v_\infty}\|_{L^1(\Omega)} < \varepsilon \qquad \text{for all } t > t_k + \tau_0,$$

which establishes (4.4) and thereby proves (4.2) with $L := \overline{v_\infty} \ge 0$. $\square$

According to Lemma 3.6 and the Arzelà-Ascoli theorem, the above convergence actually takes place in the space $W^{1,\infty}(\Omega)$.

Lemma 4.3 *With $L \ge 0$ as in Lemma 4.2, we have*

$$\|v(\cdot,t) - L\|_{W^{1,\infty}(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{4.12}$$

In particular,

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } t \to \infty. \tag{4.13}$$

PROOF. Since Lemma 3.6 asserts that $(v(\cdot,t))_{t\ge 1}$ is bounded in $C^2(\bar\Omega)$ and hence relatively compact in $C^1(\bar\Omega)$ thanks to the Arzelà-Ascoli theorem, (4.12) and thus also (4.13) immediately result from Lemma 4.2. $\square$

Having thus asserted appropriate decay of the gradient responsible for cross-diffusion in (1.3), we can proceed to make sure that u approaches its spatial mean in the large time limit.

Lemma 4.4 *The first component of the solution of (1.3) satisfies*

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t\to\infty.$$

where $\overline{u_0}$ is given by (1.5).

PROOF. In view of Lemma 3.6 and the Arzelà-Ascoli theorem, it is sufficient to show that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^2(\Omega)} \to 0 \qquad \text{as } t\to\infty. \tag{4.14}$$

To accomplish this, we first recall that if $\lambda_1>0$ denotes the first nonzero eigenvalue of the Neumann Laplacian in Ω, then

$$\|e^{\tau\Delta}\varphi\|_{L^2(\Omega)} \le e^{-\lambda_1\tau}\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \tau>0 \text{ and all } \varphi\in L^2(\Omega) \text{ fulfilling } \int_\Omega \varphi=0, \tag{4.15}$$

because for any such φ, by the variational characterization of λ_1 a standard testing procedure shows that

$$\frac{d}{d\tau}\int_\Omega |e^{\tau\Delta}\varphi|^2 = -2\int_\Omega |\nabla e^{\tau\Delta}\varphi|^2 \le -2\lambda_1\int_\Omega |e^{\tau\Delta}\varphi|^2 \qquad \text{for all } \tau>0.$$

Moreover, with some $c_1>0$ we have

$$\begin{aligned} \|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^2(\Omega)} \quad &\le \quad c_1(1+\tau^{-\frac12})\cdot e^{-\lambda_1\tau}\cdot\|\varphi\|_{L^2(\Omega)} \qquad \text{for all } \tau>0 \text{ and any } \varphi\in C^1(\bar\Omega;\mathbb{R}^n) \\ &\qquad\qquad \text{such that } \varphi\cdot\nu=0 \text{ on } \partial\Omega \end{aligned} \tag{4.16}$$

(cf. e.g. [28, Lemma 1.3]).
We next let $h(x,t):=u(x,t)\nabla v(x,t)$ for $x\in\bar\Omega$ and $t>0$, and note that according to Lemma 3.6 we can find $c_2>0$ such that

$$\|h(\cdot,t)\|_{L^2(\Omega)} \le c_2 \qquad \text{for all } t>0, \tag{4.17}$$

whereas Lemma 3.6 combined with Lemma 4.3 entails that

$$\|h(\cdot,t)\|_{L^2(\Omega)} \to 0 \qquad \text{as } t\to\infty. \tag{4.18}$$

Now in order to prove (4.14) we let $\varepsilon>0$ be given and can thereupon choose $t_0>0$ large enough such that

$$e^{-\lambda_1 t}\|u_0-\overline{u_0}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t>t_0 \tag{4.19}$$

as well as

$$c_1c_2\cdot\int_{\frac t2}^\infty (1+\sigma^{-\frac12})\cdot e^{-\lambda_1\sigma}d\sigma < \frac{\varepsilon}{3} \qquad \text{for all } t>t_0, \tag{4.20}$$

and such that furthermore

$$c_1\|h(\cdot,t)\|_{L^2(\Omega)} \cdot \int_0^\infty (1+\sigma^{-\frac{1}{2}}) \cdot e^{-\lambda_1\sigma} d\sigma < \frac{\varepsilon}{3} \qquad \text{for all } t > \frac{t_0}{2}, \tag{4.21}$$

where in achieving the latter we make use of (4.18).
Then since constants are invariant under the action of $e^{t\Delta}$, we have $e^{t\Delta}\overline{u_0} \equiv \overline{u_0}$ for all $t>0$ and thus can represent u according to

$$u(\cdot,t) - \overline{u_0} = e^{t\Delta}(u_0 - \overline{u_0}) - \int_0^t e^{(t-s)\Delta}\nabla \cdot h(\cdot,s)ds, \qquad \text{for all } t>0.$$

Here we apply (4.16) to estimate

$$\|u(\cdot,t)-\overline{u_0}\|_{L^2(\Omega)} \le \|e^{t\Delta}(u_0-\overline{u_0})\|_{L^2(\Omega)} + c_1\int_0^t \Big(1+(t-s)^{-\frac{1}{2}}\Big)\cdot e^{-\lambda_1(t-s)}\cdot\|h(\cdot,s)\|_{L^2(\Omega)}ds \qquad \text{for all } t>0, \tag{4.22}$$

where due to (4.15) and (4.19) we have

$$\|e^{t\Delta}(u_0-\overline{u_0})\|_{L^2(\Omega)} \le e^{-\lambda_1 t}\|u_0-\overline{u_0}\|_{L^2(\Omega)} < \frac{\varepsilon}{3} \qquad \text{for all } t>t_0. \tag{4.23}$$

Moreover, (4.17) and (4.20) ensure that

$$\begin{aligned} c_1\int_0^{\frac{t}{2}} \Big(1+(t-s)^{-\frac{1}{2}}\Big)\cdot e^{-\lambda_1(t-s)}\cdot\|h(\cdot,s)\|_{L^2(\Omega)}ds &\le c_1c_2\int_0^{\frac{t}{2}} \Big(1+(t-s)^{-\frac{1}{2}}\Big)\cdot e^{-\lambda_1(t-s)}ds \\ &= c_1c_2\int_{\frac{t}{2}}^t (1+\sigma^{-\frac{1}{2}})\cdot e^{-\lambda_1\sigma}d\sigma \\ &< \frac{\varepsilon}{3} \qquad \text{for all } t>t_0, \end{aligned} \tag{4.24}$$

while from (4.18) and (4.21) we infer that

$$\begin{aligned} & c_1\int_{\frac{t}{2}}^t \Big(1+(t-s)^{-\frac{1}{2}}\Big)\cdot e^{-\lambda_1(t-s)}\cdot\|h(\cdot,s)\|_{L^2(\Omega)}ds \\ &\quad\le c_1\cdot\sup_{s>\frac{t}{2}}\|h(\cdot,s)\|_{L^2(\Omega)}\cdot\int_{\frac{t}{2}}^t \Big(1+(t-s)^{-\frac{1}{2}}\Big)\cdot e^{-\lambda_1(t-s)}ds \\ &\quad= c_1\cdot\sup_{s>\frac{t}{2}}\|h(\cdot,s)\|_{L^2(\Omega)}\cdot\int_0^{\frac{t}{2}} (1+\sigma^{-\frac{1}{2}})\cdot e^{-\lambda_1\sigma}d\sigma \\ &\quad\le c_1\cdot\sup_{s>\frac{t}{2}}\|h(\cdot,s)\|_{L^2(\Omega)}\cdot\int_0^\infty (1+\sigma^{-\frac{1}{2}})\cdot e^{-\lambda_1\sigma}d\sigma \\ &\quad\le \frac{\varepsilon}{3} \qquad \text{for all } t>t_0. \end{aligned}$$

Along with (4.23), (4.24) and (4.22), this shows (4.14) and thus completes the proof. □

Now the above convergence property has a straightforward consequence for z.

Lemma 4.5 *The fourth component of the solution of (1.3) satisfies*

$$\|z(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t\to\infty \tag{4.25}$$

with $\overline{u_0}$ determined by (1.5).

PROOF. As a consequence of Lemma 3.6, we can find $c_1>0$ such that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} \le c_1 \qquad \text{for all } t>0. \tag{4.26}$$

Now Lemma 4.4 says that given $\varepsilon>0$ we can fix some sufficiently large $t_0>0$ such that

$$\|u(\cdot,t)-\overline{u_0}\|_{L^\infty(\Omega)} < \frac{\varepsilon}{4} \qquad \text{for all } t>\frac{t_0}{2}, \tag{4.27}$$

where enlarging t_0 if necessary we can also achieve that

$$\|z_0\|_{L^\infty(\Omega)}\cdot e^{-t} < \frac{\varepsilon}{4} \qquad \text{for all } t>t_0 \tag{4.28}$$

and

$$\overline{u_0}\cdot e^{-t} < \frac{\varepsilon}{4} \qquad \text{for all } t>t_0 \tag{4.29}$$

as well as

$$c_1\cdot e^{-\frac{t}{2}} < \frac{\varepsilon}{4} \qquad \text{for all } t>t_0. \tag{4.30}$$

By the variation-of-constants representation of z, we can write

$$\begin{aligned} z(\cdot,t)-\overline{u_0} &= e^{-t}e^{t\Delta}z_0 + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}\Big(u(\cdot,s)-\overline{u_0}\Big)ds \\ &\quad + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}\overline{u_0}ds \ -\overline{u_0} \qquad \text{for all } t>0, \end{aligned} \tag{4.31}$$

and use the maximum principle and (4.28) in estimating

$$\|e^{-t}e^{t\Delta}z_0\|_{L^\infty(\Omega)} \le e^{-t}\|z_0\|_{L^\infty(\Omega)} < \frac{\varepsilon}{4} \qquad \text{for all } t>t_0. \tag{4.32}$$

As $e^{(t-s)\Delta}\overline{u_0}\equiv\overline{u_0}$, by (4.29) we moreover have

$$\begin{aligned} \Big\|\int_0^t e^{-(t-s)}e^{(t-s)\Delta}\overline{u_0}ds \ -\overline{u_0}\Big\|_{L^\infty(\Omega)} &= \Big|\int_0^t e^{-(t-s)}ds-1\Big|\cdot\overline{u_0} \\ &= e^{-t}\overline{u_0} \\ &< \frac{\varepsilon}{4} \qquad \text{for all } t>t_0. \end{aligned} \tag{4.33}$$

Finally, again by means of the maximum principle we obtain

$$\Big\|\int_0^t e^{-(t-s)}e^{(t-s)\Delta}\Big(u(\cdot,s)-\overline{u_0}\Big)ds\Big\|_{L^\infty(\Omega)} \le \int_0^t e^{-(t-s)}\|u(\cdot,s)-\overline{u_0}\|_{L^\infty(\Omega)}ds \qquad \text{for all } t>0, \tag{4.34}$$

where from (4.26) and (4.30) we know that

$$\begin{aligned}\int_0^{\frac{t}{2}} e^{-(t-s)} \|u(\cdot,s)-\overline{u_0}\|_{L^\infty(\Omega)} ds &\le c_1 \int_0^{\frac{t}{2}} e^{-(t-s)} ds \\ &= c_1 (e^{-\frac{t}{2}} - e^{-t}) \\ &< \frac{\varepsilon}{4} \qquad \text{for all } t > t_0, \end{aligned} \tag{4.35}$$

and where (4.27) guarantees that

$$\begin{aligned}\int_{\frac{t}{2}}^{t} e^{-(t-s)} \|u(\cdot,s)-\overline{u_0}\|_{L^\infty(\Omega)} ds &\le \frac{\varepsilon}{4} \cdot \int_{\frac{t}{2}}^{t} e^{-(t-s)} ds \\ &= \frac{\varepsilon}{4} \cdot (1 - e^{-\frac{t}{2}}) \\ &< \frac{\varepsilon}{4} \qquad \text{for all } t > t_0. \end{aligned} \tag{4.36}$$

Inserting (4.32)-(4.36) into (4.31) yields (4.25). $\square$

Whenever the limit in Lemma 4.5 is nontrivial, we can finally show that the monotone limit of $w(\cdot,t)$ as $t \to \infty$ actually must be zero.

Lemma 4.6 *Suppose that $u_0 \not\equiv 0$. Then*

$$\|w(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{4.37}$$

Proof. Since $\int_\Omega u_0 > 0$, the uniform stabilization of z, as asserted by Lemma 4.5, enables us to find $c_1 > 0$ and $t_0 > 0$ such that

$$z(x,t) \ge c_1 \qquad \text{for all } x \in \Omega \text{ and } t > t_0.$$

Integrating the third equation in (1.3) with respect to the time variable, in view of Lemma 2.3 we thus infer that

$$\begin{aligned} w(x,t) &= w(x,t_0) \cdot \exp\Big(- \int_{t_0}^t z(x,s) ds \Big) \\ &\le \|w_0\|_{L^\infty(\Omega)} \cdot e^{-c_1(t-t_0)} \qquad \text{for all } x \in \Omega \text{ and } t > t_0, \end{aligned}$$

which immediately implies (4.37). $\square$

For completing our knowledge on the asypmtotics of solutions, it remains to determine the value of the above number L. If $u_0 \not\equiv 0$, this can easily be achieved by using Lemma 4.6 in conjunction with (2.4) and Lemma 4.3.

Lemma 4.7 *Suppose that $u_0 \not\equiv 0$. Then the number L provided by Lemma 4.2 satisfies*

$$L = \overline{v_0} + \overline{w_0}, \tag{4.38}$$

where $\overline{v_0}$ and $\overline{w_0}$ are given by (1.5).

PROOF. According to (2.4) and Lemma 4.6, we obtain that

$$\int_\Omega v(x,t)dx \to \int_\Omega v_0(x)dx + \int_\Omega w_0(x)dx \qquad \text{as } t \to \infty.$$

On the other hand, Lemma 4.3 shows that

$$\int_\Omega v(x,t)dx \to |\Omega| L \qquad \text{as } t \to \infty.$$

Combining these relations immediately yields (4.38). □

Now our main result on stabilization is evident.

PROOF of Theorem 1.2. We only need to collect Lemma 4.4, Lemma 4.3, Lemma 4.7, Lemma 4.6 and Lemma 4.5. □

Remark 4.8 By straightforward adaptation, for the corresponding variant of (1.3) given by

$$\begin{cases} u_t = d_1\Delta u - \nabla\cdot(\chi u\nabla v), \qquad x\in\Omega,\ t>0, \\ v_t = d_2\Delta v + \alpha wz, \qquad x\in\Omega,\ t>0, \\ w_t = -\beta wz, \qquad x\in\Omega,\ t>0, \\ z_t = d_4\Delta z - \gamma z + \delta u, \qquad x\in\Omega,\ t>0, \\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial z}{\partial\nu} = 0, \qquad x\in\partial\Omega,\ t>0, \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), \quad z(x,0)=z_0(x), \qquad x\in\Omega, \end{cases} \tag{4.39}$$

with positive parameters d_1, d_2, d_4, χ, α, β, γ, $\delta > 0$, one can derive similar statements on global existence and asymptotic stabilization. In this general setting, the convergence results then read

$$u(x,t)\to\overline{u_0}, \qquad v(x,t)\to\overline{v_0}+\frac{\alpha}{\beta}\overline{w_0}, \qquad w(x,t)\to 0 \qquad \text{and} \qquad z(x,t)\to\delta\overline{u_0},$$

uniformly with respect to $x\in\Omega$, whenever $u_0\not\equiv 0$.

Remark 4.9 An interesting question left open in this paper concerns the respective rates of convergence in Theorem 1.2, which is basically due to the fact that our approach is based on a compactness method. The only evident implication of our results concerns the solution component w, for which it is clear that according to the uniform convergence property of z, given any $\varepsilon>0$ one can find $C_\varepsilon>0$ such that

$$w(x,t) \quad \le \quad C_\varepsilon\cdot e^{-(\overline{u_0}-\varepsilon)t} \qquad \text{for all } t>t_0.$$

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