Blow-up prevention by quadratic degradation in a two-dimensional Keller-Segel-Navier-Stokes system

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Abstract

This paper deals with an initial-boundary value problem in a two-dimensional smoothly bounded domain for the Keller-Segel-Navier-Stokes system with logistic source, as given by

$$\begin{cases} n_t + u\cdot\nabla n &=& \Delta n - \nabla\cdot(n\nabla c) + rn - \mu n^2, \\ c_t + u\cdot\nabla c &=& \Delta c - c + n, \\ u_t + u\cdot\nabla u &=& \Delta u - \nabla P + n\nabla\phi + g, \\ \nabla\cdot u &=& 0, \end{cases}$$

which describes the mutual interaction of chemotactically moving microorganisms and their surrounding incompressible fluid.
It is shown that whenever $\mu > 0$, $r \ge 0$, $g \in C^1(\bar\Omega\times[0,\infty)) \cap L^\infty(\Omega\times(0,\infty))$ and the initial data (n_0, c_0, u_0) are sufficiently smooth fulfilling $n_0 \not\equiv 0$, the considered problem possesses a global classical solution which is bounded. Moreover, if $r = 0$, then this solution satisfies

$$n(\cdot,t)\to 0 \quad \text{and} \quad c(\cdot,t)\to 0 \qquad \text{in } L^\infty(\Omega)$$

as $t\to\infty$, and if additionally $\int_0^\infty\int_\Omega g^2(x,t)dxdt < \infty$, then all solution components decay in the sense that

$$n(\cdot,t)\to 0, \quad c(\cdot,t)\to 0 \quad \text{and} \quad u(\cdot,t)\to 0 \qquad \text{in } L^\infty(\Omega)$$

as $t\to\infty$.

Key words: chemotaxis, Navier-Stokes, boundedness, large time behavior
AMS Classification: 35B40, 35B45, 35K55, 35Q35, 35Q92, 76D05, 92C17

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1 Introduction

Chemotaxis, the biased migration of cells toward higher concentrations of a chemical, is known to play an outstanding role in a large range of biological applications ([15]). A renowned mathematical model for chemotaxis processes was proposed by Keller and Segel ([17]), in its simplest versions considering the density $n = n(x,t)$ of the cell population and the chemical concentration $c = c(x,t)$ as unknown and further system elements fixed; in particular, such two-component chemotaxis systems assume that there is neither any relevant influence of cells on the surrounding habitat, nor vice versa any relevant effect of the latter on the movement of cells or the chemical.

More recent observations, however, indicate that in certain cases of chemotactic motion in liquid environments the mutual interaction between cells and fluid may in fact be substantial. For instance, striking experimental evidence, as reported in [32], reveals dynamical generation of patterns and spontaneous emergence of turbulence in populations of aerobic bacteria suspended in sessile drops of water. Other examples arise in mechanisms of effective mixing triggered by chemotaxis in instationary fluids, e.g. in the context of broadcast spawning phenomena known to be indispensable for successful coral fertilization ([4, 22]).

As a natural generalization of the classical Keller-Segel accounting for chemotaxis-fluid interaction, in this work we shall assume the fluid flow, represented by the velocity $u = u(x,t)$ and the associated pressure $P = P(x,t)$, to be further unknown quantities and henceforth suppose that the evolution of (n, c, u, P) is governed by the Keller-Segel-Navier-Stokes system

$$\left\{ \begin{array}{rcll} n_t + u \cdot \nabla n &=& \Delta n - \nabla \cdot (n \nabla c) + rn - \mu n^2, & x \in \Omega,\ t > 0, \\ c_t + u \cdot \nabla c &=& \Delta c + n - c, & x \in \Omega,\ t > 0, \\ u_t + u \cdot \nabla u &=& \Delta u - \nabla P + n \nabla \phi + g, & x \in \Omega,\ t > 0, \\ \nabla \cdot u &=& 0, & x \in \Omega,\ t > 0, \end{array} \right. \tag{1.1}$$

in the physical domain $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Here we require that $r \geq 0$ and $\mu > 0$, thus including both the case $r > 0$ occurring when e.g. n denotes the density of a bacterial population which may proliferate according to a logistic law, and also the case $r = 0$, reflecting situations when either replication is a priori precluded but quadratic degradation occurs due to a reaction process, or reproduction is negligible at the considered time scales, with death effects due to overcrowding yet dominant at large densities. Both the cells and the chemical are assumed to be transported by the fluid, and the motion of the latter is supposed to be driven by gravitation-induced forces due to the presence of cells in a gravitational potential ϕ, and by a given external force $g = g(x,t)$ which may be relevant in larger fluids containing further components. The system (1.1) thereby covers the result of the recent modeling approaches for biomixing in [9], [18] and [19], but beyond this (1.1) also accounts for buoyant effects in the style proposed in [6] and [32] within the framework of a chemotaxis-fluid model for swimming bacteria.

A fundamental mathematical challenge arising in the analysis of (1.1) appears to consist in deciding whether or not global solutions exist, and which regularity properties can be expected. That this basic question seems far from trivial stems from the fact
that as subsystems, (1.1) contains both the Navier-Stokes equations, which themselves are yet lacking a complete existence theory ([35]), and the Keller-Segel system obtained from (1.1) on letting $g \equiv u \equiv 0$.

As for the latter, it is known that it reflects the strong destabilizing potential of chemotactic cross-diffusion in the sense that in the limit case $r = \mu = 0$ it possesses many solutions blowing up in finite time in spatially two- or higher-dimensional settings ([14], [40], [23]; cf. also [24]). Though even superlinear degradation terms in the cell evolution may be insufficient to suppress such a singularity formation ([37]), after all it is known that the presence of quadratic death terms as in (1.1) enforces the global existence of bounded solutions to the corresponding two-component chemotaxis system when either $N = 2$ and $\mu > 0$ is arbitrary ([25]), or $N \geq 3$ and μ is sufficiently large ([38]). For very large μ, solutions are even known to stabilize toward spatially homogeneous equilibria whenever $N \geq 1$ ([42]); in the case $r = 0$, even when $N = 3$, for any choice of $\mu > 0$ there exist at least global weak solutions which eventually become smooth and decay in both components ([20]).

Similar obstacles concern a widely studied variant of (1.1) in which the zero-order term $-c + n$ in the second equation is replaced by the absorption term $-nf(c)$, thus supposing that unlike in (1.1) where the signal is produced by the cells, the latter rather consume this chemical ([32]). Then still the understanding of the associated chemotaxis-only subsystem is far from complete especially in the case $N = 3$ ([29]), but the dampening effect of signal absorption at least under certain restrictions on the space dimension and the model parameter functions allows for the construction of certain global solutions to the corresponding chemotaxis-fluid system ([7], [2], [3], [39], [41], [43]), and also to some variants thereof involving e.g. nonlinear diffusion ([5], [33], [30], [31], [8]) and variants in the cross-diffusive term ([1], [34], [16]).

Main results. The only result we are aware of which addresses chemotaxis-fluid interaction in presence of a signal production mechanism is contained in [9], where a result on global existence of weak solutions has been derived for a two-dimensional simplification of (1.1), obtained on neglecting the convective term $(u \cdot \nabla)u$ and letting $r = 0$ and $g \equiv 0$. Questions concerning further boundedness and regularity properties of these solutions, and especially their large time behavior, seem to be open.

The goal of the present work is to study these questions for the full chemotaxis-Navier-Stokes system in the case $N = 2$, and to give somewhat complete answers with regard to global existence, boundedness and smoothness of solutions for general $r \geq 0, \mu > 0$ and widely arbitrary g, and with regard to convergence to zero of n and c when $r = 0$, and of even all solution components when g decays suitably.

In order to state these results more precisely, we close the system (1.1) by imposing no-flux boundary conditions for n and c and a no-slip boundary condition for u,

$$\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \quad \text{and} \quad u = 0 \qquad \text{for } x \in \partial\Omega \text{ and } t > 0, \tag{1.2}$$

and the initial conditions

$$n(x,0) = n_0(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

For simplicity we shall assume throughout this paper that the initial data are such that

$$\begin{cases} n_0 \in C^0(\bar{\Omega}), \qquad n_0 \geq 0 \quad \text{in } \bar{\Omega} \qquad \text{and} \quad n_0 \not\equiv 0, \\ c_0 \in W^{1,\infty}(\Omega), \qquad c_0 \geq 0 \quad \text{in } \bar{\Omega} \quad \text{and} \\ u_0 \in D(A^\gamma) \qquad \text{for some } \gamma \in (\frac{1}{2}, 1), \end{cases} \tag{1.4}$$

where A denotes the realization of the Stokes operator in the solenoidal subspace $L^2_\sigma(\Omega)$ of $L^2(\Omega)$, with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$ (cf. also Section 2). As to the given potential function $\phi = \phi(x)$ and the source term $g = g(x,t)$ in (1.1), we require that

$$\phi \in W^{1,\infty}(\Omega), \tag{1.5}$$

and that

$$g \in C^1(\bar\Omega \times [0,\infty)) \cap L^\infty(\Omega \times (0,\infty)). \tag{1.6}$$

Within the above framework, our main results concerning global existence and boundedness of solutions to (1.1)-(1.3) then read as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, and let $r \ge 0$ and $\mu > 0$. Assume that ϕ and g satisfy (1.5) and (1.6), and suppose that n_0, c_0 and u_0 fulfill (1.4). Then there exist functions*

$$\begin{cases} n \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ c \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ u \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \quad \textit{and} \\ P \in C^{1,0}(\bar\Omega \times (0,\infty)), \end{cases} \tag{1.7}$$

which solve (1.1)-(1.3) classically in $\Omega \times (0,\infty)$. Moreover, this solution is bounded in the sense that there exists $C > 0$ fulfilling

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{L^\infty(\Omega)} + \|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0. \tag{1.8}$$

In particular, Theorem 1.1 asserts that, as in the corresponding two-dimensional Keller-Segel system ([25]), even arbitrarily small quadratic degradation of cells is sufficient to rule out blow-up and rather ensure boundedness of solutions.

Now in several applications the limit case $r = 0$ becomes relevant. For instance, in the modeling context addressed in [18] the number r was chosen to be zero according to the natural assumption that there is no production of eggs and sperm at the considered stage of coral fertilization. Apart from this, in situations when the respective process in question proceeds at time scales significantly shorter than proliferation intervals, the production term rn in (1.1) becomes negligible. This need not be the case for the quadratic degradation term $-\mu n^2$, reflecting death of cells due to overcrowding, which can remain relevant also under such circumstances.
In this limiting situation, the total cell population can readily be seen to decay in the large time limit (cf. Lemma 4.1 below). The following result asserts that the solution possesses enough regularity properties so as to allow for the conclusion that this convergence actually occurs uniformly for the cell density function, and that moreover also the signal concentration decays asymptotically.

Theorem 1.2 *Suppose that in addition to the assumptions of Theorem 1.1 we have $r = 0$. Then the solution of (1.1)-(1.3) has the properties that*

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty \tag{1.9}$$

and

$$\|c(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{1.10}$$

Also Theorem 1.2 is in good accordance with known results for the fluid-free simplification of (1.1). Namely, it is known that in the case $r = 0$, even when $N = 3$, for any choice of $\mu > 0$ the latter chemotaxis system possesses at least global weak solutions which eventually become smooth and decay in both components ([20]).

Under a mild assumption on the decay of the external force in the fluid equation, we can finally verify that also the fluid velocity, and hence the entire solution (n, c, u), approaches zero.

Theorem 1.3 *Let the assumptions of Theorem 1.1 hold, and assume that $r = 0$ and that*

$$\int_0^\infty \int_\Omega |g|^2(x,t)dxdt < \infty. \tag{1.11}$$

Then the solution of (1.1)-(1.3) satisfies

$$\|n(\cdot,t)\|_{L^\infty(\Omega)} \to 0, \quad \|c(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \quad \textit{and} \quad \|u(\cdot,t)\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } t \to \infty. \tag{1.12}$$

We have to leave open here how far in the case $r > 0$ at least large values of μ may enforce convergence of solutions. That this cannot be expected for general $r > 0$ and $\mu > 0$ is indicated by the numerical experiments for the corresponding chemotaxis-only system presented in [26], revealing the possibility of quite colorful oscillatory dynamics even in the case $N = 1$. After all, at least for very large μ solutions of this system are known to stabilize toward spatially homogeneous equilibria whenever $N \geq 1$ ([42]), and a similar behavior might be expected in (1.1) under appropriate assumptions.

The proof of Theorem 1.1 will be based on a series of a priori estimates, to be derived in Section 3, with a crucial point of the proof consisting in a proper handling of the convective term $u \cdot \nabla u$ in the third equation of (1.1) (see Lemma 3.6 and Lemma 3.11). The regularity properties thereby additionally obtained will allow us in Section 4.1 to turn some easily achieved weak decay information on n and c (Lemma 4.1 and Lemma 4.2) into the convergence statements claimed in Theorem 1.2. In a similar manner, Theorem 1.3 will be derived in Section 4.2.

2 Preliminaries

Throughout the sequel, we let $\mathcal{P}$ denote the Helmholtz projection of $L^2(\Omega)$ onto its closed subspace $L^2_\sigma(\Omega) := \{\varphi \in L^2(\Omega) \mid \nabla \cdot \varphi = 0 \text{ in } \mathcal{D}'(\Omega)\}$ of divergence-free functions. The Stokes operator $A := -\mathcal{P}\Delta$ with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega)$ is then sectorial in $L^2_\sigma(\Omega)$ and hence generates the analytic Stokes contraction semigroup $(e^{-tA})_{t\geq 0}$ and possesses densely defined fractional powers A^α for any $\alpha \in (0, 1)$ ([27], [12]).

The following basic statement on local existence and extensibility can be proved in precisely the same manner demonstrated in [39, Lemma 2.1].

Lemma 2.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary, let $r \geq 0$ and $\mu > 0$, and suppose that (1.5) and (1.6 hold, and that n_0, c_0 and u_0 satisfy (1.4). Then there exist $T_{max} \in (0, \infty]$ and a classical solution (n, c, u, P) of (1.1)-(1.3) in $\Omega \times (0, T_{max})$ such that $n \geq 0$ and $c \geq 0$ in $\bar\Omega \times (0, T_{max})$, that*

$$\begin{cases} n \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})), \\ c \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})) \quad \textit{and} \\ u \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})), \end{cases}$$

and such that

$$\begin{aligned}&\textit{either } T_{max} = \infty, \textit{ or}\\ &\|n(\cdot,t)\|_{L^\infty(\Omega)} + \|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \to \infty \quad \textit{for all } \alpha \in (\tfrac{1}{2},1) \quad \textit{as } t \nearrow T_{max}.\end{aligned} \tag{2.1}$$

3 A priori estimates

3.1 Basic a priori bounds and decay estimates for n and c

A first basic boundedness information is essentially due to the presence of the quadratic death term in the first equation in (1.1).

Lemma 3.1 *There exist $m > 0$ and $K > 0$ such that the solution of (1.1)-(1.3) satisfies*

$$\int_\Omega n(\cdot,t) \le m \qquad \textit{for all } t \in (0,T_{max}) \tag{3.1}$$

and

$$\int_t^{t+\tau} \int_\Omega n^2 \le K \qquad \textit{for all } t \in (0,T_{max}-\tau), \tag{3.2}$$

where we have set

$$\tau := \min\Big\{1, \frac{1}{6}T_{max}\Big\}. \tag{3.3}$$

Proof. We integrate the first equation in (1.1) over Ω to obtain

$$\frac{d}{dt}\int_\Omega n = r\int_\Omega n - \mu\int_\Omega n^2 \qquad \text{for all } t \in (0,T_{max}), \tag{3.4}$$

because $\frac{\partial n}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0$ on $\partial\Omega$ and $\int_\Omega u\cdot\nabla n = -\int_\Omega n\nabla\cdot u = 0$ for all $t \in (0,T_{max})$. Since $\int_\Omega n^2 \ge \frac{1}{|\Omega|}(\int_\Omega n)^2$ for all $t \in (0,T_{max})$ by the Cauchy-Schwarz inequality, this firstly implies that $y(t) := \int_\Omega n(\cdot,t), t \in [0,T_{max})$, satisfies

$$y'(t) \le ry(t) - \frac{\mu}{|\Omega|}\cdot y^2(t) \qquad \text{for all } t \in (0,T_{max}),$$

which by an ODE comparison shows that

$$y(t) \le \max\bigg\{\int_\Omega n_0\,,\, \frac{r|\Omega|}{\mu}\bigg\} \qquad \text{for all } t \in (0,T_{max}).$$

This implies (3.1), whereupon (3.2) results upon an integration of (3.4) in time. □

In deriving some preliminary time-independent estimates for c from this, we shall make use of an auxiliary statement on boundedness in a linear differential inequality, which can be verified by a straightforward adaptation of the proof presented in [28, Lemma 3.4] for the special case $\tau = 1$.

Lemma 3.2 *Let $T > 0$, $\tau \in (0,T)$, $a > 0$ and $b > 0$, and suppose that $y : [0,T) \to [0,\infty)$ is absolutely continuous and such that*

$$y'(t) + ay(t) \le h(t) \qquad \textit{for a.e. } t \in (0,T)$$

with some nonnegative function $h \in L^1_{loc}([0,T))$ satisfying

$$\int_t^{t+\tau} h(s)ds \le b \qquad \textit{for all } t \in [0, T-\tau).$$

Then

$$y(t) \le \max\left\{ y(0) + b \,,\, \frac{b}{a\tau} + 2b \right\} \qquad \textit{for all } t \in (0,T).$$

We can thereupon deduce the following as a consequence of Lemma 3.1.

Lemma 3.3 *There exists $C > 0$ such that*

$$\int_\Omega c^2(\cdot,t) \le C \qquad \textit{for all } t \in (0, T_{max}) \tag{3.5}$$

and

$$\int_t^{t+\tau} \int_\Omega |\nabla c|^2 \le C \qquad \textit{for all } t \in (0, T_{max} - \tau) \tag{3.6}$$

with $\tau = \min\{1, \frac{1}{6} T_{max}\}$ as in (3.3).

Proof. Multiplying the second equation in (1.1) by c and integrating by parts we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega c^2 + \int_\Omega |\nabla c|^2 + \int_\Omega c^2 = \int_\Omega nc \qquad \text{for all } t \in (0, T_{max}),$$

where since

$$\int_\Omega nc \le \frac{1}{2}\int_\Omega c^2 + \frac{1}{2}\int_\Omega n^2 \qquad \text{for all } t \in (0, T_{max})$$

by the Cauchy-Schwarz inequality, we see that $y(t) := \int_\Omega c^2(\cdot,t), t \in [0, T_{max})$, satisfies

$$y'(t) + 2\int_\Omega |\nabla c|^2 + y(t) \le h(t) := \int_\Omega n^2(\cdot,t) \qquad \text{for all } t \in (0, T_{max}). \tag{3.7}$$

On dropping the second summand on the left we thus infer from an application of Lemma 3.2 relying on Lemma 3.1 that with $K > 0$ provided by the latter we have

$$y(t) \le \max\left\{ \int_\Omega c_0^2 + K \,,\, \frac{K}{\tau} + 2K \right\} \qquad \text{for all } t \in (0, T_{max}).$$

This precisely warrants (3.5), which in turn yields (3.6) after integrating (3.7) and once more employing (3.2). □

3.2 Boundedness of u in $W^{1,2}(\Omega)$

Our next goal is to make sure that the estimate (3.2) together with the assumed boundedness of g is sufficient to enforce boundedness of $\int_\Omega |\nabla u|^2$ for all $t \in (0, T_{max})$. To achieve this, we first track the evolution of the natural fluid energy functional in a standard manner to obtain a differential inequality for the latter which will immediately be used to derive some boundedness properties in Lemma 3.5, and which beyond this will be recalled later in Lemma 4.4 to assert decay of u.

Lemma 3.4 *There exists $C > 0$ such that*

$$\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 + \frac{1}{C}\int_\Omega |u|^2 \le C \cdot \left\{ \int_\Omega n^2 + \int_\Omega |g|^2 \right\} \qquad \text{for all } t \in (0, T_{max}). \tag{3.8}$$

Proof. Testing the third equation in (1.1) by u yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \int_\Omega |\nabla u|^2 = \int_\Omega n\nabla\phi \cdot u + \int_\Omega g \cdot u \qquad \text{for all } t \in (0, T_{max}), \tag{3.9}$$

because $u|_{\partial\Omega} = 0$ and $\nabla \cdot u = 0$. Here we recall that the Poincaré inequality provides $C_1 > 0$ fulfilling

$$\|\varphi\|_{L^2(\Omega)} \le C_1 \int_\Omega \|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W_0^{1,2}(\Omega; \mathbb{R}^2) \tag{3.10}$$

to obtain, using the boundedness of $\nabla\phi$ and Young's inequality, positive constants C_2 and C_3 such that

$$\begin{aligned}
\int_\Omega n\nabla\phi \cdot u + \int_\Omega g \cdot u &\le \|\nabla\phi\|_{L^\infty(\Omega)}\|n\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\|u\|_{L^2(\Omega)} \\
&\le C_2\Big\{\|n\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)}\Big\} \cdot \|\nabla u\|_{L^2(\Omega)} \\
&\le \frac{1}{4}\int_\Omega |\nabla u|^2 + C_3 \cdot \left\{ \int_\Omega n^2 + \int_\Omega |g|^2 \right\} \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Upon another application of (3.10), (3.9) therefore implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 + \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4C_1^2}\int_\Omega |u|^2 \le C_3 \cdot \left\{ \int_\Omega n^2 + \int_\Omega |g|^2 \right\} \qquad \text{for all } t \in (0, T_{max}),$$

and thereby proves (3.8). □

In light of Lemma 3.1, as a first conclusion we obtain the following time-independent bounds for u.

Lemma 3.5 *There exists $C > 0$ such that with $\tau = \min\{1, \frac{1}{6}T_{max}\}$ as given by (3.3) we have*

$$\int_\Omega |u(\cdot, t)|^2 \le C \qquad \text{for all } t \in (0, T_{max}) \tag{3.11}$$

and

$$\int_t^{t+\tau}\int_\Omega |\nabla u|^2 \le C \qquad \text{for all } t \in (0, T_{max} - \tau) \tag{3.12}$$

as well as

$$\int_t^{t+\tau}\int_\Omega |u|^4 \le C \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{3.13}$$

PROOF. By Lemma 3.4, there exist $C_1 > 0$ and $C_2 > 0$ such that $y(t) := \int_\Omega |u(\cdot,t)|^2$, $t \in [0, T_{max})$, satisfies

$$y'(t) + \int_\Omega |\nabla u|^2 + C_1 y(t) \le h(t) := C_2 \cdot \left\{ \int_\Omega n^2(\cdot,t) + \int_\Omega |g(\cdot,t)|^2 \right\} \qquad \text{for all } t \in (0, T_{max}), \quad (3.14)$$

where Lemma 3.1 and the boundedness of g assert that

$$\int_t^{t+\tau} h(s)ds \le C_3 := C_2 \cdot \left\{ K + |\Omega| \cdot \|g\|^2_{L^\infty(\Omega\times(0,\infty))} \right\} \qquad \text{for all } t \in (0, T_{max} - \tau). \quad (3.15)$$

Therefore, Lemma 3.2 becomes applicable so as to ensure that

$$y(t) \le \max \left\{ \int_\Omega |u_0|^2 + C_3 \,,\, \frac{C_3}{C_1 \tau} + 2C_3 \right\} \qquad \text{for all } t \in (0, T_{max}),$$

thus implying (3.11). Thereafter, again thanks to (3.15), an integration of (3.14) yields (3.12). Finally, combining (3.12) with (3.11) by means of a Gagliardo-Nirenberg interpolation provides $C_4 > 0$ and $C_5 > 0$ such that

$$\int_t^{t+\tau} \int_\Omega |u|^4 \le C_4 \int_t^{t+\tau} \|\nabla u(\cdot,s)\|^2_{L^2(\Omega)} \|u(\cdot,s)\|^2_{L^2(\Omega)} ds \le C_5 \qquad \text{for all } t \in (0, T_{max} - \tau)$$

and thus ensures that also (3.13) holds. □

Now by a further testing procedure, we can turn the above information into an improved estimate for u on the basis of an interpolation argument which essentially relies on our assumption that the spatial setting is two-dimensional.

Lemma 3.6 *There exists $C > 0$ such that*

$$\int_\Omega |\nabla u(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0, T_{max}). \quad (3.16)$$

PROOF. We apply the Helmholtz projector $\mathcal{P}$ to the third equation in (1.1) and multiply the resulting identity, $u_t + Au = -\mathcal{P}[(u\cdot\nabla)u] + \mathcal{P}[n\nabla\phi] + \mathcal{P}[g], t \in (0, T_{max})$, by Au. Using Young's inequality and the orthogonal projection property of $\mathcal{P}$, we thereby see that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |A^{\frac12}u|^2 + \int_\Omega |Au|^2 &= -\int_\Omega \mathcal{P}[(u\cdot\nabla)u]\cdot Au + \int_\Omega \mathcal{P}[n\nabla\phi]\cdot Au + \int_\Omega \mathcal{P}[g]\cdot Au \\ &\le \frac{3}{4}\int_\Omega |Au|^2 + \int_\Omega |(u\cdot\nabla)u|^2 + \|\nabla\phi\|^2_{L^\infty(\Omega)} \int_\Omega n^2 + \int_\Omega |g|^2 \quad (3.17) \end{aligned}$$

for all $t \in (0, T_{max})$. Here we use the Gagliardo-Nirenberg inequality, Lemma 3.5 and Young's inequality to find $C_1 > 0$ and $C_2 > 0$ such that

$$\begin{aligned} \int_\Omega |(u\cdot\nabla)u|^2 &\le C_1 \|u\|^2_{L^\infty(\Omega)} \|\nabla u\|^2_{L^2(\Omega)} \\ &\le C_1 \|Au\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla u\|^2_{L^2(\Omega)} \\ &\le C_2 \|Au\|_{L^2(\Omega)} \|\nabla u\|^2_{L^2(\Omega)} \\ &\le \frac{1}{8}\int_\Omega |Au|^2 + 2C_2^2 \left(\int_\Omega |\nabla u|^2 \right)^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

As $\nabla\phi$ is bounded, (3.17) thus entails the existence of $C_3 > 0$ such that writing

$$h_1(t) := C_3 \int_\Omega |\nabla u(\cdot,t)|^2 \quad \text{and} \quad h_2(t) := C_3 \cdot \left\{ \int_\Omega n^2(\cdot,t) + \int_\Omega |g(\cdot,t)|^2 \right\}, \qquad t \in (0,T_{max}),$$

we see that $y(t) := \int_\Omega |\nabla u(\cdot,t)|^2,\ t \in (0,T_{max})$, satisfies

$$y'(t) \le h_1(t) \cdot y(t) + h_2(t) \qquad \text{for all } t \in (0,T_{max}). \tag{3.18}$$

In order to prepare an integration thereof, we use Lemma 3.1 along with the boundedness of g as well as Lemma 3.5 to obtain $C_4 > 0$ and $C_5 > 0$ fulfilling

$$\int_t^{t+\tau} h_2(s)ds \le C_4 \qquad \text{for all } t \in (0,T_{max}-\tau) \tag{3.19}$$

and

$$\int_t^{t+\tau} \int_\Omega |\nabla u|^2 \le C_5 \qquad \text{for all } t \in (0,T_{max}-\tau) \tag{3.20}$$

with $\tau = \min\{1, \frac{1}{2}T_{max}\}$. In particular, from the latter it easily follows that if $t \in (0,T_{max})$ is arbitrary then in both cases $t \in (0,\tau)$ and $t \ge \tau$ we can find $t_0 = t_0(t) \in (t-\tau,t)$ such that $t_0 \ge 0$ and

$$\int_\Omega |\nabla u(\cdot,t_0)|^2 \le C_6 := \max\left\{ \int_\Omega |\nabla u_0|^2\,,\ \frac{C_5}{\tau} \right\}. \tag{3.21}$$

We now integrate (3.18) over (t_0,t) to infer that

$$y(t) \le y(t_0) \cdot e^{\int_{t_0}^t h_1(\sigma)d\sigma} + \int_{t_0}^t e^{\int_s^t h_1(\sigma)d\sigma} \cdot h_2(s)ds,$$

so that invoking (3.21), (3.20) and (3.19) shows that thanks to our choice of t_0,

$$\int_\Omega |\nabla u(\cdot,t)|^2 \ \le\ C_6\, e^{C_3C_5} + C_4 e^{C_3C_5},$$

which proves (3.16). □

3.3 A spatio-temporal L^2 estimate for Δc

We next turn the bounds from Lemma 3.6 and Lemma 3.1 into a higher order bound for c. Although the structure of the sources and nonlinearities in the second equation is somewhat different from those in the third, our basic strategy parallels that in the proof of Lemma 3.6 to a certain extent.

Lemma 3.7 *There exists $C > 0$ such that*

$$\int_\Omega |\nabla c(\cdot,t)|^2 \le C \qquad \textit{for all } t \in (0,T_{max}) \tag{3.22}$$

and

$$\int_t^{t+\tau} \int_\Omega |\Delta c|^2 \le C \qquad \textit{for all } t \in (0,T_{max}), \tag{3.23}$$

where $\tau = \min\{1, \frac{1}{6}T_{max}\}$ is as in (3.3).

Proof. Testing the second equation in (1.1) against $-\Delta c$ yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 = -\int_\Omega |\nabla c|^2 - \int_\Omega n\Delta c + \int_\Omega (u\cdot\nabla c)\Delta c \qquad \text{for all } t\in(0,T_{max}), \tag{3.24}$$

where by Young's inequality and the Hölder inequality,

$$-\int_\Omega n\Delta c \le \frac{1}{4}\int_\Omega |\Delta c|^2 + \int_\Omega n^2 \qquad \text{for all } t\in(0,T_{max}) \tag{3.25}$$

and

$$\int_\Omega (u\cdot\nabla c)\Delta c \le \|u\|_{L^4(\Omega)}\|\nabla c\|_{L^4(\Omega)}\|\Delta c\|_{L^2(\Omega)} \qquad \text{for all } t\in(0,T_{max}).$$

Since from the Gagliardo-Nirenberg inequality we obtain

$$\|\nabla c\|_{L^4(\Omega)} \le C_1\|\Delta c\|_{L^2(\Omega)}^{\frac{1}{2}}\|\nabla c\|_{L^2(\Omega)}^{\frac{1}{2}} \qquad \text{for all } t\in(0,T_{max}),$$

again using Young's inequality we see that

$$\begin{aligned}\int_\Omega (u\cdot\nabla c)\Delta c &\le C_1\|u\|_{L^4(\Omega)}\|\nabla c\|_{L^2(\Omega)}^{\frac{1}{2}}\|\Delta c\|_{L^2(\Omega)}^{\frac{3}{2}} \\ &\le \frac{1}{4}\|\Delta c\|_{L^2(\Omega)}^2 + C_2\|u\|_{L^4(\Omega)}^4\|\nabla c\|_{L^2(\Omega)}^2 \qquad \text{for all } t\in(0,T_{max})\end{aligned}$$

with some $C_2>0$. Combined with (3.25) and (3.24), this implies that if we let

$$h_1(t) := C_3\int_\Omega n^2(\cdot,t) \quad \text{and} \quad h_2(t) := C_3\int_\Omega |u(\cdot,t)|^4 \qquad \text{for } t\in(0,T_{max})$$

with appropriately large $C_3>0$, then

$$\frac{d}{dt}\int_\Omega |\nabla c|^2 + \int_\Omega |\Delta c|^2 \le h_1(t) + h_2(t)\cdot\int_\Omega |\nabla c|^2 \qquad \text{for all } t\in(0,T_{max}). \tag{3.26}$$

Now our basic strategy in treating this inequality parallels that in the proof of Lemma 3.6: In view of Lemma 3.3, Lemma 3.1 and Lemma 3.5 we know that there exist positive constants C_4, C_5 and C_6 such that

$$\int_t^{t+\tau}\int_\Omega |\nabla c|^2 \le C_4 \qquad \text{for all } t\in(0,T_{max}-\tau) \tag{3.27}$$

as well as

$$\int_t^{t+\tau} h_1(s)ds \le C_5 \quad \text{and} \quad \int_t^{t+\tau} h_2(s)ds \le C_6 \qquad \text{for all } t\in(0,T_{max}-\tau). \tag{3.28}$$

In particular, given $t\in(0,T_{max})$ we can use (3.27) to pick $t_0\in(t-\tau,t)\cap[0,\infty)$ fulfilling

$$\int_\Omega |\nabla c(\cdot,t_0)|^2 \le C_7 := \max\left\{\int_\Omega |\nabla c_0|^2\,,\,\frac{C_4}{\tau}\right\},$$

Therefore, on dropping the second integral on its left we infer from an integration of (3.26) that

$$\begin{aligned}\int_\Omega |\nabla c(\cdot,t)|^2 &\le \Big(\int_\Omega |\nabla c(\cdot,t_0)|^2\Big)\cdot e^{\int_{t_0}^t h_2(\sigma)d\sigma} + \int_{t_0}^t e^{\int_s^t h_2(\sigma)d\sigma}\cdot h_1(s)ds \\ &\le C_7\, e^{C_6} + C_5\, e^{C_6} \\ &=: C_8 \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

because $t-t_0\le\tau\le 1$.
Having thus verified (3.22), upon another integration of (3.26) using (3.28) we thereupon readily obtain (3.23). □

3.4 Boundedness of n in $L^p(\Omega)$ for arbitrary $p>1$

Now thanks to the bound on Δc gained Lemma 3.7, once more making strong use of the absorptive zero-order term in the first equation we can successively improve our knowledge on the regularity properties of n by means of the statement which will serve as the inductive step in a recursion.

Lemma 3.8 *Let $p\ge 2$ and $L>0$, and again abbreviate $\tau=\min\{1,\frac{1}{6}T_{max}\}$. Then there exists $C=C(p,L)>0$ with the property that if*

$$\int_t^{t+\tau}\int_\Omega n^p \le L \qquad \textit{for all } t\in(0,T_{max}-\tau), \tag{3.29}$$

then

$$\int_\Omega n^p \le C \qquad \textit{for all } t\in(0,T_{max}) \tag{3.30}$$

and

$$\int_t^{t+\tau}\int_\Omega n^{p+1} \le C \qquad \textit{for all } t\in(0,T_{max}-\tau). \tag{3.31}$$

PROOF. We first multiply the first equation in (1.1) by n^{p-1} and integrate by parts to find, again using that $\nabla\cdot u=0$, that

$$\frac{d}{dt}\int_\Omega n^p + p(p-1)\int_\Omega n^{p-2}|\nabla n|^2 = p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c + pr\int_\Omega n^p - p\mu\int_\Omega n^{p+1} \qquad \text{for all } t\in(0,T_{max}). \tag{3.32}$$

Here in the cross-diffusive integral we once more integrate by parts and use the Cauchy-Schwarz inequality to see that

$$\begin{aligned}p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c &= -(p-1)\int_\Omega n^p\Delta c \\ &\le (p-1)\Big(\int_\Omega n^{2p}\Big)^{\frac12}\Big(\int_\Omega |\Delta c|^2\Big)^{\frac12} \qquad \text{for all } t\in(0,T_{max}).\end{aligned} \tag{3.33}$$

Now the Gagliardo-Nirenberg inequality shows that with some $C_1 > 0$ and $C_2 > 0$ we have

$$\begin{aligned}
\Big(\int_\Omega n^{2p}\Big)^{\frac{1}{2}} &= \|n^{\frac{p}{2}}\|_{L^4(\Omega)}^2 \\
&\le C_1\|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}\|n^{\frac{p}{2}}\|_{L^2(\Omega)} + C_1\|n^{\frac{p}{2}}\|^2_{L^{\frac{2}{p}}(\Omega)} \\
&\le C_1\|\nabla n^{\frac{p}{2}}\|_{L^2(\Omega)}\|n^{\frac{p}{2}}\|_{L^2(\Omega)} + C_2 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

because $\|n^{\frac{p}{2}}(\cdot,t)\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} = \int_\Omega n(\cdot,t) \le \int_\Omega n_0$ for all $t \in (0, T_{max})$ by Lemma 3.1. Inserting this into (3.33) and invoking Young's inequality , we find $C_3 > 0$ fulfilling

$$\begin{aligned}
p(p-1)\int_\Omega n^{p-1}\nabla n\cdot\nabla c &\le \frac{p(p-1)C_1}{2}\Big(\int_\Omega|\Delta c|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega n^{p-2}|\nabla n|^2\Big)^{\frac{1}{2}}\Big(\int_\Omega n^p\Big)^{\frac{1}{2}} \\
&\quad +(p-1)C_2\Big(\int_\Omega|\Delta c|^2\Big)^{\frac{1}{2}} \\
&\le p(p-1)\int_\Omega n^{p-2}|\nabla n|^2 + C_3\Big(\int_\Omega|\Delta c|^2\Big)\Big(\int_\Omega n^p\Big) \\
&\quad +C_3\int_\Omega|\Delta c|^2 + C_3 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

so that since another application of Young's inequality yields $C_4 > 0$ such that

$$pr\int_\Omega n^p \le \frac{p\mu}{2}\int_\Omega n^{p+1} + C_4 \qquad \text{for all } t \in (0, T_{max}),$$

from (3.32) we infer that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega n^p + \frac{p\mu}{2}\int_\Omega n^{p+1} &\le C_3\Big(\int_\Omega|\Delta c|^2\Big)\Big(\int_\Omega n^p\Big) \\
&\quad +C_3\int_\Omega|\Delta c|^2 + C_3 + C_4 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{3.34}$$

Letting

$$h(t) := C_3\int_\Omega|\Delta c(\cdot,t)|^2, \qquad t \in (0, T_{max}),$$

denote the first factor and the second summand on the right-hand side herein, we may apply Lemma 3.7 to infer the existence of $C_5 > 0$ satisfying

$$\int_t^{t+\tau} h(s)ds \le C_5 \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{3.35}$$

Now given $t \in (0, T_{max})$, in view of (3.29) it is possible to fix $t_0 \in (t-\tau, t)$ such that $t_0 \ge 0$ and

$$\int_\Omega n^p(\cdot,t_0) \le C_6 := \max\Big\{\int_\Omega n_0^p, \frac{L}{\tau}\Big\},$$

so that integrating (3.34) shows that

$$\begin{aligned}\int_\Omega n^p(\cdot,t) &\le \Big(\int_\Omega n^p(\cdot,t_0)\Big)\cdot e^{\int_{t_0}^t h(\sigma)d\sigma} + \int_{t_0}^t e^{\int_s^t h(\sigma)d\sigma}\cdot\Big\{h(s)+C_3+C_4\Big\}ds \\ &\le C_6\,e^{C_5}+e^{C_5}\cdot\Big\{C_3+C_4+C_5\Big\}\end{aligned}$$

due to (3.35) and the fact that $t-t_0\le\tau\le 1$. This proves (3.30), and hence (3.31) results from one further integration of (3.34) if we once more make use of (3.35). □

Along with the basic estimate from Lemma 3.1, this immediately implies the following.

Lemma 3.9 *For all $p>1$ we can find $C=C(p)>0$ such that*

$$\int_\Omega n^p(\cdot,t)\le C \qquad \textit{for all } t\in(0,T_{max}). \tag{3.36}$$

Proof. This results from a straightforward induction on the basis of Lemma 3.8, using (3.2) as a starting point. □

3.5 Higher regularity of u,c and n. Global existence

We shall next finalize our collection of a priori estimates necessary to conclude global existence of (n,c,u) on the basis of (2.1). As a preparation for our results in this direction, let us in advance state the following elementary observation.

Lemma 3.10 *Suppose that $a\in(0,1)$, and that $M>0, C_1>0$ and $C_2>0$ are constants satisfying*

$$M\le C_1+C_2M^a. \tag{3.37}$$

Then

$$M\le\max\Big\{2C_1\,,\,(2C_2)^{\frac{1}{1-a}}\Big\}. \tag{3.38}$$

Proof. If $C_2M^a\le\frac{M}{2}$, then (3.37) implies that $M\le C_1+\frac{M}{2}$ and hence $M\le 2C_1$. Otherwise, however, we have $\frac{M}{2}<C_2M^a$, which means that $M<(2C_2)^{\frac{1}{1-a}}$. □

Now the previouly gained estimates for u and n imply the following.

Lemma 3.11 *For all $\alpha\in(\frac12,1)$ one can find $C(\alpha)>0$ such that*

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}\le C(\alpha) \qquad \textit{for all } t\in(\tau,T_{max}), \tag{3.39}$$

where again $\tau=\min\{1,\frac16 T_{max}\}$. In particular, there exist $\theta\in(0,1)$ and $C>0$ such that

$$\|u(\cdot,t)\|_{C^\theta(\bar\Omega)}\le C \qquad \textit{for all } t\in(\tau,T_{max}). \tag{3.40}$$

Proof. We fix $\alpha \in (\frac{1}{2}, 1)$ and first note that then the domains of the corresponding fractional power of the Stokes operator satisfy $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega) \cap L^2_\sigma(\Omega) \equiv D(A) \hookrightarrow D(A^\alpha) \hookrightarrow C^\theta(\Omega)$ for any $\theta \in (0, 2\alpha - 1)$ ([11, p.201], [13, p.77]). In particular, the regularity properties of u imply that for each $t \in (\tau, T_{max})$, the number

$$M(T) := \sup_{t \in (\tau, T)} \|A^\alpha u(\cdot, t)\|_{L^2(\Omega)}$$

is finite, and in order to prove (3.39), and hence also (3.40), it is therefore sufficient to estimate $M(T)$ from above. For this purpose, given $t \in (\tau, T_{max})$ we let $t_0 := \max\{\tau, t-1\}$ and invoke the variation-of-constants representation of u and well-known regularization estimates for the Stokes semigroup ([11]) to find $C_1 > 0$ such that

$$\begin{aligned}
\|A^\alpha u(\cdot, t)\|_{L^2(\Omega)} \ &\leq \ \Big\| A^\alpha e^{-(t-t_0)A} u(\cdot, t_0) - \int_{t_0}^t A^\alpha e^{-(t-s)A} \mathcal{P}[(u(\cdot, s) \cdot \nabla) u(\cdot, s)] ds \\
&\quad + \int_{t_0}^t A^\alpha e^{-(t-s)A} \mathcal{P}[n(\cdot, s) \nabla \phi] ds \\
&\quad + \int_{t_0}^t A^\alpha e^{-(t-s)A} \mathcal{P}[g(\cdot, s)] ds \Big\|_{L^2(\Omega)} \\
&\leq \ \|A^\alpha e^{-(t-t_0)A} u(\cdot, t_0)\|_{L^2(\Omega)} \\
&\quad + C_1 \int_{t_0}^t (t-s)^{-\alpha} \Big\| (u(\cdot, s) \cdot \nabla) u(\cdot, s) \Big\|_{L^2(\Omega)} ds \\
&\quad + C_1 \int_{t_0}^t (t-s)^{-\alpha} \Big\{ \|n(\cdot, s) \nabla \phi\|_{L^2(\Omega)} + \|g(\cdot, s)\|_{L^2(\Omega)} \Big\} ds.
\end{aligned} \tag{3.41}$$

Here in the case $t_0 = \tau$ we estimate

$$\begin{aligned}
\|A^\alpha e^{-(t-t_0)A} u(\cdot, t_0)\|_{L^2(\Omega)} \ &= \ \|e^{-(t-t_0)A} A^\alpha u(\cdot, \tau)\|_{L^2(\Omega)} \\
&\leq \ C_2 := \|A^\alpha u(\cdot, \tau)\|_{L^2(\Omega)},
\end{aligned} \tag{3.42}$$

whereas if $t_0 > \tau$ then $t - t_0 = 1$ due to the definition of t_0 and hence for some $C_3 > 0$ and $C_4 > 0$ we have

$$\begin{aligned}
\|A^\alpha e^{-(t-t_0)A} u(\cdot, t_0)\|_{L^2(\Omega)} \ &\leq \ C_3 (t-t_0)^{-\alpha} \|u(\cdot, t_0)\|_{L^2(\Omega)} \\
&\leq \ C_3 \|u(\cdot, t_0)\|_{L^2(\Omega)} \\
&\leq \ C_4
\end{aligned} \tag{3.43}$$

according to Lemma 3.5.
Moreover, since $\nabla \phi$ and g are bounded, by means of Lemma 3.9 we see that there exists $C_5 > 0$ such that

$$\begin{aligned}
C_1 \int_{t_0}^t (t-s)^{-\alpha} \Big\{ \|n(\cdot, s) \nabla \phi\|_{L^2(\Omega)} + \|g(\cdot, s)\|_{L^2(\Omega)} \Big\} ds \ &\leq \ C_5 \int_{t_0}^t (t-s)^{-\alpha} ds \\
&\leq \ \frac{C_5}{1-\alpha},
\end{aligned} \tag{3.44}$$

because $\alpha < 1$ and $t - t_0 \le 1$.
As for the second last integral in (3.41), we first pick any $\beta \in (\frac{1}{2}, \alpha)$ and then use the continuity of the embedding $D(A^\beta) \hookrightarrow L^\infty(\Omega)$ ([11], [13]) to find $C_6 > 0$ such that

$$\begin{aligned} \|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)} &\le \|u(\cdot,s)\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)} \\ &\le C_6\|A^\beta u(\cdot,s)\|_{L^2(\Omega)}\|\nabla u(\cdot,s)\|_{L^2(\Omega)} \qquad \text{for all } s\in(0,T_{max}). \end{aligned}$$

By interpolation between $D(A^\alpha)$ and $D(A^{\frac{1}{2}})$ ([10]), recalling our definition of $M(T)$ and noting that $\|A^{\frac{1}{2}}\varphi\|_{L^2(\Omega)} = \|\nabla\varphi\|_{L^2(\Omega)}$ for all $\varphi \in D(A^{\frac{1}{2}})$ we thereupon obtain $C_7 > 0$ and $C_8 > 0$ fulfilling

$$\begin{aligned} \|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)} &\le C_7\|A^\alpha u(\cdot,s)\|^a_{L^2(\Omega)}\|\nabla u(\cdot,s)\|^{2-a}_{L^2(\Omega)} \\ &\le C_8 M^a(T) \qquad \text{for all } s\in(\tau,T) \end{aligned}$$

with $a := \frac{2\beta-1}{2\alpha-1} \in (0,1)$, for we know from lemma 3.6 that ∇u belongs to $L^\infty((0,T_{max});L^2(\Omega))$. Therefore, again since $\alpha < 1$ and $t - t_0 \le 1$,

$$C_1\int_{t_0}^t (t-s)^{-\alpha}\|(u(\cdot,s)\cdot\nabla)u(\cdot,s)\|_{L^2(\Omega)}ds \le \frac{C_1C_8}{1-\alpha}\cdot M^a(T),$$

so that from (3.41)-(3.44) we all in all infer that

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C_9 + C_{10}M^a(T) \qquad \text{for all } t\in(\tau,T)$$

if we let $C_9 := \max\{C_2, C_4\} + \frac{C_5}{1-\alpha}$ and $C_{10} := \frac{C_1C_8}{1-\alpha}$. As $t \in (\tau, T)$ was arbitrary, this implies that

$$M(T) \le C_9 + C_{10}M^a(T) \qquad \text{for all } T\in(\tau,T_{max})$$

and that hence, in view of Lemma 3.10,

$$M(T) \le \max\Big\{2C_9\,,\,(2C_{10})^{\frac{1}{1-a}}\Big\}.$$

Taking $T \nearrow T_{max}$ thus yields the claim. □

This in turn provides additional information in the transport term in the equation for c, and thereby entails a pointwise bound for ∇c.

Lemma 3.12 *There exists $C > 0$ such that*

$$\|c(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C \qquad \textit{for all } t\in(2\tau,T_{max}), \tag{3.45}$$

where $\tau = \min\{1, \frac{1}{6}T_{max}\}$ is as in (3.3).

Proof. We first apply Lemma 3.3 to find $C_1 > 0$ such that

$$\|c(\cdot,t)\|_{L^2(\Omega)} \le C_1 \qquad \text{for all } t\in(0,T_{max}). \tag{3.46}$$

In order to prove the lemma, it is thus sufficient to derive a bound, independent of $T \in (2\tau, T_{max})$, for the numbers

$$M(T) := \sup_{t\in(2\tau,T)} \|\nabla c(\cdot,t)\|_{L^\infty(\Omega)}.$$

To achieve this, we use well-known smoothing properties of the Neumann heat semigroup $(e^{t\Delta})_{t\geq 0}$ in Ω, as stated e.g. in [36, Lemma 1.3] in a version covering the present situation, to see that there exists $C_2 > 0$ such that

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} &= \Big\| \nabla e^{(t-\tau)(\Delta-1)} c(\cdot,\tau) + \int_\tau^t \nabla e^{(t-s)(\Delta-1)} n(\cdot,s)ds \\
&\quad + \int_\tau^t \nabla e^{(t-s)(\Delta-1)} u(\cdot,s)\cdot\nabla c(\cdot,s)ds \Big\|_{L^\infty(\Omega)} \\
&\leq C_2 \Big\{ 1 + (t-\tau)^{-1} \Big\} \cdot e^{-(t-\tau)} \|c(\cdot,\tau)\|_{L^2(\Omega)} \\
&\quad + C_2 \int_\tau^t \Big\{ 1 + (t-s)^{-\frac{3}{4}} \Big\} \cdot e^{-(t-s)} \|n(\cdot,s)\|_{L^4(\Omega)} ds \\
&\quad + C_2 \int_\tau^t \Big\{ 1 + (t-s)^{-\frac{3}{4}} \Big\} \cdot e^{-(t-s)} \|u(\cdot,s)\cdot\nabla c(\cdot,s)\|_{L^4(\Omega)} ds \\
&\qquad\qquad \text{for all } t \in (2\tau, T_{max}).
\end{aligned} \tag{3.47}$$

Here as a particular consequence of Lemma 3.11 and the Gagliardo-Nirenberg inequality as well as (3.46), we obtain $C_3 > 0$ and $C_4 > 0$ such that

$$\begin{aligned}
\|u(\cdot,s)\nabla c(\cdot s)\|_{L^4(\Omega)} &\leq \|u(\cdot,s)\|_{L^\infty(\Omega)} \|\nabla c(\cdot,s)\|_{L^4(\Omega)} \\
&\leq C_3 \|\nabla c(\cdot,s)\|_{L^4(\Omega)} \\
&\leq C_4 \|\nabla c(\cdot,s)\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|c(\cdot,s)\|_{L^2(\Omega)}^{\frac{1}{2}} \\
&\leq C_1^{\frac{1}{2}} C_4 M^{\frac{1}{2}}(T) \qquad \text{for all } s \in (2\tau, T),
\end{aligned}$$

so that by Lemma 3.9 applied to $p := 4$ and, again, (3.46), we infer from (3.47) that with some $C_5 > 0$ we have

$$\begin{aligned}
\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} &\leq C_2 \cdot \{1 + \tau^{-1}\} \cdot C_1 \\
&\quad + C_5 \int_\tau^t \{1 + (t-s)^{-\frac{3}{4}}\} \cdot e^{-(t-s)} ds \\
&\quad + C_1^{\frac{1}{2}} C_2 C_4 M^{\frac{1}{2}}(T) \cdot \int_\tau^t \{1 + (t-s)^{-\frac{3}{4}}\} \cdot e^{-(t-s)} ds \qquad \text{for all } t \in (2\tau, T).
\end{aligned}$$

Since

$$\int_\tau^t \{1 + (t-s)^{-\frac{3}{4}}\} \cdot e^{-(t-s)} ds \leq \int_0^\infty \{1 - \sigma^{-\frac{3}{4}}\} \cdot e^{-\sigma} d\sigma \qquad \text{for all } t > \tau,$$

this implies the existence of $C_6 > 0$ and $C_7 > 0$ fulfilling

$$\|\nabla c(\cdot,t)\|_{L^\infty(\Omega)} \le C_6 + C_7 M^{\frac{1}{2}}(T) \qquad \text{for all } t \in (2\tau, T),$$

so that

$$M(T) \le C_6 + C_7 M^{\frac{1}{2}}(T) \qquad \text{for all } T \in (2\tau, T_{max})$$

and hence

$$M(T) \le \max\Big\{ 2C_6\,,\,(2C_7)^2 \Big\} \qquad \text{for all } T \in (2\tau, T_{max})$$

according to Lemma 3.10. As $T \in (2\tau, T_{max})$ was arbitrary, this completes the proof. $\Box$

It becomes thereupon possible to view the first equation in (1.1) as a forced linear heat equation in which the inhomogeneity is regular enough so as to allow for the following conclusion.

Lemma 3.13 *There exist $\lambda \in (0,1)$ and $C > 0$ such that with $\tau = \min\{1, \frac{1}{6}T_{max}\}$ as in (3.3) we have*

$$\|n(\cdot,t)\|_{C^\lambda(\bar\Omega)} \le C \qquad \textit{for all } t \in (3\tau, T_{max}). \tag{3.48}$$

Proof. We fix an arbitrary $\lambda \in (0,1)$ and can then pick $\beta \in (\frac{\lambda}{2}, \frac{1}{2})$ and thereafter $p > 1$ large enough such that $p > \frac{2}{2\beta - \lambda}$. Then $2\beta - \frac{2}{p} > \lambda$, so that if we let B denote the sectorial realization of $-\Delta + 1$ under homogeneous Neumann boundary conditions in $L^p(\Omega)$, then the domain of its fractional power B^β satisfies $D(B^\beta) \hookrightarrow C^\lambda(\bar\Omega)$ ([13]). Thus, if given $t \in (3\tau, T_{max})$ we let $t_0 := \max\{t-1, 2\tau\}$, then by means of the variation-of-constants representation of n we can estimate

$$\begin{aligned}
\|n(\cdot,t)\|_{C^\lambda(\bar\Omega)} &\le C_1 \|B^\beta n(\cdot,t)\|_{L^p(\Omega)} \\
&\le C_1 \|B^\beta e^{(t-t_0)\Delta} n(\cdot,t_0)\|_{L^p(\Omega)} \\
&\quad + C_1 \int_{t_0}^t \Big\| B^\beta e^{(t-s)\Delta} \nabla \cdot \Big(n(\cdot,s)\nabla c(\cdot,s) + n(\cdot,s) u(\cdot,s) \Big) \Big\|_{L^p(\Omega)} ds \\
&\quad + C_1 \int_{t_0}^t \Big\| B^\beta e^{(t-s)\Delta} \Big\{ r n(\cdot,s) - \mu n^2(\cdot,s) \Big\} \Big\|_{L^p(\Omega)} ds
\end{aligned} \tag{3.49}$$

with some $C_1 > 0$, because $\nabla \cdot u = 0$. Here since $1 \ge t - t_0 \ge \tilde\tau := \min\{1,\tau\}$, invoking standard regularization properties of the analytic semigroup $(e^{-\sigma B})_{\sigma \ge 0}$ ([10]) and Lemma 3.9 provides $C_2 > 0$ and $C_3 > 0$ such that

$$\begin{aligned}
\|B^\beta e^{(t-t_0)\Delta} n(\cdot,t_0)\|_{L^p(\Omega)} &= e^{t-t_0} \|B^\beta e^{-(t-t_0)B} n(\cdot,t_0)\|_{L^p(\Omega)} \\
&\le C_2\, e^{t-t_0} (t-t_0)^{-\beta} \|n(\cdot,t_0)\|_{L^p(\Omega)} \\
&\le C_2 e \tilde\tau^{-\beta} \|n(\cdot,t_0)\|_{L^p(\Omega)} \\
&\le C_3.
\end{aligned} \tag{3.50}$$

We next use that e.g. according to [36, Lemma 1.3] there exists $C_4 > 0$ with the property that for each $\varphi \in C^1(\bar\Omega);\mathbb{R}^2)$ such that $\varphi\cdot\nu = 0$ on $\partial\Omega$ we have

$$\|e^{\sigma\Delta}\nabla\cdot\varphi\|_{L^p(\Omega)} \le C_4\sigma^{-\frac12}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \sigma\in(0,1),$$

to estimate

$$\begin{aligned}
&\int_{t_0}^t \Big\|B^\beta e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)\nabla c(\cdot,s)+n(\cdot,s)u(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds && (3.51)\\
&\quad= \int_{t_0}^t e^{\frac{t-s}{2}}\Big\{\Big\|B^\beta e^{-\frac{t-s}{2}B}e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(n(\cdot,s)\nabla c(\cdot,s)+n(\cdot,s)u(\cdot,s)\Big)\Big\|_{L^p(\Omega)}\Big\}ds\\
&\quad\le C_5\int_{t_0}^t e^{\frac{t-s}{2}}\cdot\Big(\frac{t-s}{2}\Big)^{-\beta}\Big\|e^{\frac{t-s}{2}\Delta}\nabla\cdot\Big(n(\cdot,s)\nabla c(\cdot,s)+n(\cdot,s)u(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds\\
&\quad\le C_4C_5\int_{t_0}^t e^{\frac{t-s}{2}}\cdot\Big(\frac{t-s}{2}\Big)^{-\beta-\frac12}\Big\{\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^p(\Omega)}+\|n(\cdot,s)u(\cdot,s)\Big\|_{L^p(\Omega)}\Big\}ds && (3.52)
\end{aligned}$$

with appropriately large $C_5 > 0$. As Lemma 3.9, Lemma 3.12 and Lemma 3.11 yield $C_6 > 0$ and $C_7 > 0$ such that

$$\|n(\cdot,s)\nabla c(\cdot,s)\|_{L^p(\Omega)} \le \|n(\cdot,s)\|_{L^p(\Omega)}\|\nabla c(\cdot,s)\|_{L^\infty(\Omega)} \le C_6 \qquad \text{for all } s\in(2\tau,T_{max})$$

and

$$\|n(\cdot,s)u(\cdot,s)\|_{L^p(\Omega)} \le \|n(\cdot,s)\|_{L^p(\Omega)}\|u(\cdot,s)\|_{L^\infty(\Omega)} \le C_7 \qquad \text{for all } s\in(\tau,T_{max}),$$

from (3.51) we infer that

$$\begin{aligned}
&\int_{t_0}^t \Big\|B^\beta e^{(t-s)\Delta}\nabla\cdot\Big(n(\cdot,s)\nabla c(\cdot,s)+n(\cdot,s)u(\cdot,s)\Big)\Big\|_{L^p(\Omega)}ds\\
&\qquad\le C_4C_5(C_6+C_7)\cdot e^{\frac12}\cdot 2^{\beta+\frac12}\cdot\int_{t_0}^t (t-s)^{-\beta-\frac12}ds\\
&\qquad\le C_4C_5(C_6+C_7)\cdot e^{\frac12}\cdot 2^{\beta+\frac12}\cdot\frac{1}{\frac12-\beta}, && (3.53)
\end{aligned}$$

because $\beta<\frac12$, and again because $t-t_0\le 1$.

Proceeding similarly, upon one further application of Lemma 3.9 we obtain positive constants C_8, C_9 and C_{10} such that once more since $t-t_0\le1$ we have

$$\begin{aligned}
&\int_{t_0}^t \Big\|B^\beta e^{(t-s)\Delta}\Big\{rn(\cdot,s)-\mu n^2(\cdot,s)\Big\}\Big\|_{L^p(\Omega)}ds\\
&\quad\le C_8\int_{t_0}^t (t-s)^{-\beta}\Big\|rn(\cdot,s)-\mu n^2(\cdot,s)\Big\|_{L^p(\Omega)}ds\\
&\quad\le C_9\int_{t_0}^t (t-s)^{-\beta}\cdot\Big\{\|n(\cdot,s)\|_{L^p(\Omega)}+\|n(\cdot,s)\|^2_{L^{2p}(\Omega))}\Big\}ds\\
&\quad\le C_{10}\int_{t_0}^t (t-s)^{-\beta}ds\\
&\quad\le \frac{C_{10}}{1-\beta}.
\end{aligned}$$

Together with (3.50) and (3.53) inserted into (3.49), this establishes (3.48). $\square$

Collecting the previous three lemmata directly leads to our main result on global existence and boundedness.

Proof of Theorem 1.1. In view of the extensibility criterion (2.1) in Lemma 2.1, as an immediate consequence of Lemma 3.13, Lemma 3.12 and Lemma 3.11 we firstly obtain that $T_{max} = \infty$. Since $D(A^\alpha) \hookrightarrow L^\infty(\Omega)$ for any $\alpha > \frac{1}{2}$ ([13], [11]), the bounds asserted therein, secondly, in particular prove the claimed boundedness properties. $\square$

4 Decay

4.1 Decay of n and c in the case $r = 0$. Proof of Theorem 1.2

The following variant of Lemma 3.1 forms the basis of our approach toward proving the decay properties in Theorem 1.2. The proof of Lemma 4.1 can be obtained by straightforward adaptation of the reasoning in Lemma 3.1, so that we may omit repeating the arguments here.

Lemma 4.1 *Let $r = 0$. Then the solution of (1.1)-(1.3) satisfies*

$$\int_0^t \int_\Omega n^2 \le \frac{1}{\mu} \int_\Omega n_0 \qquad \text{for all } t \in (0, T_{max}) \tag{4.1}$$

and

$$\int_\Omega n(\cdot, t) \le \left\{ \frac{1}{\int_\Omega n_0} + \frac{t}{|\Omega|} \right\}^{-1} \qquad \text{for all } t \in (0, T_{max}). \tag{4.2}$$

As a consequence, we obtain a basic decay property also for the second solution component.

Lemma 4.2 *There exists $C > 0$ such that*

$$\int_\Omega c(\cdot, t) \le \frac{C}{t+1} \qquad \text{for all } t \in (0, T_{max}). \tag{4.3}$$

Proof. Integrating the second equation in (1.1) we see that there exists $C_1 > 0$ such that $y(t) := \int_\Omega c(\cdot, t)$, $t \in [0, T_{max})$, satisfies

$$\begin{aligned} y'(t) &= -y(t) + \int_\Omega n(\cdot, t) \\ &\le -y(t) + \frac{C_1}{t+1} \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

We now let

$$C_2 := \max \left\{ 2 \int_\Omega c_0 \, , \, 4C_1 \right\}$$

and define

$$\overline{y}(t) := \frac{C_2}{t+2} \qquad \text{for } t \ge 0.$$

Then $\overline{y}(0) = \frac{C_2}{2} \ge \int_\Omega c_0 = y(0)$ and

$$\begin{aligned}
\overline{y}'(t) + \overline{y}(t) - \frac{C_1}{t+1} &= -\frac{C_2}{(t+2)^2} + \frac{C_2}{t+2} - \frac{C_1}{t+1} \\
&= \frac{C_2}{t+2} \cdot \Big\{ \frac{1}{2} - \frac{1}{t+2} \Big\} + \frac{1}{2(t+2)} \cdot \Big\{ C_2 - 2C_1 \cdot \frac{t+2}{t+1} \Big\} \\
&\ge \frac{C_2}{t+2} \cdot \Big\{ \frac{1}{2} - \frac{1}{2} \Big\} + \frac{1}{2(t+2)} \cdot \{ C_2 - 4C_1 \} \\
&\ge 0 \qquad \text{for all } t > 0.
\end{aligned}$$

By comparison, we thus infer that $y(t) \le \overline{y}(t)$ for all $t \in (0, T_{max})$, which directly establishes (4.3). $\Box$

Now thanks to the precompactness of both $(n(\cdot,t))_{t>1}$ and $(c(\cdot,t))_{t>1}$ implied by Lemma 3.13 and Lemma 3.12, the latter two decay properties readily entail the first of our main results on large time behavior, addressing the case when $r = 0$ but yet g is widely arbitrary.

Proof of Theorem 1.2. Supposing that (1.9) be false, we could find $C_1 > 0$ and $(t_j)_{j\in\mathbb{N}} \subset (3,\infty)$ such that $t_j \to \infty$ as $j \to \infty$ and

$$\|n(\cdot,t_j)\|_{L^\infty(\Omega)} \ge C_1 \qquad \text{for all } j \in \mathbb{N}. \tag{4.4}$$

Since $(n(\cdot,t_j))_{j\in\mathbb{N}}$ is relatively compact in $C^0(\bar{\Omega})$ according to Lemma 3.13 and the Arzelà-Ascoli theorem, on extracting a subsequence we may assume that

$$n(\cdot,t_j) \to n_\infty \qquad \text{in } L^\infty(\Omega) \qquad \text{as } j \to \infty$$

with some nonnegative $n_\infty \in C^0(\bar{\Omega})$. But from Lemma 3.1 we already know that $n(\cdot,t) \to 0$ in $L^1(\Omega)$ as $t \to \infty$, whence necessarily $n_\infty \equiv 0$, which is evidently incompatible with (4.4) and thereby proves (1.9).

In quite a similar manner, (1.10) results from combining Lemma 4.2 with Lemma 3.12. $\Box$

4.2 Decay of u when $r = 0$ and $\int_0^\infty \int_\Omega |g|^2 < \infty$. Proof of Theorem 1.3

In the case when r vanishes and g belongs to $L^2(\Omega \times (0,\infty))$, Lemma 4.1 asserts that the time-dependent function appearing on the right-hand side of the differential inequality for $\int_\Omega |u|^2$ in Lemma 3.4 is integrable over $(0,\infty)$. In exploiting this in Lemma 4.4 we will refer to the following result from elementary calculus.

Lemma 4.3 *Let $\kappa > 0$ and $h \in L^1((0,\infty))$ be nonnegative, and suppose that $y \in C^0([0,\infty)) \cap C^1((0,\infty))$ is nonnegative and such that*

$$y'(t) + \kappa y(t) \le h(t) \qquad \textit{for all } t > 0. \tag{4.5}$$

Then

$$y(t) \to 0 \qquad \textit{as } t \to \infty. \tag{4.6}$$

Proof. By an ODE comparison argument, y satisfies

$$y(t) \le e^{-\kappa t} y(0) + \int_0^t e^{-\kappa(t-s)} h(s) ds \qquad \text{for all } t > 0, \tag{4.7}$$

where clearly $e^{-\kappa t} y(0) \to 0$ as $t \to \infty$. As furthermore

$$\begin{aligned} \int_0^t e^{-\kappa(t-s)} h(s) ds &= \int_0^{\frac{t}{2}} e^{-\kappa(t-s)} h(s) ds + \int_{\frac{t}{2}}^t e^{-\kappa(t-s)} h(s) ds \\ &\le e^{-\frac{\kappa t}{2}} \int_0^\infty h(s) ds + \int_{\frac{t}{2}}^\infty h(s) ds \qquad \text{for all } t > 0, \end{aligned}$$

the inclusion $h \in L^1((0,\infty))$ warrants that (4.7) implies (4.6). $\Box$

We can thereby go beyond Lemma 3.5 and derive a basic decay property of u.

Lemma 4.4 *Suppose that $r = 0$ and $\int_0^\infty \int_\Omega |g|^2 < \infty$. Then*

$$\int_\Omega |u(\cdot,t)|^2 \to 0 \qquad \textit{as } t \to \infty. \tag{4.8}$$

Proof. From Lemma 3.4 we know that there exist $C_1 > 0$ and $C_2 > 0$ such that $y(t) := \int_\Omega |u(\cdot,t)|^2$, $t \ge 0$, satisfies

$$y'(t) + C_1 y(t) \le h(t) := C_2 \cdot \left\{ \int_\Omega n^2(\cdot,t) + \int_\Omega |g(\cdot,t)|^2 \right\} \qquad \text{for all } t > 0.$$

As

$$\int_0^\infty h(t) dt \le C_2 \cdot \left\{ \frac{1}{\mu} \int_\Omega n_0 + \int_0^\infty \int_\Omega |g|^2 \right\} < \infty$$

according to Lemma 4.1 and our assumption on g, an application of Lemma 4.3 thus yields (4.8). $\Box$

As before combining this decay information with our previous higher-order estimates, without substantial further efforts we arrive at our final convergence result.

Proof of Theorem 1.3. We can either repeat the argument from the proof of Theorem 1.2, or alternatively proceed as follows.
Taking $\alpha \in (\frac{1}{2}, 1)$ and $\beta \in (\frac{1}{2}, \alpha)$, from Lemma 3.11 we obtain $C_1 > 0$ such that

$$\|A^\alpha u(\cdot,t)\|_{L^2(\Omega)} \le C_1 \qquad \text{for all } t > 0,$$

so that upon interpolation we infer that with some $C_2 > 0$ we have

$$\begin{aligned} \|A^\beta u(\cdot,t)\|_{L^2(\Omega)} &\le C_2 \|A^\alpha u(\cdot,t)\|_{L^2(\Omega)}^{\frac{\beta}{\alpha}} \|u(\cdot,t)\|_{L^2(\Omega)}^{\frac{\beta-\alpha}{\alpha}} \\ &\le C_1^{\frac{\beta}{\alpha}} C_2 \|u(\cdot,t)\|_{L^2(\Omega)}^{\frac{\beta-\alpha}{\alpha}} \qquad \text{for all } t > 0. \end{aligned}$$

Since $D(A^\beta) \hookrightarrow L^\infty(\Omega)$, the claim is therefore a consequence of Lemma 4.4. $\Box$

Acknowledgments. The author is supported by the National Natural Science Foundation of China (No. 11171061).

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