Optimal rates of convergence to the singular Barenblatt profile for the fast diffusion equation

Marek Fila
Department of Applied Mathematics and Statistics, Comenius University, 84248 Bratislava, Slovakia (fila@fmph.uniba.sk)

Michael Winkler
Institut für Mathematik, Universität Paderborn, 33098 Paderborn, Germany (michael.winkler@math.uni-paderborn.de)

We study the asymptotic behaviour of solutions of the fast diffusion equation near extinction. For a class of initial data, the asymptotic behaviour is described by a singular Barenblatt profile. We complete previous results on rates of convergence to the singular Barenblatt profile by describing a new phenomenon concerning the difference between the rates in time and space.

1. Introduction

We consider the Cauchy problem for the fast diffusion equation:

\[
\begin{cases}
  u_\tau = \nabla \cdot (u^{m-1} \nabla u), & y \in \mathbb{R}^n, \tau \in (0, T), \\
  u(y, 0) = u_0(y) \geq 0, & y \in \mathbb{R}^n,
\end{cases}
\]  

where \( m < 1, \ T > 0 \) and \( u_0 \) is continuous and bounded. It is known that for \( m \) below the critical exponent \( m_c := (n - 2)/n \) all solutions with initial data in some suitable space, like \( L^p(\mathbb{R}^n) \) with \( p := n(1 - m)/2 \), vanish in finite time. We consider such solutions and study the rates of their extinction in the range

\[-\infty < m < m_* := \frac{n - 4}{n - 2}, \quad n > 2.\]  

The exponent \( m_* \) plays an important role in the results on asymptotic behaviour near extinction in \([1, 2, 3, 6, 8, 9, 10]\).

The book [15] contains a general description of the phenomenon of extinction, even for \( m \leq 0 \). It is explained there that the size of the initial data at infinity (the tail of \( u_0 \)) is very important in determining both the extinction time and the extinction rates. For \( 0 < m < 1 \), problem (1.1) is well-posed (see [5, 14, 15]) while for \( m \leq 0 \) neither existence nor uniqueness hold, in general, but it is known (see [5, 15]) that a solution exists if \( u_0 \) is “large enough”. We shall only consider such initial data \( u_0 \) for \( m \leq 0 \). For more recent results on the fast diffusion equation which include also the case \( m \leq 0 \) we refer to [4].
For $m < m_c$ we have explicit self-similar solutions $U_{D,T}$ called generalized Barenblatt solutions, given by the formula

$$U_{D,T}(y,\tau) := \frac{1}{R(\tau)^n} \left( D + \frac{\beta(1-m)}{2} \left| \frac{y}{R(\tau)} \right|^2 \right)^{-\frac{1}{1-m}},$$

(1.3)

where

$$R(\tau) := (T-\tau)^{-\beta}, \quad \beta := \frac{1}{n(1-m) - 2} = \frac{1}{n(m_c - m)}.$$ 

Here $T \geq 0$ (extinction time) and $D > 0$ are free parameters. These solutions have a decay rate near extinction of the form $\|u(\cdot,\tau)\|_\infty = O((T-\tau)^n\beta)$.

A very interesting limit case occurs if we take $D = 0$ in formula (1.3), and we find the singular solution

$$U_{0,T}(y,\tau) := k_* (T-\tau)^{\mu/2}|y|^{-\mu}, \quad k_* := \frac{2(n-\mu)}{(n-\mu)(n-2)}, \quad \mu := \frac{2}{1-m},$$

whose attracting properties were studied in [9] where we obtained a continuum of extinction rates for suitable bounded data $u_0$. More precisely, the following was shown in [9].

**Theorem 1.1.** Assume that

$$n \geq 5 \quad \text{and} \quad 0 < m < m_* = \frac{2(n-4)}{n-2},$$

(1.4)

and let the initial function $u_0$ be continuous, bounded, and satisfy the conditions:

$$0 \leq u_0(y) \leq A |y|^{-\mu} \quad \text{for all } y \neq 0$$

and

$$A |y|^{-\mu} - c_1 |y|^{-l} \leq u_0(y) \leq A |y|^{-\mu} - c_2 |y|^{-l} \quad \text{for } |y| \geq 1$$

for some $A, c_1, c_2 > 0$, and

$$\mu + 2 < l \leq L := \mu + \sqrt{2(n-\mu)}.$$ 

(1.5)

Then the solution $u$ of problem (1.1) has complete extinction precisely at the time $T := (A/k_*)^{-1-m} > 0$, and the following holds:

(i) There are positive constants $K_1, K_2$ such that for $0 < \tau < T$ we have

$$K_1(T-\tau)^{\theta_1} \leq \|u(\cdot,\tau)\|_\infty \leq K_2(T-\tau)^{\theta_1},$$

where

$$\theta_1 := \frac{n\mu - \gamma_1}{2(n-\mu)} > 0, \quad \gamma_1 := \frac{\mu \alpha_1}{l - \mu}, \quad \alpha_1 := (l - \mu - 2)(n - l).$$ 

(1.6)

(ii) For every $r_0 > 0$ there exist positive constants $C_1, C_2$ such that

$$C_1(T-\tau)^{\theta_1} \leq A \left| \frac{y}{R(\tau)} \right|^{-\mu} - R^n(\tau)u(y,\tau) \leq C_2(T-\tau)^{\theta_1}$$

for $0 < \tau < T$, $|y| \geq r_0 R(\tau)$, where $\theta_1 := \beta \alpha_1/\mu$. 
One of the main aims of the present paper is to show that Theorem 1.1 (i) does not hold for $l > L$ while Theorem 1.1 (ii) holds for a larger range of $l$. The meaning of Theorem 1.1 (ii) becomes clear after a suitable reformulation (see Theorem 1.2 (ii)).

To study the behaviour of solutions near extinction one can rewrite (1.1) by introducing the change of variables

$$t := \frac{1 - m}{2} \log \left( \frac{R(\tau)}{R(0)} \right) \quad \text{and} \quad x := \sqrt{\frac{3(1-m)}{2}} \frac{y}{R(\tau)}.$$  \hspace{1cm} (1.7)

with $R$ as above, and the rescaled function

$$v(x, t) := R(\tau)^n u(y, \tau).$$ \hspace{1cm} (1.8)

If $u$ is a solution of (1.1) then $v$ solves the equation

$$v_t = \nabla \cdot (v^{m-1} \nabla v) + \mu \nabla \cdot (x v), \quad t > 0, \quad x \in \mathbb{R}^n,$$ \hspace{1cm} (1.9)

which is a nonlinear Fokker-Planck equation. The generalized Barenblatt solutions $U_{D,T}$ are transformed into generalized Barenblatt profiles $V_D$ which are stationary solutions of (1.9):

$$V_D(x) := (D + |x|^2)^{\frac{1}{m-1}}, \quad x \in \mathbb{R}^n.$$  

The singular Barenblatt solution becomes

$$V_0(x) = |x|^{-\mu}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$ 

The main result from [9] can now be formulated as follows.

**Theorem 1.2.** Let (1.4) hold. Assume that $v_0 \geq 0$ is continuous, bounded and such that

$$|x|^{-\mu} - c_1|x|^{-l} \leq v_0(x) \leq |x|^{-\mu} - c_2|x|^{-l} \quad \text{for} \ |x| \geq 1,$$

where $l$ is as in (1.5) and $c_1, c_2 > 0$. Assume also that $v_0(x) \leq |x|^{-\mu}$ for all $x \neq 0$. Let $v$ denote the solution of (1.9) with initial condition

$$v(x, 0) = v_0(x), \quad x \in \mathbb{R}^n.$$ \hspace{1cm} (1.10)

Then:

(i) There exist $K_1, K_2 > 0$ such that for $t \geq 1$ we have

$$K_1 e^{-\gamma_l t} \leq \|v(\cdot, t)\|_{\infty} \leq K_2 e^{-\gamma_l t},$$ \hspace{1cm} (1.11)

where $\gamma_l$ is as in (1.6).

(ii) For every $r_0 > 0$ one can find $C_1, C_2 > 0$ such that for $t \geq 1$ and $|x| \geq r_0$ the following holds

$$C_1 e^{-\alpha_l t} \leq |x|^{-\mu} - v(x, t) \leq C_2 e^{-\alpha_l t},$$ \hspace{1cm} (1.12)

where $\alpha_l$ is as in (1.6).
The reason why we assume that $l > \mu + 2$ is that the difference $|x|^{-\mu} - V_D(x)$ behaves like $|x|^{-\mu+2}$ as $|x| \to \infty$. In this paper we show that the condition $\mu + 2 < l \leq L$ is optimal for Theorem 1.2 (i) but not for Theorem 1.2 (ii) which holds for a larger range

$$l \in (\mu + 2, l_*), \quad l_* := \frac{1}{2}(n + \mu + 2). \tag{1.13}$$

More precisely, we prove the following:

**Theorem 1.3.** Let (1.2) hold and assume that $v_0 \geq 0$ is continuous.

(i) If

$$v_0(x) \leq |x|^{-\mu}, \quad x \neq 0, \tag{1.14}$$

and

$$v_0(x) \leq |x|^{-\mu} - c|x|^{-l}, \quad |x| > 1,$$

with some $l$ as in (1.13) and $c > 0$ then for any $r_0 > 0$ there exists $C(r_0) > 0$ such that the solution of (1.9), (1.10) satisfies

$$v(x, t) \leq |x|^{-\mu} - C(r_0)e^{-\alpha_l t}|x|^{-l}, \quad |x| \geq r_0, \quad t \geq 0,$$

where $\alpha_l$ is as in (1.6).

(ii) Assume that $v_0(x) > 0$ for $x \in \mathbb{R}^n$ and

$$v_0(x) \geq |x|^{-\mu} - c|x|^{-l}, \quad |x| > 1,$$

with some $l$ as in (1.13) and $c > 0$. Then one can find $C > 0$ such that the solution of (1.9), (1.10) satisfies

$$v(x, t) \geq |x|^{-\mu} - Ce^{-\alpha_l t}|x|^{-l}, \quad x \neq 0, \quad t > 0.$$

(iii) Set

$$\alpha_* := \alpha_{l_*} = \frac{(n - \mu - 2)^2}{4}. \tag{1.15}$$

If (1.14) holds then for any $\alpha > \alpha_*$ and each $r_0 > 0$ there exists $C(\alpha, r_0) > 0$ such that the solution of (1.9), (1.10) satisfies

$$\sup_{|x| \geq r_0} \left(|x|^{-\mu} - v(x, t)\right) \geq Ce^{-\alpha t}, \quad t > 0.$$

**Theorem 1.4.** Let (1.2), (1.14) hold and assume that $v_0 \geq 0$ is continuous. Then for any

$$\gamma > \gamma_L = \frac{\mu(L - \mu - 2)(n - L)}{L - \mu} = \mu \left(n + 2 - \mu - 2\sqrt{2(n - \mu)}\right) \tag{1.16}$$

there exists $C(\gamma) > 0$ such that the solution of (1.9), (1.10) satisfies

$$v(x, t) \leq C(\gamma)e^{\gamma t}, \quad x \in \mathbb{R}^n, \quad t > 0.$$
We find the fact that the optimal condition on $l$ is different for (1.11) and (1.12) remarkable. It is in contrast with corresponding results for the equation $u_t = \Delta u + u^p$, see [7, 11, 12].

The threshold value $l_*$ appeared before in [10] where we studied the rates of convergence to Barenblatt profiles $V_D$ with $D > 0$. Rates of convergence to the singular Barenblatt profile $V_0$ were found in [8] for $m = m_*$. The rates in [8] are algebraic while in Theorems 1.3 and 1.4 they are exponential.

To prove our results we construct suitable radial sub- and supersolutions in a spirit similar to [9, 10]. Radial barriers have also been used recently in [13] to investigate the fast diffusion equation on hyperbolic space.

In Section 2 we prove Theorem 1.3 (i), (ii). Section 3 is devoted to the proof of Theorem 1.3 (iii) and Section 4 to Theorem 1.4.

2. Convergence rate for $l \in (\mu + 2, l_*)$

Throughout the paper we shall assume that (1.2) holds. The radial version of the nonlinear Fokker-Planck equation (1.9) reads

$$v_t = (v^{m-1}v_r)_r + \frac{n-1}{r}v^{m-1}v_r + \mu rv_r + \mu rv, \quad r > 0, \ t > 0. \quad (2.1)$$

In this section we shall construct sub- and supersolutions thereof with a particular structure. The action of the operator $\mathcal{P}$ defined by

$$\mathcal{P}w := w_t - (w^{m-1}w_r)_r - \frac{n-1}{r}w^{m-1}w_r - \mu rw_r - \mu nw, \quad r > 0, \ t > 0, \quad (2.2)$$

on such functions is described by the following.

Lemma 2.1. Let $0 \leq r_0 < r_1 \leq \infty$, $y : [0, \infty) \to \mathbb{R}$ and $\varphi : (r_0, r_1) \to (0, \infty)$ be smooth functions. Then

$w(r, t) := \left(r^2 + y(t)\varphi(r)\right)^{-\frac{\mu}{2}}, \quad r \in (r_0, r_1), \ t > 0,$

satisfies

$$\mathcal{P}w = \mu \frac{y(t)\left(r^2 + y(t)\varphi(r)\right)^{-\frac{\mu+2}{2}}}{2}A[y(t)]\varphi \quad \text{for } r \in (r_0, r_1) \text{ and } t > 0, \quad (2.3)$$

where

$$A[y(t)]\varphi := r^2 \left(\varphi_{rr} + \frac{n-1}{r} \varphi_r\right) - \mu r \varphi_r - \frac{y(t)}{y(t)} \varphi - y(t) \left\{-\varphi \left(\varphi_{rr} + \frac{n-1}{r} \varphi_r\right) + \frac{\mu}{2} \varphi_2^2\right\}$$

for $r \in (r_0, r_1)$ and $t > 0$.

Proof. The formula (2.3) can be derived by a straightforward computation (cf. [10, Lemma 3.5] for details). \qed
Our first choice of comparison functions will involve solutions \( \varphi \) of the linear initial value problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
(r^2 + 1) \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) - \mu r \varphi_r + \alpha \varphi = 0, \quad r > 0, \\
\varphi(0) = 1, \quad \varphi_r(0) = 0,
\end{array} \right.
\end{aligned}
\]
(2.4)
where \( \alpha > 0 \).

The following statements concerning (2.4) are contained in [10, Lemma 3.3].

**Lemma 2.2.** Let \( \alpha \in (0, \alpha_*) \) with \( \alpha_* \) as in (1.15). Let \( l \) denote the smaller positive root of the equation
\[
\alpha = (l - \mu - 2)(n - l).
\]
Then the solution \( \varphi \) of (2.4) is positive and decreasing on \([0, \infty)\), and there exist positive constants \( c_1, c_2 \) and \( c_3 \) such that
\[
c_1 r^{-(l-\mu - 2)} \leq \varphi(r) \leq c_2 r^{-(l-\mu - 2)} \quad \text{for all } r \geq 1
\]
as well as
\[
\frac{\varphi_r(r)}{\varphi(r)} \geq -\frac{c_3 r}{r^2 + 1} \quad \text{for all } r > 0.
\]

These functions \( \varphi \) form the core of our upper estimate for \( v \):

**Lemma 2.3.** Suppose that
\[
v_0(r) < r^\mu \quad \text{for all } r > 0,
\]
and
\[
v_0(r) \leq r^\mu - cr^{-l} \quad \text{for all } r > 1
\]
with some \( l \in (\mu + 2, l_*) \) and \( c > 0 \). Then for any \( r_0 > 0 \) there exists \( C(r_0) > 0 \) such that the solution of (2.1) satisfies
\[
v(r, t) \leq r^{-\mu} - C(r_0) e^{-\alpha_l t} r^{-l} \quad \text{for all } r \geq r_0 \text{ and } t \geq 0.
\]
(2.7)

**Proof.** We may assume that \( r_0 \leq 1 \). Since \( \mu + 2 < l < l_* \), the number \( \alpha_l \) in (1.6) satisfies \( 0 < \alpha_l < \alpha_* \) with \( \alpha_* \) as in (1.15), and hence Lemma 2.2 says that the corresponding solution \( \varphi \) of (2.4) is positive and decreasing on \([0, \infty)\) and satisfies
\[
c_1 r^{-(l-\mu - 2)} \leq \varphi(r) \leq c_2 r^{-(l-\mu - 2)} \quad \text{for all } r > r_0
\]
as well as
\[
-\varphi_r(r) \leq c_3 \frac{r}{r^2 + 1} \varphi(r) \quad \text{for all } r > 0
\]
with certain positive constants \( c_1, c_2 \) and \( c_3 \). Moreover, due to the continuity of \( v_0 \) and (2.5) we can fix \( c_4 > 0 \) such that
\[
v_0(r) \leq (r^2 + c_4)^{-\frac{\mu}{2}} \quad \text{for all } r \in [0, 1].
\]
(2.10)
Taking $c > 0$ as in (2.6), we now choose $B > 0$ satisfying

$$B \leq \min \left\{ \frac{2}{c_3 + 2}, \frac{2c}{\mu c_2}, c_4 \right\}$$

(2.11)

and define

$$\pi(r, t) := \left( r^2 + y(t)\varphi(r) \right)^{-\frac{2}{n}}, \quad y(t) := Be^{-\alpha t}, \quad r \geq 0, \ t \geq 0.$$ 

We claim that then

$$\pi \geq 0 \quad \text{for } r > 0 \text{ and } t > 0,$$

(2.12)

which in view of Lemma 2.1 is equivalent to the inequality $A[y(t)]\varphi \geq 0$ for $r > 0$ and $t > 0$ with $A$ as defined in Lemma 2.1.

Using (2.4) and the fact that $y'/y \equiv -\alpha$, we compute

$$A[y(t)]\varphi = r^2 \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) - \mu \varphi_r + \alpha \varphi$$

$$= -y(t) \left\{ \varphi \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) + \frac{\mu}{2} \varphi_r^2 \right\}$$

$$= -\left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) - y(t) \left\{ \varphi \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) + \frac{\mu}{2} \varphi_r^2 \right\}$$

$$= -\left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) \left\{ 1 - y(t) \left[ \varphi - \frac{\mu}{2} \frac{\varphi_r^2}{\varphi_{rr} + \frac{n-1}{r} \varphi_r} \right] \right\}$$

(2.13)

for $r > 0$ and $t > 0$. Here we note that, again by (2.4),

$$-\left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) = \frac{\alpha \varphi - \mu \varphi_r}{r^2 + 1} > -\frac{\mu \varphi_r}{r^2 + 1} \quad \text{for all } r > 0,$$

(2.14)

hence invoking (2.9) we obtain

$$-\frac{\mu}{2} \frac{\varphi_r^2}{\varphi_{rr} + \frac{n-1}{r} \varphi_r} < \frac{(r^2 + 1)(-\varphi_r)}{2r} \leq \frac{c_3}{2} \varphi(r) \quad \text{for all } r > 0.$$ 

Since (2.14) also implies that $-(\varphi_{rr} + \frac{n-1}{r} \varphi_r) \geq 0$ on $(0, \infty)$ by monotonicity of $\varphi$, (2.13) yields that for all $r > 0$ and $t > 0$ we have

$$A[y(t)]\varphi \geq -\left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) \left\{ 1 - y(t) \left[ \frac{c_3 + 2}{2} \varphi(r) \right] \right\}$$

$$\geq -\left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) \left\{ 1 - B \frac{c_3 + 2}{2} \right\} \geq 0,$$

because $y(t) \leq y(0) = B, \varphi(r) \leq \varphi(0) = 1$ and $B \leq 2/(c_3 + 2)$ by (2.11).

Having thus proved (2.12), we proceed to check that

$$\pi(r, 0) \geq \nu_0(r) \quad \text{for all } r \geq 0.$$ 

(2.15)

To this end, we first consider the case when $r \in [0, 1]$, in which we use (2.10) and the restriction $B \leq c_4$ asserted by (2.11) to estimate

$$\pi(r, 0) = \left( r^2 + B \varphi(r) \right)^{-\frac{2}{n}} \geq \left( r^2 + B \right)^{-\frac{2}{n}} \geq \nu_0(r) \quad \text{for all } r \in [0, 1].$$
Lemma 2.4. Assume that $v_0 > 0$ on $[0, \infty)$, and that

$$v_0(r) \geq r^{-\mu} - cr^{-l} \quad \text{for all } r > 1$$

(2.16)

with some $l \in (\mu + 2, l_*)$ and $c > 0$. Then one can find $C > 0$ such that the solution of (2.1) satisfies

$$v(r, t) \geq r^{-\mu} - C e^{-\alpha t} r^{-l} \quad \text{for all } r > 0 \text{ and } t > 0.$$  

(2.17)

Proof. We define

$$k_1 := (2^{\frac{2}{l+2}} - 1)^{-\frac{l+2}{l}} \quad \text{and} \quad r_1 := \max \left\{1, (2c)^{-\frac{l+2}{l}} \right\}$$

with $c$ as in (2.16), and fix $c_1 > 0$ such that

$$(1 + z)^{-\frac{l}{l+2}} \leq 1 - c_1 z \quad \text{for all } z \in \left[0, k_1^{-(l+\mu)} \right].$$

(2.18)

Then choosing $B > 0$ satisfying

$$B^{-\frac{2}{l+2}} r_1^{\frac{l(l-\mu)-2}{2}} \leq \min_{r \in [0, r_1]} v_0(r), \quad B \geq \frac{c}{c_1} \quad \text{and} \quad B \geq c_1,$$

(2.19)

we for $r > 0$ and $t \geq 0$ set

$$v(r, t) := \left(r^2 + y(t) \varphi(r) \right)^{-\frac{l}{l+2}}, \quad \varphi(r) := r^{-(l-\mu-2)}, \quad y(t) := B e^{-\alpha t}.$$
Then it can be easily verified that
\[ \varphi_{rr} + \frac{n-1}{r} \varphi_r = \mu r \varphi_r + \alpha \varphi = 0, \quad r > 0, \]
and that \( \varphi_{rr} + \frac{n-1}{r} \varphi_r < 0 \) on \((0, \infty)\). Accordingly, using Lemma 2.1 we see that
\[
\mathcal{P}_v = \frac{\mu}{2} y(t) \left( r^2 + y(t) \varphi(r) \right)^{-\frac{n+2}{2}} \left\{ r^2 \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) - \mu r \varphi_r + \alpha \varphi \right\} 
\]
\[
- y(t) \left[ - \varphi \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) + \frac{\mu}{2} \varphi^2 \right] 
\]
\[
= - \frac{\mu}{2} y(t) \left( r^2 + y(t) \varphi(r) \right)^{-\frac{n+2}{2}} \left[ - \varphi \left( \varphi_{rr} + \frac{n-1}{r} \varphi_r \right) + \frac{\mu}{2} \varphi^2 \right] 
\]
\[
\leq 0 \quad \text{for } r > 0 \text{ and } t > 0. 
\]

In order to check that
\[
\mathcal{V}(r, 0) \leq v_0(r) \quad \text{for all } r > 0, \quad (2.20) 
\]
we first consider the case when \( r \leq r_1 \). Then we obtain
\[
\mathcal{V}(r, 0) \leq \left( Br^{-1(l-\mu)-2} \right)^{-\frac{n}{2}} \leq B^{-\frac{n}{2}} r_1^{\frac{\mu(l-\mu)}{2}} \leq v_0(r) \quad \text{for all } r \leq r_1 \quad (2.21) 
\]
due to the first requirement in (2.19).

Next, if \( r \) is large such that both \( r \geq k_1 B^{1/(l-\mu)} \) and \( r > r_1 \) hold, then \( Br^{1-\mu} \leq k_1^{1-\mu} \), so that (2.18) applies to ensure that
\[
\mathcal{V}(r, 0) = r^{-\mu} \left( 1 + Br^{1-\mu(l-\mu)} \right)^{-\frac{n}{2}} \leq r^{-\mu} - c_1 B r^{-l}. 
\]

On the other hand, since \( r > r_1 \) entails that \( r \geq 1 \), we may invoke (2.16) to achieve
\[
v_0(r) \geq r^{-\mu} - c r^{-l} \geq r^{-\mu} - c_1 B^{-l} \geq \mathcal{V}(r, 0), \quad r \geq \max\{k_1 B^{\frac{1-\mu}{l}}, r_1\}, \quad (2.22) 
\]
in view of the second condition in (2.19).

Finally, if \( r > r_1 \) is such that \( r < k_1 B^{\frac{1-\mu}{l}} \), then \( k := B^{-\frac{1-\mu}{r}} r \) satisfies \( k < k_1 \). Moreover, by definition of \( r_1 \) and (2.16) we know that \( r > r_1 \) entails the inequality
\[
v_0(r) \geq r^{-\mu} \left( 1 - c r^{1(1-\mu)} \right) \geq \frac{1}{2} r^{-\mu} = \frac{1}{2} k^{-\mu} B^{-\frac{\mu}{1+\mu}}. 
\]

Since \( k < k_1 \) and the definition of \( k_1 \) imply that
\[
\left( 1 + k^{-1(1-\mu)} \right)^{-\frac{n}{2}} \leq \left( 1 + k_1^{-1(1-\mu)} \right)^{-\frac{n}{2}} = \frac{1}{2}, 
\]
we obtain that
\[
\mathcal{V}(r, 0) = \left( 1 + k^{-1(1-\mu)} \right)^{-\frac{n}{2}} k^{-\mu} B^{-\frac{\mu}{1+\mu}} \leq \frac{1}{2} k^{-\mu} B^{-\frac{\mu}{1+\mu}} \leq v_0(r) 
\]
whenever \( r_1 < r < k_1 B^{1/(l-\mu)} \). In conjunction with (2.21) and (2.22) this proves (2.20), so that the comparison principle becomes applicable to guarantee that \( u(r,t) \leq v(r,t) \) for all \( r > 0 \) and \( t \geq 0 \). In particular, by convexity of \( 0 \leq z \mapsto (1 + z)^{-\frac{\mu}{2}} \) this shows that

\[
v(r,t) \geq v(r,t) \equiv r^{-\mu} \left( 1 + B e^{-\alpha(t-r^{-l-\mu})} \right)^{-\frac{\mu}{2}} \geq r^{-\mu} - \frac{\mu B}{2} e^{-\alpha t} r^{-l}
\]

for all \( r,t > 0 \), and thereby establishes (2.17).

\[\square\]

**Proof of Theorem 1.3 (i).** For radial solutions, Lemma 2.3 yields the claim. If \( v_0 \) is not radial then we choose a radial function \( v_0^+ \) satisfying the assumptions of Lemma 2.3 such that

\[v_0(x) \leq v_0^+ (|x|), \quad x \in \mathbb{R}^n,\]

and argue by comparison.

\[\square\]

**Proof of Theorem 1.3 (ii).** Analogously, for radial solutions the conclusion is a consequence of Lemma 2.4 and in the non-radial case we compare with a radial solution emanating from \( v_0^- (|x|) \) satisfying the assumptions of Lemma 2.4 such that

\[v_0^- (|x|) \leq v_0(x), \quad x \in \mathbb{R}^n.\]

\[\square\]

### 3. Universal lower bound for the convergence rate

In this section we prove Theorem 1.3 (iii). As a first preliminary, an important observation is contained in the following lemma which asserts oscillatory behaviour in a linear Euler-type ODE, provided that a certain parameter is supercritical.

**Lemma 3.1.** Let \( \tilde{\mu} \in (0, n-2) \) and \( \tilde{\alpha} > \tilde{\alpha}_* := \frac{(n-2-\tilde{\mu})^2}{4} \). Then

\[\tilde{\varphi}(r) := r^{-\frac{n-2-\tilde{\mu}}{2}} \cos \left( \sqrt{\tilde{\alpha} - \tilde{\alpha}_*} \ln r \right), \quad r > 0,
\]

satisfies

\[r^2 \left( \tilde{\varphi}_{rr} + \frac{n-1}{r} \tilde{\varphi}_r \right) - \tilde{\mu} r \tilde{\varphi}_r + \tilde{\alpha} \tilde{\varphi} = 0 \quad \text{for all } r > 0.
\]

**Proof.** Writing \( \zeta := -(n-2-\tilde{\mu})/2 + i \sqrt{\tilde{\alpha} - \tilde{\alpha}_*} \) and \( \Phi(r) := r^\zeta \) for \( r > 0 \), we have \( \tilde{\varphi}(r) = Re \Phi(r) \) for \( r > 0 \). Since it can easily be computed that

\[r^2 \left( \Phi_{rr} + \frac{n-1}{r} \Phi_r \right) - \tilde{\mu} \Phi_r + \tilde{\alpha} \Phi = p(\zeta) r^\zeta \quad \text{for all } r > 0
\]

with \( p(\zeta) := \zeta^2 + (n-2-\tilde{\mu}) \zeta + \tilde{\alpha} \), the validity of (3.1) follows from the observation that according to our choice of \( \zeta \) we actually have \( p(\zeta) = 0 \). \[\square\]

Functions of the above type play a key role in the construction of supersolutions of (2.1), the initial data of which are compact perturbations of the singular steady state.
Suppose that (2.5) holds. Then for any $\alpha > \alpha_* \text{ and each } r_0 > 0$ there exists $C(\alpha, r_0) > 0$ such that the solution of (2.1) satisfies
\begin{equation}
\sup_{r \geq r_0} \left( r^{-\mu} - v(r, t) \right) \geq Ce^{-\alpha t} \quad \text{for all } t > 0.
\end{equation}

Proof. Recalling the notation from Lemma 3.1, from the fact that $\alpha > \alpha_* = \alpha_*(\mu)$ we obtain that there exists $\hat{\mu} \in (0, \mu)$ close enough to $\mu$ such that still $\alpha > \alpha_*(\hat{\mu})$. We can then fix any $\tilde{\alpha} \in (\alpha_*(\hat{\mu}), \alpha)$ and let $\tilde{\varphi}$ denote the corresponding function defined in Lemma 3.1. Since $r_0 > 0$, the oscillatory behaviour of $\tilde{\varphi}$ allows us to find two zeros $r_-$ and $r_+$ of $\tilde{\varphi}$ such that $r_0 < r_- < r_+$. It is then clear that for some $r_1 \in (r_-, r_+)$ we have $\tilde{\varphi}(r_1) = 0$ and $\tilde{\varphi} > 0$ on $(r_1, r_+)$. As evidently $\tilde{\varphi} > 0$ on $[r_1, r_+]$, along with the facts that $\alpha > \tilde{\alpha}$ and $\mu > \tilde{\mu}$ this entails that
\begin{align*}
c_1 := \min_{r \in [r_1, r_+]} \left\{ (\alpha - \tilde{\alpha})\tilde{\varphi}(r) - (\mu - \tilde{\mu})r\tilde{\varphi}(r) \right\}
\end{align*}
is positive, and since $\tilde{\varphi}$ is smooth,
\begin{align*}
c_2 := \max_{r \in [r_1, r_+]} \left\{ -\tilde{\varphi}(r)\left( \frac{1}{r}\tilde{\varphi}_r(r) + \frac{n-1}{r}\tilde{\varphi}(r) \right) + \frac{\mu^2}{2}\tilde{\varphi}_r(r) \right\}
\end{align*}
is finite. Next, using that $v_0$ is continuous and satisfies (2.5), we easily obtain $c_3 > 0$ fulfilling
\begin{align*}
v_0(r) \leq (R^2 + c_3)^{-\frac{\tilde{\varphi}}{2}} \quad \text{for all } r \in [0, r_+].
\end{align*}
We then fix $B > 0$ small enough such that
\begin{align*}
B \leq \min \left\{ \frac{c_1}{c_2}, \frac{c_3}{\varphi(r_1)} \right\}
\end{align*}
and write
\begin{align*}
y(t) := Be^{-\alpha t}, \quad t \geq 0.
\end{align*}
We finally define a continuous function $\varphi : [0, \infty) \to [0, \infty)$ by setting
\begin{align*}
\varphi(r) := \begin{cases} 
\tilde{\varphi}(r_1), & r \in [0, r_1], \\
\tilde{\varphi}(r), & r \in (r_1, r_+], \\
0, & r > r_+,
\end{cases}
\end{align*}
and let
\begin{align*}
\varphi(r, t) := \left( r^2 + y(t)\varphi(r) \right)^{-\frac{\mu}{2}}, \quad r \geq 0, \ t \geq 0.
\end{align*}
Then clearly $\varphi = 0$ for all $r > r_+$ and $t > 0$, because $(r, t) \mapsto r^{-\mu}$ solves (2.1). Furthermore, since for small $r$ we have $\varphi_r(r) \equiv 0$, Lemma 2.1 says that
\begin{align*}
\varphi(\mu) = \frac{\mu}{2}y(t)\left( r^2 + y(t)\varphi(r) \right)^{-\frac{\mu+2}{2}}/\alpha \varphi(r) > 0 \quad \text{for all } r < r_1 \text{ and } t > 0.
\end{align*}
Finally, in the intermediate range where \( r \in (r_1, r_+) \) we recall Lemma 3.1 to see that with \( A \) as defined in Lemma 2.1 we have

\[
A[y(t)]\phi = \nu^2 \left( \phi_{rr} + \frac{n-1}{r} \phi_r \right) - \mu r \phi_r + \alpha \phi - Be^{-\alpha t} \left\{ -\phi \left( \phi_{rr} + \frac{n-1}{r} \phi_r \right) + \frac{\mu}{2} \phi_r^2 \right\}
\]

for all \( r \in (r_1, r_+) \) and \( t > 0 \), so that from the definition of \( c_1, c_2 \) and (3.4) we infer that \( A[y(t)]\phi \geq c_1 - Be^{-\alpha t}c_2 \geq 0 \) for all \( r \in (r_1, r_+) \) and \( t > 0 \). In light of Lemma 2.1, this shows that \( \varpi \geq 0 \) for \( r \in (r_1, r_+) \) and \( t > 0 \), so that since

\[
\lim_{r \to r_1^+} \phi_r(r) = \lim_{r \to r_1^-} \phi_r(r) = 0 \quad \text{and} \quad \lim_{r \to r_+} \phi_r(r) < 0 = \lim_{r \to r_+} \phi_r(r),
\]

it follows that \( \varpi \) is a supersolution of (2.1).

In order to check that \( \varpi(r, 0) \geq v_0(r) \) for all \( r \geq 0 \), (3.5) we go back to (3.3) and use the second restriction in (3.4) to observe that indeed

\[
\varpi(r, 0) = \left( r^2 + B \phi(r) \right)^{-\frac{n}{2}} \geq \left( r^2 + B \tilde{\phi}(r_1) \right)^{-\frac{n}{2}} \geq (r^2 + c_3)^{-\frac{n}{2}} \geq v_0(r)
\]

for \( r \in [0, r_+] \), because evidently \( \phi(r) \leq \tilde{\phi}(r_1) \) for all \( r \geq 0 \). As for large \( r \), however, from (2.5) and the definition of \( \phi \) we immediately obtain the estimate

\[
\varpi(r, 0) = r^{-\mu} \geq v_0(r) \quad \text{for} \quad r > r_+.
\]

This proves (3.5). Since \( \varpi \) is a supersolution, the comparison principle ensures that \( \varpi(r, t) \geq v(r, t) \) for all \( r \geq 0 \) and \( t \geq 0 \). If we take \( c_4 > 0 \) small enough satisfying

\[
(1 + z)^{-\frac{n}{2}} \leq 1 - c_4 z \quad \text{for} \quad z \in [0, B r_0^{-2} \tilde{\phi}(r_1)],
\]

then evaluating the inequality obtained above at \( r = r_0 \) we conclude that

\[
\begin{align*}
\nu^{-\mu} - v(r_0, t) & \geq \nu^{-\mu} - \varpi(r_0, t) = \nu^{-\mu} - r_0^{-\mu} \left( 1 + y(t) r_0^{-2} \tilde{\phi}(r_1) \right)^{-\frac{n}{2}} \\
& \geq c_4 y(t) r_0^{-\mu - 2} \tilde{\phi}(r_1) \quad \text{for all} \quad t \geq 0,
\end{align*}
\]

which implies (3.2).

**Proof of Theorem 1.3 (iii).** The statement follows from Lemma 3.2 and a simple comparison argument as at the end of the previous section.
4. Universal upper bound for the grow-up rate

In order to describe the behaviour of solutions near the spatial origin in more detail, we shall use a comparison function with a slightly different structure (cf. (4.1) below). The following lemma provides a formula which shows how the parabolic operator \( \mathcal{P} \) introduced in (2.2) acts on a function of this form. Its proof is based on straightforward computations, and details can be found in [9, Lemma 3.2].

**Lemma 4.1.** Let \( \kappa > 0 \) and \( \sigma_0 > 0 \), and set

\[
\sigma(t) := \sigma_0 e^{\mu t}, \quad \xi(r,t) := \sigma(t)r, \quad r, t \geq 0.
\]

Suppose that \( \psi : [0, \infty) \to [0, \infty) \) is twice continuously differentiable in \( (\xi_0, \xi_1) \) with some \( \xi_0 \) and \( \xi_1 \) satisfying \( 0 \leq \xi_0 < \xi_1 \). Then for

\[
v(r,t) := \sigma(t)\left(\xi^2(r,t) + \psi(\xi(r,t))\right)^{-\frac{1}{2}}, \quad r, t \geq 0,
\]

we have the identity

\[
\mathcal{P}v(r,t) = \frac{\mu}{2}\sigma(t)\left(\xi^2(r,t) + \psi(\xi(r,t))\right)^{-\frac{3}{2}}B\psi(\xi(r,t))
\]

for all \((r,t) \in S := \{(\rho, \tau) \in (0, \infty)^2 \mid \xi(\rho, \tau) \in (\xi_0, \xi_1)\}\), where

\[
B\psi(\xi) := \left(\xi^2 + \psi\right)\left(\psi_{\xi\xi} + \frac{n-1}{\xi}\psi_{\xi}\right) - (\mu + \kappa)\xi\psi_{\xi} + 2\kappa\psi - \frac{\mu}{2}\psi_{\xi}, \quad \xi \in (\xi_0, \xi_1).
\]

The next lemma again describes oscillatory behaviour in a linear ODE of Euler type, and may be viewed as a counterpart of Lemma 3.1.

**Lemma 4.2.** Let \( m < m_\ast \). Then \( \kappa_\ast := n + 1 - \mu - 2\sqrt{2(n - \mu)} \) satisfies \( \kappa_\ast < n - 2 - \mu \), and for each \( \kappa \in (\kappa_\ast, n - 2 - \mu) \) the numbers

\[
a(\kappa) := \frac{n - 2 - \mu - \kappa}{2} \quad \text{and} \quad b(\kappa) := \frac{\sqrt{8\kappa - (n - 2 - \mu - \kappa)^2}}{2}
\]

are real and positive. Moreover, \( \psi : (0, \infty) \to \mathbb{R} \) defined by

\[
\psi(\xi) := \xi^{-a(\kappa)} \cos\left(b(\kappa) \ln \xi\right), \quad \xi > 0,
\]

is a solution of

\[
\xi^2\left(\psi_{\xi\xi} + \frac{n-1}{\xi}\psi_{\xi}\right)\psi_{\xi} - (\mu + \kappa)\xi\psi_{\xi} + 2\kappa\psi = 0, \quad \xi > 0.
\]

**Proof.** Since \( m < m_\ast \) implies that \( \mu + 2 < n \), we have \( \sqrt{2(n - \mu)} > 2 \) and hence indeed

\[
n - 2 - \mu - \kappa_\ast = -4 + 2\sqrt{2(n - \mu)} > 0.
\]

We rewrite the radicand in the definition of \( b(\kappa) \) according to

\[
R(\kappa) := 8\kappa - (n - 2 - \mu - \kappa)^2 = -\kappa^2 + 2(n + 2 - \mu)\kappa - (n - 2 - \mu)^2,
\]
Lemma 4.3. Assume (2.5). Then for any \( \gamma \) satisfying (1.16) there exists \( C(\gamma) > 0 \) such that the solution of (2.1) satisfies

\[
v(r, t) \leq C(\gamma)e^{\gamma t} \quad \text{for all } r > 0 \text{ and } t > 0.
\]

**Proof.** Since \( \gamma > \gamma_L \), the number \( \kappa := \gamma/\mu \) satisfies \( \kappa > \kappa_L \), so that in view of Lemma 4.2 we may pick some \( \tilde{\kappa} < \kappa \) such that \( \tilde{\kappa} < n - 2 - \mu \) and \( \kappa > \kappa_L \). We let

\[
\tilde{\psi}(\xi) := \xi^{-a(\tilde{\kappa})} \cos \left( b(\tilde{\kappa}) \ln \xi \right), \quad \xi > 0,
\]

with \( a(\tilde{\kappa}) > 0 \) and \( b(\tilde{\kappa}) > 0 \) as defined in (4.2). Then \( \tilde{\psi} \) has infinitely many zeros, which makes it possible to fix \( \xi_+ \) and \( \xi_- \) such that \( 0 < \xi_- < \xi_+ \), \( \tilde{\psi}(\xi_+) = \tilde{\psi}(\xi_-) = 0 \) and \( \tilde{\psi} > 0 \) on \((\xi_-, \xi_+)\). Next, taking \( \xi_1 \in (\xi_-, \xi_+) \) to be the unique zero of \( \tilde{\psi}_\xi \) in \((\xi_-, \xi_+)\), we obtain that \( \tilde{\psi} > 0 \) in \([\xi_1, \xi_+)\) and \( \tilde{\psi}_\xi < 0 \) in \((\xi_1, \xi_+)\), so that

\[
-\xi \tilde{\psi}_\xi(\xi) + 2\tilde{\psi}(\xi) \geq c_1 \quad \text{for all } \xi \in (\xi_1, \xi_+)
\]

holds with some \( c_1 > 0 \). Moreover, since \( \tilde{\psi} \) is smooth, we can find \( c_2 > 0 \) with the property

\[
-\tilde{\psi}(\xi) \left( \tilde{\psi}_{\xi\xi}(\xi) + \frac{n - 1}{\xi} \tilde{\psi}_\xi(\xi) \right) + \frac{\mu}{2} \tilde{\psi}_\xi^2(\xi) \leq c_2 \quad \text{for all } \xi \in (\xi_1, \xi_+).
\]

Finally, in view of (2.5) we can fix \( c_3 > 0 \) such that

\[
v_0(r) \leq (r^2 + c_3)^{-2} \quad \text{for all } r \in [0, \xi_+]
\]

and then pick \( \eta > 0 \) small fulfilling

\[
\eta \leq \min \left\{ \frac{(\kappa - \tilde{\kappa})c_1}{c_2}, \frac{c_3}{\tilde{\psi}(\xi_1)} \right\}.
\]
Upon these choices,

\[
\psi(\xi) := \begin{cases} 
\eta \psi_\xi(\xi_1), & \xi \in [0, \xi_1], \\
\eta \psi_\xi(\xi), & \xi \in (\xi_1, \xi_+], \\
0, & \xi > \xi_+, 
\end{cases}
\]

defines a nonnegative continuous function \( \psi \) on \([0, \infty) \) which satisfies

\[
\lim_{\xi \searrow \xi_1} \psi(\xi) = \lim_{\xi \nearrow \xi_1} \psi(\xi) = 0 \quad \text{and} \quad \lim_{\xi \searrow \xi_+} \psi(\xi) = \lim_{\xi \nearrow \xi_+} \psi(\xi) = \frac{\kappa}{\mu} \psi_\xi(\xi_+), \quad (4.10)
\]

as well as

\[
\psi(\xi) \leq \eta \psi(\xi_1) \quad \text{for all} \quad \xi > 0. \quad (4.11)
\]

In particular, if we set

\[
\tau(r, t) := \sigma(t) \left( \xi^2(r, t) + \psi(\xi(r, t)) \right)^{-\frac{\eta}{2}}, \quad r \geq 0, \ t \geq 0,
\]

with \( \sigma(t) := e^{\mu t} \) and \( \xi(r, t) := \sigma^{1/\mu}(t)r \), then \( \tau \) is continuous in \([0, \infty)^2 \). Obviously,

\[
\tau_{\tau} = 0 \quad \text{whenever} \quad \xi(r, t) > \xi_+ + \tau, \quad (4.12)
\]

for at such points we have \( \tau(r, t) = r^{-\mu} \). Next, if \((r, t) \in (0, \infty)^2 \) is such that \( \xi(r, t) < \xi_1 \) then with \( B \) as defined in Lemma 4.1 we have \( B\psi(\xi(r, t)) = 2\kappa \psi(\xi(r, t)) \geq 0 \), which by Lemma 4.1 implies that

\[
\tau_{\tau} \geq 0 \quad \text{if} \quad \xi(r, t) < \xi_1. \quad (4.13)
\]

Finally, in the intermediate region where \( \xi_1 < \xi < \xi_+ \) we use Lemma 4.2 to compute, partially dropping the argument \((r, t)\) of \( \xi \) for simplicity,

\[
B\psi(\xi(r, t)) = \left( \xi^2 + \psi(\xi) \right) \left( \psi_\xi + \frac{n-1}{\xi} \psi_\xi(\xi) \right) - (\mu + \kappa) \psi_\xi(\xi) + 2\kappa \psi(\xi) - \frac{\mu}{2} \psi^2_\xi(\xi) \\
= -\eta(\kappa - \bar{\kappa}) \psi_\xi(\xi) + 2\eta(\kappa - \bar{\kappa}) \psi(\xi) + \eta^2 \psi(\xi) \left( \psi_\xi + \frac{n-1}{\xi} \psi_\xi(\xi) \right) - \frac{\mu}{2} \eta^2 \psi^2_\xi(\xi) \quad \text{if} \quad (r, t) \in (\xi_1, \xi_+).
\]

Recalling (4.6), (4.7) and the first requirement contained in (4.9), we deduce that \( B\psi(\xi(r, t)) \geq \eta(\kappa - \bar{\kappa}) c_1 - \eta^2 c_2 \geq 0 \) if \((r, t) \in (\xi_1, \xi_+)\), which together with (4.12), (4.13) and (4.10) shows that \( \tau \) is a supersolution of (2.1).

Furthermore, at \( t = 0 \) we have \( \sigma(t) = 1 \) and thus \( \tau(r, 0) = (r^2 + \psi(r))^{-\mu/2} \) for all \( r \geq 0 \), so that for small \( r \) we obtain from (4.11), (4.9) and (4.8) that

\[
\tau(r, 0) \geq \left( r^2 + \eta \psi(\xi_1) \right)^{-\frac{\eta}{2}} \geq (r^2 + c_3)^{-\frac{\eta}{2}} \geq v_0(r) \quad \text{for all} \quad r \in [0, \xi_+].
\]

Since (2.5) implies that \( v_0(r) \leq r^{-\mu} \tau(r, 0) \) if \( r > \xi_+ \), we see that \( \tau(r, 0) \geq v_0(r) \) for all \( r \geq 0 \). Therefore, \( \tau(r, t) \geq v(r, t) \) for all \( r \geq 0 \) and \( t \geq 0 \) by comparison. In
particular, using that $\xi^2 + \psi(\xi) \geq c_4 := \min \{ \xi_1^2, \eta \tilde{\psi}(\xi_1) \}$ for all $\xi \geq 0$, we conclude that

$$v(r, t) \leq \sigma(t) \left( \xi^2 (r, t) + \psi(\xi(r, t)) \right)^{-\frac{c}{2}} \leq c_4^{-\frac{c}{2}} \sigma(t) \quad \text{for all } r \geq 0 \text{ and } t \geq 0.$$

This shows that (4.5) holds if we set $C(\gamma) := c_4^{-\mu/2}$. \hfill \Box

**Proof of Theorem 1.4.** Lemma 4.3 and comparison with radial solutions yield the claim. \hfill \Box

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