# Optimal rates of convergence to the singular Barenblatt profile for the fast diffusion equation

**Marek Fila**
Department of Applied Mathematics and Statistics, Comenius University, 84248 Bratislava, Slovakia (fila@fmph.uniba.sk)

**Michael Winkler**
Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany (michael.winkler@math.uni-paderborn.de)

(MS received 'Received date'; Accepted date')

We study the asymptotic behaviour of solutions of the fast diffusion equation near extinction. For a class of initial data, the asymptotic behaviour is described by a singular Barenblatt profile. We complete previous results on rates of convergence to the singular Barenblatt profile by describing a new phenomenon concerning the difference between the rates in time and space.

## 1. Introduction

We consider the Cauchy problem for the fast diffusion equation:

$$\begin{cases} u_\tau = \nabla \cdot (u^{m-1}\,\nabla u), & y \in \mathbb{R}^n,\ \tau \in (0,T), \\ u(y,0) = u_0(y) \geq 0, & y \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $m < 1$, $T > 0$ and $u_0$ is continuous and bounded. It is known that for $m$ below the critical exponent $m_c := (n-2)/n$ all solutions with initial data in some suitable space, like $L^p(\mathbb{R}^n)$ with $p := n(1-m)/2$, vanish in finite time. We consider such solutions and study the rates of their extinction in the range

$$-\infty < m < m_* := \frac{n-4}{n-2}, \qquad n > 2. \tag{1.2}$$

The exponent $m_*$ plays an important role in the results on asymptotic behaviour near extinction in [1, 2, 3, 6, 8, 9, 10].

The book [15] contains a general description of the phenomenon of extinction, even for $m \leq 0$. It is explained there that the size of the initial data at infinity (the tail of $u_0$) is very important in determining both the extinction time and the extinction rates. For $0 < m < 1$, problem (1.1) is well-posed (see [5, 14, 15]) while for $m \leq 0$ neither existence nor uniqueness hold, in general, but it is known (see [5, 15]) that a solution exists if $u_0$ is "large enough". We shall only consider such initial data $u_0$ for $m \leq 0$. For more recent results on the fast diffusion equation which include also the case $m \leq 0$ we refer to [4].

For $m < m_c$ we have explicit self-similar solutions $U_{D,T}$ called *generalized Barenblatt solutions*, given by the formula

$$U_{D,T}(y,\tau) := \frac{1}{R(\tau)^n}\left(D + \frac{\beta(1-m)}{2}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}, \tag{1.3}$$

where

$$R(\tau) := (T-\tau)^{-\beta}, \qquad \beta := \frac{1}{n(1-m)-2} = \frac{1}{n\,(m_c - m)}.$$

Here $T \geq 0$ (extinction time) and $D > 0$ are free parameters. These solutions have a decay rate near extinction of the form $\|u(\cdot,\tau)\|_\infty = O((T-\tau)^{n\beta})$.

A very interesting limit case occurs if we take $D = 0$ in formula (1.3), and we find the singular solution

$$U_{0,T}(y,\tau) := k_*\,(T-\tau)^{\mu/2}|y|^{-\mu}, \qquad k_* := (2(n-\mu))^{\mu/2}, \qquad \mu := \frac{2}{1-m}.$$

whose attracting properties were studied in [9] where we obtained a continuum of extinction rates for suitable bounded data $u_0$. More precisely, the following was shown in [9].

**Theorem 1.1.** *Assume that*

$$n \geq 5 \qquad \text{and} \qquad 0 < m < m_* = \frac{n-4}{n-2}, \tag{1.4}$$

*and let the initial function $u_0$ be continuous, bounded, and satisfy the conditions:*

$$0 \leq u_0(y) \leq A\,|y|^{-\mu} \qquad \text{for all } y \neq 0$$

*and*

$$A\,|y|^{-\mu} - c_1|y|^{-l} \leq u_0(y) \leq A\,|y|^{-\mu} - c_2|y|^{-l} \qquad \text{for } \ |y| \geq 1$$

*for some $A, c_1, c_2 > 0$, and*

$$\mu + 2 < l \leq L := \mu + \sqrt{2(n-\mu)}. \tag{1.5}$$

*Then the solution $u$ of problem* (1.1) *has complete extinction precisely at the time $T := (A/k_*)^{1-m} > 0$, and the following holds:*

*(i) There are positive constants $K_1, K_2$ such that for $0 < \tau < T$ we have*

$$K_1(T-\tau)^{\theta_l} \leq \|u(\cdot,\tau)\|_\infty \leq K_2(T-\tau)^{\theta_l},$$

*where*

$$\theta_l := \frac{n\mu - \gamma_l}{2(n-\mu)} > 0, \qquad \gamma_l := \frac{\mu\alpha_l}{l-\mu}, \qquad \alpha_l := (l-\mu-2)(n-l). \tag{1.6}$$

*(ii) For every $r_0 > 0$ there exist positive constants $C_1, C_2$ such that*

$$C_1(T-\tau)^{\vartheta_l} \leq A\left|\frac{y}{R(\tau)}\right|^{-\mu} - R^n(\tau)u(y,\tau) \leq C_2(T-\tau)^{\vartheta_l}$$

*for $0 < \tau < T$, $|y| \geq r_0 R(\tau)$, where $\vartheta_l := \beta\alpha_l/\mu$.*

One of the main aims of the present paper is to show that Theorem 1.1 (i) does not hold for $l > L$ while Theorem 1.1 (ii) holds for a larger range of $l$. The meaning of Theorem 1.1 (ii) becomes clear after a suitable reformulation (see Theorem 1.2 (ii)).

To study the behaviour of solutions near extinction one can rewrite (1.1) by introducing the change of variables

$$t := \frac{1-m}{2} \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{\beta(1-m)}{2}}\, \frac{y}{R(\tau)}\,, \tag{1.7}$$

with $R$ as above, and the rescaled function

$$v(x,t) := R(\tau)^n\, u(y,\tau). \tag{1.8}$$

If $u$ is a solution of (1.1) then $v$ solves the equation

$$v_t = \nabla\cdot(v^{m-1}\nabla v) + \mu\,\nabla\cdot(x\,v), \quad t > 0\,, \quad x \in \mathbb{R}^n\,, \tag{1.9}$$

which is a nonlinear Fokker-Planck equation. The generalized Barenblatt solutions $U_{D,T}$ are transformed into *generalized Barenblatt profiles* $V_D$ which are stationary solutions of (1.9):

$$V_D(x) := (D + |x|^2)^{\frac{1}{m-1}}, \quad x \in \mathbb{R}^n\,.$$

The singular Barenblatt solution becomes

$$V_0(x) = |x|^{-\mu}, \quad x \in \mathbb{R}^n \setminus \{0\}\,.$$

The main result from [9] can now be formulated as follows.

**Theorem 1.2.** *Let* (1.4) *hold. Assume that* $v_0 \geq 0$ *is continuous, bounded and such that*

$$|x|^{-\mu} - c_1|x|^{-l} \leq v_0(x) \leq |x|^{-\mu} - c_2|x|^{-l} \qquad \text{for } |x| \geq 1,$$

*where* $l$ *is as in* (1.5) *and* $c_1, c_2 > 0$. *Assume also that* $v_0(x) \leq |x|^{-\mu}$ *for all* $x \neq 0$. *Let* $v$ *denote the solution of* (1.9) *with initial condition*

$$v(x,0) = v_0(x), \qquad x \in \mathbb{R}^n. \tag{1.10}$$

*Then:*
*(i) There exist* $K_1, K_2 > 0$ *such that for* $t \geq 1$ *we have*

$$K_1\, e^{\gamma_l t} \leq \|v(\cdot,t)\|_\infty \leq K_2\, e^{\gamma_l t}, \tag{1.11}$$

*here* $\gamma_l$ *is as in* (1.6).
*(ii) For every* $r_0 > 0$ *one can find* $C_1, C_2 > 0$ *such that for* $t \geq 1$ *and* $|x| \geq r_0$ *the following holds*

$$C_1\, e^{-\alpha_l t} \leq |x|^{-\mu} - v(x,t) \leq C_2\, e^{-\alpha_l t}, \tag{1.12}$$

*where* $\alpha_l$ *is as in* (1.6).

The reason why we assume that $l > \mu + 2$ is that the difference $|x|^{-\mu} - V_D(x)$ behaves like $|x|^{-(\mu+2)}$ as $|x| \to \infty$. In this paper we show that the condition $\mu+2 < l \le L$ is optimal for Theorem 1.2 (i) but not for Theorem 1.2 (ii) which holds for a larger range

$$l \in (\mu+2, l_\star), \qquad l_\star := \frac{1}{2}(n+\mu+2). \tag{1.13}$$

More precisely, we prove the following:

**Theorem 1.3.** *Let* (1.2) *hold and assume that* $v_0 \ge 0$ *is continuous.*
*(i) If*

$$v_0(x) \le |x|^{-\mu}, \qquad x \ne 0, \tag{1.14}$$

*and*

$$v_0(x) \le |x|^{-\mu} - c|x|^{-l}, \qquad |x| > 1,$$

*with some* $l$ *as in* (1.13) *and* $c > 0$ *then for any* $r_0 > 0$ *there exists* $C(r_0) > 0$ *such that the solution of* (1.9), (1.10) *satisfies*

$$v(x,t) \le |x|^{-\mu} - C(r_0)e^{-\alpha_l t}|x|^{-l}, \qquad |x| \ge r_0, \quad t \ge 0,$$

*where* $\alpha_l$ *is as in* (1.6).
*(ii) Assume that* $v_0(x) > 0$ *for* $x \in \mathbb{R}^n$ *and*

$$v_0(x) \ge |x|^{-\mu} - c|x|^{-l}, \qquad |x| > 1,$$

*with some* $l$ *as in* (1.13) *and* $c > 0$*. Then one can find* $C > 0$ *such that the solution of* (1.9), (1.10) *satisfies*

$$v(x,t) \ge |x|^{-\mu} - Ce^{-\alpha_l t}|x|^{-l}, \qquad x \ne 0, \quad t > 0.$$

*(iii) Set*

$$\alpha_\star := \alpha_{l_\star} = \frac{(n-\mu-2)^2}{4}. \tag{1.15}$$

*If* (1.14) *holds then for any* $\alpha > \alpha_\star$ *and each* $r_0 > 0$ *there exists* $C(\alpha, r_0) > 0$ *such that the solution of* (1.9), (1.10) *satisfies*

$$\sup_{|x| \ge r_0} \Big(|x|^{-\mu} - v(x,t)\Big) \ge Ce^{-\alpha t}, \qquad t > 0.$$

**Theorem 1.4.** *Let* (1.2), (1.14) *hold and assume that* $v_0 \ge 0$ *is continuous. Then for any*

$$\gamma > \gamma_L = \frac{\mu(L-\mu-2)(n-L)}{L-\mu} = \mu\left(n+2-\mu-2\sqrt{2(n-\mu)}\right) \tag{1.16}$$

*there exists* $C(\gamma) > 0$ *such that the solution of* (1.9), (1.10) *satisfies*

$$v(x,t) \le C(\gamma)e^{\gamma t}, \qquad x \in \mathbb{R}^n, \quad t > 0.$$

We find the fact that the optimal condition on $l$ is different for (1.11) and (1.12) remarkable. It is in contrast with corresponding results for the equation $u_t = \Delta u + u^p$, see [7, 11, 12].

The threshold value $l_*$ appeared before in [10] where we studied the rates of convergence to Barenblatt profiles $V_D$ with $D > 0$. Rates of convergence to the singular Barenblatt profile $V_0$ were found in [8] for $m = m_*$. The rates in [8] are algebraic while in Theorems 1.3 and 1.4 they are exponential.

To prove our results we construct suitable radial sub- and supersolutions in a spirit similar to [9, 10]. Radial barriers have also been used recently in [13] to investigate the fast diffusion equation on hyperbolic space.

In Section 2 we prove Theorem 1.3 (i), (ii). Section 3 is devoted to the proof of Theorem 1.3 (iii) and Section 4 to Theorem 1.4.

## 2. Convergence rate for $l \in (\mu + 2, l_\star)$

Throughout the paper we shall assume that (1.2) holds. The radial version of the nonlinear Fokker-Planck equation (1.9) reads

$$v_t = (v^{m-1}v_r)_r + \frac{n-1}{r}v^{m-1}v_r + \mu r v_r + \mu n v, \qquad r > 0,\ t > 0. \tag{2.1}$$

In this section we shall construct sub- and supersolutions thereof with a particular structure. The action of the operator $\mathcal{P}$ defined by

$$\mathcal{P}w := w_t - (w^{m-1}w_r)_r - \frac{n-1}{r}w^{m-1}w_r - \mu r w_r - \mu n w, \qquad r > 0,\ t > 0, \tag{2.2}$$

on such functions is described by the following.

**Lemma 2.1.** *Let $0 \le r_0 < r_1 \le \infty$, $y : [0,\infty) \to \mathbb{R}$ and $\varphi : (r_0, r_1) \to (0,\infty)$ be smooth functions. Then*

$$w(r,t) := \Big(r^2 + y(t)\varphi(r)\Big)^{-\frac{\mu}{2}}, \qquad r \in (r_0, r_1),\ t > 0,$$

*satisfies*

$$\mathcal{P}w = \frac{\mu}{2} y(t)\Big(r^2 + y(t)\varphi(r)\Big)^{-\frac{\mu+2}{2}} \mathcal{A}[y(t)]\varphi \qquad \text{for } r \in (r_0, r_1) \text{ and } t > 0, \tag{2.3}$$

*where*

$$\mathcal{A}[y(t)]\varphi := r^2\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) - \mu r\varphi_r - \frac{y'(t)}{y(t)}\varphi - y(t)\Big\{-\varphi\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) + \frac{\mu}{2}\varphi_r^2\Big\}$$

*for $r \in (r_0, r_1)$ and $t > 0$.*

*Proof.* The formula (2.3) can be derived by a straightforward computation (cf. [10, Lemma 3.5] for details). □

Our first choice of comparison functions will involve solutions $\varphi$ of the linear initial value problem

$$\begin{cases} (r^2+1)\Big(\varphi_{rr}+\frac{n-1}{r}\varphi_r\Big)-\mu r\varphi_r+\alpha\varphi=0, & r>0,\\ \varphi(0)=1, \quad \varphi_r(0)=0, \end{cases} \tag{2.4}$$

where $\alpha>0$.

The following statements concerning (2.4) are contained in [10, Lemma 3.3].

**Lemma 2.2.** *Let $\alpha\in(0,\alpha_\star)$ with $\alpha_\star$ as in* (1.15). *Let $l$ denote the smaller positive root of the equation*

$$\alpha=(l-\mu-2)(n-l).$$

*Then the solution $\varphi$ of* (2.4) *is positive and decreasing on $[0,\infty)$, and there exist positive constants $c_1, c_2$ and $c_3$ such that*

$$c_1 r^{-(l-\mu-2)}\le\varphi(r)\le c_2 r^{-(l-\mu-2)} \qquad \textit{for all } r\ge 1$$

*as well as*

$$\frac{\varphi_r(r)}{\varphi(r)}\ge-\frac{c_3 r}{r^2+1} \qquad \textit{for all } r>0.$$

These functions $\varphi$ form the core of our upper estimate for $v$:

**Lemma 2.3.** *Suppose that*

$$v_0(r)<r^{-\mu} \qquad \textit{for all } r>0, \tag{2.5}$$

*and*

$$v_0(r)\le r^{-\mu}-cr^{-l} \qquad \textit{for all } r>1 \tag{2.6}$$

*with some $l\in(\mu+2,l_\star)$ and $c>0$. Then for any $r_0>0$ there exists $C(r_0)>0$ such that the solution of* (2.1) *satisfies*

$$v(r,t)\le r^{-\mu}-C(r_0)e^{-\alpha_l t}r^{-l} \qquad \textit{for all } r\ge r_0 \textit{ and } t\ge 0. \tag{2.7}$$

*Proof.* We may assume that $r_0\le 1$. Since $\mu+2<l<l_\star$, the number $\alpha_l$ in (1.6) satisfies $0<\alpha_l<\alpha_\star$ with $\alpha_\star$ as in (1.15), and hence Lemma 2.2 says that the corresponding solution $\varphi$ of (2.4) is positive and decreasing on $[0,\infty)$ and satisfies

$$c_1 r^{-(l-\mu-2)}\le\varphi(r)\le c_2 r^{-(l-\mu-2)} \qquad \text{for all } r>r_0 \tag{2.8}$$

as well as

$$-\varphi_r(r)\le c_3\frac{r}{r^2+1}\varphi(r) \qquad \text{for all } r>0 \tag{2.9}$$

with certain positive constants $c_1, c_2$ and $c_3$. Moreover, due to the continuity of $v_0$ and (2.5) we can fix $c_4>0$ such that

$$v_0(r)\le(r^2+c_4)^{-\frac{\mu}{2}} \qquad \text{for all } r\in[0,1]. \tag{2.10}$$

Taking $c > 0$ as in (2.6), we now choose $B > 0$ satisfying

$$B \le \min\Big\{\frac{2}{c_3+2}, \frac{2c}{\mu c_2}, c_4\Big\} \tag{2.11}$$

and define

$$\overline{v}(r,t) := \Big(r^2 + y(t)\varphi(r)\Big)^{-\frac{\mu}{2}}, \qquad y(t) := Be^{-\alpha_l t}, \qquad r \ge 0,\ t \ge 0.$$

We claim that then

$$\mathcal{P}\overline{v} \ge 0 \qquad \text{for } r > 0 \text{ and } t > 0, \tag{2.12}$$

which in view of Lemma 2.1 is equivalent to the inequality $\mathcal{A}[y(t)]\varphi \ge 0$ for $r > 0$ and $t > 0$ with $\mathcal{A}$ as defined in Lemma 2.1.

Using (2.4) and the fact that $y'/y \equiv -\alpha_l$, we compute

$$\begin{aligned}
\mathcal{A}[y(t)]\varphi &= r^2\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) - \mu r\varphi_r + \alpha_l\varphi \\
&\quad - y(t)\Big\{\varphi\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) + \frac{\mu}{2}\varphi_r^2\Big\} \\
&= -\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) - y(t)\Big\{\varphi\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) + \frac{\mu}{2}\varphi_r^2\Big\} \\
&= -\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big)\Big\{1 - y(t)\Big[\varphi - \frac{\mu}{2}\frac{\varphi_r^2}{\varphi_{rr} + \frac{n-1}{r}\varphi_r}\Big]\Big\}
\end{aligned} \tag{2.13}$$

for $r > 0$ and $t > 0$. Here we note that, again by (2.4),

$$-\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) = \frac{\alpha_l\varphi - \mu r\varphi_r}{r^2+1} > -\frac{\mu r\varphi_r}{r^2+1} \qquad \text{for all } r > 0, \tag{2.14}$$

hence invoking (2.9) we obtain

$$-\frac{\mu}{2}\frac{\varphi_r^2}{\varphi_{rr} + \frac{n-1}{r}\varphi_r} < \frac{(r^2+1)(-\varphi_r)}{2r} \le \frac{c_3}{2}\varphi(r) \qquad \text{for all } r > 0.$$

Since (2.14) also implies that $-(\varphi_{rr} + \frac{n-1}{r}\varphi_r) \ge 0$ on $(0,\infty)$ by monotonicity of $\varphi$, (2.13) yields that for all $r > 0$ and $t > 0$ we have

$$\begin{aligned}
\mathcal{A}[y(t)]\varphi &\ge -\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big)\Big\{1 - y(t)\frac{c_3+2}{2}\varphi(r)\Big\} \\
&\ge -\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big)\Big\{1 - B\frac{c_3+2}{2}\Big\} \ge 0,
\end{aligned}$$

because $y(t) \le y(0) = B, \varphi(r) \le \varphi(0) = 1$ and $B \le 2/(c_3+2)$ by (2.11).

Having thus proved (2.12), we proceed to check that

$$\overline{v}(r,0) \ge v_0(r) \qquad \text{for all } r \ge 0. \tag{2.15}$$

To this end, we first consider the case when $r \in [0,1]$, in which we use (2.10) and the restriction $B \le c_4$ asserted by (2.11) to estimate

$$\overline{v}(r,0) = \Big(r^2 + B\varphi(r)\Big)^{-\frac{\mu}{2}} \ge \big(r^2 + B\big)^{-\frac{\mu}{2}} \ge v_0(r) \qquad \text{for all } r \in [0,1],$$

again due to the fact that $\varphi \le 1$. Conversely, if $r > 1$ then $r \ge r_0$ and hence from the convexity of $0 \le z \mapsto (1+z)^{-\frac{\mu}{2}}$ and (2.8) we infer that

$$\overline{v}(r,0) \ge r^{-\mu} - \frac{\mu B}{2} r^{-\mu-2}\varphi(r) \ge r^{-\mu} - \frac{\mu B c_2}{2} r^{-l} \qquad \text{for all } r > 1.$$

In view of (2.6), this easily yields $\overline{v}(r,0) \ge v_0(r)$ for such $r$, because (2.11) ensures that $\mu B c_2/2 \le c$.

As a consequence of (2.12) and (2.15), the comparison principle states that $\overline{v}(r,t) \ge v(r,t)$ for all $r \ge 0$ and $t \ge 0$, which can be turned into (2.7) as follows. We let $z_0 := B r_0^{-2}\varphi(r_0)$ and take $c_5 > 0$ small enough such that $(1+z)^{-\mu/2} \le 1 - c_5 z$ for all $z \in [0, z_0]$. Then, since $\varphi(r) \le \varphi(r_0)$ for $r \ge r_0$, we have $Be^{-\alpha_l t} r^{-2}\varphi(r) \le z_0$ for $r \ge r_0$ and $t \ge 0$, so that indeed

$$\begin{aligned} v(r,t) &\le \overline{v}(r,t) = r^{-\mu}\Big(1 + Be^{-\alpha_l t} r^{-2}\varphi(r)\Big)^{-\frac{\mu}{2}} \\ &\le r^{-\mu}\Big(1 - c_5 B e^{-\alpha_l t} r^{-2}\varphi(r)\Big) \le r^{-\mu} - c_1 c_5 B e^{-\alpha_l t} r^{-l} \end{aligned}$$

for all $r \ge r_0$ and $t \ge 0$, according to the first inequality in (2.8). $\square$

Upon a different – actually more explicit – choice of $\varphi$, we next establish a corresponding lower bound for the solution of (2.1).

**Lemma 2.4.** *Assume that $v_0 > 0$ on $[0,\infty)$, and that*

$$v_0(r) \ge r^{-\mu} - c r^{-l} \qquad \textit{for all } r > 1 \tag{2.16}$$

*with some $l \in (\mu+2, l_\star)$ and $c > 0$. Then one can find $C > 0$ such that the solution of* (2.1) *satisfies*

$$v(r,t) \ge r^{-\mu} - Ce^{-\alpha_l t} r^{-l} \qquad \textit{for all } r > 0 \textit{ and } t > 0. \tag{2.17}$$

*Proof.* We define

$$k_1 := (2^{\frac{2}{\mu}} - 1)^{-\frac{1}{l-\mu}}, \qquad r_1 := \max\Big\{1, (2c)^{-\frac{1}{l-\mu}}\Big\}$$

with $c$ as in (2.16), and fix $c_1 > 0$ such that

$$(1+z)^{-\frac{\mu}{2}} \le 1 - c_1 z \qquad \text{for all } z \in \Big[0, k_1^{-(l-\mu)}\Big]. \tag{2.18}$$

Then choosing $B > 0$ satisfying

$$B^{-\frac{\mu}{2}} r_1^{\frac{\mu(l-\mu-2)}{2}} \le \min_{r\in[0,r_1]} v_0(r), \qquad B \ge \frac{c}{c_1} \qquad \text{and} \qquad B \ge c_1, \tag{2.19}$$

we for $r > 0$ and $t \ge 0$ set

$$\underline{v}(r,t) := \Big(r^2 + y(t)\varphi(r)\Big)^{-\frac{\mu}{2}}, \qquad \varphi(r) := r^{-(l-\mu-2)}, \qquad y(t) := Be^{-\alpha_l t}.$$

Then it can be easily verified that

$$r^2\Big(\varphi_{rr}+\frac{n-1}{r}\varphi_r\Big)-\mu r\varphi_r+\alpha_l\varphi=0, \qquad r>0,$$

and that $\varphi_{rr}+\frac{n-1}{r}\varphi_r<0$ on $(0,\infty)$. Accordingly, using Lemma 2.1 we see that

$$\begin{aligned}
\mathcal{P}\underline{v} &= \frac{\mu}{2}y(t)\Big(r^2+y(t)\varphi(r)\Big)^{-\frac{\mu+2}{2}}\Big\{r^2\Big(\varphi_{rr}+\frac{n-1}{r}\varphi_r\Big)-\mu r\varphi_r+\alpha_l\varphi \\
&\qquad -y(t)\Big[-\varphi\Big(\varphi_{rr}+\frac{n-1}{r}\varphi_r\Big)+\frac{\mu}{2}\varphi_r^2\Big]\Big\} \\
&= -\frac{\mu}{2}y^2(t)\Big(r^2+y(t)\varphi(r)\Big)^{-\frac{\mu+2}{2}}\Big[-\varphi\Big(\varphi_{rr}+\frac{n-1}{r}\varphi_r\Big)+\frac{\mu}{2}\varphi_r^2\Big] \\
&\le 0 \qquad \text{for } r>0 \text{ and } t>0.
\end{aligned}$$

In order to check that

$$\underline{v}(r,0)\le v_0(r) \qquad \text{for all } r>0, \tag{2.20}$$

we first consider the case when $r\le r_1$. Then we obtain

$$\underline{v}(r,0)\le\Big(Br^{-(l-\mu-2)}\Big)^{-\frac{\mu}{2}}\le B^{-\frac{\mu}{2}}r_1^{\frac{\mu(l-\mu-2)}{2}}\le v_0(r) \qquad \text{for all } r\le r_1 \tag{2.21}$$

due to the first requirement in (2.19).

Next, if $r$ is large such that both $r\ge k_1B^{1/(l-\mu)}$ and $r>r_1$ hold, then $Br^{-(l-\mu)}\le k_1^{-(l-\mu)}$, so that (2.18) applies to ensure that

$$\underline{v}(r,0)=r^{-\mu}\Big(1+Br^{-(l-\mu)}\Big)^{-\frac{\mu}{2}}\le r^{-\mu}-c_1Br^{-l}.$$

On the other hand, since $r>r_1$ entails that $r\ge 1$, we may invoke (2.16) to achieve

$$v_0(r)\ge r^{-\mu}-cr^{-l}\ge r^{-\mu}-c_1Br^{-l}\ge\underline{v}(r,0), \qquad r\ge\max\{k_1B^{\frac{1}{l-\mu}},r_1\}, \tag{2.22}$$

in view of the second condition in (2.19).

Finally, if $r>r_1$ is such that $r<k_1B^{\frac{1}{l-\mu}}$, then $k:=B^{-\frac{1}{l-\mu}}r$ satisfies $k<k_1$. Moreover, by definition of $r_1$ and (2.16) we know that $r>r_1$ entails the inequality

$$v_0(r)\ge r^{-\mu}\Big(1-cr^{-(l-\mu)}\Big)\ge\frac{1}{2}r^{-\mu}=\frac{1}{2}k^{-\mu}B^{-\frac{\mu}{l-\mu}}.$$

Since $k<k_1$ and the definition of $k_1$ imply that

$$\Big(1+k^{-(l-\mu)}\Big)^{-\frac{\mu}{2}}\le\Big(1+k_1^{-(l-\mu)}\Big)^{-\frac{\mu}{2}}=\frac{1}{2},$$

we obtain that

$$\underline{v}(r,0)=\Big(1+k^{-(l-\mu)}\Big)^{-\frac{\mu}{2}}k^{-\mu}B^{-\frac{\mu}{l-\mu}}\le\frac{1}{2}k^{-\mu}B^{-\frac{\mu}{l-\mu}}\le v_0(r)$$

whenever $r_1 < r < k_1 B^{1/(l-\mu)}$. In conjunction with (2.21) and (2.22) this proves (2.20), so that the comparison principle becomes applicable to guarantee that $\underline{v}(r,t) \le v(r,t)$ for all $r > 0$ and $t \ge 0$. In particular, by convexity of $0 \le z \mapsto (1+z)^{-\frac{\mu}{2}}$ this shows that

$$v(r,t) \ge \underline{v}(r,t) = r^{-\mu}\Big(1 + Be^{-\alpha_l t} r^{-(l-\mu)}\Big)^{-\frac{\mu}{2}} \ge r^{-\mu} - \frac{\mu B}{2} e^{-\alpha_l t} r^{-l}$$

for all $r, t > 0$, and thereby establishes (2.17). □

*Proof of Theorem 1.3 (i).* For radial solutions, Lemma 2.3 yields the claim. If $v_0$ is not radial then we choose a radial function $v_0^+$ satisfying the assumptions of Lemma 2.3 such that

$$v_0(x) \le v_0^+(|x|), \qquad x \in \mathbb{R}^n,$$

and argue by comparison. □

*Proof of Theorem 1.3 (ii).* Analogously, for radial solutions the conclusion is a consequence of Lemma 2.4 and in the non-radial case we compare with a radial solution emanating from $v_0^-(|x|)$ satisfying the assumptions of Lemma 2.4 such that

$$v_0^-(|x|) \le v_0(x), \qquad x \in \mathbb{R}^n.$$

□

## 3. Universal lower bound for the convergence rate

In this section we prove Theorem 1.3 (iii). As a first preliminary, an important observation is contained in the following lemma which asserts oscillatory behaviour in a linear Euler-type ODE, provided that a certain parameter is supercrtical.

**Lemma 3.1.** *Let $\tilde{\mu} \in (0, n-2)$ and $\tilde{\alpha} > \tilde{\alpha}_\star := \frac{(n-2-\tilde{\mu})^2}{4}$. Then*

$$\tilde{\varphi}(r) := r^{-\frac{n-2-\tilde{\mu}}{2}} \cos\Big(\sqrt{\tilde{\alpha} - \tilde{\alpha}_\star}\,\ln r\Big), \qquad r > 0,$$

*satisfies*

$$r^2\Big(\tilde{\varphi}_{rr} + \frac{n-1}{r}\tilde{\varphi}_r\Big) - \tilde{\mu} r \tilde{\varphi}_r + \tilde{\alpha}\tilde{\varphi} = 0 \qquad \textit{for all } r > 0. \tag{3.1}$$

*Proof.* Writing $\zeta := -(n-2-\tilde{\mu})/2 + i\sqrt{\tilde{\alpha} - \tilde{\alpha}_\star}$ and $\Phi(r) := r^\zeta$ for $r > 0$, we have $\tilde{\varphi}(r) = Re\Phi(r)$ for $r > 0$. Since it can easily be computed that

$$r^2\Big(\Phi_{rr} + \frac{n-1}{r}\Phi_r\Big) - \tilde{\mu}\Phi_r + \tilde{\alpha}\Phi = p(\zeta) r^\zeta \qquad \text{for all } r > 0$$

with $p(\zeta) := \zeta^2 + (n-2-\tilde{\mu})\zeta + \tilde{\alpha}$, the validity of (3.1) follows from the observation that according to our choice of $\zeta$ we actually have $p(\zeta) = 0$. □

Functions of the above type play a key role in the construction of supersolutions of (2.1), the initial data of which are compact perturbations of the singular steady state.

**Lemma 3.2.** *Suppose that* (2.5) *holds. Then for any* $\alpha > \alpha_\star$ *and each* $r_0 > 0$ *there exists* $C(\alpha, r_0) > 0$ *such that the solution of* (2.1) *satisfies*

$$\sup_{r \geq r_0} \Big( r^{-\mu} - v(r,t) \Big) \geq Ce^{-\alpha t} \qquad \textit{for all } t > 0. \tag{3.2}$$

*Proof.* Recalling the notation from Lemma 3.1, from the fact that $\alpha > \alpha_\star = \alpha_\star(\mu)$ we obtain that there exists $\tilde{\mu} \in (0, \mu)$ close enough to $\mu$ such that still $\alpha > \alpha_\star(\tilde{\mu})$. We can then fix any $\tilde{\alpha} \in (\alpha_\star(\tilde{\mu}), \alpha)$ and let $\tilde{\varphi}$ denote the corresponding function defined in Lemma 3.1. Since $r_0 > 0$, the oscillatory behaviour of $\tilde{\varphi}$ allows us to find two zeros $r_-$ and $r_+$ of $\tilde{\varphi}$ such that $r_0 < r_- < r_+$ and $\tilde{\varphi} > 0$ in $(r_-, r_+)$. It is then clear that for some $r_1 \in (r_-, r_+)$ we have $\tilde{\varphi}_r(r_1) = 0$ and $\tilde{\varphi}_r < 0$ on $(r_1, r_+]$. As evidently $\tilde{\varphi} > 0$ on $[r_1, r_+)$, along with the facts that $\alpha > \tilde{\alpha}$ and $\mu > \tilde{\mu}$ this entails that

$$c_1 := \min_{r \in [r_1, r_+]} \Big\{ (\alpha - \tilde{\alpha})\tilde{\varphi}(r) - (\mu - \tilde{\mu}) r \tilde{\varphi}_r(r) \Big\}$$

is positive, and since $\tilde{\varphi}$ is smooth,

$$c_2 := \max_{r \in [r_1, r_+]} \Big\{ - \tilde{\varphi}(r) \Big( \tilde{\varphi}_{rr}(r) + \frac{n-1}{r} \tilde{\varphi}_r(r) \Big) + \frac{\mu}{2} \tilde{\varphi}_r^2(r) \Big\}$$

is finite. Next, using that $v_0$ is continuous and satisfies (2.5), we easily obtain $c_3 > 0$ fulfilling

$$v_0(r) \leq (R^2 + c_3)^{-\frac{\mu}{2}} \qquad \text{for all } r \in [0, r_+]. \tag{3.3}$$

We then fix $B > 0$ small enough such that

$$B \leq \min \Big\{ \frac{c_1}{c_2}, \frac{c_3}{\tilde{\varphi}(r_1)} \Big\} \tag{3.4}$$

and write

$$y(t) := Be^{-\alpha t}, \qquad t \geq 0.$$

We finally define a continuous function $\varphi : [0, \infty) \to [0, \infty)$ by setting

$$\varphi(r) := \begin{cases} \tilde{\varphi}(r_1), & r \in [0, r_1], \\ \tilde{\varphi}(r), & r \in (r_1, r_+], \\ 0, & r > r_+, \end{cases}$$

and let

$$\overline{v}(r,t) := \Big( r^2 + y(t)\varphi(r) \Big)^{-\frac{\mu}{2}}, \qquad r \geq 0, \ t \geq 0.$$

Then clearly $\mathcal{P}\overline{v} = 0$ for all $r > r_+$ and $t > 0$, because $(r,t) \mapsto r^{-\mu}$ solves (2.1). Furthermore, since for small $r$ we have $\varphi_r(r) \equiv 0$, Lemma 2.1 says that

$$\mathcal{P}\overline{v} = \frac{\mu}{2} y(t) \Big( r^2 + y(t)\varphi(r) \Big)^{-\frac{\mu+2}{2}} \alpha\varphi(r) > 0 \qquad \text{for all } r < r_1 \text{ and } t > 0.$$

Finally, in the intermediate range where $r \in (r_1, r_+)$ we recall Lemma 3.1 to see that with $\mathcal{A}$ as defined in Lemma 2.1 we have

$$\begin{aligned}
\mathcal{A}[y(t)]\varphi &= r^2\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) - \mu r\varphi_r + \alpha\varphi \\
&\quad -Be^{-\alpha t}\Big\{-\varphi\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) + \frac{\mu}{2}\varphi_r^2\Big\} \\
&= (\alpha - \tilde{\alpha})\varphi - (\mu - \tilde{\mu})r\varphi_r - Be^{-\alpha t}\Big\{-\varphi\Big(\varphi_{rr} + \frac{n-1}{r}\varphi_r\Big) + \frac{\mu}{2}\varphi_r^2\Big\}
\end{aligned}$$

for all $r \in (r_1, r_+)$ and $t > 0$, so that from the definition of $c_1$, $c_2$ and (3.4) we infer that $\mathcal{A}[y(t)]\varphi \ge c_1 - Be^{-\alpha t}c_2 \ge 0$ for all $r \in (r_1, r_+)$ and $t > 0$. In light of Lemma 2.1, this shows that $\mathcal{P}\overline{v} \ge 0$ for $r \in (r_1, r_+)$ and $t > 0$, so that since

$$\lim_{r \nearrow r_1} \varphi_r(r) = \lim_{r \searrow r_1} \varphi_r(r) = 0 \qquad \text{and} \qquad \lim_{r \nearrow r_+} \varphi_r(r) < 0 = \lim_{r \searrow r_+} \varphi_r(r),$$

it follows that $\overline{v}$ is a supersolution of (2.1).

In order to check that

$$\overline{v}(r,0) \ge v_0(r) \qquad \text{for all } r \ge 0, \tag{3.5}$$

we go back to (3.3) and use the second restriction in (3.4) to observe that indeed

$$\overline{v}(r,0) = \Big(r^2 + B\varphi(r)\Big)^{-\frac{\mu}{2}} \ge \Big(r^2 + B\tilde{\varphi}(r_1)\Big)^{-\frac{\mu}{2}} \ge (r^2 + c_3)^{-\frac{\mu}{2}} \ge v_0(r)$$

for $r \in [0, r_+]$, because evidently $\varphi(r) \le \tilde{\varphi}(r_1)$ for all $r \ge 0$. As for large $r$, however, from (2.5) and the definition of $\varphi$ we immediately obtain the estimate

$$\overline{v}(r,0) = r^{-\mu} \ge v_0(r) \qquad \text{for } r > r_+.$$

This proves (3.5). Since $\overline{v}$ is a supersolution, the comparison principle ensures that $\overline{v}(r,t) \ge v(r,t)$ for all $r \ge 0$ and $t \ge 0$. If we take $c_4 > 0$ small enough satisfying

$$(1+z)^{-\frac{\mu}{2}} \le 1 - c_4 z \qquad \text{for all } z \in [0, Br_0^{-2}\tilde{\varphi}(r_1)],$$

then evaluating the inequality obtained above at $r = r_0$ we conclude that

$$\begin{aligned}
r_0^{-\mu} - v(r_0,t) &\ge r_0^{-\mu} - \overline{v}(r_0,t) = r_0^{-\mu} - r_0^{-\mu}\Big(1 + y(t)r_0^{-2}\tilde{\varphi}(r_1)\Big)^{-\frac{\mu}{2}} \\
&\ge c_4 y(t) r_0^{-\mu-2}\tilde{\varphi}(r_1) \qquad \text{for all } t \ge 0,
\end{aligned}$$

which implies (3.2). □

*Proof of Theorem 1.3 (iii).* The statement follows from Lemma 3.2 and a simple comparison argument as at the end of the previous section. □

## 4. Universal upper bound for the grow-up rate

In order to describe the behaviour of solutions near the spatial origin in more detail, we shall use a comparison function with a slightly different structure (cf. (4.1) below). The following lemma provides a formula which shows how the parabolic operator $\mathcal{P}$ introduced in (2.2) acts on a function of this form. Its proof is based on straightforward computations, and details can be found in [9, Lemma 3.2].

**Lemma 4.1.** *Let $\kappa > 0$ and $\sigma_0 > 0$, and set*

$$\sigma(t) := \sigma_0 \, e^{\mu\kappa t}, \qquad \xi(r,t) := \sigma^{\frac{1}{\mu}}(t) r, \qquad r, t \geq 0.$$

*Suppose that $\psi : [0,\infty) \to [0,\infty)$ is twice continuously differentiable in $(\xi_0, \xi_1)$ with some $\xi_0$ and $\xi_1$ satisfying $0 \leq \xi_0 < \xi_1$. Then for*

$$v(r,t) := \sigma(t)\Big(\xi^2(r,t) + \psi(\xi(r,t))\Big)^{-\frac{\mu}{2}}, \qquad r, t \geq 0, \tag{4.1}$$

*we have the identity*

$$\mathcal{P}v(r,t) = \frac{\mu}{2}\sigma(t)\Big(\xi^2(r,t) + \psi(\xi(r,t))\Big)^{-\frac{\mu}{2}-1}\mathcal{B}\psi(\xi(r,t))$$

*for all $(r,t) \in S := \{(\rho,\tau) \in (0,\infty)^2 \mid \xi(\rho,\tau) \in (\xi_0,\xi_1)\}$, where*

$$\mathcal{B}\psi(\xi) := \Big(\xi^2 + \psi\Big)\Big(\psi_{\xi\xi} + \frac{n-1}{\xi}\psi_\xi\Big) - (\mu+\kappa)\xi\psi_\xi + 2\kappa\psi - \frac{\mu}{2}\psi_\xi^2, \quad \xi \in (\xi_0,\xi_1).$$

The next lemma again describes oscillatory behaviour in a linear ODE of Euler type, and may be viewed as a counterpart of Lemma 3.1.

**Lemma 4.2.** *Let $m < m_\star$. Then $\kappa_L := n + 2 - \mu - 2\sqrt{2(n-\mu)}$ satisfies $\kappa_L < n - 2 - \mu$, and for each $\kappa \in (\kappa_L, n-2-\mu)$ the numbers*

$$a(\kappa) := \frac{n-2-\mu-\kappa}{2} \qquad \text{and} \qquad b(\kappa) := \frac{\sqrt{8\kappa - (n-2-\mu-\kappa)^2}}{2} \tag{4.2}$$

*are real and positive. Moreover, $\psi : (0,\infty) \to \mathbb{R}$ defined by*

$$\psi(\xi) := \xi^{-a(\kappa)} \cos\Big(b(\kappa)\ln\xi\Big), \qquad \xi > 0, \tag{4.3}$$

*is a solution of*

$$\xi^2\Big(\psi_{\xi\xi} + \frac{n-1}{\xi}\Big)\psi_\xi - (\mu+\kappa)\xi\psi_\xi + 2\kappa\psi = 0, \qquad \xi > 0. \tag{4.4}$$

*Proof.* Since $m < m_\star$ implies that $\mu + 2 < n$, we have $\sqrt{2(n-\mu)} > 2$ and hence indeed

$$n - 2 - \mu - \kappa_L = -4 + 2\sqrt{2(n-\mu)} > 0.$$

We rewrite the radicand in the definition of $b(\kappa)$ according to

$$R(\kappa) := 8\kappa - (n-2-\mu-\kappa)^2 = -\kappa^2 + 2(n+2-\mu)\kappa - (n-2-\mu)^2,$$

and thereby see that its roots are precisely the numbers $\kappa_+$ and $\kappa_-$ with

$$\kappa_\pm = n+2-\mu \pm \sqrt{(n+2-\mu)^2-(n-2-\mu)^2} = n+2-\mu \pm \sqrt{8(n-\mu)}.$$

Thus, $\kappa_- = \kappa_L$ and $\kappa_+ > n-2-\mu$. It follows that whenever $\kappa \in (\kappa_L, n-2-\mu)$, the function $\psi$ defined by (4.3) satisfies $\psi(\xi) = Re\Psi(\xi)$, where $\Psi(\xi) := \xi^{-\zeta}, \xi > 0$, with $\zeta := a(\kappa) + ib(\kappa)$. Now it can easily be verified that

$$\xi^2\Big(\Psi_{\xi\xi} + \frac{n-1}{\xi}\Psi_\xi\Big) - (\mu+\kappa)\xi\Psi_\xi + 2\kappa\Psi = Q(\zeta)\xi^{-\zeta-2} \qquad \text{for all } \xi > 0$$

with $Q(\zeta) := \zeta^2 - (n-2-\mu-\kappa)\zeta + 2\kappa$. Since actually $Q(\zeta) = 0$ by definition of $\zeta$, we conclude that (4.4) holds. □

We are now in the position to derive an upper bound for the grow-up rate of solutions to (2.1) by constructing appropriate supersolutions, again emanating from compact perturbations of the singular equilibrium.

**Lemma 4.3.** *Assume* (2.5). *Then for any $\gamma$ satisfying* (1.16) *there exists $C(\gamma) > 0$ such that the solution of* (2.1) *satisfies*

$$v(r,t) \le C(\gamma)e^{\gamma t} \qquad \textit{for all } r > 0 \textit{ and } t > 0. \tag{4.5}$$

*Proof.* Since $\gamma > \gamma_L$, the number $\kappa := \gamma/\mu$ satisfies $\kappa > \kappa_L$, so that in view of Lemma 4.2 we may pick some $\tilde\kappa < \kappa$ such that $\tilde\kappa < n-2-\mu$ and $\tilde\kappa > \kappa_L$. We let

$$\tilde\psi(\xi) := \xi^{-a(\tilde\kappa)}\cos\Big(b(\tilde\kappa)\ln\xi\Big), \qquad \xi > 0,$$

with $a(\tilde\kappa) > 0$ and $b(\tilde\kappa) > 0$ as defined in (4.2). Then $\tilde\psi$ has infinitely many zeros, which makes it possible to fix $\xi_+$ and $\xi_-$ such that $0 < \xi_- < \xi_+$, $\tilde\psi(\xi_+) = \tilde\psi(\xi_-) = 0$ and $\tilde\psi > 0$ on $(\xi_-, \xi_+)$. Next, taking $\xi_1 \in (\xi_-, \xi_+)$ to be the unique zero of $\tilde\psi_\xi$ in $(\xi_-, \xi_+)$, we obtain that $\tilde\psi > 0$ in $[\xi_1, \xi_+)$ and $\tilde\psi_\xi < 0$ in $(\xi_1, \xi_+]$, so that

$$-\xi\tilde\psi_\xi(\xi) + 2\tilde\psi(\xi) \ge c_1 \qquad \text{for all } \xi \in (\xi_1, \xi_+) \tag{4.6}$$

holds with some $c_1 > 0$. Moreover, since $\tilde\psi$ is smooth, we can find $c_2 > 0$ with the property

$$-\tilde\psi(\xi)\Big(\tilde\psi_{\xi\xi}(\xi) + \frac{n-1}{\xi}\tilde\psi_\xi(\xi)\Big) + \frac{\mu}{2}\tilde\psi_\xi^2(\xi) \le c_2 \qquad \text{for all } \xi \in (\xi_1, \xi_+). \tag{4.7}$$

Finally, in view of (2.5) we can fix $c_3 > 0$ such that

$$v_0(r) \le (r^2 + c_3)^{-\frac{\mu}{2}} \qquad \text{for all } r \in [0, \xi_+] \tag{4.8}$$

and then pick $\eta > 0$ small fulfilling

$$\eta \le \min\Big\{\frac{(\kappa-\tilde\kappa)c_1}{c_2}, \frac{c_3}{\tilde\psi(\xi_1)}\Big\}. \tag{4.9}$$

Upon these choices,

$$\psi(\xi) := \begin{cases} \eta\tilde{\psi}(\xi_1), & \xi \in [0,\xi_1], \\ \eta\tilde{\psi}(\xi), & \xi \in (\xi_1,\xi_+], \\ 0, & \xi > \xi_+, \end{cases}$$

defines a nonnegative continuous function $\psi$ on $[0,\infty)$ which satisfies

$$\lim_{\xi\nearrow\xi_1} \psi_\xi(\xi) = \lim_{\xi\searrow\xi_1} \psi_\xi(\xi) = 0 \quad \text{and} \quad \lim_{\xi\nearrow\xi_+} \psi_\xi(\xi) < 0 = \lim_{\xi\searrow\xi_+} \psi_\xi(\xi) \tag{4.10}$$

as well as

$$\psi(\xi) \le \eta\tilde{\psi}(\xi_1) \qquad \text{for all } \xi > 0. \tag{4.11}$$

In particular, if we set

$$\overline{v}(r,t) := \sigma(t)\Big(\xi^2(r,t) + \psi(\xi(r,t))\Big)^{-\frac{\mu}{2}}, \qquad r \ge 0,\ t \ge 0,$$

with $\sigma(t) := e^{\mu\kappa t}$ and $\xi(r,t) := \sigma^{1/\mu}(t)r$, then $\overline{v}$ is continuous in $[0,\infty)^2$. Obviously,

$$\mathcal{P}\overline{v} = 0 \qquad \text{whenever } \xi(r,t) > \xi_+, \tag{4.12}$$

for at such points we have $\overline{v}(r,t) = r^{-\mu}$. Next, if $(r,t) \in (0,\infty)^2$ is such that $\xi(r,t) < \xi_1$ then with $\mathcal{B}$ as defined in Lemma 4.1 we have $\mathcal{B}\psi(\xi(r,t)) = 2\kappa\psi(\xi(r,t)) \ge 0$, which by Lemma 4.1 implies that

$$\mathcal{P}\overline{v} \ge 0 \qquad \text{if } \xi(r,t) < \xi_1. \tag{4.13}$$

Finally, in the intermediate region where $\xi_1 < \xi < \xi_+$ we use Lemma 4.2 to compute, partially dropping the argument $(r,t)$ of $\xi$ for simplicity,

$$\begin{aligned} \mathcal{B}\psi(\xi(r,t)) &= \Big(\xi^2 + \psi(\xi)\Big)\Big(\psi_{\xi\xi} + \frac{n-1}{\xi}\psi_\xi(\xi)\Big) \\ &\quad -(\mu+\kappa)\xi\psi_\xi(\xi) + 2\kappa\psi(\xi) - \frac{\mu}{2}\psi_\xi^2(\xi) \\ &= -\eta(\kappa-\tilde{\kappa})\xi\tilde{\psi}_\xi(\xi) + 2\eta(\kappa-\tilde{\kappa})\tilde{\psi}(\xi) \\ &\quad +\eta^2\tilde{\psi}(\xi)\Big(\tilde{\psi}_{\xi\xi}(\xi) + \frac{n-1}{\xi}\tilde{\psi}_\xi(\xi)\Big) - \frac{\mu}{2}\eta^2\tilde{\psi}_\xi^2(\xi) \quad \text{if } \xi(r,t) \in (\xi_1,\xi_+). \end{aligned}$$

Recalling (4.6), (4.7) and the first requirement contained in (4.9), we deduce that $\mathcal{B}\psi(\xi(r,t)) \ge \eta(\kappa-\tilde{\kappa})c_1 - \eta^2 c_2 \ge 0$ if $\xi(r,t) \in (\xi_1,\xi_+)$, which together with (4.12), (4.13) and (4.10) shows that $\overline{v}$ is a supersolution of (2.1).

Furthermore, at $t = 0$ we have $\sigma(t) = 1$ and thus $\overline{v}(r,0) = (r^2 + \psi(r))^{-\mu/2}$ for all $r \ge 0$, so that for small $r$ we obtain from (4.11), (4.9) and (4.8) that

$$\overline{v}(r,0) \ge \Big(r^2 + \eta\tilde{\psi}(\xi_1)\Big)^{-\frac{\mu}{2}} \ge (r^2 + c_3)^{-\frac{\mu}{2}} \ge v_0(r) \qquad \text{for all } r \in [0,\xi_+].$$

Since (2.5) implies that $v_0(r) \le r^{-\mu} = \overline{v}(r,0)$ if $r > \xi_+$, we see that $\overline{v}(r,0) \ge v_0(r)$ for all $r \ge 0$. Therefore, $\overline{v}(r,t) \ge v(r,t)$ for all $r \ge 0$ and $t \ge 0$ by comparison. In particular, using that $\xi^2 + \psi(\xi) \ge c_4 := \min\left\{\xi_1^2, \eta\tilde{\psi}(\xi_1)\right\}$ for all $\xi \ge 0$, we conclude that

$$v(r,t) \le \sigma(t)\Big(\xi^2(r,t) + \psi(\xi(r,t))\Big)^{-\frac{\mu}{2}} \le c_4^{-\frac{\mu}{2}}\sigma(t) \qquad \text{for all } r \ge 0 \text{ and } t \ge 0.$$

This shows that (4.5) holds if we set $C(\gamma) := c_4^{-\mu/2}$. □

*Proof of Theorem 1.4.* Lemma 4.3 and comparison with radial solutions yield the claim. □

## Acknowledgment

We thank the referee for careful reading and many useful comments which helped improve the presentation. The first author was supported in part by the Slovak Research and Development Agency under the contract No. APVV-0134-10 and by the VEGA grant 1/0319/15. This work was initiated while the second author visited the Comenius University. He is grateful for the warm hospitality there.

## References

1 A. Blanchet, M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. *Arch. Rat. Mech. Anal.* **191** (2009), 347–385.

2 M. Bonforte, J. Dolbeault, G. Grillo and J. L. Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities. *Proc. Nat. Acad. Sciences* **107** (2010), 16459–16464.

3 M. Bonforte, G. Grillo and J. L. Vázquez. Special fast diffusion with slow asymptotics. Entropy method and flow on a Riemannian manifold. *Arch. Rat. Mech. Anal.* **196** (2010), 631–680.

4 M. Bonforte and J. L. Vázquez. Positivity, local smoothing, and Harnack inequalities for very fast diffusion equations. *Adv. Math.* **223** (2010), 529–578.

5 P. Daskalopoulos and C. E. Kenig. *Degenerate diffusions. Initial value problems and local regularity theory.* EMS Tracts in Mathematics 1 (European Mathematical Society, Zürich, 2007).

6 M. Fila, J. King and M. Winkler. Rate of convergence to Barenblatt profiles for the fast diffusion equation with a critical exponent. *J. London Math. Soc.* **90** (2014), 167–183.

7 M. Fila, J. King, M. Winkler and E. Yanagida. Optimal lower bound of the grow-up rate for a supercritical parabolic equation. *J. Diff. Equations* **228** (2006), 339–356.

8 M. Fila and H. Stuke. Special asymptotics for a critical fast diffusion equation. *Discr. Cont. Dyn. Systems Series S* **7** (2014), 725–735.

9 M. Fila, J. L. Vázquez and M. Winkler. A continuum of extinction rates for the fast diffusion equation. *Comm. Pure Appl. Anal.* **10** (2011), 1129–1147.

10 M. Fila, J. L. Vázquez, M. Winkler and E. Yanagida. Rate of convergence to Barenblatt profiles for the fast diffusion equation. *Arch. Rat. Mech. Anal.* **204** (2012), 599–625.

11 M. Fila and M. Winkler. Rate of convergence to a singular steady state of a supercritical parabolic equation. *J. Evol. Equations* **8** (2008), 673–692.

12 M. Fila, M. Winkler and E. Yanagida. Grow-up rate of solutions for a supercritical semilinear diffusion equation. *J. Diff. Equations* **205** (2004), 365–389.

13 G. Grillo and M. Muratori, Radial fast diffusion on the hyperbolic space. *Proc. London Math. Soc.* **109** (2014), 283–317.

14 M. A. Herrero and M. Pierre, The Cauchy problem for $u_t = \Delta u^m$ when $0 < m < 1$. *Trans. Amer. Math. Soc.* **291** (1985), 145–158.

15 J. L. Vázquez. *Smoothing and decay estimates for nonlinear diffusion equations.* Oxford Lecture Notes in Maths. and its Applications, vol. 33 (Oxford University Press, Oxford, 2006).