The two-dimensional Keller-Segel system with singular sensitivity and signal absorption: Global large-data solutions and their relaxation properties

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Abstract

We consider the chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot \left(\frac{u}{v}\nabla v\right), \\ v_t = \Delta v - uv, \end{cases}$$

as originally introduced in 1971 by Keller and Segel in the second of their seminal works. This system constitutes a prototypical model for taxis-driven pattern formation and front propagation in various biological context such as tumor angiogenesis, but in the higher-dimensional context any global existence theory for large-data solutions is yet lacking.

In the present work it is shown that in bounded planar domains Ω with smooth boundary, for all reasonably regular initial data $u_0 \geq 0$ and $v_0 > 0$, the corresponding Neumann initial-boundary value problem possesses a global generalized solution. Thus particularly addressing arbitrarily large initial data, this goes beyond previously gained results asserting global existence of solutions only in spatial one-dimensional problems, or under certain smallness conditions on the initial data. The derivation of this result is based on a priori estimates for the quantities $\nabla \ln(u+1)$ and ∇v in spatio-temporal L^2 spaces, where further boundedness and compactness properties are derived from the former by relying on the planar spatial setting in using an associated Moser-Trudinger inequality.

Furthermore, some further boundedness and relaxation propeties are derived, inter alia indicating that for any such solution we have $v(\cdot,t) \to 0$ in $L^p(\Omega)$ as $t \to \infty$ for all finite $p > 1$, and that in an appropriate generalized sense the quantities u and $\nabla \ln v$ eventually enter bounded sets in $L^p(\Omega)$ and $L^2(\Omega)$, respectively, with diameters only determined by the total population size $\int_\Omega u_0$.

Finally, some numerical experiments illustrate the analytically obtained results.

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1 Introduction

The Keller-Segel system with singular sensitivity and signal consumption. The theoretical understanding of taxis-driven spontaneous emergence of spatial structures in biological systems has attracted great interest in the past decades. In their seminal works ([19], [20]), Keller and Segel achieved a breakthrough in the mathematical modeling by proposing two-component cross-diffusive parabolic systems for the description of such processes in two types of situations: The first of these is characterized by the ability of individuals to actively secrete a chemoattractive signal, as present in numerous types of cell populations, with the prototypical case of starving *Dictyostelium discoideum* colonies forming the probably most prevalent representative. The correspondingly obtained PDE model, widely referred to as the classical Keller-Segel system, has been studied intensely in the mathematical literature, and thereby substantial knowledge could be achieved, especially with regard to the occurrence of blow-up as a mathematical counterpart of aggregation processes revealed by experiments (cf. the surveys [15], [14] and [2], for instance).

Contrary to this, in the second biological framework addressed by Keller and Segel the considered individual cells are yet much more primitive in that they merely follow a chemical cue which they are unable to produce, but which they rather consume as a nutrient, and which they thus nevertheless influence in their concentration. As indicated by striking experimental evidence, even such simple situations may support the onset of structures in that sharp fronts may spontaneously develop from originally almost homogeneously distributed populations, e.g. of *Escherichia coli* ([1]). In order to capture such phenomena theoretically, Keller and Segel proposed the system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{1.1}$$

with $\chi > 0$, for the unknown cell density $u = u(x,t)$ and signal concentration $v = v(x,t)$, where the second equation models consumption of the signal upon contact with cells, and where in the first equation it is assumed that the chemotactic stimulus is perceived in accordance with the Weber-Fechner law, thus requiring the chemotactic sensitivity $S(u,v) := \chi \frac{u}{v}$ to be chosen proportional to the reciprocal signal density. Indeed, the ability of this particular type of absorption-taxis interplay to generate wave-like solution behavior, as discussed already in [20], formed a motivation for several analytical studies on the existence and stability properties of traveling wave solutions to (1.1) ([17], [26], [27]), and also to some closely related systems ([35], [30], [32]). In fact, the biological importance of singular chemotactic sensitivities as in (1.1), apparently of substantial mathematical relevance to the occurrence of such wave-like behavior, has been further underlined independently in more thorough modeling approaches ([33], [23], [18], cf. also the survey [14]), also in more complex biological frameworks such as especially in tumor angiogenesis ([36], [24]), but also e.g. in taxis-driven morphogen transport ([5]).

The challenges of proving global existence in (1.1). Beyond the detection of solutions with particular wave-like structures, however, only little seems known with regard to a rigorous mathematical theory of global existence and qualitative behavior of solutions to (1.1) emanating from general initial data. This may be viewed as reflecting the circumstance that on the one hand, it should be expected that the absorption mechanism in the second equation in (1.1) will force the solution component v to attain small values in a significant part of an associated space-time region, whereas on the other hand the destabilizing cross-diffusive action expressed in the first equation in (1.1) is significantly enhanced precisely at points where this concentration v is small. Accordingly, up to now a comprehensive existence theory apparently is available only in the spatially one-dimensional case where results on global existence for appropriate initial-boundary value problems for (1.1) with widely arbitrary initial data have been derived in [39] and [28]. As for higher-dimensional settings, the only result we are aware of has recently been obtained in [42], where the Cauchy problem for (1.1) in $\mathbb{R}^n$ has been addressed for $n \in \{2,3\}$, and where under appropriate smallness conditions on the initial data the existence of globally defined classical solutions has been established. Even in the simplified variant of (1.1) obtained on replacing the second equation therein by the ODE $v_t = -uv$, global existence results seem restricted to one-dimensional cases so far, whereas corresponding statements in higher-dimensional situations have been derived only under suitable smallness conditions on the data ([48], [25]).

In contrast to this, in the related chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), \\ v_t = \Delta v - v + u \end{cases} \tag{1.2}$$

the signal production mechanism expressed in the second equation is known to inhibit an evolution of v toward small values in such an effective manner that global existence of bounded smooth solutions can be achieved for reasonably smooth but arbitrarily large data in initial-boundary value problems in bounded n-dimensional domains under the assumption that $\chi < \sqrt{\frac{2}{n}}$ ([3], [44]), with a slight relaxation of this condition recently achieved when $n = 2$ ([22]). Within larger ranges of χ, at least weak solutions exist globally ([44]), where in the two-dimensional radially symmetric case certain generalized solutions can be constructed actually without any restriction on the size of χ ([37]).

On the other hand, the signal absorption mechanism of the type considered in (1.1), inter alia through its evident consequence on boundedness of v throughout evolution, is known to have a significant smoothing effect in related chemotaxis systems with regular sensitivity functions. For instance, the Neumann initial-boundary value problem for

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), \\ v_t = \Delta v - uv, \end{cases} \tag{1.3}$$

when posed in bounded domains in $\mathbb{R}^n$ with arbitrarily large initial data (u_0, v_0), possesses global bounded classical solutions in the case $n = 2$ and global weak solutions if $n = 3$ ([40]), whereas if $n \geq 3$ then global bounded classical solutions can be found whenever $\|v_0\|_{L^\infty(\Omega)}$ is suitably small ([38]). In comparison with known results on the occurrence of exploding solutions in the classical Keller-Segel system containing $v_t = \Delta v - v + u$ as its second equation ([13], [45], [31]), this indeed underlines that signal absorption in fact may entirely suppress blow-up, at least in presence of non-singular chemotactic sensitivities.

Main results. It is the goal of the present paper to address the questions of global solvability and large time behavior in the two-dimensional version of (1.1) for arbitrarily large initial data in an appropriate framework. In order to accomplish this, it will be necessary to develop an approach which is entirely different from those used in the precedent literature: In fact, the derivation of the small-data global existence results in e.g. the works [42], [48] and [25] exclusively follow the intuitively nearby strategy to show that if u and $\nabla \ln v$ are appropriately small at the initial time, then the smoothing effect of diffusion will overbalance the destabilizing action of cross-diffusion throughout evolution, because in (1.1) the latter can then be viewed as an essentially quadratic and hence small deviation of a linear parabolic problem. Similar perturbation arguments have been used in detecting global existence and essentially diffusive behavior of small-data solutions in various related chemotaxis systems ([38], [6], [43]), but as evident from complementary results on the occurrence of explosions e.g. in the classical Keller-Segel system, such a reasoning may blend out possibly singular behavior of large-data solutions, and thereby be insufficient to unveil the global dynamics.

As described in more detail below, contrary to these previous approaches our analysis will hence be based on a priori information on regularity properties of solutions which rely on very mild assumptions on the initial constellation only, and particularly allow for large data. To make this more precise, let us consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (\frac{u}{v} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - uv, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$, where for convenience in presentation the initial data in (1.4) are supposed to satisfy

$$\begin{cases} u_0 \in C^0(\bar{\Omega}) \quad \text{with } u_0 \ge 0 \text{ in } \Omega \quad \text{and} \quad u_0 \not\equiv 0 \qquad \text{as well as} \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{with } v_0 > 0 \text{ in } \bar{\Omega}. \end{cases} \tag{1.5}$$

Under these assumptions, we shall firstly see that actually no further requirements on the size of the data are necessary for the construction of certain globally defined generalized solutions:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Then for all u_0 and v_0 satisfying (1.5), the problem (1.4) possesses at least one global generalized solution in the sense of Definition 2.1 below.*

According to the respective requirements (2.1), (2.2 and (2.3) made in Definition 2.1, the regularity properties of such generalized solutions may be rather poor. We therefore believe that within the concept pursued here, solutions will in general not be unique; however, an in-depth analysis of this mathematically delicate issue would significantly go beyond the scope of the present work, so that we have to leave a detailed discussion of the uniqueness question as an interesting topic for future research.

After all, all of the solutions obtained in Theorem 1.1 enjoy further boundedness and relaxation properties:

Theorem 1.2 *Suppose that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary. Then given any $m > 0$ and $p > 1$ one can find $K_1(m,p) > 0$ and $K_2(m) > 0$ with the property that for each $M > 0$ there exist $T_1(m,M) > 0$ and $T_2(M) > 0$ such that whenever u_0 and v_0 satisfy (1.5) with*

$$\int_\Omega u_0 \le m \qquad \textit{and} \qquad -\int_\Omega \ln\Big(\frac{v_0}{\|v_0\|_{L^\infty(\Omega)}}\Big) \le M, \tag{1.6}$$

then the global generalized solution (u,v) of (1.4) from Theorem 1.1 satisfies

$$\frac{1}{T}\int_0^T \ln\Big\{\frac{1}{|\Omega|}\int_\Omega \Big(u(x,s)+1\Big)^p dx\Big\}dt \le K_1(m,p) \qquad \textit{for all } T\ge T_1(m,M) \tag{1.7}$$

and

$$\frac{1}{T}\int_0^T\int_\Omega \frac{|\nabla v|^2}{v^2} \le K_2(m) \qquad \textit{for all } T\ge T_2(M). \tag{1.8}$$

In particular, (1.7) firstly implies that given any finite $p>1$, the function $u(\cdot,t)$ belongs to $L^p(\Omega)$ for a.e. $t>0$. But beyond this, from (1.7) it also follows that in the generalized sense of temporal averages appearing therein, for any $p>1$ the first component u of each individual solution will eventually enter a bounded set in $L^p(\Omega)$, the diameter of which depends only on the size of the total mass $\int_\Omega u_0$ of cells as a biologically relevant quantity. Likewise, (1.8) may be interpreted as reflecting a similar property of the signal gradient $\nabla \ln v$ in (1.4), expressing eventual absorption of the latter in a ball in $L^2(\Omega)$, with radius again determined by $\int_\Omega u_0$ only.

Finally, the solution component v always approaches the steady-state limit zero asymptotically, regardless of the size of the initial data:

Theorem 1.3 *Assume that $\Omega\subset\mathbb{R}^2$ is a bounded domain with smooth boundary, and that (1.5) holds. Then the global generalized solution (u,v) of (1.4) from Theorem 1.1 has the additional properties that*

$$v\in L^\infty(\Omega\times(0,\infty)) \qquad \textit{as well as} \qquad v\in C^0_{w\star}([0,\infty);L^\infty(\Omega)), \tag{1.9}$$

where the latter is to be understood in the sense that possibly after redefinition on a null set of times, v is continuous on $[0,\infty)$ as an $L^\infty(\Omega)$-valued function with respect to the weak-$\star$ topology.
Moreover,

$$v(\cdot,t)\overset{\star}{\rightharpoonup} 0 \qquad \textit{in } L^\infty(\Omega) \qquad \textit{as } t\to\infty \tag{1.10}$$

and

$$v(\cdot,t)\to 0 \qquad \textit{in } L^p(\Omega) \qquad \textit{as } t\to\infty \tag{1.11}$$

for any $p\in[1,\infty)$.

Main ideas. As compared to previous studies on (1.1), the main novelty in our approach appears to consist in the circumstance that at its core it resorts to the apparently only evident global quasi-dissipative structure inherent to (1.4), as given by the identity

$$\frac{d}{dt}\int_\Omega\Big(-\ln v\Big)+\int_\Omega\frac{|\nabla v|^2}{v^2}=m:=\int_\Omega u_0, \tag{1.12}$$

which can formally be obtained on testing the second equation in (1.4) by $\frac{1}{v}$, and using the mass conservation property $\frac{d}{dt}\int_\Omega u=0$. In fact, a corresponding variant thereof that can rigorously be derived at the level of solutions $(u_\varepsilon,v_\varepsilon)$ to suitably regularized problems (see (2.10)) can be used to establish the inequality

$$\int_0^t\int_\Omega\frac{|\nabla u|^2}{(u+1)^2}\le -\int_\Omega \ln\Big(\frac{v_0}{\|v_0\|_{L^\infty(\Omega)}}\Big)+2m+mt, \tag{1.13}$$

by using $\frac{1}{u+1}$ as a test function in the first equation of (1.4) (Lemma 2.3 and Lemma 2.4). Together with suitable time regularity properties thereby implied (Lemma 2.5), this will allow for passing to limits through an appropriate sequence $\varepsilon = \varepsilon_k \searrow 0$, and for the identification of corresponding limit functions u and v as potential candidates for a generalized solution (Lemma 2.6). The verification of the fact that this limit indeed satisfies (1.4) in the claimed generalized sense proceeds in several steps: Firstly, by means of the Moser-Trudinger inequality it can be shown that the estimate associated with (1.13) entails a bound for the corresponding quantity appearing on the left of (1.7) (Lemma 2.7), which in conjunction with the Vitali convergence theorem can be seen to imply strong convergence of the respective first solution components in $L^1(\Omega \times (0,T))$ for arbitrary $T > 0$ (Lemma 2.8). This will enable us to conclude in Lemma 2.9 that indeed the initial-boundary value problem for the second equation in (1.4) is satisfied by (u, v) in the natural weak sense specified in Definition 2.1. Thereupon, an appropriate testing procedure applied to this weak identity shows that the accordingly defined function w satisfies (1.12) at least in the sense of an inequality, with "=" replaced by "$\geq$", for almost every $t > 0$ (Lemma 2.10). Hence implying strong convergence of the corresponding sequence $(\nabla w_{\varepsilon_k})_{k\in\mathbb{N}}$ in $L^2(\Omega \times (0,T))$ for all $T > 0$ (Corollary 2.11), this will enable us to show that also the first sub-problem in (1.4) is satisfied in a suitably generalized sense, and thus to complete the proof of Theorem 1.1.

Mainly based on the estimates collected in the course of our analysis, the boundedness and regularity properties from Theorem 1.2 and Theorem 1.3 will thereafter be derived in Section 3 and Section 4, respectively. Finally, Section 5 contains some numerical simulations illustrating the results previously gained, as well as possible types of solution behavior.

As can already be conjectured from the above outline, by relying on very basic structural properties of the system our strategy of constructing solutions enjoys a certain robustness with respect to variations in the model. In particular, the main steps of our analysis readily extend to more realistic models derived from (1.4) on accounting for couplings to further components and mechanisms such as haptotactic interactions, for instance. Rather than providing a corresponding result for the greatest possible class of systems, however, for reasons of clarity in presentation the present work aims at concretizing our apparently new method of analyzing the interaction of singular chemotaxis with signal absorption to the prototypical model (1.4).

2 Construction of global generalized solutions

2.1 A generalized solution concept and a family of approximate solutions

Let us first specify the particular notion of solution that will be pursued in the sequel, guided by the motivation to employ a concept which is weak enough so as to be compatible with the rather weak a priori information on solution regularity to be gained below. To this end, we will follow an approach partially addressing suitably transformed versions of the functions in question, which may be viewed as a far relative of the well-known concept of renormalized solutions ([8]), and which resembles concepts which have previously been used in the context of certain types of chemotaxis problems ([46], [47]).

Definition 2.1 *Assume that u_0 and v_0 satisfy (1.5). Then a pair (u, v) of functions*

$$\begin{cases} u \in L^1_{loc}(\bar\Omega \times [0,\infty)), \\ v \in L^\infty_{loc}(\bar\Omega \times [0,\infty)) \cap L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \end{cases} \tag{2.1}$$

with
$$u \geq 0 \quad a.e.\ in\ \Omega \times (0,\infty) \qquad and \qquad v > 0 \quad a.e.\ in\ \Omega \times (0,\infty) \tag{2.2}$$
as well as
$$\nabla \ln(u+1) \in L^2_{loc}(\bar{\Omega} \times [0,\infty)) \qquad and \qquad \nabla \ln v \in L^2_{loc}(\bar{\Omega} \times [0,\infty)), \tag{2.3}$$
will be called a global generalized solution *of (1.4) if u has the mass conservation property*
$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x) \qquad for\ a.e.\ t > 0, \tag{2.4}$$
if the inequality
$$\begin{aligned} -\int_0^\infty \int_\Omega \ln(u+1)\varphi_t - \int_\Omega \ln(u_0+1)\varphi(\cdot,0) \quad \geq \quad & \int_0^\infty \int_\Omega |\nabla \ln(u+1)|^2 \varphi - \int_0^\infty \int_\Omega \nabla \ln(u+1) \cdot \nabla \varphi \\ & - \int_0^\infty \int_\Omega \frac{u}{u+1} \Big(\nabla \ln(u+1) \cdot \nabla \ln v \Big) \varphi \\ & + \int_0^\infty \int_\Omega \frac{u}{u+1} \nabla \ln v \cdot \nabla \varphi \end{aligned} \tag{2.5}$$
holds for each nonnegative $\varphi \in C_0^\infty(\bar{\Omega} \times [0,\infty))$, and if moreover the identity
$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla \varphi + \int_0^\infty \int_\Omega uv\varphi \tag{2.6}$$
is valid for any $\varphi \in L^\infty(\Omega \times (0,\infty)) \cap L^2((0,\infty); W^{1,2}(\Omega))$ having compact support in $\bar{\Omega} \times [0,\infty)$ with $\varphi_t \in L^2(\Omega \times (0,\infty))$.

Remark. i) Using that $0 \leq \ln(\xi+1) \leq \xi$ for all $\xi \geq 0$, the requirements in (2.1), (2.2) and (2.3) can easily be verified to warrant that each of the summands in (2.5) and (2.6) is well-defined.

ii) Along the lines demonstrated in [46, Lemma 2.1], in conjunction with the mass conservation law (2.4), the weak supersolution property (2.5) of u with respect to the first equation in (1.4) can be seen to be actually a genuine generalization of a respective solution property; that is, if $(u,v) \in (C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)))^2$ is such that $u \geq 0$ and $v > 0$ in $\bar{\Omega} \times [0,\infty)$ and such that (u,v) is a generalized solution of (1.4) in the above sense, then (u,v) also solves (1.4) in the classical sense.

In order to construct such generalized solutions by means of an approximation procedure, throughout the sequel we fix a nonincreasing cut-off function $\rho \in C^\infty([0,\infty))$ fulfilling $\rho \equiv 1$ in $[0,1]$ and $\rho \equiv 0$ in $[2,\infty)$, and define $f_\varepsilon \in C^\infty([0,\infty))$ by letting
$$f_\varepsilon(s) := \int_0^s \rho(\varepsilon\sigma)d\sigma, \qquad s \geq 0, \tag{2.7}$$
for $\varepsilon \in (0,1)$. Then for any such ε, the properties of ρ imply that f_ε evidently satisfies
$$f_\varepsilon(0) = 0 \qquad \text{and} \qquad 0 \leq f'_\varepsilon \leq 1 \quad \text{on } [0,\infty) \tag{2.8}$$
as well as
$$f_\varepsilon(s) = s \quad \text{for all } s \in \Big[0, \frac{1}{\varepsilon}\Big] \qquad \text{and} \qquad f'_\varepsilon(s) = 0 \quad \text{for all } s \geq \frac{2}{\varepsilon}, \tag{2.9}$$

and moreover we have

$$f_\varepsilon(s) \nearrow s \quad \text{and} \quad f'_\varepsilon(s) \nearrow 1 \qquad \text{as } \varepsilon \searrow 0$$

for each $s \geq 0$.
Now as a consequence of (2.9), each of the approximate problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \nabla \cdot (\frac{u_\varepsilon f'_\varepsilon(u_\varepsilon)}{v_\varepsilon} \nabla v_\varepsilon), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - f_\varepsilon(u_\varepsilon) v_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{2.10}$$

is in fact globally solvable:

Lemma 2.2 *Assume that (1.5) holds, and let $\varepsilon \in (0,1)$. Then (2.10) possesses a global classical solution $(u_\varepsilon, v_\varepsilon)$, for each $\vartheta > 2$ uniquely determined by the inclusions*

$$\begin{cases} u_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ v_\varepsilon \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\vartheta}(\Omega)), \end{cases}$$

which is such that $u_\varepsilon > 0$ in $\bar\Omega \times (0,\infty)$ and

$$\int_\Omega u_\varepsilon(x,t)dx = \int_\Omega u_0(x)dx \qquad \textit{for all } t > 0 \tag{2.11}$$

as well as

$$0 < v_\varepsilon \leq \|v_0\|_{L^\infty(\Omega)} \qquad \textit{in } \bar\Omega \times [0,\infty). \tag{2.12}$$

Proof. By straightforward adaptation of well-known arguments, as detailed e.g. in [16] and [10] for closely related situations, it can be verified that for each $\varepsilon \in (0,1)$ there exist $T_{max,\varepsilon} \in (0,\infty]$ and a unique couple of functions

$$\begin{cases} u_\varepsilon \in C^0(\bar\Omega \times [0,T_{max,\varepsilon})) \cap C^{2,1}(\bar\Omega \times (0,T_{max,\varepsilon})), \\ v_\varepsilon \in C^0(\bar\Omega \times [0,T_{max,\varepsilon})) \cap C^{2,1}(\bar\Omega \times (0,T_{max,\varepsilon})) \cap L^\infty_{loc}([0,T_{max,\varepsilon}); W^{1,\vartheta}(\Omega)), \end{cases}$$

with $u_\varepsilon > 0$ in $\bar\Omega \times (0,T_{max,\varepsilon})$ and $v_\varepsilon > 0$ in $\bar\Omega \times [0,T_{max,\varepsilon})$, such that $(u_\varepsilon, v_\varepsilon)$ is a classical solution of (2.10) in $\Omega \times (0,T_{max,\varepsilon})$, and such that

$$\text{either } T_{max,\varepsilon} = \infty, \quad \text{or} \quad \limsup_{t \nearrow T_{max,\varepsilon}} \Big(\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} + \|v_\varepsilon(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} \Big) = \infty, \quad \text{or} \quad \liminf_{t \nearrow T_{max,\varepsilon}} \inf_{x \in \Omega} v_\varepsilon(x,t) = 0. \tag{2.13}$$

Moreover, an integration of the first equation in (2.10) over $x \in \Omega$ shows that this solution enjoys the mass conservation property

$$\frac{d}{dt} \int_\Omega u_\varepsilon = 0 \qquad \text{for all } t \in (0, T_{max,\varepsilon}),$$

whereas from the nonnegativity of f_ε and the maximum principle it follows that

$$\|v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le \|v_0\|_{L^\infty(\Omega)} \qquad \text{in } \bar\Omega \times (0,T_{max,\varepsilon}).$$

To prove the lemma, we thus only need to verify that for any fixed $\varepsilon \in (0,1)$, the corresponding maximal existence time $T_{max,\varepsilon}$ cannot be finite, which amounts to showing that in (2.13), neither the second nor the third alternative can occur under the contrary hypothesis that $T_{max,\varepsilon} < \infty$. But since supp $f_\varepsilon' \subset [0,\frac{2}{\varepsilon}]$ by (2.9), an application of the maximum principle to the first equation in (2.10) shows that

$$u_\varepsilon(x,t) \le c_1(\varepsilon) := \max\left\{\|u_0\|_{L^\infty(\Omega)}\,,\,\frac{2}{\varepsilon}\right\} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T_{max,\varepsilon}). \tag{2.14}$$

Together with (2.12) and an argument from parabolic regularity theory (see e.g.[16, Lemma 4.1]), this firstly ensures that for each $\tau \in (0,T_{max,\varepsilon})$ the number $\sup_{t\in(\tau,T_{max,\varepsilon})}\|v_\varepsilon(\cdot,t)\|_{W^{1,\vartheta}(\Omega)}$ is finite. Secondly, by comparison in the second equation in (2.10) we moreover obtain from (2.14) that

$$v_\varepsilon(x,t) \ge \left\{\min_{y\in\bar\Omega} v_0(y)\right\}\cdot e^{-c_1(\varepsilon)t} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T_{max,\varepsilon}),$$

which also excludes the rightmost alternative in (2.13) and thereby completes the proof. $\square$

Now following a standard procedure of changing variables in (1.4), taking u_ε and v_ε from Lemma 2.2 we substitute

$$w_\varepsilon(x,t) := -\ln\Big(\frac{v_\varepsilon(x,t)}{\|v_0\|_{L^\infty(\Omega)}}\Big), \qquad (x,t)\in\bar\Omega\times[0,\infty),\ \varepsilon\in(0,1), \tag{2.15}$$

and thus infer from the latter that each of the problems

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon + \nabla\cdot(u_\varepsilon f_\varepsilon'(u_\varepsilon)\nabla w_\varepsilon), & x\in\Omega,\ t>0,\\ w_{\varepsilon t} = \Delta w_\varepsilon - |\nabla w_\varepsilon|^2 + f_\varepsilon(u_\varepsilon), & x\in\Omega,\ t>0,\\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial w_\varepsilon}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u_\varepsilon(x,0)=u_0(x), \quad w_\varepsilon(x,0)=w_0(x) := -\ln\Big(\frac{v_0(x)}{\|v_0\|_{L^\infty(\Omega)}}\Big), & x\in\Omega, \end{cases} \tag{2.16}$$

possesses a global classical solution $(u_\varepsilon,w_\varepsilon)$ for which both u_ε and w_ε are nonnegative.

2.2 A priori estimates. Preliminary compactness properties of $((u_\varepsilon,w_\varepsilon))_{\varepsilon\in(0,1)}$

We next collect some ε-independent a priori estimates for the solutions of (2.16), where in view of the requirements from Definition 2.1 it will be indispensable to obtain appropriate bounds especially for the respective spatial gradients. As for the second solution component, this can easily be achieved by combining the presence of the dissipative term $|\nabla w_\varepsilon|^2$ in the second equation of (2.16) with the mass control expressed in (2.11).

Lemma 2.3 *For all $\varepsilon\in(0,1)$, we have*

$$\int_\Omega w_\varepsilon(\cdot,t) + \int_0^t\int_\Omega |\nabla w_\varepsilon|^2 \le \int_\Omega w_0 + mt \qquad \textit{for all } t>0, \tag{2.17}$$

where $m := \int_\Omega u_0$. *In particular,*

$$\|w_\varepsilon(\cdot,t)\|_{L^1(\Omega)} \le \int_\Omega w_0 + mt \qquad \textit{for all } t>0 \tag{2.18}$$

and

$$\int_0^t \int_\Omega |\nabla w_\varepsilon|^2 \le \int_\Omega w_0 + mt \qquad \textit{for all } t>0, \tag{2.19}$$

where $m := \int_\Omega u_0$.

PROOF. An integration of the second equation in (2.16) over Ω shows that

$$\frac{d}{dt}\int_\Omega w_\varepsilon = -\int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega f_\varepsilon(u_\varepsilon) \qquad \text{for all } t>0.$$

Since $\int_\Omega f_\varepsilon(u_\varepsilon(\cdot,t)) \le \int_\Omega u_\varepsilon(\cdot,t) = m$ for all $t>0$ by (2.7) and (2.11), this immediately yields (2.17), whereupon thanks to the nonnegativity of w_ε, both (2.18) and (2.19) are evident consequences thereof. $\square$

In comparison to this, deriving bounds for ∇u_ε seems more delicate. In view of the sparse information available so far, it seems adequate to therefore resort to estimating $\nabla \ln(u_\varepsilon+1)$ instead, thus following approaches in related situations characterized by a similar lack of evident regularity information ([46], [47]).

Lemma 2.4 *For each* $\varepsilon \in (0,1)$, *we have*

$$\int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} \le \int_\Omega w_0 + 2m + mt \qquad \textit{for all } t>0, \tag{2.20}$$

where $m := \int_\Omega u_0$.

PROOF. We multiply the first equation in (2.16) by $\frac{1}{u_\varepsilon+1}$ and integrate by parts to find that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \ln(u_\varepsilon+1) &= \int_\Omega \frac{u_{\varepsilon t}}{u_\varepsilon+1} \\
&= \int_\Omega \frac{1}{u_\varepsilon+1}\Delta u_\varepsilon + \int_\Omega \frac{1}{u_\varepsilon+1}\nabla\cdot\Big(u_\varepsilon f_\varepsilon'(u_\varepsilon)\nabla w_\varepsilon\Big) \\
&= \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \int_\Omega \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\nabla w_\varepsilon \qquad \text{for all } t>0,
\end{aligned} \tag{2.21}$$

and using Young's inequality we see that here

$$\begin{aligned}
\left|\int_\Omega \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{(u_\varepsilon+1)^2}\nabla u_\varepsilon\cdot\nabla w_\varepsilon\right| &\le \frac12\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \frac12\int_\Omega \frac{u_\varepsilon^2 f_\varepsilon'^2(u_\varepsilon)}{(u_\varepsilon+1)^2}|\nabla w_\varepsilon|^2 \\
&\le \frac12\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \frac12\int_\Omega |\nabla w_\varepsilon|^2 \qquad \text{for all } t>0,
\end{aligned}$$

because $f_\varepsilon'^2 \le 1$. As $0 \le \ln(u_\varepsilon+1) \le u_\varepsilon$, on integration in time we thus obtain from (2.21) that

$$\begin{aligned}
\frac{1}{2}\int_0^t \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} &\le \frac{1}{2}\int_0^t \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega \ln\Big(u_\varepsilon(\cdot,t)+1\Big) - \int_\Omega \ln(u_0+1) \\
&\le \frac{1}{2}\int_0^t \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega u_\varepsilon(\cdot,t) \\
&\le \frac{1}{2}\int_0^t \int_\Omega |\nabla w_\varepsilon|^2 + m \qquad \text{for all } t>0
\end{aligned}$$

according to (2.11). An application of Lemma 2.3 therefore yields (2.20). $\square$

Inter alia to eventually achieve suitable pointwise convergence properties through applications of the Aubin-Lions lemma, let us state some straightforward consequences of the above estimates for the regularity of the solutions to (2.16) with respect to the time variable. Independently, we shall derive the corresponding inequality (2.24) for the component v_ε which will become relevant in the proof of the continuity property in (1.9).

Lemma 2.5 *For all $T>0$ there exists $C(T)>0$ such that for any $\varepsilon \in (0,1)$,*

$$\int_0^T \Big\|\partial_t \ln\big(u_\varepsilon(\cdot,t)+1\big)\Big\|_{(W^{2,2}(\Omega))^\star} dt \le C(T) \tag{2.22}$$

and

$$\int_0^T \|w_{\varepsilon t}(\cdot,t)\|_{(W^{2,2}(\Omega))^\star} dt \le C(T). \tag{2.23}$$

Moreover, for each $p>2$ and any $T>0$ one can find $C(p,T)>0$ fulfilling

$$\int_0^T \|v_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,p}(\Omega))^\star} dt \le C(p,T) \tag{2.24}$$

for each $\varepsilon \in (0,1)$.

Proof. On testing the first equation in (2.16) by $\frac{\psi}{u_\varepsilon(\cdot,t)+1}$ for fixed $t>0$ and arbitrary $\psi \in C^\infty(\bar\Omega)$, we obtain

$$\begin{aligned}
\int_\Omega \partial_t \ln(u_\varepsilon(\cdot,t)+1)\cdot\psi &= -\int_\Omega \frac{1}{u_\varepsilon+1}\nabla u_\varepsilon\cdot\nabla\psi + \int_\Omega \frac{1}{(u_\varepsilon+1)^2}|\nabla u_\varepsilon|^2\psi \\
&\quad -\int_\Omega \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{u_\varepsilon+1}\nabla w_\varepsilon\cdot\nabla\psi \\
&\quad +\int_\Omega \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{(u_\varepsilon+1)^2}(\nabla u_\varepsilon\cdot\nabla w_\varepsilon)\psi,
\end{aligned}$$

which by several applications of the Cauchy-Schwarz inequality and Young's inequality implies that

$$\begin{aligned}
\bigg|\int_\Omega \partial_t \ln(u_\varepsilon(\cdot,t)+1)\cdot\psi\bigg| &\le \bigg\{\bigg(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\bigg)^{\frac12} + \int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \bigg(\int_\Omega |\nabla w_\varepsilon|^2\bigg)^{\frac12} \\
&\qquad + \bigg(\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\bigg)^{\frac12}\cdot\bigg(\int_\Omega |\nabla w_\varepsilon|^2\bigg)^{\frac12}\bigg\}\cdot\Big\{\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\Big\} \\
&\le \bigg\{2\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \int_\Omega |\nabla w_\varepsilon|^2 + 1\bigg\}\cdot\Big\{\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)}\Big\}
\end{aligned}$$

for all $\psi \in C^\infty(\bar\Omega)$, where again we have used that $0 \le f'_\varepsilon \le 1$ by (2.7). Since in view of the fact that $W^{2,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ we can fix $c_1 > 0$ such that $\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \le c_1\|\psi\|_{W^{2,2}(\Omega)}$ for any such ψ, this entails that

$$\Big\|\partial_t \ln(u_\varepsilon(\cdot,t)+1)\Big\|_{(W^{2,2}(\Omega))^\star} \le c_1 \cdot \left\{ 2\int_\Omega \frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2} + \int_\Omega |\nabla w_\varepsilon|^2 + 1 \right\} \qquad \text{for all } t>0,$$

so that after an integration in time, thanks to Lemma 2.4 and Lemma 2.3 this implies (2.22).

Similarly, for $\psi \in C^\infty(\bar\Omega)$ and $t > 0$ we obtain from the second equation in (2.16) together with (2.8) and (2.11) that

$$\begin{aligned}
\left|\int_\Omega w_{\varepsilon t}(\cdot,t)\psi dx\right| &= \left| -\int_\Omega \nabla w_\varepsilon\cdot\nabla\psi - \int_\Omega |\nabla w_\varepsilon|^2\psi + \int_\Omega f_\varepsilon(u_\varepsilon)\psi\right| \\
&\le \left\{ \left\{\int_\Omega |\nabla w_\varepsilon|^2\right\}^{\frac12} + \int_\Omega |\nabla w_\varepsilon|^2 + \int_\Omega f_\varepsilon(u_\varepsilon)\right\} \cdot \Big\{ \|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \Big\} \\
&\le \left\{ 2\int_\Omega |\nabla w_\varepsilon|^2 + \frac14 + \int_\Omega u_0 \right\} \cdot c_1 \|\psi\|_{W^{2,2}(\Omega)}.
\end{aligned}$$

Therefore,

$$\|w_{\varepsilon t}(\cdot,t)\|_{(W^{2,2}(\Omega))^\star} \le c_1 \cdot \left\{ 2\int_\Omega |\nabla w_\varepsilon|^2 + \frac14 + \int_\Omega u_0 \right\} \qquad \text{for all } t>0,$$

from which (2.23) readily follows.

Finally, for fixed $p > 2$ we have $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, which allows us to pick $c_2 > 0$ such that for all $\psi \in C^\infty(\bar\Omega)$ we have $\|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \le c_2\|\psi\|_{W^{1,p}(\Omega)}$ and hence

$$\begin{aligned}
\left|\int_\Omega v_{\varepsilon t}(\cdot,t)\psi dx\right| &= \left| -\int_\Omega \nabla v_\varepsilon\cdot\nabla\psi - \int_\Omega f_\varepsilon(u_\varepsilon)v_\varepsilon\psi\right| \\
&\le \left\{ \left\{\int_\Omega |\nabla v_\varepsilon|^2\right\}^{\frac12} + \int_\Omega f_\varepsilon(u_\varepsilon)v_\varepsilon\right\} \cdot \Big\{ \|\nabla\psi\|_{L^2(\Omega)} + \|\psi\|_{L^\infty(\Omega)} \Big\} \\
&\le \left\{ \left\{\int_\Omega |\nabla v_\varepsilon|^2\right\}^{\frac12} + m\|v_0\|_{L^\infty(\Omega)} \right\} \cdot c_2 \|\psi\|_{W^{1,p}(\Omega)}
\end{aligned}$$

by (2.7), (2.11) and (2.12). As a consequence,

$$\|v_{\varepsilon t}(\cdot,t)\|^2_{(W^{1,p}(\Omega))^\star} \le 2c_2^2 \left\{ \int_\Omega |\nabla v_\varepsilon|^2 + m^2\|v_0\|^2_{L^\infty(\Omega)} \right\} \qquad \text{for all } t>0,$$

so that also (2.24) results from Lemma 2.3, because $|\nabla v_\varepsilon| \le |\nabla w_\varepsilon| \cdot \|v_0\|_{L^\infty(\Omega)}$ by (2.12). □

Now a straightforward reasoning involving the extraction of suitable sequences, based on the last three lemmata and standard compactness arguments, readily leads to the following result identifying a candidate (u, v) for a generalized solution, as well as some first approximation properties thereof with respect to the solutions of (2.10).

Lemma 2.6 *There exist nonnegative functions u and w defined on $\Omega \times (0,\infty)$ as well as a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$, and such that as $\varepsilon = \varepsilon_k \searrow 0$,*

$$u_\varepsilon \to u \qquad \text{a.e. in } \Omega \times (0,\infty), \tag{2.25}$$

$$\ln(u_\varepsilon + 1) \rightharpoonup \ln(u+1) \qquad \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{2.26}$$

$$w_\varepsilon \to w \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \quad \text{and a.e. in } \Omega \times (0,\infty), \tag{2.27}$$

$$w_\varepsilon \rightharpoonup w \qquad \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)) \qquad \text{and} \tag{2.28}$$

$$w_\varepsilon(\cdot,t) \to w(\cdot,t) \qquad \text{in } L^1(\Omega) \quad \text{for a.e. } t>0 \tag{2.29}$$

as well as

$$v_\varepsilon \to v \qquad \text{in } L^1_{loc}(\bar\Omega \times [0,\infty)) \quad \text{and a.e. in } \Omega \times (0,\infty), \tag{2.30}$$

$$v_\varepsilon \overset{\star}{\rightharpoonup} v \qquad \text{in } L^\infty(\Omega \times (0,\infty)), \tag{2.31}$$

$$v_\varepsilon \rightharpoonup v \qquad \text{in } L^2_{loc}([0,\infty); W^{1,2}(\Omega)), \tag{2.32}$$

$$v_\varepsilon(\cdot,t) \to v(\cdot,t) \qquad \text{in } L^1(\Omega) \quad \text{for a.e. } t>0 \qquad \text{and} \tag{2.33}$$

$$v_{\varepsilon t} \rightharpoonup v_t \qquad \text{in } L^2_{loc}([0,\infty); (W^{1,p}(\Omega))^\star) \quad \text{for all } p>2 \tag{2.34}$$

with $v := \|v_0\|_{L^\infty(\Omega)} \cdot e^{-w}$. Moreover, the pair (u,v) has the properties (2.1), (2.2) and (2.3) in Definition 2.1.

Proof. In view of Lemma 2.4, Lemma 2.3 and Lemma 2.5, the properties (2.25)-(2.29) can be achieved on choosing an appropriate sequence by means of two straightforward applications of an Aubin-Lions lemma ([41]). By nonnegativity of w_ε, (2.27), (2.28) and (2.29) thereafter imply (2.30), (2.32) and (2.33), whereas (2.12) guarantees that on extraction of a suitable subsequence, also (2.31) is valid. That finally also (2.34) can be achieved is a direct consequence of the estimate (2.24) in Lemma 2.5.
Now the properties (2.2) and (2.3) immediately follow from (2.25), (2.30) and the finiteness of w a.e. in $\Omega \times (0,\infty)$ as well as (2.26) and (2.28), while the second inclusion in (2.1) is obvious from (2.31) and the first follows from Fatou's lemma, which in conjunction with (2.11) implies that

$$\int_0^T \int_\Omega u \le \liminf_{\varepsilon=\varepsilon_k \searrow 0} \int_0^T \int_\Omega u_\varepsilon \le mT$$

for all $T>0$. □

2.3 Strong convergence of $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ in $L^1_{loc}(\bar\Omega \times [0,\infty))$. Validity of (2.6)

Up to now, our knowledge on approximation of u by u_ε is essentially restricted to information on the corresponding convergence properties of $\ln(u_\varepsilon+1)$, as described in Lemma 2.6. For adequately passing to the limit in various expressions related to those appearing in (2.5) and (2.6) and also in the crucial mass conservation law (2.4), however, it seems indispensable to derive some further compactness properties addressing the quantity u_ε itself in appropriate Lebesgue spaces. In the presently considered spatially two-dimensional setting, we shall see that in fact strong precompactness of the sequence

$(u_{\varepsilon k})_{k\in\mathbb{N}}$ in $L^1_{loc}(\bar\Omega\times[0,\infty))$ can be achieved by an argument based on exploiting (2.20) by means of the Moser-Trudinger inequality, and a subsequent application of the Vitali convergence theorem. A similar overall strategy has previously been applied in the different context of a chemotaxis-fluid system ([47]).

For this purpose, we shall first derive from Lemma 2.4 the following inequality which will, independently from our present purpose, moreover form the source of the relaxation property (1.7).

Lemma 2.7 *For all $p>1$ there exists $\Lambda(p)>0$ such that whenever u_0 and v_0 satisfy (1.5), given any $\varepsilon\in(0,1)$ we have*

$$\int_0^t \ln\left\{\frac{1}{|\Omega|}\int_\Omega \Big(u_\varepsilon(x,s)+1\Big)^p dx\right\}ds \le \Lambda(p)\cdot(1+m)t+\Lambda(p)\cdot\left\{\int_\Omega w_0+m\right\} \qquad \textit{for all } t>0, \quad (2.35)$$

where again $m:=\int_\Omega u_0$.

PROOF. According to the Moser-Trudinger inequality ([7]), we can find positive constats c_1,c_2 and c_3 such that

$$\int_\Omega e^\varphi \le c_1\, e^{c_2\int_\Omega|\nabla\varphi|^2+c_3\int_\Omega\varphi} \qquad \text{for all nonnegative } \varphi\in W^{1,2}(\Omega).$$

Applying this to $\varphi:=p\ln(u_\varepsilon(\cdot,t)+1)$ for fixed $p>1$ and $t>0$, we see that

$$\frac{1}{|\Omega|}\int_\Omega\Big(u_\varepsilon(\cdot,t)+1\Big)^p \le \frac{c_1}{|\Omega|}\cdot e^{p^2c_2\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+pc_3\int_\Omega\ln(u_\varepsilon+1)} \qquad \text{for all } t>0,$$

so that since

$$\int_\Omega \ln(u_\varepsilon+1)\le\int_\Omega u_\varepsilon = m \qquad \text{for all } t>0$$

by (2.11), we obtain that

$$\ln\left\{\frac{1}{|\Omega|}\int_\Omega(u_\varepsilon+1)^p\right\}\le \ln\frac{c_1}{|\Omega|}+p^2c_2\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}+pc_3m \qquad \text{for all } t>0.$$

On integration in time using (2.20), we therefore conclude that for all $t>0$,

$$\begin{aligned}\int_0^t\ln\left\{\frac{1}{|\Omega|}\int_\Omega\Big(u_\varepsilon(\cdot,s)+1\Big)^p\right\}ds &\le \left\{\ln\frac{c_1}{|\Omega|}+pc_3m\right\}\cdot t+p^2c_2\int_0^t\int_\Omega\frac{|\nabla u_\varepsilon|^2}{(u_\varepsilon+1)^2}\\ &\le \left\{\ln\frac{c_1}{|\Omega|}+pc_3m\right\}\cdot t+p^2c_2\cdot\left\{\int_\Omega w_0+2m+mt\right\}\end{aligned}$$

which immediately yields (2.35) if we let $\Lambda(p):=\max\{\ln\frac{c_1}{|\Omega|},2p^2c_2,pc_3+p^2c_2\}$, for instance. □

Now thanks to the Vitali convergence theorem, the desired strong precompactness property of the particular sequence $(u_{\varepsilon_k})_{k\in\mathbb{N}}$ can be derived from this in quite a straightforward manner.

Lemma 2.8 *Let u and $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ be as obtained in Lemma 2.6. Then*

$$u_\varepsilon \to u \quad in\ L^1_{loc}(\bar\Omega \times [0,\infty)) \qquad as\ \varepsilon = \varepsilon_k \searrow 0. \tag{2.36}$$

In particular,

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad for\ a.e.\ t > 0. \tag{2.37}$$

Proof. This can be derived from Lemma 2.6 and Lemma 2.7 by means of the Vitali convergence theorem. Since details on this have been given in [47, Lemma 4.2] for a closely related situation, we may confine ourselves with sketching the main steps here.
We fix $T > 0$, and taking $\Lambda := \Lambda(2) > 0$ from Lemma 2.7 we abbreviate $c_1 := \Lambda\cdot(1+m)T + \Lambda(\int_\Omega w_0 + m)$, again with $m := \int_\Omega u_0$. Given $\eta > 0$, we can then pick $\Sigma > 1$ large enough and thereafter $\delta > 0$ suitably small such that

$$\frac{mc_1}{\ln\frac{\Sigma}{|\Omega|}} < \frac{\eta}{2} \qquad \text{and} \qquad \sqrt{\Sigma T\delta} < \frac{\eta}{2}. \tag{2.38}$$

Then introducing the sets

$$\mathcal{S}_1(\varepsilon) := \Big\{ t\in(0,T) \ \Big| \ \int_\Omega u_\varepsilon^2(\cdot,t) \le \Sigma \Big\} \qquad \text{and}$$
$$\mathcal{S}_2(\varepsilon) := \Big\{ t\in(0,T) \ \Big| \ \int_\Omega u_\varepsilon^2(\cdot,t) > \Sigma \Big\},$$

for $\varepsilon\in(0,1)$, we first use (2.35) to estimate

$$c_1 \ge \int_{\mathcal{S}_2(\varepsilon)} \ln\Big\{ \frac{1}{|\Omega|}\int_\Omega \Big(u_\varepsilon(\cdot,t)+1\Big)^2 \Big\} dt \ge \int_{\mathcal{S}_2(\varepsilon)} \ln\Big\{ \frac{1}{|\Omega|}\int_\Omega u_\varepsilon^2(\cdot,t)\Big\} dt \ge \int_{\mathcal{S}_2(\varepsilon)} \ln\frac{\Sigma}{|\Omega|}$$

and hence

$$|\mathcal{S}_2(\varepsilon)| \le \frac{c_1}{\ln\frac{\Sigma}{|\Omega|}}$$

for any such ε.
Therefore, given an arbitrary measurable $E \subset \Omega\times(0,T)$ such that $|E| < \delta$, writing $E(t) := \{x\in\Omega \mid (x,t)\in E\}$ we may recall (2.11) and twice apply the Cauchy-Schwarz inequality to see that for each $\varepsilon\in(0,1)$ we have

$$\begin{aligned}
\iint_E u_\varepsilon &\le \int_{\mathcal{S}_1(\varepsilon)}\int_{E(t)} u_\varepsilon(x,t)dxdt + \int_{\mathcal{S}_2(\varepsilon)}\int_{E(t)} u_\varepsilon(x,t)dxdt \\
&\le \int_{\mathcal{S}_1(\varepsilon)} |E(t)|^{\frac12} \Big\{ \int_\Omega u_\varepsilon^2(x,t)dx \Big\}^{\frac12} dt + m|\mathcal{S}_2(\varepsilon)| \\
&\le \sqrt{\Sigma}\cdot|\mathcal{S}_1(\varepsilon)|^{\frac12} \Big\{ \int_{\mathcal{S}_1(\varepsilon)} |E(t)|dt \Big\}^{\frac12} + m|\mathcal{S}_2(\varepsilon)| \\
&\le \sqrt{\Sigma}\sqrt{T}\sqrt{|E|} + m\cdot\frac{c_1}{\ln\frac{\Sigma}{|\Omega|}}
\end{aligned}$$

$$\leq \sqrt{\Sigma T\delta} + m \cdot \frac{c_1}{\ln \frac{\Sigma}{|\Omega|}}$$
$$\leq \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

according to (2.38) and our assumption on the size of E. Since we already know from Lemma 2.6 that $u_\varepsilon \to u$ a.e. in $\Omega \times (0,T)$ as $\varepsilon = \varepsilon_k \searrow 0$, along with the Vitali theorem this shows that in fact $u_\varepsilon \to u$ in $L^1(\Omega \times (0,T))$ and thereby establishes (2.36). The mass conservation property (2.37) is thereafter an obvious consequence of (2.11) and (2.36). □

As a first consequence thereof, we can pass to the limit in the weak formulation of the second equation in (2.10), thereby showing that v indeed is a weak solution of the respective sub-problem of (1.4), as required in Definition 2.1.

Lemma 2.9 *The functions u and v obtained in Lemma 2.6 satisfy the identity (2.6) in Definition 2.1 for all test functions from the class indicated there.*

PROOF. For each φ from the class in question, in light of Lemma 2.8 combined with (2.31) we know that if $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ is as in Lemma 2.6 then

$$\int_0^\infty \int_\Omega f_\varepsilon(u_\varepsilon)v_\varepsilon\varphi \to \int_0^\infty \int_\Omega uv\varphi \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0, \tag{2.39}$$

because the definition of f_ε warrants that for each $T > 0$ we have

$$\begin{aligned}
\int_0^T \int_\Omega |f_\varepsilon(u_\varepsilon) - u| &\leq \int_0^T \int_\Omega |f_\varepsilon(u_\varepsilon) - f_\varepsilon(u)| + \int_0^T \int_\Omega |f_\varepsilon(u) - u| \\
&\leq \int_0^T \int_\Omega |u_\varepsilon - u| + \int_0^T \int_{\{u(\cdot,t)\geq\frac{1}{\varepsilon}\}} |f_\varepsilon(u) - u| \\
&\leq \int_0^T \int_\Omega |u_\varepsilon - u| + 2\int_0^T \int_{\{u(\cdot,t)\geq\frac{1}{\varepsilon}\}} u \\
&\to 0 \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0
\end{aligned}$$

due to Lemma 2.8 and the dominated convergence theorem. Thanks to (2.39) as well as (2.30) and (2.32), it readily follows on taking $\varepsilon = \varepsilon_k \searrow 0$ in each of the integrals making up the corresponding weak formulation of the respective initial-boundary value subproblem of (2.10) satisfied by v_ε that indeed (2.6) is satisfied. □

2.4 Strong convergence of $(\nabla w_{\varepsilon_k})_{k\in\mathbb{N}}$ in $L^2_{loc}(\bar\Omega \times [0,\infty))$. Proof of Theorem 1.1

In order to be able to pass to the limit also in the first equation in (2.10) in an appropriate manner, in view of the integrals making up (2.5) and its approximate analogue, the currently gained weak precompactness properties of $(\nabla \ln(u_\varepsilon + 1))_{\varepsilon\in(0,1)}$ and $(\nabla w_\varepsilon)_{\varepsilon\in(0,1)}$ in $L^2_{loc}(\bar\Omega \times [0,\infty))$ seem yet insufficient. To overcome this shortage, our next goal will consist in showing that $(\nabla w_{\varepsilon_k})_{k\in\mathbb{N}}$ is convergent actually with respect to the strong topology in $L^2(\Omega \times (0,T))$ for any $T > 0$. The crucial step toward this will be established in the following lemma in which we shall make use of the fact that, as we already know from Lemma 2.9, v satisfies the weak identity (2.6) for all corresponding test functions φ indicated there. On suitable choices of adequately regularized variants of

$$\varphi(x,t) := \frac{1}{v(x,t)}, \qquad x \in \Omega,\ t > 0,$$

namely, we shall see that the regularity properties of v gained up to now are sufficient to justify the identity

$$-\int_\Omega \ln\Big(\frac{v(x,t_0)}{\|v_0\|_{L^\infty(\Omega)}}\Big)dx + \int_0^{t_0}\int_\Omega \frac{|\nabla v(x,t)|^2}{v^2(x,t)}dxdt = -\int_\Omega \ln\Big(\frac{v_0(x)}{\|v_0\|_{L^\infty(\Omega)}}\Big)dx,$$

as formally obtained from (1.4) by means of a corresponding testing procedure, at least in form of an inequality for almost all times:

Lemma 2.10 *Let w denote the limit function gained in Lemma 2.6. Then there exists a null set $N \subset (0,\infty)$ such that*

$$\int_0^{t_0}\int_\Omega |\nabla w|^2 \ge \int_\Omega w_0 - \int_\Omega w(\cdot,t_0) + mt_0 \qquad \textit{for all } t_0 \in (0,\infty)\setminus N, \tag{2.40}$$

where $m := \int_\Omega u_0$.

PROOF. We fix any sequence $(\eta_k)_{k\in\mathbb{N}} \subset (0,1)$ such that $\eta_k \searrow 0$ as $k \to \infty$, and for each $k \in \mathbb{N}$ we can then pick a null set $N_k \subset (0,\infty)$ such that each $t_0 \in (0,\infty)\setminus N_k$ is a Lebesgue point of $0 < t \mapsto \int_\Omega \ln\Big\{v(x,t)+\eta_k\Big\}dx$. Moreover, Lemma 2.6 provides a null set $N_\star \subset (0,\infty)$ with the property that $w(\cdot,t_0) \in L^1(\Omega)$ for all $t_0 \in (0,\infty)\setminus N_\star$.
Now by concavity of $0 < \xi \mapsto \ln\xi$, for all $\eta \in (\eta_k)_{k\in\mathbb{N}}$ and any $h > 0$, the limit function v from Lemma 2.6 satisfies

$$\ln\Big\{v(x,t)+\eta\Big\} - \ln\Big\{v(x,t-h)+\eta\Big\} \le \frac{1}{v(x,t-h)+\eta}\cdot\Big\{v(x,t)-v(x,t-h)\Big\} \qquad \text{for all } x\in\Omega \text{ and } t>0, \tag{2.41}$$

where we have extended v so as to exist on all of $\Omega\times\mathbb{R}$ by letting

$$v(x,t) := v_0(x) \qquad \text{for } x\in\Omega \text{ and } t \le 0. \tag{2.42}$$

In order to exploit (2.41) appropriately, we fix $t_0 \in N := N_\star \cup \bigcup_{k\in\mathbb{N}} N_k$, and for $\delta \in (0,1)$ we let $\zeta_\delta \in W^{1,\infty}(\mathbb{R})$ be defined by

$$\zeta_\delta(t) := \begin{cases} 1 & \text{if } t \le t_0, \\ \frac{t_0+\delta-t}{\delta} & \text{if } t \in (t_0, t_0+\delta), \\ 0 & \text{if } t \ge t_0+\delta, \end{cases}$$

and multiply (2.41) by $\frac{1}{h}\zeta_\delta(t)$ for $h \in (0,t_0)$ to see that for any such δ we have

$$\begin{aligned} I(\delta,h) &:= \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\ln\Big\{v(\cdot,t)+\eta\Big\} - \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\ln\Big\{v(\cdot,t-h)+\eta\Big\} \\ &\le \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)\cdot\Big\{v(\cdot,t)-v(\cdot,t-h)\Big\}\cdot\frac{1}{v(\cdot,t-h)+\eta} \\ &=: J(\delta,h). \end{aligned} \tag{2.43}$$

Here substituting $t-h$ by t shows that

$$\begin{aligned}
I(\delta,h) &= \frac{1}{h}\int_0^\infty \int_\Omega \zeta_\delta(t)\ln\Big\{v(\cdot,t)+\eta\Big\} \\
&\quad -\frac{1}{h}\int_0^\infty \int_\Omega \zeta_\delta(t+h)\ln\Big\{v(\cdot,t)+\eta\Big\} - \frac{1}{h}\int_{-h}^0 \int_\Omega \zeta_\delta(t+h)\ln\Big\{v(\cdot,t)+\eta\Big\} \\
&= -\int_0^\infty \int_\Omega \frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h}\cdot \ln\Big\{v(\cdot,t)+\eta\Big\} - \int_\Omega \ln(v_0+\eta),
\end{aligned}$$

because for $h<t_0$ we know that $\zeta_\delta\equiv 1$ on $(0,h)\subset(0,t_0)$, and because $v(\cdot,t)\equiv v_0$ for $t<0$ due to (2.42). Since evidently

$$\frac{\zeta_\delta(\cdot+h)-\zeta_\delta}{h} \overset{\star}{\rightharpoonup} \zeta_\delta' \quad \text{in } L^\infty(\mathbb{R}) \qquad \text{as } h\searrow 0, \tag{2.44}$$

from this and the boundedness of $\ln(v+\eta)$, as guaranteed for each fixed $\eta\in(\eta_k)_{k\in\mathbb{N}}$ by the boundedness of v, it follows that

$$I(\delta,h)\to \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \ln\Big\{v(\cdot,t)+\eta\Big\} - \int_\Omega \ln(v_0+\eta) \qquad \text{as } h\searrow 0. \tag{2.45}$$

On the right-hand side of (2.43) we also partially substitute as before to see, once more using (2.42), that

$$\begin{aligned}
J(\delta,h) &= \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)v(\cdot,t)\cdot\frac{1}{v(\cdot,t-h)+\eta} - \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)v(\cdot,t-h)\cdot\frac{1}{v(\cdot,t-h)+\eta} \\
&= \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t)v(\cdot,t)\cdot\frac{1}{v(\cdot,t-h)+\eta} - \frac{1}{h}\int_0^\infty\int_\Omega \zeta_\delta(t+h)v(\cdot,t)\cdot\frac{1}{v(\cdot,t)+\eta} \\
&\quad -\frac{1}{h}\int_{-h}^0\int_\Omega \zeta_\delta(t+h)v(\cdot,t)\cdot\frac{1}{v(\cdot,t)+\eta} \\
&= -\int_0^\infty\int_\Omega \frac{\zeta_\delta(t+h)-\zeta_\delta(t)}{h}\cdot v(\cdot,t)\cdot\frac{1}{v(\cdot,t)+\eta} \\
&\quad -\int_0^\infty\int_\Omega \zeta_\delta(t)v(\cdot,t)\cdot\frac{1}{h}\bigg\{\frac{1}{v(\cdot,t)+\eta}-\frac{1}{v(\cdot,t-h)+\eta}\bigg\} \\
&\quad -\int_\Omega \frac{v_0}{v_0+\eta} \\
&=: J_1(\delta,h)+J_2(\delta,h)+J_3(\delta,h) \qquad \text{for all } \delta\in(0,1) \text{ and } h\in(0,t_0),
\end{aligned} \tag{2.46}$$

where again by (2.44) and the boundedness of v,

$$J_1(\delta,h)\to\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \frac{v}{v+\eta} \qquad \text{as } h\searrow 0. \tag{2.47}$$

In order to prepare an analysis of $J_2(\delta,h)$, we note that

$$\varphi(x,t) := \zeta_\delta(t)\cdot S_h\Big[\frac{1}{v+\eta}\Big](x,t), \qquad (x,t)\in\Omega\times(0,\infty),$$

with the action of the backward averaging operator S_h given by

$$S_h[\psi](x,t) := \frac{1}{h}\int_{t-h}^t \psi(x,s)ds, \qquad (x,t)\in\Omega\times(0,\infty), \qquad \psi\in L^1_{loc}(\bar\Omega\times\mathbb{R};\mathbb{R}^d), \quad d\ge 1,$$

satisfies

$$\varphi_t = \zeta_\delta'(t)\cdot S_h\Big[\frac{1}{v+\eta}\Big] + \zeta_\delta(t)\cdot\frac{1}{h}\Big\{\frac{1}{v(\cdot,t)+\eta} - \frac{1}{v(\cdot,t-h)+\eta}\Big\} \qquad \text{in } \Omega\times(0,\infty)$$

and

$$\nabla\varphi = -\zeta_\delta(t)\cdot S_h\Big[\frac{1}{(v+\eta)^2}\nabla v\Big] \qquad \text{in } \Omega\times(0,\infty),$$

so that in particular the regularity properties of v obtained in Lemma 2.6 warrant that φ is an admissible test function in (2.6). An evaluation of the latter, again using (2.42), thus shows that with u taken from Lemma 2.6 we have

$$\begin{aligned} J_2(\delta,h) &= -\int_0^\infty\int_\Omega v\varphi_t + \int_0^\infty\int_\Omega \zeta_\delta'(t)v(\cdot,t)S_h\Big[\frac{1}{v+\eta}\Big](\cdot,t) \\ &= \int_\Omega v_0\varphi(\cdot,0) - \int_0^\infty\int_\Omega \nabla v\cdot\nabla\varphi - \int_\Omega uv\varphi + \int_0^\infty\int_\Omega \zeta_\delta'(t)v(\cdot,t)S_h\Big[\frac{1}{v+\eta}\Big](\cdot,t) \\ &= \int_\Omega \frac{v_0}{v_0+\eta} + \int_0^\infty\int_\Omega \zeta_\delta(t)\nabla v(\cdot,t)\cdot S_h\Big[\frac{1}{(v+\eta)^2}\nabla v\Big](\cdot,t) \\ &\quad -\int_0^\infty\int_\Omega \zeta_\delta(t)u(\cdot,t)v(\cdot,t)S_h\Big[\frac{1}{v+\eta}\Big](\cdot,t) \\ &\quad +\int_0^\infty\int_\Omega \zeta_\delta'(t)v(\cdot,t)S_h\Big[\frac{1}{v+\eta}\Big](\cdot,t) \qquad \text{for all } \delta\in(0,1) \text{ and } h\in(0,t_0). \end{aligned} \tag{2.48}$$

Since $\frac{1}{(v+\eta)^2}\nabla v$ belongs to $L^2_{loc}(\bar\Omega\times\mathbb{R})$ and $\frac{1}{v+\eta}$ lies in $L^\infty(\Omega\times\mathbb{R})$, by means of a standard reasoning (cf. e.g. the argument in [46, Lemma 10.2]) it follows that herein

$$\int_0^\infty\int_\Omega \zeta_\delta(t)\nabla v(\cdot,t)\cdot S_h\Big[\frac{1}{(v+\eta)^2}\nabla v\Big](\cdot,t) \to \int_0^\infty\int_\Omega \zeta_\delta(t)\frac{|\nabla v|^2}{(v+\eta)^2} \qquad \text{as } h\searrow 0$$

and that

$$-\int_0^\infty\int_\Omega \zeta_\delta(t)u(\cdot,t)v(\cdot,t)S_h\Big[\frac{1}{v+\eta}\Big](\cdot,t) \to -\int_0^\infty\int_\Omega \zeta_\delta(t)\frac{uv}{v+\eta} \qquad \text{as } h\searrow 0$$

as well as

$$\int_0^\infty\int_\Omega \zeta_\delta'(t)v(\cdot,t)S_h\Big[\frac{1}{v+\eta}\Big](\cdot,t) \to \int_0^\infty\int_\Omega \zeta_\delta'(t)\frac{v}{v+\eta} = -\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \frac{v}{v+\eta} \qquad \text{as } h\searrow 0,$$

whence in the limit $h\searrow 0$ from (2.43), (2.45), (2.46) (2.47) and (2.48) we infer that

$$\frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \ln\Big\{v(\cdot,t)+\eta\Big\} - \int_\Omega \ln(v_0+\eta)$$

$$
\begin{aligned}
&\leq \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \frac{v}{v+\eta} \\
&\quad +\Big\{\int_\Omega \frac{v_0}{v_0+\eta} + \int_0^\infty\int_\Omega \zeta_\delta(t)\frac{|\nabla v|^2}{(v+\eta)^2} - \int_0^\infty\int_\Omega \zeta_\delta(t)\frac{uv}{v+\eta} - \frac{1}{\delta}\int_{t_0}^{t_0+\delta}\int_\Omega \frac{v}{v+\eta}\Big\} \\
&\quad - \int_\Omega \frac{v_0}{v_0+\eta} \\
&= \int_0^\infty\int_\Omega \zeta_\delta(t)\frac{|\nabla v|^2}{(v+\eta)^2} - \int_0^\infty\int_\Omega \zeta_\delta(t)\frac{uv}{v+\eta} \qquad \text{for all } \delta\in(0,1).
\end{aligned}
$$

Now twice using the Beppo Levi theorem, thanks to the Lebesgue point property of t_0 we obtain on taking $\delta \searrow 0$ that

$$
\int_\Omega \ln\Big\{v(\cdot,t_0)+\eta\Big\} - \int_\Omega \ln(v_0+\eta) \leq \int_0^{t_0}\int_\Omega \frac{|\nabla v|^2}{(v+\eta)^2} - \int_0^{t_0}\int_\Omega \frac{uv}{v+\eta} \qquad \text{for all } \eta\in(\eta_k)_{k\in\mathbb{N}}. \tag{2.49}
$$

Since again by Beppo Levi's theorem, as $\eta=\eta_k \searrow 0$ we have

$$
\int_0^{t_0}\int_\Omega \frac{|\nabla v|^2}{(v+\eta)^2} \nearrow \int_0^{t_0}\int_\Omega \frac{|\nabla v|^2}{v^2} = \int_0^{t_0}\int_\Omega |\nabla w|^2
$$

and, thanks to the mass conservation property (2.37), also

$$
\int_0^{t_0}\int_\Omega \frac{uv}{v+\eta} \nearrow \int_0^{t_0}\int_\Omega u = mt_0
$$

as well as

$$
\int_\Omega \ln(v_0+\eta) \searrow \int_\Omega \ln v_0 = -\int_\Omega w_0 + |\Omega| \ln \|v_0\|_{L^\infty(\Omega)}
$$

and

$$
\int_\Omega \ln\Big\{v(\cdot,t_0)+\eta\Big\} \searrow \int_\Omega \ln v(\cdot,t_0) = -\int_\Omega w(\cdot,t_0) + |\Omega|\ln\|v_0\|_{L^\infty(\Omega)},
$$

with both $\int_\Omega w_0$ and $\int_\Omega w(\cdot,t_0)$ being finite according to the fact that $t_0 \notin N_\star$. In consequence, (2.49) hence establishes (2.40) and thereby proves the lemma, because N clearly has measure zero. $\square$

In view of the elementary criterion on strong convergence in L^2 spaces, as implied by weak convergence and convergence in $\mathbb{R}$ of the associated sequence of norms, the readily entails the desired convergence property of $(\nabla w_{\varepsilon_k})_{k\in\mathbb{N}}$.

Corollary 2.11 *Let w and $(\varepsilon_k)_{k\in\mathbb{N}}$ by as given by Lemma 2.6. Then for each $T>0$ we have*

$$
\nabla w_\varepsilon \to \nabla w \quad \text{in } L^2(\Omega\times(0,T)) \qquad \text{as } \varepsilon=\varepsilon_k\searrow 0. \tag{2.50}
$$

PROOF. Given $T > 0$, in view of Lemma 2.6 and Lemma 2.10 we can fix $t_0 \ge T$ such that $\int_\Omega w_\varepsilon(\cdot,t_0) \to \int_\Omega w(\cdot,t_0)$ as $\varepsilon = \varepsilon_k \searrow 0$, and such that the inequality in (2.40) is valid. Recalling Lemma 2.3, we therefore obtain

$$\begin{aligned}
\limsup_{\varepsilon=\varepsilon_k\searrow 0} \int_0^{t_0}\int_\Omega |\nabla w_\varepsilon|^2 &= \limsup_{\varepsilon=\varepsilon_k-\searrow 0} \Big\{ \int_\Omega w_0 - \int_\Omega w_\varepsilon(\cdot,t_0) + mt_0 \Big\} \\
&= \int_\Omega w_0 - \int_\Omega w(\cdot,t_0) + mt_0 \\
&\le \int_0^{t_0}\int_\Omega |\nabla w|^2,
\end{aligned}$$

which together with the weak convergence property in (2.28) ensures that $\nabla w_\varepsilon \to \nabla w$ in $L^2(\Omega\times(0,t_0))$ and hence implies (2.50), because $t_0 \ge T$. $\Box$

We are now in the position to accomplish the last step toward the existence result in Theorem 1.1. For later reference, we formulate this in a separate lemma which also implicitly includes a statement on how this particular solution can be found via approximation.

Lemma 2.12 *The couple (u,v) provided by Lemma 2.6 is a global generalized solution of (1.4) in the sense of Definition 2.1.*

PROOF. In view of Lemma 2.6, Lemma 2.8 and Lemma 2.9, we only need to verify (2.5). To this end, we fix an arbitrary nonnegative $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ and then obtain on multiplying the first equation in (2.16) by $\frac{\varphi}{u_\varepsilon+1}$ that

$$\begin{aligned}
I_1(\varepsilon) &:= \int_0^\infty\int_\Omega |\nabla \ln(u_\varepsilon+1)|^2\varphi \\
&= -\int_0^\infty\int_\Omega \ln(u_\varepsilon+1)\varphi_t - \int_\Omega \ln(u_0+1)\varphi(\cdot,0) + \int_0^\infty\int_\Omega \nabla\ln(u_\varepsilon+1)\cdot\nabla\varphi \\
&\quad - \int_0^\infty\int_\Omega \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{u_\varepsilon+1}\Big(\nabla\ln(u_\varepsilon+1)\cdot\nabla w_\varepsilon\Big)\varphi + \int_0^\infty\int_\Omega \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{u_\varepsilon+1}\nabla w_\varepsilon\cdot\nabla\varphi \\
&=: I_2(\varepsilon)+I_3(\varepsilon)+I_4(\varepsilon)+I_5(\varepsilon)+I_6(\varepsilon) \qquad (2.51)
\end{aligned}$$

for all $\varepsilon \in (0,1)$. Here choosing $T > 0$ large enough such that $\varphi \equiv 0$ on $\Omega \times (T,\infty)$, we obtain from (2.26) that $\ln(u_\varepsilon+1) \rightharpoonup \ln(u+1)$ in $L^2((0,T);W^{1,2}(\Omega))$ as $\varepsilon = \varepsilon_k \searrow 0$, which ensures that

$$I_2(\varepsilon) \to -\int_0^\infty\int_\Omega \ln(u+1)\varphi_t \qquad\text{and}\qquad I_4(\varepsilon) \to \int_0^\infty\int_\Omega \nabla\ln(u+1)\cdot\nabla\varphi \qquad (2.52)$$

as $\varepsilon = \varepsilon_k \searrow 0$, and that

$$\int_0^\infty\int_\Omega |\nabla\ln(u+1)|^2\varphi \le \liminf_{\varepsilon=\varepsilon_k\searrow 0} I_1(\varepsilon). \qquad (2.53)$$

Moreover, using that $\nabla w_\varepsilon \to \nabla w$ in $L^2(\Omega\times(0,T))$ as $\varepsilon = \varepsilon_k \searrow 0$ by Corollary 2.11, and that this combined with the observations that $0 \le \frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{u_\varepsilon+1} \le 1$ for all $\varepsilon \in (0,1)$ and $\frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{u_\varepsilon+1} \to \frac{u}{u+1}$ a.e. in $\Omega\times(0,T)$ as $\varepsilon = \varepsilon_k \searrow 0$ warrants that

$$\frac{u_\varepsilon f_\varepsilon'(u_\varepsilon)}{u_\varepsilon+1}\nabla w_\varepsilon \to \frac{u}{u+1}\nabla w = -\frac{u}{u+1}\nabla\ln v \qquad \text{in } L^2(\Omega\times(0,T))$$

as $\varepsilon = \varepsilon_k \searrow 0$ ([46, Lemma 10.4]), we also find that

$$I_5(\varepsilon) \to \int_0^\infty \int_\Omega \frac{u}{u+1} \Big(\nabla \ln(u+1) \cdot \nabla \ln v \Big) \varphi \qquad \text{and} \qquad I_6(\varepsilon) \to - \int_0^\infty \int_\Omega \frac{u}{u+1} \nabla \ln v \cdot \nabla \varphi \quad (2.54)$$

as $\varepsilon = \varepsilon_k \searrow 0$. Collecting (2.52), (2.53) and (2.54), we readily see that (2.5) results from (2.51). $\square$

Our main result on global existence of generalized solutions actually reduces to a corollary.

Proof of Theorem 1.1. The claim is an obvious consequence of Lemma 2.12. $\square$

3 Relaxation. Proof of Theorem 1.2

Proof of Theorem 1.2. Given $m > 0$ and $p > 1$, we let $K_1(m,p) := \Lambda(p) \cdot (2+m)$ and $K_2(m) := 1+m$, where $\Lambda(p) > 0$ is as provided by Lemma 2.7. Then fixing $M > 0$ and assuming u_0 and v_0 to satisfy (1.5) as well as (1.6), we set $T_1(m,M) := M + m$ and $T_2(M) := M$. For each $\varepsilon \in (0,1)$, Lemma 2.7 thereupon warrants that

$$\begin{aligned} \frac{1}{T} \int_0^T \ln \bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(u_\varepsilon(x,t) + 1 \Big)^p dx \bigg\} dt &\le \Lambda(p) \cdot (1+m) + \frac{\Lambda(p)}{T} \cdot \bigg\{ \int_\Omega w_0 + m \bigg\} \\ &\le \Lambda(p) \cdot (1+m) + \frac{\Lambda(p)}{T} \cdot (M+m) \qquad \text{for all } T > 0, \end{aligned}$$

so that since $\frac{\Lambda(p)}{T_1(M,m)} \cdot (M+m) = \Lambda(p)$ we obtain

$$\frac{1}{T} \int_0^T \ln \bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(u_\varepsilon(x,t) + 1 \Big)^p dx \bigg\} dt \le \Lambda(p) \cdot (2+m) \qquad \text{for all } T > T_1(M,m).$$

In order to derive (1.7) from this, in accordance with the convergence property (2.25) we fix a null set $N \subset (0,\infty)$ such that as $\varepsilon = \varepsilon_k \searrow 0$ we have $u_\varepsilon(\cdot,t) \to u(\cdot,t)$ a.e. in Ω for all $t \in (0,\infty) \setminus N$. Since $\frac{1}{|\Omega|} \int_\Omega (\varphi+1)^p \ge 1$ for any nonnegative measurable function φ on Ω, writing

$$g_\varepsilon(t) := \ln \bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(u_\varepsilon(\cdot,t) + 1 \Big)^p \bigg\} \qquad \text{and} \qquad g(t) := \ln \bigg\{ \frac{1}{|\Omega|} \int_\Omega \Big(u(\cdot,t) + 1 \Big)^p \bigg\}$$

for $t > 0$ and $\varepsilon \in (0,1)$, we see that both g_ε and g are nonnegative, and from a first application of Fatou's lemma and the monotonicity of ln we conclude that

$$g(t) \le \liminf_{\varepsilon=\varepsilon_k \searrow 0} g_\varepsilon(t) \qquad \text{for all } t \in (0,\infty) \setminus N.$$

Once more invoking Fatou's lemma, we thus infer that

$$\begin{aligned} \frac{1}{T} \int_0^T \ln \bigg\{ \int_\Omega \Big(u(x,t) + 1 \Big)^p \bigg\} dt &= \frac{1}{T} \int_0^T g(t) dt \\ &\le \liminf_{\varepsilon=\varepsilon_k \searrow 0} \frac{1}{T} \int_0^T g_\varepsilon(t) dt \\ &= \liminf_{\varepsilon=\varepsilon_k \searrow 0} \frac{1}{T} \int_0^T \ln \bigg\{ \int_\Omega \Big(u_\varepsilon(x,t) + 1 \Big)^p \bigg\} dt \\ &\le \Lambda(p) \cdot (2+m) \qquad \text{for all } T > T_1(M,m), \end{aligned}$$

and that thus indeed (1.7) is valid. Similarly, (2.19) shows that

$$\begin{aligned}\frac{1}{T}\int_0^T\int_\Omega |\nabla w_\varepsilon|^2 &\le m+\frac{1}{T}\int_\Omega w_0\\ &\le m+\frac{M}{T}\\ &\le m+1 \qquad \text{for all } T>T_2(M),\end{aligned}$$

and thereby proves (1.8) by means of Lemma 2.6 and a standard argument. □

4 Decay of v. Proof of Theorem 1.3

In order to finally prove asymptotic decay of v, we shall first apply two standard testing procedures to the second equation in (2.10) to obtain two preliminary weak decay properties, as expressed in terms of finiteness of the spatio-temporal integrals in (4.1) and (4.2), as well as a favorable monotonicity feature of the spatial L^1 norm of v with respect to time. We formulate the latter in the version (4.3) directly suitable for our later purpose, involving integrals over past time intervals rather than evaluations at particular times therein, thereby also circumventing any difficulties possibly arising from our lack of knowledge on whether in the convergence statement (2.33) from Lemma 2.6 the exclusion of an exceptional null set can be removed.

Lemma 4.1 *Assume (1.5), and let (u,v) denote the global generalized solution of (1.4) from Theorem 1.1. Then*

$$\int_0^t\int_\Omega uv \le \int_\Omega v_0 \qquad \textit{for all } t>0 \tag{4.1}$$

as well as

$$\int_0^t\int_\Omega |\nabla v|^2 \le \frac{1}{2}\int_\Omega v_0^2 \qquad \textit{for all } t>0, \tag{4.2}$$

and moreover ther exists a null set $N\subset(0,\infty)$ with the property that whenever $t\in(0,\infty)\setminus N$, we have

$$\int_{\mathcal{S}}\int_\Omega v \ge |\mathcal{S}|\int_\Omega v(\cdot,t) \qquad \textit{for all measurable } \mathcal{S}\subset(0,t). \tag{4.3}$$

Proof. On testing the second equation in (2.10) against 1 and v_ε, respectively, we see that for all $\varepsilon\in(0,1)$ we have

$$\int_\Omega v_\varepsilon(\cdot,t)+\int_s^t\int_\Omega f_\varepsilon(u_\varepsilon)v_\varepsilon = \int_\Omega v_\varepsilon(\cdot,s) \qquad \text{whenever } 0\le s<t<\infty, \tag{4.4}$$

that is,

$$\int_0^t\int_\Omega f_\varepsilon(u_\varepsilon)v_\varepsilon = \int_\Omega v_0 - \int_\Omega v_\varepsilon(\cdot,t) \le \int_\Omega v_0 \qquad \text{for all } t>0 \tag{4.5}$$

as well as

$$\int_0^t\int_\Omega |\nabla v_\varepsilon|^2 = \frac{1}{2}\int_\Omega v_0^2 - \frac{1}{2}\int_\Omega v_\varepsilon^2(\cdot,t) - \int_0^t\int_\Omega f_\varepsilon(u_\varepsilon)v_\varepsilon^2 \le \frac{1}{2}\int_\Omega v_0^2 \qquad \text{for all } t>0. \tag{4.6}$$

Thanks to the pointwise convergence statemets in (2.25) and (2.30), (4.1) results from (4.5) due to Fatou's lemma, whereas the weak convergence property (2.32) in conjunction with (4.6) entails (4.2) by a standard argument based on lower semicontinuity of the norm in $L^2(\Omega\times(0,t))$ with respect to weak convergence.

To verify (4.3), recalling (2.33) we fix a null set $N\subset(0,\infty)$ such that for any $t\in(0,\infty)\setminus N$ we have $v_\varepsilon(\cdot,t)\to v(\cdot,t)$ in $L^1(\Omega)$ as $\varepsilon=\varepsilon_k\searrow 0$, where $(\varepsilon_k)_{k\in\mathbb{N}}\subset(0,1)$ denotes the sequence provided by Lemma 2.6. Then given an arbitrary $t\in(0,\infty)\setminus N$ we first observe that $\int_\Omega v_\varepsilon(\cdot,s)\ge\int_\Omega v_\varepsilon(\cdot,t)$ for all $s\in(0,t)$ by (4.4), so that if $\mathcal{S}\subset(0,t)$ is measurable, then

$$\int_{\mathcal{S}}\int_\Omega v_\varepsilon(x,s)dxds\ge\int_{\mathcal{S}}\int_\Omega v_\varepsilon(x,t)dxds=|\mathcal{S}|\int_\Omega v_\varepsilon(\cdot,t)$$

for all $\varepsilon\in(0,1)$. In view of (2.30) and our choice of t, we may take $\varepsilon=\varepsilon_k\searrow 0$ on both sides here to infer that indeed (4.3) is valid. $\square$

With this information at hand, we can now pass to the proof of our main result on additional regularity and asymptotic decay of v.

PROOF of Theorem 1.3. The first inclusion in (1.9) is obvious from (2.29), while the continuity feature in (1.9) can be seen by means of a standard argument relying on the boundedness of v and an application of the regularity property implied by (2.34) for arbitrary fixed $p>2$: For any such p, namely, the latter ensures that upon modification on a null set of times if necessary, v becomes an element of $C^0([0,\infty);(W^{1,p}(\Omega))^\star)$, which can easily be seen to satisfy $\|v(\cdot,t)\|_{L^\infty(\Omega)}\le c_1:=\|v\|_{L^\infty(\Omega\times(0,\infty))}$ actually for all $t\ge 0$, because the complement of null sets in $(0,\infty)$ is dense in $(0,\infty)$. Thus, if $t\in[0,\infty)$ and $(t_k)_{k\in\mathbb{N}}\subset[0,\infty)$ are such that $t_k\to t$ as $k\to\infty$, then $v(\cdot,t_k)\to v(\cdot,t)$ in $(W^{1,p}(\Omega))^\star$ as $k\to\infty$. Since $W^{1,p}(\Omega)$ is dense in $(L^\infty(\Omega))^\star\simeq L^1(\Omega)$, and since from the above we know that $v(\cdot,t_k)$ is bounded and hence relatively compact in $L^\infty(\Omega)$ with respect to the weak-$\star$ topology, this readily implies that $\int_\Omega v(\cdot,t_k)\varphi\to\int_\Omega v(\cdot,t)\varphi$ for all $\varphi\in L^1(\Omega)$ as $k\to\infty$, thus establishing (1.9).

Next, to verify (1.11) we first use Lemma 4.1 to see that

$$I_1:=\int_0^\infty\int_\Omega uv\qquad\text{and}\qquad I_2:=\int_0^\infty\int_\Omega|\nabla v|^2 \tag{4.7}$$

are both finite, and taking $K_1(m,2)>0$ and $T_1(M,m)>0$ from (1.7), with $m:=\int_\Omega u_0$ and $M:=\int_\Omega w_0$, we fix $\Sigma>0$ large such that $\ln\frac{\Sigma}{|\Omega|}\ge 2K_1(m,2)$ and define

$$\mathcal{S}(t):=\left\{s\in(0,t)\ \middle|\ \int_\Omega u^2(\cdot,s)\le\Sigma\right\}\qquad\text{for } t>T_1(M,m).$$

Then for any such t, (1.7) entails that

$$\frac{\ln\frac{\Sigma}{|\Omega|}}{2}\cdot t\ge K_1(m,2)t\ge\int_0^t\ln\left\{\frac{1}{|\Omega|}\int_\Omega\Big(u(\cdot,s)+1\Big)^2\right\}ds\ge\Big|(0,t)\setminus\mathcal{S}(t)\Big|\cdot\ln\frac{\Sigma}{|\Omega|},$$

from which it follows that

$$|\mathcal{S}(t)|\ge\frac{t}{2}\qquad\text{for all } t>T_1(M,m). \tag{4.8}$$

Now in order to derive (1.11) from the basic decay properties implicitly contained in (4.7), we recall that if we write $\overline{\varphi} := \frac{1}{|\Omega|}\int_\Omega \varphi$ for $\varphi \in L^1(\Omega)$, then the Poincaré inequality provides $c_2 > 0$ fulfilling

$$\|\varphi - \overline{\varphi}\|_{L^2(\Omega)} \le c_2 \|\nabla\varphi\|_{L^2(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

so that in

$$\begin{aligned} I_1 &\ge \int_{\mathcal{S}(t)}\int_\Omega uv \\ &= \int_{\mathcal{S}(t)}\int_\Omega u(x,s)\cdot\Big\{v(x,s) - \overline{v(\cdot,s)}\Big\}dxds + \int_{\mathcal{S}(t)}\int_\Omega \overline{v(\cdot,s)}\cdot u(x,s)dxds, \quad t > T_1(M,m), \end{aligned} \tag{4.9}$$

we can control the first summand on the right, also using the Cauchy-Schawrz inequality and the definition of $\mathcal{S}(t)$, so as to obtain

$$\begin{aligned} &\left|\int_{\mathcal{S}(t)}\int_\Omega u(x,s)\cdot\Big\{v(x,s) - \overline{v(\cdot,s)}\Big\}dxds\right| \\ &\qquad\le \int_{\mathcal{S}(t)}\Big\{\int_\Omega u^2(x,s)dx\Big\}^{\frac{1}{2}}\cdot\Big\{\int_\Omega \Big|v(x,s) - \overline{v(\cdot,s)}\Big|^2dx\Big\}^{\frac{1}{2}}ds \\ &\qquad\le \sqrt{\Sigma}\int_{\mathcal{S}(t)}\Big\|v(\cdot,s) - \overline{v(\cdot,s)}\Big\|_{L^2(\Omega)}ds \\ &\qquad\le c_2\sqrt{\Sigma}\int_{\mathcal{S}(t)}\int_{\mathcal{S}(t)}\|\nabla v(\cdot,s)\|_{L^2(\Omega)}ds \qquad \text{for all } t > T_1(M,m). \end{aligned} \tag{4.10}$$

As $\mathcal{S}(t)$ is contained in $(0,t)$ for all $t > T_1(M,m)$, another application of the Cauchy-Schwarz inequality shows that herein

$$c_2\sqrt{\Sigma}\int_{\mathcal{S}(t)}\|\nabla v(\cdot,s)\|_{L^2(\Omega)}ds \le c_2\sqrt{\Sigma}\cdot\sqrt{I_2|\mathcal{S}(t)|} \le c_2\sqrt{\Sigma I_2 t} \qquad \text{for all } t > T_1(M,m). \tag{4.11}$$

On the other hand, thanks to the mass conservation law (2.4), the monotonicity property (4.3) and (4.8), the last summand in (4.9) can be estimated from below according to

$$\begin{aligned} \int_{\mathcal{S}(t)}\int_\Omega \overline{v(\cdot,s)}\cdot u(x,s)dxds &= \int_{\mathcal{S}(t)}\overline{v(\cdot,s)}\cdot\Big\{\int_\Omega u(x,s)dx\Big\}ds \\ &= m\int_{\mathcal{S}(t)}\overline{v(\cdot,s)}ds \\ &= \frac{m}{|\Omega|}\int_{\mathcal{S}(t)}\int_\Omega v(x,s)dxds \\ &\ge \frac{m}{|\Omega|}\cdot\Big\{\int_\Omega v(x,t)dx\Big\}\cdot|\mathcal{S}(t)| \\ &\ge \frac{mt}{2|\Omega|}\cdot\int_\Omega v(x,t)dx \qquad \text{for all } t > T_1(M,m). \end{aligned}$$

Together with (4.10) and (4.11) inserted into (4.9), this ensures that

$$\frac{mt}{2|\Omega|} \cdot \int_\Omega v(x,t)dx \le I_1 + c_2\sqrt{\Sigma I_2 t} \qquad \text{for all } t > T_1(M,m)$$

and thereby shows that

$$\|v(\cdot,t)\|_{L^1(\Omega)} \le \frac{c_3}{\sqrt{t}} \qquad \text{for all } t > T_1(M,m) \tag{4.12}$$

with $c_3 := \frac{2|\Omega|}{m} \cdot (\frac{I_1}{\sqrt{T_1(M,m)}} + c_2\sqrt{\Sigma I_2})$. As for $p \in [1,\infty)$ we have

$$\|v(\cdot,t)\|_{L^p(\Omega)} \le \|v(\cdot,t)\|_{L^\infty(\Omega)}^{\frac{p-1}{p}} \|v(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{p}} \le c_1^{\frac{p-1}{p}} \|v(\cdot,t)\|_{L^1(\Omega)}^{\frac{1}{p}} \qquad \text{for all } t > T_1(M,m)$$

by the Hölder inequality and our definition of c_1, (4.12) implies (1.11).

Finally, if (1.10) was false, then once more by boundedness of $(v(\cdot,t))_{t>0}$ in $L^\infty(\Omega)$ we could find $v_\infty \in L^\infty(\Omega) \setminus \{0\}$ and $(\tilde{t}_k)_{k\in\mathbb{N}} \subset (0,\infty)$ such that $\tilde{t}_k \to \infty$ and $v(\cdot,\tilde{t}_k) \stackrel{\star}{\rightharpoonup} v_\infty$ in $L^\infty(\Omega)$ as $k \to \infty$. But then (1.11) applied to e.g. $p := 1$ would entail that on passing to a subsequence we could assume that $v(\cdot,\tilde{t}_k) \to 0$ a.e. in Ω as $k \to \infty$, so that by Egorov's theorem we could draw the absurd conclusion that $v_\infty = 0$ a.e. in Ω. $\Box$

5 Numerical illustrations

In this section we shall present some numerical experiments in order to illustrate the behavior of radially symmetric solutions to the Neumann initial-value problem for (1.1) in the unit disk $B_1(0) \subset \mathbb{R}^2$. To this end, we consider the correspondingly transformed problem

$$\begin{cases} u_t = \frac{1}{r}(ru_r)_r + \frac{\chi}{r}(ruw_r)_r, \qquad r \in (0,1),\ t>0, \\ w_t = \frac{1}{r}(rw_r)_r - w_r^2 + u, \qquad r \in (0,1),\ t>0, \\ u_r(0,t) = u_r(1,t) = w_r(0,t) = w_r(1,t) = 0, \qquad t>0, \\ u(r,0) = u_0(r), \quad w(r,0) = w_0(r), \qquad r \in (0,1), \end{cases} \tag{5.1}$$

for the unknown functions $u = u(r,t)$ and $w = w(r,t) := -\ln v(r,t)$, where $r = |x| \in [0,1]$, and where for definiteness we have chosen $\chi = 20$. The subsequent simulations were carried out using a time-explicit finite difference scheme on an equidistant grid in the variable r, with grid size 0.0002 and time step size 10^{-10}.

We first study the behavior of the solution emanating from initial data which are given by

$$u_0(r) := (1-10r)_+^2 \quad \text{and} \quad w_0(r) := 100 \cdot (1-e^{-5r}), \qquad r \in [0,1], \tag{5.2}$$

and plotted in Figure 1.

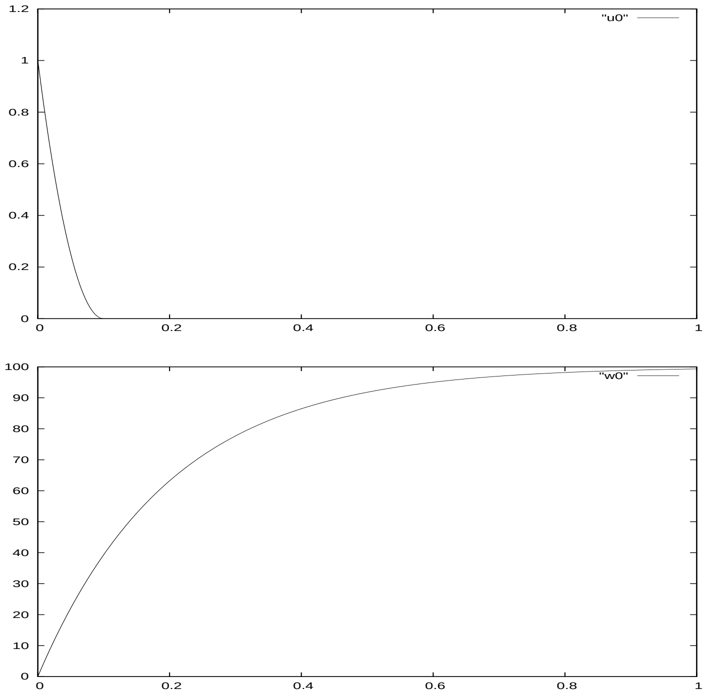
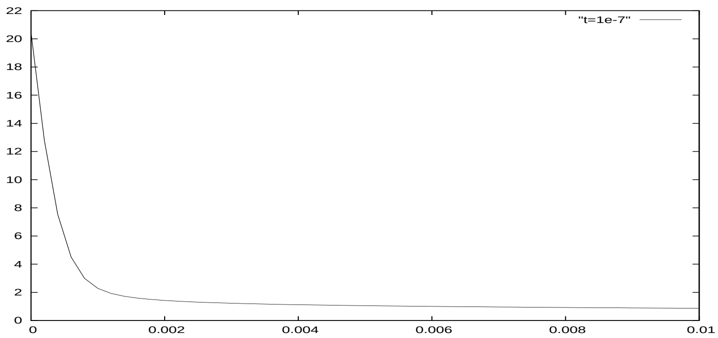


Figure 1: Initial data $u_0(r) := (1-10r)_+^2$ and $w_0(r) := 100(1-e^{-5r})$
Horizontal axis: $r = |x|$; vertical axis: $u_0(r)$ (left), $w_0(r)$ (right)

As Figure 2 illustrates, the initially present signal gradient, according to the positivity of w_{0r} hence reflecting attraction toward the origin, in this case seems sufficient to enforce a substantial mass aggregation during evolution: In fact, the conserved total mass $\int_{B_1(0)} u_0$, though already originally concentrated near $r = 0$, but yet to a rather mild extent, undergoes a considerable accumulation during an initial period. This may be interpreted as indicating that for more extreme choices of the initial data one may even expect the occurrence of genuine explosion phenomena, characterized by finite-time blow-up of e.g. the norm of $u(\cdot, t)$ in $L^\infty(B_1(0))$ and thus going beyond the capability of the numerical scheme used here.



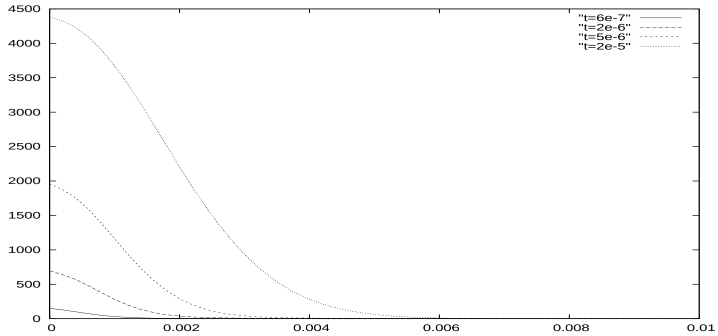


Figure 2: Solution behavior for $\chi = 20$, $u_0(r) := (1 - 10r)_+^2$ and $w_0(r) := 100(1 - e^{-5r})$ at times $t = 10^{-7}$ (left) as well as $t = 6 \cdot 10^{-7}, t = 2 \cdot 10^{-6}, t = 5 \cdot 10^{-6}, t = 2 \cdot 10^{-5}$ (right). Horizontal axis: $r = |x|$, restricted to the range $[0, 0.01]$; vertical axis: $u(r, t)$

On the other hand, Figure 3 combined with Figure 4 underlines that in accordance with the analytical results on relaxation obtained in Theorem 1.2, such types of singular behavior indeed should be transient: After a drastic initial increase, the spatial L^∞ norm of u settles down to much smaller magnitudes on larger time scales. Figure 3 moreover illustrates the convergence statement from Theorem 1.3 by indicating that whereas $v = e^{-w}$ initially satisfies $\|v(\cdot, 0)\|_{L^\infty(B_1(0))} = 1$ by (5.2), already at time $t = 0.1$ we have $\|v(\cdot, t)\|_{L^\infty(B_1(0))} \le e^{-11}$.

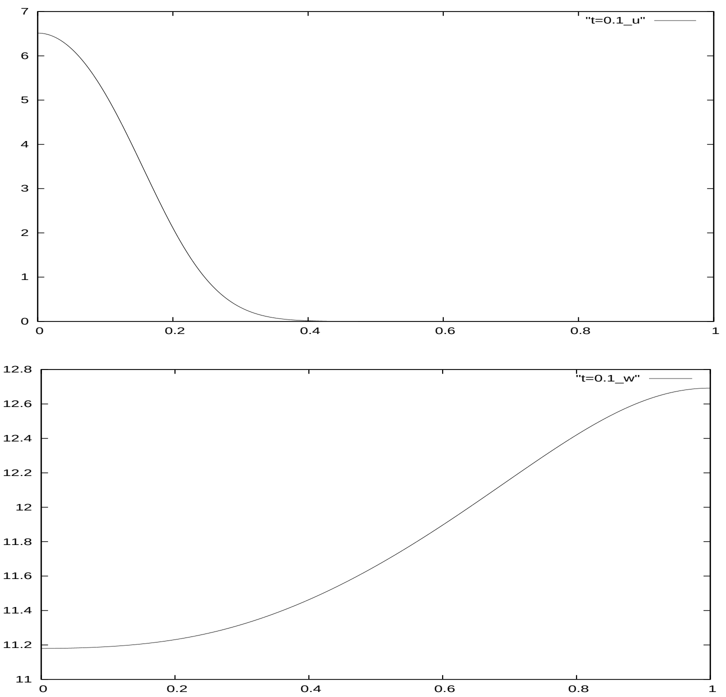


Figure 3: Behavior of the solution from Figure 2 at $t = 0.1$. Horizontal axis: $r = |x|$; vertical axis: $u(r, t)$ (left), $w(r, t)$ (right)

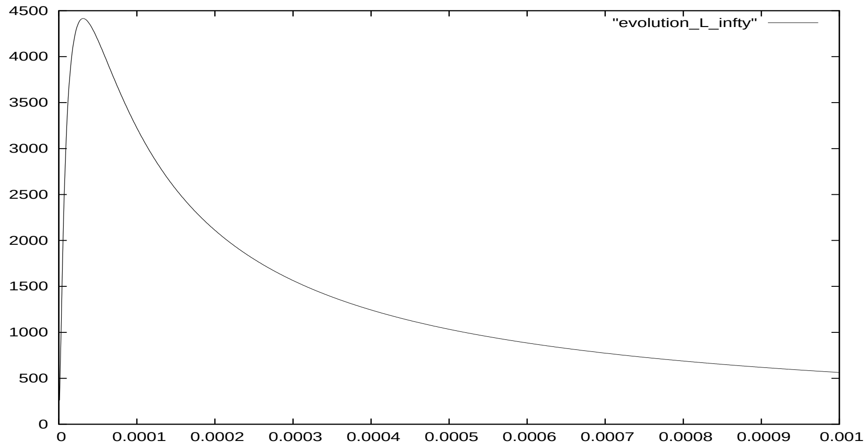


Figure 4: Time evolution of the norm of $u(\cdot,t)$ in $L^\infty(B_1(0))$ of the solution from Figure 2. Horizontal axis: t; vertical axis: $\|u(\cdot,t)\|_{L^\infty(B_1(0)))}$

In sharp contrast to the above, the solution behavior may be much more regular but yet nontrivial when the initial setting reflects a repulsive signal gradient near the origin. To illustrate this, we consider initial data with opposite monotonicity in the second component but essentially unchanged otherwise, as given by

$$u_0(r) := (1-10r)_+^2 \quad \text{and} \quad w_0(r) := 100e^{-5r}, \qquad r \in [0,1], \tag{5.3}$$

and depicted in Figure 5.

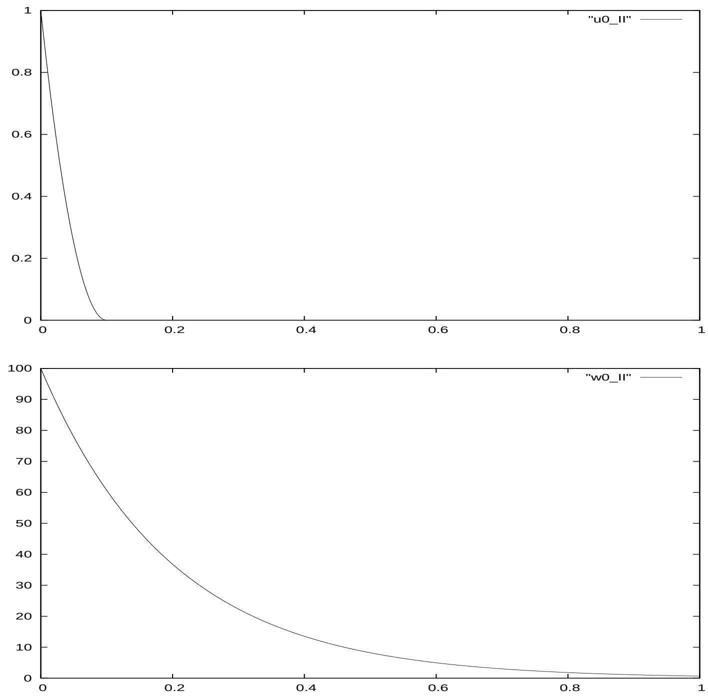


Figure 5: Initial data $u_0(r) := (1-10r)_+^2$ and $w_0(r) := 100e^{-5r}$
Horizontal axis: $r = |x|$; vertical axis: $u_0(r)$ (left), $w_0(r)$ (right)

According to the initially repulsive character of the origin, it may then be not surprising that at least on short time scales, the spatial maximum $\|u(\cdot,t)\|_{L^\infty(B_1(0))}$ does not exhibit any type of singular behavior comparable to that in the above situation, and in fact Figure 6 indicates that this quantity even decreases monotonically in time.

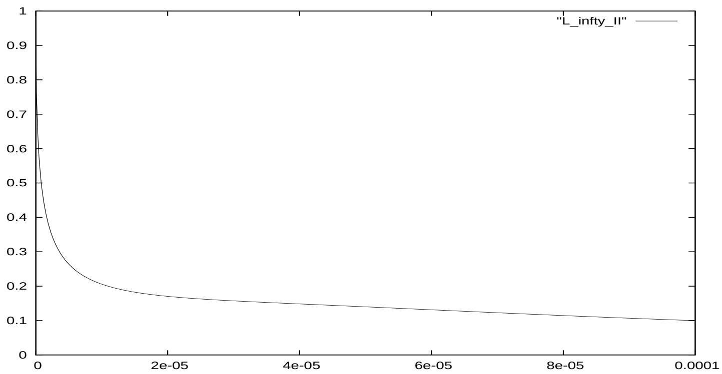


Figure 6: Time evolution of the norm of $u(\cdot,t)$ in $L^\infty(B_1(0)))$ of the solution from Figure 5.
Horizontal axis: t; vertical axis: $\|u(\cdot,t)\|_{L^\infty(B_1(0)))}$

As shown in Figure 7, however, contrary to the behavior seen in Figure 2 and Figure 3 the first solution component now loses its spatial monotonicity with respect to r; instead, it rather exhibits a wave-like evolution reflecting mass transport away from the origin, with the total mass essentially concentrated in thin annuli at each fixed time, and with these annuli veering away from the origin at a speed which apparently decreases with time.

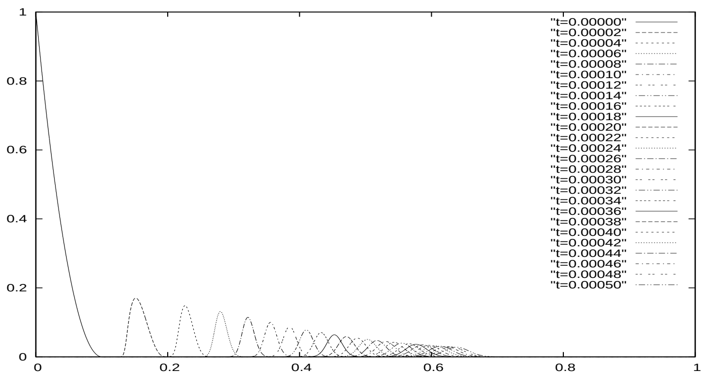


Figure 7: Solution behavior for the initial data from (5.3) at times $t = 0.00000, t = 0.00002, ..., t = 0.00050$.
Horizontal axis: $r = |x|$; vertical axis: $u(r,t)$

In accordance with Theroem 1.2, Figure 8 finally inter alia confirms that the signal gradient, considerably large at the original state, substantially relaxes during this process.

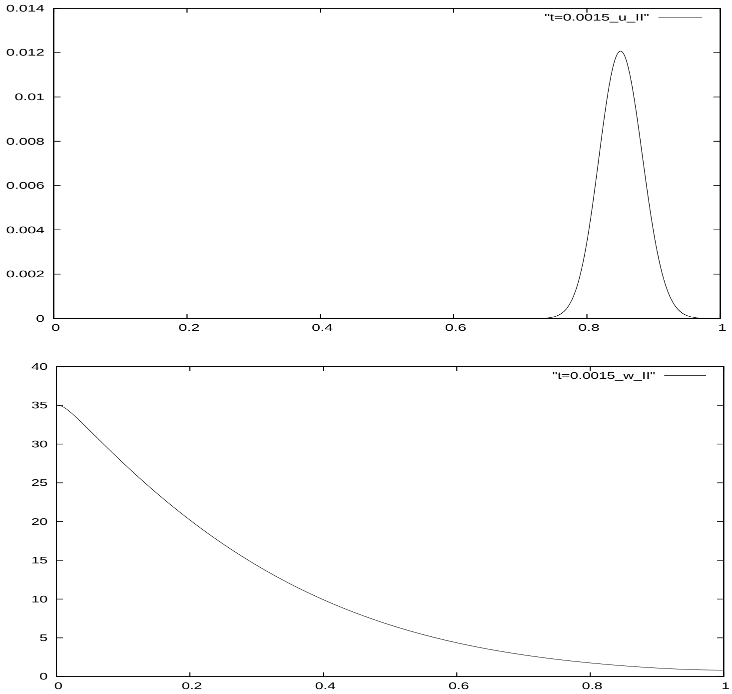


Figure 8: Behavior of the solution from Figure 5 at $t = 0.0015$.
Horizontal axis: $r = |x|$; vertical axis: $u(r, t)$ (left), $w(r, t)$ (right)

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