A degenerate chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up

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Abstract

This paper aims at providing a first step toward a qualitative theory for a new class of chemotaxis models derived from the celebrated Keller-Segel system, with the main novelty being that diffusion is nonlinear with flux delimiter features. More precisely, as a prototypical representative of this class we study radially symmetric solutions of the parabolic-elliptic system

$$\begin{cases} u_t = \nabla \cdot \Big(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big) - \chi\,\nabla \cdot \Big(\dfrac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\Big), \\ 0 = \Delta v - \mu + u, \end{cases}$$

under the initial condition $u|_{t=0} = u_0 > 0$ and no-flux boundary conditions in balls $\Omega \subset \mathbb{R}^n$, where $\chi > 0$ and $\mu := \frac{1}{|\Omega|}\int_\Omega u_0$.

The main results assert the existence of a unique classical solution, extensible in time up to a maximal $T_{max} \in (0,\infty]$ which has the property that

$$\text{if} \quad T_{max} < \infty \quad \text{then} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \qquad (\star)$$

The proof of this is mainly based on comparison methods, which firstly relate pointwise lower and upper bounds for the spatial gradient u_r to L^∞ bounds for u and to *upper bounds* for $z := \frac{u_t}{u}$; secondly, another comparison argument involving nonlocal nonlinearities provides an appropriate control of z_+ in terms of bounds for u and $|u_r|$, with suitably mild dependence on the latter.

As a consequence of $(\star)$, by means of suitable a priori estimates it is moreover shown that the above solutions are global and bounded when either

$$n \ge 2 \ \text{ and } \chi < 1, \qquad \text{or} \qquad n = 1,\ \chi > 0 \ \text{ and } m < m_c,$$

with $m_c := \frac{1}{\sqrt{\chi^2-1}}$ if $\chi > 1$ and $m_c := \infty$ if $\chi \le 1$.
That these conditions are essentially optimal will be shown in a forthcoming paper in which $(\star)$ will be used to derive complementary results on the occurrence of solutions blowing up in finite time with respect to the norm of u in $L^\infty(\Omega)$.

Key words: chemotaxis; flux limitation; degenerate diffusion
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1 Introduction

Keller-Segel systems with flux limitation. The celebrated model by Keller and Segel [24, 25] was heuristically derived to model growth phenomena mediated by a chemoattractant, specifically the aggregation of dictyostelium discoideum due to an attractive chemical substance. The general structure of the model is as follows:

$$\begin{cases} u_t = \nabla\left(D_u(u,v)\nabla u - S(u,v)u\nabla v\right) + H_1(u,v), \\ \\ v_t = D_v\Delta v + H_2(u,v), \end{cases} \tag{1.1}$$

where $u = u(x,t)$ denotes the cell (or organism) density at position x and time t, and $v = v(x,t)$ is the density of the chemoattractant. Here the function S measures the chemotactic sensitivity, the positive functions D_u and D_v represent the diffusivity of the cells and of the chemoattractant, respectively, and H_1 and H_2 model source terms related to interactions. In a more general framework in which diffusions are not isotropic, D_u and S can be positive definite matrices. See the survey by Hillen and Painter [21] for a review of modeling issues based on the classical approach of continuum mechanics closed by empirical models for the closure of conservation equations. The essay by Horstmann [22] provides an additional source of information concerning modeling and applications in biology. The recent survey [9] provides a review and qualitative analysis of a variety of mathematical problems and multiscale derivations of the original model as well as of some recent developments such as the specific one treated in this paper.

On the other hand, a natural question can be posed, namely if the use of parabolic models is consistent with the physics of the phenomena under consideration, or if, for instance, the use of hyperbolic models can be more appropriate. Or even within the approach by parabolic equations, if linear models are acceptable, while in the nonlinear case whether one should consider degenerate parabolic equations characterized by a finite propagation velocity. Intuitively, the answer is that phenomena with finite propagation velocity should be captured by an appropriate choice of nonlinear diffusion terms [31]. A conceivable approach leads to consider functions $D_u(u,v)$ and $D_v(u,v)$ not only depending on u and v, but also on their derivatives in space and time. A recent study in this direction [7] has shown that macroscopic models can be obtained from the underlying description at the scale of cells delivered by suitable developments of kinetic theory methods. More in details, appropriate models of cell-cell interaction lead to macroscopic expressions for diffusion and cross-diffusion with nonlinear limited flux terms of the type

$$\nabla\cdot\left(D_u(u,v)\frac{u\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}}\right) \qquad \text{and} \qquad \nabla\cdot\left(S(u,v)\frac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\right),$$

respectively, with ν denoting the kinematic viscosity and c the maximum speed of propagation, so that in combination with an adequate equation for the evolution of the chemoattractant, this appraoch suggests to consider models of type

$$\begin{cases} u_t = \nabla\cdot\left(D_u(u,v)\dfrac{u\nabla u}{\sqrt{u^2+\frac{\nu^2}{c^2}|\nabla u|^2}} - S(u,v)\dfrac{u\nabla v}{\sqrt{1+|\nabla v|^2}}\right) + H_1(u,v), \\ v_t = D_v\Delta v + H_2(u,v), \end{cases} \tag{1.2}$$

as consistent modifications of the classical Keller-Segel system. This idea, which is somehow related to the optimal transport framework [12], can be motivated by a natural assumption of cell dynamics, where overcrowding is naturally avoided [13]. Furthermore, the introduction of this type of terms is founded in the assumption that particles do not diffuse arbitrarily in space but, on the contrary, move through some privileged ways such as the border of cells. Moreover, in this new approach the non-physical diffusion is eliminated and the population moves with a finite speed of propagation, which is one of the intrinsic characteristics. Indeed, the qualitative analysis of related systems with limited flux [3, 4] as well as some extensions to biological contexts (transport of morphogens) has been recently explored [2], inter alia confirming the expected movement of fronts at finite speeds.

Boundedness vs. blow-up. In the framework of chemotaxis systems, however, a different qualitative aspect seems even more important, namely the ability of the respective system to spontaneously generate structures. In this regard, the classical Keller-Segel system, as obtained from (1.1) on letting $D_u \equiv D_v \equiv S \equiv 1$, $H \equiv 0$ and $K(u,v) = u - v$, is known to have the property that some solutions reflect such aggregation processes even in the extreme mathematical sense of finite-time blow-up of some solutions when either the spatial dimension n satisfies $n \geq 3$ ([34]), or when $n = 2$ and the total mass of cells is suitably large ([20, 27]); on the other hand, if either $n \geq 3$ and the initial data fulfill appropriate smallness conditions, or $n = 2$ and $\int_\Omega u(\cdot, 0)$ is small, or if $n = 1$, then for various types of initial-boundary value problems, global bounded solutions are known to exist ([15, 28, 33, 29, 14]).

As for Keller-Segel-type models with flux limitations, the corresponding problem appears to be unsolved, and it is the goal of the present paper to present a first step into a qualitative theory for such systems, with a particular focus on the question whether solutions exist globally, as conjectured in [8], or whether blow-up in finite time may occur for some initial data. Specifically, we will consider the apparently most prototypical among the systems (1.2) in its parabolic-elliptic simplification, as suggested in [23]); more precisely, we shall be concerned with the initial-boundary value problem

$$\begin{cases} u_t = \nabla \cdot \left(\dfrac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} \right) - \chi \nabla \cdot \left(\dfrac{u \nabla v}{\sqrt{1 + |\nabla v|^2}} \right), \qquad x \in \Omega, \ t > 0, \\ 0 = \Delta v - \mu + u, \qquad x \in \Omega, \ t > 0, \\ \left(\dfrac{u \nabla u}{\sqrt{u^2 + |\nabla u|^2}} - \chi \dfrac{u \nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \cdot \nu = 0, \qquad x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{1.3}$$

in a ball $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 1$, where $\chi > 0$ indicates the strength of chemotactic cross-diffusion. In order to further simplify the analysis, we shall assume the initial data to satisfy

$$u_0 \in C^3(\bar{\Omega}) \quad \text{is radially symmetric and positive in } \bar{\Omega} \text{ with } \tfrac{\partial u_0}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1.4}$$

so that the spatial average

$$\mu := \frac{1}{|\Omega|} \int_\Omega u_0(x) dx \tag{1.5}$$

is positive.

Main results. In this framework, the first of our main results asserts local existence of a uniquely determined classical solution. In its most crucial part, however, the following theorem furthermore provides the extensibility criterion (1.6) which will be of great importance both for deriving global existence in Theorem 1.2 below, as well as for characterizing the asymptotic behavior of non-global solutions near their blow-up time [10].

Theorem 1.1 *Suppose that u_0 complies with (1.4). Then there exist $T_{max} \in (0, \infty]$ and a uniquely determined pair (u, v) of positive radially symmetric functions $u \in C^{2,1}(\bar{\Omega} \times [0, T_{max}))$ and $v \in C^{2,0}(\bar{\Omega} \times [0, T_{max}))$ which solve (1.3) classically in $\Omega \times (0, T_{max})$, and which are such that*

$$\text{if} \quad T_{max} < \infty \quad \text{then} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{1.6}$$

In particular, (1.6) rules out the occurrence of any *gradient blow-up* phenomenon in the present framework; it is thus impossible that ∇u becomes unbounded in finite time, whereas u itself remains bounded. In view of the complex evolution mechanism in (1.3), inter alia involving doubly degenerate diffusion, this conclusion seems far from trivial; indeed, various types of gradient-dependent nonlinearities and degeneracies are known to enforce unboundedness of gradients for some solutions even in scalar reaction-diffusion equations [5, 26, 30]. Moreover, the additionally present cross-diffusive interaction apparently rules out the accessibility of (1.3) to most of the techniques well-established in contexts of scalar parabolic equations with diffusion degeneracies of related type, such as e.g. the mean curvature flow equation and derivatives thereof, among others [17, 18], or [11].

A natural next goal consists in identifying circumstances under which the above solutions are global. Going in this direction, the second of our main results provides conditions on the parameter χ in (1.3) and, when $n = 1$, on the mass level m, which turn out to be sufficient not only for global extensibility, but also for uniform boundedness of all solutions emanating from initial data u_0 with $\int_\Omega u_0 = m$.

Theorem 1.2 *Assume that u_0 satisfies (1.4), and that either*

$$n \geq 2 \qquad \text{and} \qquad \chi < 1, \tag{1.7}$$

or

$$n = 1, \quad \chi > 0 \qquad \text{and} \qquad \int_\Omega u_0 < m_c, \tag{1.8}$$

where in the case $n = 1$ we have set

$$m_c := \begin{cases} \frac{1}{\sqrt{\chi^2 - 1}} & \text{if } \chi > 1, \\ +\infty & \text{if } \chi \leq 1. \end{cases} \tag{1.9}$$

Then the problem (1.3) possesses a unique global classical solution $(u, v) \in C^{2,1}(\bar{\Omega} \times [0, \infty)) \times C^{2,0}(\bar{\Omega} \times [0, \infty))$ which is radially symmetric and such that for some $C > 0$ we have

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \|v(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all } t > 0. \tag{1.10}$$

As a first and immediate conclusion thereof, we underline that when $\chi < 1$, in stark contrast to the original Keller-Segel model, the system (1.3) does not exhibit any critical mass phenomenon, nor any phenomenon of critical sizes of initial data with respect to global existence of solutions. Let us secondly mention that the conditions (1.7) and (1.8), as identified above, are in fact essentially optimal for the obtained conclusion: Indeed, in [10] the picture will in this respect be basically completed by showing that if $\chi > 1$ then in both cases $n = 1$ with $m > m_c$, and $n \geq 2$, some initial data can be constructed such that the corresponding solutions will blow up in finite time. Together with the latter, our results thus indicate that in comparison to the original Keller-Segel system, the occurrence of a critical mass phenomenon is shifted from the two-dimensional to the one-dimensional setting, whereas in the case $n \geq 2$ we rather encounter a *critical sensitivity phenomenon* in that the size of χ becomes the crucial quantity to determine whether or not blow-up may happen.

Main ideas. Excluding gradient blow-up. In view of the doubly degenerate structure of the diffusion operator $\nabla \cdot \left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right)$ in (1.3), standard theory yields local existence and extensibility as long as u remains uniformly positive and both u and ∇u remain bounded (Lemma 2.1), where thanks to our positivity assumption on u_0, a corresponding lower bound for u can readily be obtained (Lemma 3.2).

The crucial part in the derivation of Theorem 1.1 will thus consist in ruling out the possibility of gradient blow-up, and in our approach toward this we will substantially make use of the radial symmetry of our solutions: Based on two different interpretations of the equation satisfied by u_r as linear inhomogeneous parabolic equations (Lemma 2.3), under the standing assumption that u is non-global but remains bounded we will first obtain a uniform lower bound for u_r by a comparison argument (Lemma 2.3), and thereafter develop this into a bound for $|u_r|$ in Section 5.

The latter step itself will involve the quantity $z := \frac{u_t}{u}$, as known to be of great importance on various types of different nonlinear diffusion equations [6, 31]. In the present context, we shall see that u_r can indeed be controlled in terms of the *positive part* z_+ of z through an inequality of the form

$$\|u_r(\cdot,t)\|_{L^\infty((0,R))} \leq C \cdot \Big(1 + \|z_+\|_{L^\infty((0,R)\times(0,t))}\Big) \qquad \text{for all } t \in (0, T_{max}), \tag{1.11}$$

where $T_{max} \in (0, \infty)$ denotes the maximal existence time (Corollary 5.3). This will be achieved by splitting the interval $(0, R)$ in two parts and first performing a testing procedure to estimate u_r in the corresponding inner region in certain weighted Lebesgue spaces and taking limits appropriately (Lemma 5.1), whereupon a comparison argument in the associated outer region will complete the proof of (1.11) (Lemma 5.2).

In order to complete the proof of Theorem 1.1 by providing a suitable estimate for z_+, we shall make use of the observation that z satisfies the *one-sided* nonlocal parabolic inequality

$$z_t(r,t) \leq \mathcal{L}z + d \cdot \Big(1 + \|z_+\|_{L^\infty((0,R)\times(0,t))}\Big)$$

with some $d > 0$ and some homogeneous linear elliptic operator $\mathcal{L}$ (Lemma 5.5). In fact, employing a maximum principle-type argument will show that this implies a pointwise upper bound for z (Lemma 5.6), which in conjunction with (1.11) will prove Theorem 1.1.

Thanks to the mild extensibility criterion (1.6) thus gained, the proof of Theorem 1.2 thus actually reduces to the derivation of suitable a priori bounds for solutions with respect to the norm of u in

$L^\infty(\Omega)$. This will be accomplished in the respective cases detailed in Theorem 1.2 by means of an essentially straightforward adaptation of the Moser-Alikakos iteration technique to the present setting in Section 6.

2 Preliminaries

2.1 Local existence and a first extensibility criterion

To begin with, let us suitably reduce (1.3), locally in time, so as to become accessible to standard existence theory. We thereby obtain the following result on local existence of a smooth solution to (1.3), extensible as long as such a reduction is possible. As a by-product, this procedure yields the first basic extensibility criterion (2.2) the improvement of which will be the main objective of the subsequent Sections 3-5.

Lemma 2.1 *Suppose that u_0 satisfies (1.4). Then there exist $T_{max} \in (0,\infty]$ and a uniquely determined pair $(u.v)$ of radially symmetric positive functions*

$$u \in C^{2,1}(\bar\Omega \times [0,T_{max})), \qquad v \in C^{2,0}(\bar\Omega \times [0,T_{max})), \tag{2.1}$$

which solve (1.3) classically in $\Omega \times (0,T_{max})$, and which are such that

$$\textit{if } T_{max} < \infty \quad \textit{then either} \quad \liminf_{t \nearrow T_{max}} \inf_{x\in\Omega} u(x,t) = 0 \quad \textit{or} \quad \limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \infty. \tag{2.2}$$

Proof. We let

$$\varepsilon := \min\left\{ \frac{1}{2}\inf_{x\in\Omega} u_0(x)\,,\, \frac{1}{2\|u_0\|_{L^\infty(\Omega)}}\,,\, \frac{1}{2\|\nabla u_0\|_{L^\infty(\Omega)}} \right\} \tag{2.3}$$

and take cut-off functions $\psi_\varepsilon \in C^\infty(\mathbb{R})$ and $\phi_\varepsilon \in C^\infty(\mathbb{R})$ satisfying

$$\frac{\varepsilon}{2} \le \psi_\varepsilon(s) \le \frac{2}{\varepsilon} \quad \text{for all } s\in\mathbb{R} \qquad \text{and} \qquad \psi_\varepsilon(s) = s \quad \text{for all } s \in \Big(\varepsilon\,,\,\frac{1}{\varepsilon}\Big),$$

as well as

$$\phi_\varepsilon(s) \le \frac{2}{\varepsilon} \quad \text{for all } s\in\mathbb{R} \qquad \text{and} \qquad \phi_\varepsilon(s) = s \quad \text{for all } s \le \frac{1}{\varepsilon}\cdot$$

Then

$$a_\varepsilon(s,p) := \frac{\psi_\varepsilon(s)}{\sqrt{\psi_\varepsilon^2(s) + \phi_\varepsilon^2(|p|)}}, \qquad s\in\mathbb{R},\ p\in\mathbb{R}^n,$$

defines a function $a_\varepsilon \in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ fulfilling

$$a_\varepsilon(s,p) \le \frac{\psi_\varepsilon(s)}{\sqrt{\psi_\varepsilon^2(s)}} = 1 \quad \text{for all } s\in\mathbb{R} \text{ and } p\in\mathbb{R}^n$$

and

$$a_\varepsilon(s,p) \ge \frac{\frac{\varepsilon}{2}}{\sqrt{(\frac{2}{\varepsilon})^2 + (\frac{2}{\varepsilon})^2}} = \frac{\varepsilon^2}{4\sqrt{2}} \quad \text{for all } s\in\mathbb{R} \text{ and } p\in\mathbb{R}^n.$$

We can therefore adapt a fixed point argument which is well-established in the existence theory of parabolic-elliptic chemotaxis systems (cf. [16] or [19], for instance) to find $T_\varepsilon > 0$ such that the problem

$$\begin{cases} u_t = \nabla \cdot \Big(\dfrac{\psi_\varepsilon(u)\nabla u}{\sqrt{\psi_\varepsilon^2(u) + \phi_\varepsilon^2(|\nabla u|)}} \Big) - \chi \nabla \cdot \Big(\dfrac{u \nabla v}{\sqrt{1+|\nabla v|^2}} \Big), \qquad x \in \Omega,\ t \in (0, T_\varepsilon), \\ 0 = \Delta v - \mu + u, \qquad x \in \Omega,\ t \in (0, T_\varepsilon), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t \in (0, T_\varepsilon), \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases}$$

possesses a unique classical solution $(u_\varepsilon, v_\varepsilon)$ such that $u_\varepsilon \in C^{2,1}(\bar\Omega \times [0, T_\varepsilon))$ and $v_\varepsilon \in C^{2,0}(\bar\Omega \times [0, T_\varepsilon))$, and such that both u_ε and v_ε are radially symmetric and positive. Furthermore, since $2\varepsilon \le u_0 \le \frac{1}{2\varepsilon}$ and $|\nabla u_0| \le \frac{1}{2\varepsilon}$ in Ω according to our choice of ε, by continuity of u_ε and ∇u_ε in $\bar\Omega \times [0, T_\varepsilon)$ we can find $\tilde T_\varepsilon \in (0, T_\varepsilon)$ such that

$$\varepsilon \le u_\varepsilon \le \frac{1}{\varepsilon} \quad \text{and} \quad |\nabla u_\varepsilon| \le \frac{1}{\varepsilon} \qquad \text{in } \Omega \times (0, \tilde T_\varepsilon).$$

In particular, this implies that $\psi_\varepsilon(u_\varepsilon) = u_\varepsilon$ and $\phi_\varepsilon(|\nabla u_\varepsilon|) = |\nabla u_\varepsilon|$ in $\Omega \times (0, \tilde T_\varepsilon)$, and that thus $a_\varepsilon(u_\varepsilon) = \frac{u_\varepsilon}{\sqrt{u_\varepsilon^2 + |\nabla u_\varepsilon|^2}}$ in this region, meaning that $(u_\varepsilon, v_\varepsilon)$ actually solves the original problem (1.3) in $\Omega \times (0, \tilde T_\varepsilon)$.
Finally, in view of the dependence of ε on u_0 as expressed in (2.3), a standard extensibility argument yields that the above solution can be continued so as to exist up to some maximal time $T_{max} \le \infty$ in such a way that (2.2) is valid. □

2.2 Radial solutions

Since all our solutions are radially symmetric, whenever this appears convenient me may without any danger of confusion utilize the notation $u(r,t)$ and $v(r,t)$ instead of $u(x,t)$ and $v(x,t)$, respectively, where $r = |x| \in (0, R)$.

In this particular radial setting, u actually fulfills a favorable parabolic equation specified in the following lemma.

Lemma 2.2 *Assume (1.4). Then the solution of (1.3) satisfies*

$$\begin{aligned} u_t &= \frac{u^3 u_{rr}}{\sqrt{u^2+u_r^2}^3} + \frac{u_r^4}{\sqrt{u^2+u_r^2}^3} + \frac{n-1}{r} \cdot \frac{u u_r}{\sqrt{u^2+u_r^2}} \\ &\quad -\chi \frac{u_r v_r}{\sqrt{1+v_r^2}} - \chi \frac{u(\mu - u)}{\sqrt{1+v_r^2}^3} - \chi \cdot \frac{n-1}{r} \cdot \frac{u v_r^3}{\sqrt{1+v_r^2}^3}, \end{aligned} \tag{2.4}$$

for all $r \in (0, R)$ and $t \in (0, T_{max})$.

PROOF. We differentiate on the right-hand side of the first equation in (1.3) to obtain

$$
\begin{aligned}
u_t &= \frac{1}{r^{n-1}} \cdot \Big(r^{n-1} \frac{u u_r}{\sqrt{u^2+u_r^2}}\Big)_r - \frac{\chi}{r^{n-1}} \cdot \Big(r^{n-1} \frac{u v_r}{\sqrt{1+v_r^2}}\Big)_r \\
&= \frac{u u_{rr}}{\sqrt{u^2+u_r^2}} + \frac{u_r^2}{\sqrt{u^2+u_r^2}} - \frac{1}{2} \cdot \frac{u u_r (2u u_r + 2u_r u_{rr})}{\sqrt{u^2+u_r^2}^3} + \frac{n-1}{r} \cdot \frac{u u_r}{\sqrt{u^2+u_r^2}} \\
&\quad -\chi \frac{u v_{rr}}{\sqrt{1+v_r^2}} - \chi \frac{u_r v_r}{\sqrt{1+v_r^2}} + \frac{1}{2} \cdot \chi \frac{u v_r \cdot 2 v_r v_{rr}}{\sqrt{1+v_r^2}^3} - \chi \cdot \frac{n-1}{r} \cdot \frac{u v_r}{\sqrt{1+v_r^2}}
\end{aligned}
\tag{2.5}
$$

for all $r \in (0,R)$ and $t \in (0,T_{max})$. Here we can rearrange

$$
\begin{aligned}
&\frac{u u_{rr}}{\sqrt{u^2+u_r^2}} + \frac{u_r^2}{\sqrt{u^2+u_r^2}} - \frac{1}{2} \cdot \frac{u u_r (2u u_r + 2u_r u_{rr})}{\sqrt{u^2+u_r^2}^3} \\
&\qquad = \frac{u u_{rr}}{\sqrt{u^2+u_r^2}^3} \cdot \Big\{(u^2+u_r^2) - u_r^2\Big\} + \frac{u_r^2}{\sqrt{u^2+u_r^2}^3} \cdot \Big\{(u^2+u_r^2) - u^2\Big\} \\
&\qquad = \frac{u^3 u_{rr}}{\sqrt{u^2+u_r^2}^3} + \frac{u_r^4}{\sqrt{u^2+u_r^2}^3}
\end{aligned}
$$

and, similarly,

$$
\begin{aligned}
&-\chi \frac{u v_{rr}}{\sqrt{1+v_r^2}} + \frac{1}{2} \cdot \chi \frac{u v_r \cdot 2 v_r v_{rr}}{\sqrt{1+v_r^2}^3} - \chi \cdot \frac{n-1}{r} \cdot \frac{u v_r}{\sqrt{1+v_r^2}} \\
&\qquad = -\chi \frac{u v_{rr}}{\sqrt{1+v_r^2}^3} \cdot \Big\{(1+v_r^2) - v_r^2\Big\} - \chi \cdot \frac{n-1}{r} \cdot \frac{u v_r}{\sqrt{1+v_r^2}^3} \cdot (1+v_r^2) \\
&\qquad = -\chi \frac{u(v_{rr} + \frac{n-1}{r} v_r)}{\sqrt{1+v_r^2}^3} - \chi \cdot \frac{n-1}{r} \cdot \frac{u v_r^3}{\sqrt{1+v_r^2}^3}
\end{aligned}
$$

for $r \in (0,R)$ and $t \in (0,T_{max})$. Since $v_{rr} + \frac{n-1}{r} v_r = \mu - u$ by (1.3), the identity (2.4) thus results from (2.5). $\square$

We next differentiate (2.4) to obtain a corresponding equation for u_r. Here suitable arrangements will lead to the two alternative interpretations (2.7) and (2.10) thereof as linear inhomogeneous parabolic equations. The first of these will be used to establish an estimate *from below* for u_r in Lemma 4.1 by a straightforward comparison argument, whereas upon some more involved preparations, on the basis of the latter we will apply another comparison procedure to derive a certain *upper* bound for u_r in Lemma 5.2.

Lemma 2.3 *Assume (1.4). Then*

$$
\begin{aligned}
u_{rt} &= \frac{u^3 u_{rrr}}{\sqrt{u^2+u_r^2}^3} + 3\frac{u^2 u_r^3 u_{rr}}{\sqrt{u^2+u_r^2}^5} - 3\frac{u^3 u_r u_{rr}^2}{\sqrt{u^2+u_r^2}^5} \\
&\quad +4\frac{u^2 u_r^3 u_{rr}}{\sqrt{u^2+u_r^2}^5} + \frac{u_r^5 u_{rr}}{\sqrt{u^2+u_r^2}^5} - 3\frac{u u_r^5}{\sqrt{u^2+u_r^2}^5}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{n-1}{r^2}\cdot\frac{uu_r}{\sqrt{u^2+u_r^2}}+\frac{n-1}{r}\cdot\frac{u^3u_{rr}}{\sqrt{u^2+u_r^2}^3}+\frac{n-1}{r}\frac{u_r^4}{\sqrt{u^2+u_r^2}^3}\\
&-\chi\mu\frac{u_r}{\sqrt{1+v_r^2}^3}+2\chi\frac{uu_r}{\sqrt{1+v_r^2}^3}+3\chi\mu\frac{uv_rv_{rr}}{\sqrt{1+v_r^2}^5}-3\chi\frac{u^2v_rv_{rr}}{\sqrt{1+v_r^2}^5}\\
&-\chi\frac{u_{rr}v_r}{\sqrt{1+v_r^2}}-\chi\frac{u_rv_{rr}}{\sqrt{1+v_r^2}}+\chi\frac{u_rv_r^2v_{rr}}{\sqrt{1+v_r^2}^3}\\
&+\chi\cdot\frac{n-1}{r^2}\cdot\frac{uv_r^3}{\sqrt{1+v_r^2}^3}-\chi\cdot\frac{n-1}{r}\cdot\frac{u_rv_r^3}{\sqrt{1+v_r^2}^3}-3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^5}
\end{aligned}
\tag{2.6}
$$

for all $r\in(0,R)$ and $t\in(0,T_{max})$. In particular,

$$
(\mathcal{P}u_r)(r,t)=0\quad\text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}),
\tag{2.7}
$$

where the inhomogeneous linear parabolic operator $\mathcal{P}$ is defined by

$$
(\mathcal{P}\varphi)(r,t):=\varphi_t-A_1(r,t)\varphi_{rr}-A_2(r,t)\varphi_r-A_3(r,t)\varphi-A_4(r,t),\qquad r\in(0,R),\ t\in(0,T_{max}),
\tag{2.8}
$$

with

$$
\begin{aligned}
A_1(r,t) &:= \frac{u^3}{\sqrt{u^2+u_r^2}^3},\\
A_2(r,t) &:= 3\frac{u^2u_r^3}{\sqrt{u^2+u_r^2}^5}-3\frac{u^3u_ru_{rr}}{\sqrt{u^2+u_r^2}^5}+4\frac{u^2u_r^3}{\sqrt{u^2+u_r^2}^5}+\frac{u_r^5}{\sqrt{u^2+u_r^2}^5}+\frac{n-1}{r}\cdot\frac{u^3}{\sqrt{u^2+u_r^2}^3}\\
&\quad -\chi\frac{v_r}{\sqrt{1+v_r^2}},\\
A_3(r,t) &:= -3\frac{uu_r^4}{\sqrt{u^2+u_r^2}^5}-\frac{n-1}{r^2}\frac{u}{\sqrt{u^2+u_r^2}}\\
&\quad -\chi\mu\frac{1}{\sqrt{1+v_r^2}^3}+2\chi\frac{u}{\sqrt{1+v_r^2}^3}-\chi\frac{v_{rr}}{\sqrt{1+v_r^2}}+\chi\frac{v_r^2v_{rr}}{\sqrt{1+v_r^2}^3}-\chi\cdot\frac{n-1}{r}\cdot\frac{v_r^3}{\sqrt{1+v_r^2}^3}\qquad\text{and}\\
A_4(r,t) &:= \frac{n-1}{r}\cdot\frac{u_r^4}{\sqrt{u^2+u_r^2}^3}\\
&\quad +3\chi\mu\frac{uv_rv_{rr}}{\sqrt{1+v_r^2}^5}-3\chi\frac{u^2v_rv_{rr}}{\sqrt{1+v_r^2}^5}+\chi\cdot\frac{n-1}{r^2}\cdot\frac{uv_r^3}{\sqrt{1+v_r^2}^3}-3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^5}
\end{aligned}
\tag{2.9}
$$

for $r\in(0,R)$ and $t\in(0,T_{max})$. Likewise,

$$
(\mathcal{Q}u_r)(r,t)=0\qquad\text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}),
\tag{2.10}
$$

with $\mathcal{Q}$ given by

$$
(\mathcal{Q}\varphi)(r,t):=\varphi_t-A_1(r,t)\varphi_{rr}-A_2(r,t)\varphi_r-\tilde{A}_3(r,t)\varphi-\tilde{A}_4(r,t),\qquad r\in(0,R),\ t\in(0,T_{max}),
\tag{2.11}
$$

where

$$
\begin{aligned}
\tilde{A}_3(r,t) &:= \frac{n-1}{r}\cdot\frac{u_r^3}{\sqrt{u^2+u_r^2}^3} \\
&\quad -\chi\mu\frac{1}{\sqrt{1+v_r^2}^3}+2\chi\frac{u}{\sqrt{1+v_r^2}^3}-\chi\frac{v_{rr}}{\sqrt{1+v_r^2}}+\chi\frac{v_r^2v_{rr}}{\sqrt{1+v_r^2}^3}-\chi\cdot\frac{n-1}{r}\cdot\frac{v_r^3}{\sqrt{1+v_r^2}^3} \qquad \textit{and} \\
\tilde{A}_4(r,t) &:= -3\frac{uu_r^5}{\sqrt{u^2+u_r^2}^5}-\frac{n-1}{r^2}\cdot\frac{uu_r}{\sqrt{u^2+u_r^2}} \\
&\quad +3\chi\mu\frac{uv_rv_{rr}}{\sqrt{1+v_r^2}^5}-3\chi\frac{u^2v_rv_{rr}}{\sqrt{1+v_r^2}^5}+\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^3}{\sqrt{1+v_r^2}^3}-3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^5} \qquad (2.12)
\end{aligned}
$$

for $r\in(0,R)$ and $t\in(0,T_{max})$.

Proof. Differentiation of (2.4) with respect to r yields

$$
\begin{aligned}
u_{rt} &= \frac{u^3u_{rrr}}{\sqrt{u^2+u_r^2}^3}+3\cdot\frac{u^2u_ru_{rr}}{\sqrt{u^2+u_r^2}^3}-\frac{3}{2}\cdot\frac{u^3u_{rr}\cdot(2uu_r+2u_ru_{rr})}{\sqrt{u^2+u_r^2}^5} \\
&\quad +4\cdot\frac{u_r^3u_{rr}}{\sqrt{u^2+u_r^2}^3}-\frac{3}{2}\cdot\frac{u_r^4\cdot(2uu_r+2u_ru_{rr})}{\sqrt{u^2+u_r^2}^5} \\
&\quad -\frac{n-1}{r^2}\cdot\frac{uu_r}{\sqrt{u^2+u_r^2}}+\frac{n-1}{r}\cdot\frac{uu_{rr}}{\sqrt{u^2+u_r^2}} \\
&\quad +\frac{n-1}{r}\cdot\frac{u_r^2}{\sqrt{u^2+u_r^2}}-\frac{1}{2}\cdot\frac{n-1}{r}\cdot\frac{uu_r\cdot(2uu_r+2u_ru_{rr})}{\sqrt{u^2+u_r^2}^3} \\
&\quad -\chi\mu\cdot\frac{u_r}{\sqrt{1+v_r^2}^3}+2\chi\cdot\frac{uu_r}{\sqrt{1+v_r^2}^3}+\frac{3}{2}\chi\cdot\frac{u(\mu-u)\cdot 2v_rv_{rr}}{\sqrt{1+v_r^2}^5} \\
&\quad -\chi\cdot\frac{u_{rr}v_r}{\sqrt{1+v_r^2}}-\chi\cdot\frac{u_rv_{rr}}{\sqrt{1+v_r^2}}+\frac{1}{2}\chi\cdot\frac{u_rv_r\cdot 2v_rv_{rr}}{\sqrt{1+v_r^2}^3} \\
&\quad +\chi\cdot\frac{n-1}{r^2}\cdot\frac{uv_r^3}{\sqrt{1+v_r^2}^3}-\chi\cdot\frac{n-1}{r}\cdot\frac{u_rv_r^3}{\sqrt{1+v_r^2}^3} \\
&\quad -3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^3}+\frac{3}{2}\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^3\cdot 2v_rv_{rr}}{\sqrt{1+v_r^2}^5} \qquad (2.13)
\end{aligned}
$$

for $r\in(0,R)$ and $t\in(0,T_{max})$, where

$$
\begin{aligned}
3\cdot\frac{u^2u_ru_{rr}}{\sqrt{u^2+u_r^2}^3}-\frac{3}{2}\cdot\frac{u^3u_{rr}\cdot(2uu_r+2u_ru_{rr})}{\sqrt{u^2+u_r^2}^5} &= 3\cdot\frac{u^2u_ru_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot\Big\{(u^2+u_r^2)-u^2\Big\}-3\cdot\frac{u^3u_ru_{rr}^2}{\sqrt{u^2+u_r^2}^5} \\
&= 3\cdot\frac{u^2u_r^3u_{rr}}{\sqrt{u^2+u_r^2}^5}-3\cdot\frac{u^3u_ru_{rr}^2}{\sqrt{u^2+u_r^2}^5}
\end{aligned}
$$

and

$$
\begin{aligned}
4\cdot\frac{u_r^3u_{rr}}{\sqrt{u^2+u_r^2}^3}-\frac{3}{2}\cdot\frac{u_r^4\cdot(2uu_r+2u_ru_{rr})}{\sqrt{u^2+u_r^2}^5} &= \frac{u_r^3u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot\Big\{4(u^2+u_r^2)-3u_r^2\Big\}-3\cdot\frac{uu_r^5}{\sqrt{u^2+u_r^2}^5}\\
&= 4\cdot\frac{u^2u_r^3u_{rr}}{\sqrt{u^2+u_r^2}^5}+\frac{u_r^5u_{rr}}{\sqrt{u^2+u_r^2}^5}-3\cdot\frac{uu_r^5}{\sqrt{u^2+u_r^2}^5}
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\frac{n-1}{r}\cdot\frac{uu_{rr}}{\sqrt{u^2+u_r^2}}+\frac{n-1}{r}\cdot\frac{u_r^2}{\sqrt{u^2+u_r^2}}-\frac{1}{2}\cdot\frac{n-1}{r}\cdot\frac{uu_r\cdot(2uu_r+2u_ru_{rr})}{\sqrt{u^2+u_r^2}^3}\\
&\qquad= \frac{n-1}{r}\cdot\frac{uu_{rr}}{\sqrt{u^2+u_r^2}^3}\cdot\Big\{(u^2+u_r^2)-u_r^2\Big\}+\frac{n-1}{r}\cdot\frac{u_r^2}{\sqrt{u^2+u_r^2}^3}\cdot\Big\{(u^2+u_r^2)-u^2\Big\}\\
&\qquad= \frac{n-1}{r}\cdot\frac{u^3u_{rr}}{\sqrt{u^2+u_r^2}^3}+\frac{n-1}{r}\cdot\frac{u_r^4}{\sqrt{u^2+u_r^2}^3}
\end{aligned}
$$

for $r\in(0,R)$ and $t\in(0,T_{max})$. Finally simplifying the last two summands in (2.13) according to

$$
\begin{aligned}
&-3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^3}+\frac{3}{2}\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^3\cdot 2v_rv_{rr}}{\sqrt{1+v_r^2}^5}\\
&\qquad= -3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^5}\cdot\Big\{(1+v_r^2)-v_r^2\Big\}\\
&\qquad= -3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^5},
\end{aligned}
$$

from (2.13) we easily obtain (2.6), and thus also (2.7) and (2.10). □

Thanks to the favorable structure of the equation for v in (1.3), this second solution component can be expressed explicitly in terms of u. This leads to the following observations which will frequently be referred to throughout the sequel.

Lemma 2.4 *Assume (1.4). Then*

$$
v_r(r,t)=\frac{\mu r}{n}-r^{1-n}\cdot\int_0^r\rho^{n-1}u(\rho,t)d\rho\quad \textit{for all } r\in(0,R) \textit{ and } t\in(0,T_{max}) \tag{2.14}
$$

and

$$
v_{rr}(r,t)=\frac{\mu}{n}-u+\frac{n-1}{r^n}\cdot\int_0^r\rho^{n-1}u(\rho,t)d\rho\quad \textit{for all } r\in(0,R) \textit{ and } t\in(0,T_{max}). \tag{2.15}
$$

Moreover we have

$$
v_{rt}=-\frac{uu_r}{\sqrt{u^2+u_r^2}}+\chi\cdot\frac{uv_r}{\sqrt{1+v_r^2}}\quad \textit{in } (0,R)\times(0,T_{max}). \tag{2.16}
$$

PROOF. Since by the second equation in (1.3) we have

$$(r^{n-1}v_r)_r = r^{n-1}(\mu - u) \quad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}),$$

the identity (2.14) easily results by integration, whereupon a differentiation of (2.14) with respect to r yields (2.15).
Next we differentiate (2.14) with respect to t and use the first equation in (1.3) to see that

$$\begin{aligned}
v_{rt}(r,t) &= -\frac{1}{r^{n-1}} \cdot \int_0^r \rho^{n-1} u_t(\rho,t)d\rho \\
&= -\frac{1}{r^{n-1}} \cdot \int_0^r \rho^{n-1} \cdot \left\{ \frac{1}{\rho^{n-1}} \Big(\rho^{n-1} \frac{uu_r}{\sqrt{u^2+u_r^2}} - \chi \rho^{n-1} \frac{uv_r}{\sqrt{1+v_r^2}} \Big) \right\}_r (\rho,t)d\rho \\
&= -\frac{1}{r^{n-1}} \cdot \left\{ r^{n-1} \frac{uu_r}{\sqrt{u^2+u_r^2}} - \chi r^{n-1} \frac{uv_r}{\sqrt{1+v_r^2}} \right\}
\end{aligned}$$

for all $r \in (0,R)$ and $t \in (0,T_{max})$, which shows (2.16). □

Let us note some pointwise estimates resulting from Lemma 2.5 in a straightforward manner.

Lemma 2.5 *Let (1.4) hold. Then for each $t \in (0,T_{max})$ and any $r \in (0,R)$ we have*

$$-\frac{\mu R^n}{n} \cdot r^{1-n} \le v_r(r,t) \le \frac{\mu}{n} \cdot r \tag{2.17}$$

and

$$|v_r(r,t)| \le \frac{\|u(\cdot,t)\|_{L^\infty((0,R))}}{n} \cdot r \tag{2.18}$$

as well as

$$|v_{rr}(r,t)| \le \|u(\cdot,t)\|_{L^\infty((0,R))}. \tag{2.19}$$

PROOF. Fixing $t \in (0,T_{max})$ and writing $M := \|u(\cdot,t)\|_{L^\infty((0,R))}$, we clearly have $\mu \le M$, so that since from Lemma 2.4 we know that

$$v_r(r,t) \le \frac{\mu}{n} \cdot r \quad \text{for all } r \in (0,R)$$

and

$$v_r(r,t) \ge -\frac{1}{r^{n-1}} \cdot \int_0^r \rho^{n-1} \cdot M d\rho = -\frac{Mr}{n} \quad \text{for all} r \in (0,R),$$

both the right inequality in (2.17) as well as (2.18) are immediate. Similarly, using (2.15) we can estimate

$$\begin{aligned}
v_{rr}(r,t) &\le \frac{\mu}{n} + \frac{n-1}{r^n} \cdot \int_0^r \rho^{n-1} \cdot M d\rho = \frac{\mu}{n} + \frac{(n-1)M}{n} \\
&\le M \quad \text{for all } r \in (0,R)
\end{aligned}$$

and

$$v_{rr}(r,t) \ge -M \quad \text{for all } r \in (0,R),$$

which yields (2.19). Finally, to derive the left inequality in (2.17) we observe that $\int_0^r \rho^{n-1}u(\rho,t)d\rho \le \frac{m}{\omega_n}$ for all $r \in (0,R)$ and recall that $\frac{m}{\omega_n} = \frac{\mu R^n}{n}$ by (1.5) to obtain from (2.14) that

$$v_r(r,t) \ge -r^{1-n}\int_0^r \rho^{n-1}u(\rho,t)d\rho \ge -r^{1-n}\cdot\frac{m}{\omega_n} = -r^{1-n}\cdot\frac{\mu R^n}{n} \quad \text{for all } r \in (0,R).$$

This completes the proof. $\Box$

3 A pointwise estimate from below for u

In order to show that (2.2) actually reduces to (1.6), let us first rule out the occurrence of the first alternative in (2.2). In proving this, we shall make use of the following elementary inequality.

Lemma 3.1 *We have*

$$\frac{\xi}{\sqrt{1+\xi}^3} \le \frac{2}{3\sqrt{3}} \qquad \textit{for all } \xi \ge 0.$$

Proof. Since $\varphi(\xi) := \frac{\xi}{\sqrt{1+\xi}^3}$, $\xi \ge 0$, satisfies $\varphi(0) = 0, \varphi(\xi) \to 0$ as $\xi \to \infty$ and $\varphi'(\xi) = (1+\xi)^{-\frac{5}{2}} \cdot (1-\frac{\xi}{2})$ for all $\xi > 0$, it follows that $\varphi(\xi) \le \varphi(2) = \frac{2}{\sqrt{3}^3}$ for all $\xi \ge 0$. $\Box$

By means of a comparison argument applied to (2.4), we can now in fact exclude that solutions attain zeros within finite time.

Lemma 3.2 *If (1.4) holds, then*

$$u(r,t) \ge \Big(\inf_{r\in(0,R)} u_0(r)\Big)\cdot e^{-\kappa t} \qquad \textit{for all } r \in (0,R) \textit{ and } t \in (0,T_{max}), \tag{3.1}$$

where

$$\kappa := \chi\mu + \frac{2(n-1)\chi\mu}{3\sqrt{3}n}. \tag{3.2}$$

Proof. We rewrite (2.4) in the form

$$u_t = a_1(r,t)u_{rr} + a_{21}(r,t)u_r + \frac{a_{22}(r,t)}{r}\cdot u_r - \chi\cdot\frac{u(\mu-u)}{\sqrt{1+v_r^2}^3} - \frac{n-1}{r}\cdot\chi\frac{uv_r^3}{\sqrt{1+v_r^2}^3} \tag{3.3}$$

for all $r \in (0,R)$ and $t \in (0,T_{max})$, where

$$a_1(r,t) := \frac{u^3}{\sqrt{u^2+u_r^2}^3}$$

and

$$a_{21}(r,t) := \frac{u_r^3}{\sqrt{u^2+u_r^2}^3} - \chi\cdot\frac{v_r}{\sqrt{1+v_r^2}}$$

as well as

$$a_{22}(r,t) := (n-1)\cdot \frac{u}{\sqrt{u^2+u_r^2}}$$

define continuous functions in $[0,R]\times(0,T_{max})$. In (3.3), we can estimate

$$-\chi\cdot\frac{u(\mu-u)}{\sqrt{1+v_r^2}^3} \ge -\chi\mu\cdot\frac{u}{\sqrt{1+v_r^2}^3} \ge -\chi\mu u \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}),$$

and in order to control the last term in (3.3) we use the one-sided inequality $v_r \le \frac{\mu r}{n}$ provided by Lemma 2.5, which in conjunction with Lemma 3.1 entails that

$$\begin{aligned} -\frac{n-1}{r}\cdot\chi\frac{uv_r^3}{\sqrt{1+v_r^2}^3} &= -(n-1)\chi\cdot\frac{v_r^2}{\sqrt{1+v_r^2}^3}\cdot\frac{v_r}{r}\cdot u \\ &\ge -(n-1)\chi\cdot\frac{2}{3\sqrt{3}}\cdot\frac{\mu}{n}\cdot u \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}). \end{aligned}$$

Accordingly, from (3.3) we infer that with κ as in (3.2) we have

$$u_t \ge a_1(r,t)u_{rr} + a_{21}(r,t)u_r + \frac{a_{22}(r,t)}{r}\cdot u_r - \kappa u \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}),$$

so that for all $\varepsilon>0$, writing $\varphi(r,t) := e^{(\kappa+\varepsilon)t}u(r,t)$ we see that

$$\begin{aligned} \varphi_t &\ge e^{(\kappa+\varepsilon)t}\cdot\Big\{a_1(r,t)u_{rr} + a_{21}(r,t)u_r + a_{22}(r,t)u_r - \kappa u + (\kappa+\varepsilon)u\Big\} \\ &= a_1(r,t)\varphi_{rr} + a_{21}(r,t)\varphi_r + a_{22}(r,t)\varphi_r + \varepsilon e^{(\kappa+\varepsilon)t}u \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}) \end{aligned} \tag{3.4}$$

Now if for some $T\in(0,T_{max})$, φ attains its minimum over $[0,R]\times[0,T]$ at some $(r_0,t_0)\in[0,R]\times[0,T]$, then necessarily

$$\varphi_r(r_0,t_0)=0, \quad \varphi_{rr}(r_0,t_0)\ge 0 \quad \text{and} \quad \varphi_t(r_0,t_0)\le 0. \tag{3.5}$$

Therefore, in the case $t_0>0$ and $r_0>0$ we may directly apply (3.4) to obtain

$$\begin{aligned} 0\ge\varphi_t(r_0,t_0) &\ge a_1(r_0,t_0)\varphi_{rr}(r_0,t_0) + a_{21}(r_0,t_0)\varphi_r(r_0,t_0) + \frac{a_{22}(r_0,t_0)}{r_0}\cdot\varphi_r(r_0,t_0) + \varepsilon e^{(\kappa+\varepsilon)t_0}u(r_0,t_0) \\ &\ge \varepsilon e^{(\kappa+\varepsilon)t_0}u(r_0,t_0) > 0, \end{aligned}$$

which is impossible.
However, if $t_0>0$ and $r_0=0$, then there must exist a sequence $(r_j)_{j\in\mathbb{N}}$ of numbers $r_j\in(0,R)$ such that $r_j\searrow 0$ as $j\to\infty$ and $\varphi_r(r_j,t_0)\ge 0$ for all $j\in\mathbb{N}$, because otherwise $\varphi(\cdot,t_0)$ would have a strict local maximum at $r=0$. Since $a_3\ge 0$, evaluating (3.4) at $r=r_j=$ we would thus obtain that

$$\begin{aligned} \varphi_t(r_j,t_0) &\ge a_1(r_j,t_0)\varphi_{rr}(r_j,t_0) + a_{21}(r_j,t_0)\varphi_r(r_j,t_0) + \frac{a_{22}(r_j,t_0)}{r_j}\cdot\varphi_r(r_j,t_0) + \varepsilon e^{(\kappa+\varepsilon)t_0}u(r_j,t_0) \\ &\ge a_1(r_j,t_0)\varphi_{rr}(r_j,t_0) + a_{21}(r_j,t_0)\varphi_r(r_j,t_0) + \varepsilon e^{(\kappa+\varepsilon)t_0}u(r_j,t_0) \qquad \text{for all } j\in\mathbb{N}, \end{aligned}$$

so that since $\varphi(\cdot, t_0)$ is smooth in $[0, R]$, we may let $j \to \infty$ here to infer using (3.5) that

$$\begin{array}{rcl} 0 \geq \varphi_t(0, t_0) & \geq & a_1(0, t_0)\varphi_{rr}(0, t_0) + a_{21}(0, t_0)\varphi_r(0, t_0) + \varepsilon e^{(\kappa+\varepsilon)t_0}u(0, t_0) \\ & \geq & \varepsilon e^{(\kappa+\varepsilon)t_0}u(0, t_0) > 0. \end{array}$$

This absurd conclusion shows that actually $t_0 = 0$, which implies that $\varphi \geq \inf_{r \in (0,R)} \varphi(r, 0) = \inf_{r \in (0,R)} u_0(r)$ throughout $[0, R] \times [0, T]$ for any $T \in (0, T_{max})$. Taking $T \nearrow T_{max}$ and $\varepsilon \searrow 0$ we thereby obtain (3.1). $\Box$

4 A pointwise lower estimate for u_r

It remains to exclude the possibility of finite-time blow-up of u_r despite boundedness of u. A first step toward this can accomplished by invoking parabolic comparison to derive the following lower bound for u_r from (2.7). Let us emphasize that our argument makes essential use of the fact that on the right-hand side of (2.6), the most singular term $\frac{n-1}{r^2} \cdot \frac{uu_r}{\sqrt{u^2+u_r^2}}$ therein appears with a negative sign, and that in consequence a corresponding upper estimate for u_r can apparently not be obtained by a direct approach of the type pursued here, at least not when $n \geq 2$.

Lemma 4.1 *Assume that $T_{max} < \infty$, but that $\sup_{(r,t) \in (0,R) \times (0,T_{max})} u(r, t) < \infty$. Then there exists $C > 0$ such that*

$$u_r(r, t) \geq -C \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}). \tag{4.1}$$

Proof. According to our hypothesis, we can find $c_1 > 0$ such that

$$u(r, t) \leq c_1 \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}), \tag{4.2}$$

so that Lemma 2.5 provides $c_2 > 0$ and $c_3 > 0$ such that

$$|v_r(r, t)| \leq c_2 r \quad \text{and} \quad |v_{rr}(r, t)| \leq c_3 \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}). \tag{4.3}$$

We now take $D \geq 1$ and $\alpha > 0$ large enough fulfilling

$$u_{0r}(r) > -D \qquad \text{for all } r \in (0, R) \tag{4.4}$$

and

$$\alpha > c_4 + \frac{c_5}{D}, \tag{4.5}$$

where

$$c_4 := 2c_1\chi + c_3\chi + c_2^2 c_3 \chi R^2 + (n-1)c_2^3 \chi R^2 \tag{4.6}$$

and

$$c_5 := 3c_1c_2c_3\chi\mu + 3c_1^2c_2c_3\chi R + (n-1)c_1c_2^3\chi R + 3(n-1)c_1c_2^2c_3\chi R, \tag{4.7}$$

and define a comparison function $\underline{\varphi}$ by letting

$$\underline{\varphi}(r, t) := -De^{\alpha t} \qquad \text{for } r \in [0, R] \text{ and } t \geq 0.$$

Then since $\underline{\varphi}_r = \underline{\varphi}_{rr} \equiv 0$, with $\mathcal{P}$ as in (2.8) we have

$$
\begin{aligned}
(\mathcal{P}\underline{\varphi})(r,t) \;=\;& -\alpha D e^{\alpha t} \\
& -3\frac{u_r^4 \cdot De^{\alpha t}}{\sqrt{u^2+u_r^2}^5} - \frac{n-1}{r^2}\cdot\frac{u\cdot De^{\alpha t}}{\sqrt{u^2+u_r^2}} - \chi\mu\frac{De^{\alpha t}}{\sqrt{1+v_r^2}^3} \\
& +2\chi\frac{u\cdot De^{\alpha t}}{\sqrt{1+v_r^2}^3} - \chi\frac{v_{rr}\cdot De^{\alpha t}}{\sqrt{1+v_r^2}} + \chi\frac{v_r^2 v_{rr}\cdot De^{\alpha t}}{\sqrt{1+v_r^2}^3} \\
& -\chi\cdot\frac{n-1}{r}\cdot\frac{v_r^3\cdot De^{\alpha t}}{\sqrt{1+v_r^2}^3} \\
& -\frac{n-1}{r}\cdot\frac{u_r^4}{\sqrt{u^2+u_r^2}^3} - 3\chi\mu\frac{uv_rv_{rr}}{\sqrt{1+v_r^2}^5} + 3\chi\frac{u^2v_rv_{rr}}{\sqrt{1+v_r^2}^5} \\
& -\chi\cdot\frac{n-1}{r^2}\cdot\frac{uv_r^3}{\sqrt{1+v_r^2}^3} + 3\chi\cdot\frac{n-1}{r}\cdot\frac{uv_r^2v_{rr}}{\sqrt{1+v_r^2}^5}
\end{aligned}
\tag{4.8}
$$

for all $r \in (0,R)$ and $t \in (0,T_{max})$. Here the second, third, fourth and ninth term on the right are nonpositive, and we claim that each of the remaining summands containing χ can be controlled in modulus by the first term on the right-hand side suitably. Indeed, repeatedly using (4.2), (4.3) and (4.4), we can estimate

$$
\left|2\chi\frac{u\cdot De^{\alpha t}}{\sqrt{1+v_r^2}^3}\right| \le 2\chi\cdot c_1\cdot De^{\alpha t},
$$

$$
\left|-\chi\frac{v_{rr}\cdot De^{\alpha t}}{\sqrt{1+v_r^2}}\right| \le \chi\cdot c_3\cdot De^{\alpha t},
$$

$$
\left|\chi\frac{v_r^2v_{rr}\cdot De^{\alpha t}}{\sqrt{1+v_r^2}^3}\right| \le \chi\cdot c_2^2r^2\cdot c_3\cdot De^{\alpha t} \le c_2^2c_3\chi R^2\cdot De^{\alpha t},
$$

$$
\left|-3\chi\mu\frac{uv_rv_{rr}}{\sqrt{1+v_r^2}^5}\right| \le 3\chi\mu\cdot c_1\cdot c_2r\cdot c_3 \le 3c_1c_2c_3\chi\mu R
$$

as well as

$$
\left|3\chi\frac{u^2v_rv_{rr}}{\sqrt{1+v_r^2}^5}\right| \le 3\chi\cdot c_1^2\cdot c_2r\cdot c_3 \le 3c_1^2c_2c_3\chi R
$$

and finally,

$$
\left|-\chi\cdot\frac{n-1}{r^2}\cdot\frac{uv_r^3}{\sqrt{1+v_r^2}^3}\right| \le \chi\cdot\frac{n-1}{r^2}\cdot c_1\cdot c_2^3r^3 \le (n-1)c_1c_2^3\chi R,
$$

as well as

$$\left| -\chi \cdot \frac{n-1}{r} \cdot \frac{v_r^3 \cdot De^{\alpha t}}{\sqrt{1+v_r^2}^3} \right| \le \chi \cdot \frac{n-1}{r} \cdot c_2^3 r^3 \cdot De^{\alpha t} \le (n-1) c_2^3 \chi R^2 \cdot De^{\alpha t}$$

and

$$\left| 3\chi \cdot \frac{n-1}{r} \cdot \frac{u v_r^2 v_{rr}}{\sqrt{1+v_r^2}^5} \right| \le 3\chi \cdot \frac{n-1}{r} \cdot c_1 \cdot c_2^2 r^2 \cdot c_3 \le 3(n-1) c_1 c_2^2 c_3 \chi R.$$

Therefore, (4.8) implies that with c_4 and c_5 as in (4.6) and (4.7) we have

$$\begin{aligned} (\mathcal{P}\underline{\varphi})(r,t) &\le -\alpha De^{\alpha t} + c_4 \cdot De^{\alpha t} + c_5 \\ &\le -\alpha De^{\alpha t} + \Big(c_4 + \frac{c_5}{D}\Big) \cdot De^{\alpha t} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}), \end{aligned}$$

whence our assumption (4.5) on α ensures that $(\mathcal{P}\underline{\varphi})(r,t) < 0$ for all $r \in (0,R)$ and $t \in (0,T_{max})$. Since $(\mathcal{P}u_r)(r,t) = 0$ for all $(r,t) \in (0,R) \times (0,T_{max})$ by Lemma 2.3, and since moreover

$$\underline{\varphi}(r,0) = -D < u_{0r}(r) = u_r(0,r) \qquad \text{for all } r \in [0,R]$$

and, clearly,

$$\underline{\varphi}_r(0,t) = u_r(0,t) = 0 \quad \text{as well as} \quad \underline{\varphi}_r(R,t) = u_r(R,t) = 0 \qquad \text{for all } t \in (0,T_{max}),$$

from the comparison principle we conclude that $u_r(r,t) \ge \underline{\varphi}(r,t)$ for all $r \in (0,R)$ and $t \in (0,T_{max})$, and that hence

$$u_r(r,t) \ge -De^{\alpha T_{max}} \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}),$$

which proves the claim. □

5 A bound for $|u_r|$. Proof of Theorem 1.1

The goal of this section is to complete the proof of Theorem 1.1 by further developing the one-sided inequality for u_r from Lemma 4.1 into a bound for $|u_r|$ in modulus, provided that T_{max} is finite but u itself remains bounded (Corollary 5.7). An important role in our analysis in this direction will be played by the function $z := \frac{u_t}{u}$, which is indeed well-defined and continuous in $[0,R] \times [0,T_{max})$ by Lemma 2.1. Furthermore, according to Lemma 2.2 we have the representation

$$\begin{aligned} z \;=\; & \frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3} + \frac{u_r^4}{u\sqrt{u^2+u_r^2}^3} + \frac{n-1}{r} \cdot \frac{u_r}{\sqrt{u^2+u_r^2}} \\ & -\chi \frac{u_r v_r}{u\sqrt{1+v_r^2}} - \chi \frac{\mu - u}{\sqrt{1+v_r^2}^3} - \chi \cdot \frac{n-1}{r} \cdot \frac{v_r^3}{\sqrt{1+v_r^2}^3}, \end{aligned} \tag{5.1}$$

for $r \in (0,R)$ and $t \in (0,T_{max})$.

Now in a first key observation, to be presented in Corollary 5.3, we will establish a useful relationship between u_r and the function z, essentially controlling $\|u_r(\cdot,t)\|_{L^\infty((0,R))}$ for any fixed $t \in (0,T_{max})$ by the maximum of the *positive part* z_+ of z over the *whole memory region* $(0,R)\times(0,t)$. To achieve this, we will foremost use an integral technique to estimate $|u_r|$ in terms of z_+ on the basis of (5.1) and Lemma 4.1 in a suitably small subinterval $(0,R_0)$ of $(0,R)$ (Lemma 5.1). This will in particular imply an upper bound for u_r at $r = R_0$ and therefore allow for applying a comparison argument to (2.10) which will yield a pointwise upper estimate for u_r in the corresponding outer region (R_0,R) (Lemma 5.2).

The second essential step will thereafter consist in deriving a nonlocal parabolic inequality for z with a memory-type nonlinearity (Lemma 5.5). Upon another comparison, this will entail a pointwise upper bound for z (Lemma 5.6) and hence also for u_r.

5.1 A bound for $|u_r|$ in terms of z_+

5.1.1 Estimating $|u_r|$ near the origin

Let us first apply an appropriate testing procedure to (5.1) to find some small $R_0 \in (0,R)$ with the property that $u_r(\cdot,t)$ can be bounded in certain weighted Lebesgue spaces over $(0,R_0)$ in such a way that on taking limits we can derive a respective L^∞ estimate from this.

Lemma 5.1 *Assume that $T_{max} < \infty$, but that $\sup_{(r,t)\in(0,R)\times(0,T_{max})} u(r,t) < \infty$. Then there exist $R_0 \in (0,R)$ and $C > 0$ such that*

$$\|u_r(\cdot,t)\|_{L^\infty((0,R_0))} \le C\cdot\Big(1+\|z_+(\cdot,t)\|_{L^\infty((0,R_0))}\Big) \qquad \textit{for all } t\in(0,T_{max}). \tag{5.2}$$

Proof. We first rearrange (5.1) to obtain

$$\begin{aligned}\frac{u_r^4}{u^3} &= \frac{\sqrt{u^2+u_r^2}}{u^2}\cdot z - u_{rr} - \frac{n-1}{r}\cdot\frac{\sqrt{u^2+u_r^2}^2 u_r}{u^2}\\ &\quad +\chi\frac{(\mu-u)\sqrt{u^2+u_r^2}^3}{\sqrt{1+v_r^2}^3} + \chi\frac{\sqrt{u^2+u_r^2}^3 u_r v_r}{u^3\sqrt{1+v_r^2}} + \chi\cdot\frac{n-1}{r}\cdot\frac{\sqrt{u^2+u_r^2}^3 v_r^3}{u^2\sqrt{1+v_r^2}^3},\end{aligned} \tag{5.3}$$

where

$$\begin{aligned}-u_{rr} - \frac{n-1}{r}\cdot\frac{\sqrt{u^2+u_r^2}^2 u_r}{u^2} &= -\Big(u_{rr}+\frac{n-1}{r}u_r\Big) - \frac{n-1}{r}\cdot\frac{u_r^3}{u^2}\\ &= -\frac{1}{r^{n-1}}(r^{n-1}u_r)_r - \frac{n-1}{r}\cdot\frac{u_r^3}{u^2}\end{aligned} \tag{5.4}$$

for $r \in (0,R)$ and $t \in (0,T_{max})$. In order to choose R_0 appropriately, we use our boundedness assumption on u to fix $c_1 \ge \mu$ and $c_2 > 0$ such that

$$u(r,t) \le c_1 \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}) \tag{5.5}$$

and

$$\sqrt{u^2+u_r^2}^3 \le c_2\cdot(1+|u_r|^3) \qquad \text{in } (0,R)\times(0,T_{max}), \tag{5.6}$$

and recall Lemma 3.2 to find $c_3 > 0$ fulfilling

$$u(r,t) \geq c_3 \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}). \tag{5.7}$$

We claim that then the conclusion of the lemma holds if we pick any $R_0 \in (0,R)$ satisfying

$$R_0 \leq \frac{nc_3^3}{4c_1^3 c_2 \chi\mu}. \tag{5.8}$$

To see this, we take an arbitrary even integer $m \geq 0$, multiply (5.3) by $r^{n-1}u_r^m$ and integrate over $(0,R_0)$ to see using (5.4) that

$$\begin{aligned}
I(t) \;&:=\; \int_0^{R_0} r^{n-1}\frac{u_r^{m+4}}{u^3}dr \\
&= \int_0^{R_0} r^{n-1}\frac{\sqrt{u^2+u_r^2}^3}{u^2}\cdot u_r^m z dr - \int_0^{R_0}(r^{n-1}u_r)_r \cdot u_r^m dr - (n-1)\int_0^{R_0} r^{n-2}\frac{u_r^{m+3}}{u^2}dr \\
&\quad +\chi\int_0^{R_0} r^{n-1}\frac{(\mu-u)\sqrt{u^2+u_r^2}^3 u_r^m}{u^2\sqrt{1+v_r^2}^3}dr + \chi\int_0^{R_0} r^{n-1}\frac{\sqrt{u^2+u_r^2}^3 u_r^{m+1}v_r}{u^3\sqrt{1+v_r^2}}dr \\
&\quad +(n-1)\chi\int_0^{R_0} r^{n-2}\frac{\sqrt{u^2+u_r^2}^3 u_r^m v_r^3}{u^2\sqrt{1+v_r^2}^3}dr \\
&=: J_1(t)+...+J_6(t) \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{5.9}$$

Here by (5.5) we have

$$I(t) \geq \frac{1}{c_1^3}\cdot\int_0^{R_0} r^{n-1}u_r^{m+4}dr \qquad \text{for all } t\in(0,T_{max}), \tag{5.10}$$

and our goal is to show that the sum on the right-hand side of (5.9) can be controlled adequately by the term on the right of (5.10).
For this purpose, we first use Lemma 2.4 in rewriting $J_5(t)$ to obtain

$$\begin{aligned}
J_5(t) \;&=\; \frac{\chi\mu}{n}\int_0^{R_0} r^n\cdot\frac{\sqrt{u^2+u_r^2}^3 u_r^{m+1}}{u^3\sqrt{1+v_r^2}}dr - \chi\int_0^{R_0}\frac{\sqrt{u^2+u_r^2}^3 u_r^{m+1}}{u^3\sqrt{1+v_r^2}}\cdot\Big(\int_0^r \rho^{n-1}u(\rho,t)d\rho\Big)dr \\
&=: J_{51}(t)+J_{52}(t) \qquad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{5.11}$$

where (5.6), (5.7) and Young's inequality enable us to infer that

$$\begin{aligned}
J_{51}(t) \;&\leq\; \frac{\chi\mu}{n}\int_0^{R_0} r^n\cdot\frac{c_2\cdot(1+|u_r|^3)\cdot|u_r|^{m+1}}{c_3^3}dr \\
&= \frac{c_2\chi\mu}{nc_3^3}\int_0^{R_0} r^n|u_r|^{m+1}dr + \frac{c_2\chi\mu}{nc_3^3}\int_0^{R_0} r^n u_r^{m+4}dr \\
&\leq \frac{c_2\chi\mu}{nc_3^3}\int_0^{R_0} r^n(1+u_r^{m+4})dr + \frac{c_2\chi\mu}{nc_3^3}\int_0^{R_0} r^n u_r^{m+4}dr \\
&= \frac{c_2\chi\mu R_0^{n+1}}{n(n+1)c_3^3} + \frac{2c_2\chi\mu}{nc_3^3}\int_0^{R_0} r^n u_r^{m+4}dr \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

Trivially estimating

$$\int_0^{R_0} r^n u_r^{m+4} dr \le R_0 \int_0^{R_0} r^{n-1} u_r^{m+4},$$

according to (5.10) and our restriction (5.8) on R_0 we thus conclude that

$$J_{51}(t) \le c_4 + \frac{1}{2c_1^3} \int_0^{R_0} r^{n-1} u_r^{m+4} dr \qquad \text{for all } t \in (0, T_{max}) \tag{5.12}$$

with $c_4 := \frac{c_2 \chi \mu R^{n+1}}{n(n+1)c_3^3}$.
In order to derive an appropriate upper bound for the second term on the right of (5.11), let us apply Lemma 4.1 to fix a constant $L \ge 1$ such that

$$u_r(r,t) \ge -L \qquad \text{for all } r \in (0,R) \text{ and } t \in (0, T_{max}). \tag{5.13}$$

Then since $-u_r^{m+1} = (-u_r)^{m+1}$ due to the fact that $m+1$ is odd, this in conjunction with (5.6), (5.7) and (5.3) implies that

$$\begin{aligned}
J_{52}(t) &\le \chi L^{m+1} \int_0^{R_0} \frac{\sqrt{u^2+u_r^2}^3}{u^3\sqrt{1+v_r^2}} \cdot \Big(\int_0^r \rho^{n-1} u(\rho,t) d\rho \Big) dr \\
&\le \frac{c_1 c_2 \chi}{c_3^3} L^{m+1} \int_0^{R_0} (1+|u_r|^3) \cdot \Big(\int_0^r \rho^{n-1} d\rho \Big) dr \\
&= \frac{c_1 c_2 \chi}{n c_3^3} L^{m+1} \int_0^{R_0} r^n (1+|u_r|^3) dr \\
&= \frac{c_1 c_2 \chi R_0^{n+1}}{n(n+1) c_3^3} L^{m+1} + \frac{c_1 c_2 \chi}{n c_3^3} \int_0^{R_0} r^n L^{m+1} |u_r|^3 dr \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{5.14}$$

As by Young's inequality we have

$$\begin{aligned}
\int_0^{R_0} r^n L^{m+1} |u_r|^3 dr &\le \frac{3}{m+4} \int_0^{R_0} r^n u_r^{m+4} dr + \frac{m+1}{m+4} \int_0^{R_0} r^n L^{m+4} dr \\
&\le \frac{3R}{m+4} \int_0^{R_0} r^{n-1} u_r^{m+4} dr + \frac{R^{n+1}}{n+1} L^{m+4},
\end{aligned}$$

in view of the fact that $L \ge 1$ we obtain from (5.14) that

$$J_{52}(t) \le c_5 L^{m+4} + \frac{c_6}{m+4} \int_0^{R_0} r^{n-1} u_r^{m+4} dr \qquad \text{for all } t \in (0, T_{max}) \tag{5.15}$$

with $c_5 := \frac{2c_1 c_2 \chi R^{n+1}}{n(n+1)c_3^3}$ and $c_6 := \frac{3 c_1 c_2 \chi R}{n c_3^3}$.
Going back to (5.9), we next apply (5.6), (5.7) and Young's and Hölder's inequalities to estimate

$$J_1(t) \le \frac{c_2}{c_3^2} \int_0^{R_0} r^{n-1} (1+|u_r|^3) \cdot u_r^m z_+ dr$$

$$
\begin{aligned}
&= \frac{c_2}{c_3^2}\int_0^{R_0} r^{n-1}u_r^m z_+ dr + \frac{c_2}{c_3^2}\int_0^{R_0} r^{n-1}|u_r|^{m+3}z_+ dr \\
&\le \frac{c_2}{c_3^2}\int_0^{R_0} r^{n-1}z_+ dr + \frac{2c_2}{c_3^2}\int_0^{R_0} r^{n-1}|u_r|^{m+3}z_+ dr \\
&\le \frac{c_2}{c_3^2}\Big(\int_0^{R_0} r^{n-1}z_+^{m+4}dr\Big)^{\frac{1}{m+4}}\cdot\Big(\frac{R_0^n}{n}\Big)^{\frac{m+3}{m+4}} + \frac{2c_2}{c_3^2}\Big(\int_0^{R_0} r^{n-1}u_r^{m+4}dr\Big)^{\frac{m+3}{m+4}}\cdot\Big(\int_0^{R_0} r^{n-1}z_+^{m+4}dr\Big)^{\frac{1}{m+4}},
\end{aligned}
$$

which means that if we let $R_1 := \max\{1, R\}$ and $c_7 := \max\{\frac{c_2}{c_3^2}\cdot\frac{R_1^n}{n}\,,\,\frac{2c_2}{c_3^2}\}$, then

$$
J_1(t) \le c_7\cdot\Bigg\{1+\Big(\int_0^{R_0} r^{n-1}u_r^{m+4}dr\Big)^{\frac{m+3}{m+4}}\Bigg\}\cdot\Big(\int_0^{R_0} r^{n-1}z_+^{m+4}dr\Big)^{\frac{1}{m+4}} \qquad \text{for all } t\in(0,T_{max}). \tag{5.16}
$$

As for the second term on the right of (5.9), in order to remove second-order derivatives, in the case $n \ge 2$ we twice integrate by parts to obtain

$$
\begin{aligned}
J_2(t) &= m\int_0^{R_0} r^{n-1}u_r^m u_{rr}dr - R_0^{n-1}u_r^{m+1}(R_0,t) \\
&= -\frac{(n-1)m}{m+1}\int_0^{R_0} r^{n-2}u_r^{m+1}dr + \frac{m}{m+1}R_0^{n-1}u_r^{m+1}(R_0,t) - R_0^{n-1}u_r^{m+1}(R_0,t) \\
&= -\frac{(n-1)m}{m+1}\int_0^{R_0} r^{n-2}u_r^{m+1}dr - \frac{1}{m+1}R_0^{n-1}u_r^{m+1}(R_0,t) \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}
$$

Once more since $m+1$ is odd, (5.13) again becomes applicable to provide the one-sided estimate

$$
\begin{aligned}
J_2(t) &\le \frac{(n-1)m}{m+1}L^{m+1}\int_0^{R_0} r^{n-2} + \frac{1}{m+1}R_0^{n-1}L^{m+1} \\
&\le c_8 L^{m+1} \qquad \text{for all } t\in(0,T_{max})
\end{aligned} \tag{5.17}
$$

with $c_8 := \frac{(n-1)m}{m+1}\cdot\frac{R^{n-1}}{n-1} + \frac{1}{m+1}R^{n-1} \equiv R^{n-1}$ when $n \ge 2$, and it can easily be verified that this conclusion can be extended so as to include the case $n = 1$ as well.

Similarly, since also $m+3$ is odd, we may invoke (5.13) and then (5.7) to see that in the case $n \ge 2$,

$$
\begin{aligned}
J_3(t) &\le (n-1)\int_0^{R_0} r^{n-2}\frac{L^{m+3}}{u^2}dr \\
&\le \frac{n-1}{c_3^2}L^{m+3}\frac{R_0^{n-1}}{n-1} \\
&\le c_9 L^{m+3} \qquad \text{for all } t\in(0,T_{max})
\end{aligned} \tag{5.18}
$$

with $c_9 := \frac{R^{n-1}}{c_3^2}$, and note that (5.18) trivially holds when $n = 1$.

We next estimate $J_4(t)$ by first using (5.6), (5.7) and Young's inequality according to

$$\begin{aligned} J_4(t) &\le \frac{c_2\chi\mu}{c_3^2}\int_0^{R_0} r^{n-1}(1+|u_r|^3)u_r^m dr \\ &\le \frac{c_2\chi\mu}{c_3^2}\int_0^{R_0} r^{n-1}dr + \frac{2c_2\chi\mu}{c_3^2}\int_0^{R_0} r^{n-1}|u_r|^{m+3}dr \\ &= \frac{c_2\chi\mu R_0^n}{nc_3^2} + \frac{2c_2\chi\mu}{c_3^2}\int_0^{R_0} r^{n-1}|u_r|^{m+3}dr \qquad \text{for all } t\in(0,T_{max}). \end{aligned}$$

Since thanks to the Hölder inequality we know that

$$\begin{aligned} \int_0^{R_0} r^{n-1}|u_r|^{m+3}dr &\le \Big(\frac{R_0^n}{n}\Big)^{\frac{1}{m+4}}\cdot\Big(\int_0^{R_0} r^{n-1}u_r^{m+4}dr\Big)^{\frac{m+3}{m+4}} \\ &\le \frac{R_1^n}{n}\cdot\Big(\int_0^{R_0} r^{n-1}u_r^{m+4}dr\Big)^{\frac{m+3}{m+4}}, \end{aligned} \tag{5.19}$$

again with $R_1=\max\{1,R\}$, this entails that

$$J_4(t)\le c_{10}\cdot\left\{1+\Big(\int_0^{R_0} r^{n-1}u_r^{m+4}dr\Big)^{\frac{m+3}{m+4}}\right\} \qquad \text{for all } t\in(0,T_{max}) \tag{5.20}$$

if we let $c_{10}:=\max\{\frac{c_2\chi\mu R^n}{nc_3^2}, \frac{2c_2\chi\mu}{c_3^2}\cdot\frac{R_1^n}{n}\}$.

Finally, in treating the last integral in (5.9) we make use of the upper estimate for v_r in (2.17) to see, again recalling (5.6), (5.7) and using Young's inequality, that

$$\begin{aligned} J_6(t) &\le \frac{(n-1)\chi\mu^3}{n^3}\int_0^{R_0}\frac{r^{n+1}\sqrt{u^2+u_r^2}^3u_r^m}{u^2\sqrt{1+v_r^2}^3}dr \\ &\le \frac{(n-1)c_2\chi\mu^3}{n^3c_3^2}\int_0^{R_0} r^{n+1}(1+|u_r|^3)u_r^m dr \\ &\le \frac{(n-1)c_2\chi\mu^3}{n^3c_3^2}\int_0^{R_0} r^{n+1}dr + \frac{2(n-1)c_2\chi\mu^3}{n^3c_3^2}\int_0^{R_0} r^{n+1}|u_r|^{m+3}dr \\ &= \frac{(n-1)c_2\chi\mu^3R_0^{n+2}}{n^3(n+2)c_3^2} + \frac{2(n-1)c_2\chi\mu^3}{n^3c_3^2}\int_0^{R_0} r^{n+1}|u_r|^{m+3}dr \qquad \text{for all } t\in(0,T_{max}). \end{aligned}$$

Now due to (5.19),

$$\begin{aligned} \int_0^{R_0} r^{n+1}|u_r|^{m+3}dr &\le R_0^2\int_0^{R_0} r^{n-1}|u_r|^{m+3}dr \\ &\le \frac{R_1^{n+2}}{n}\Big(\int_0^{R_0} r^{n-1}u_r^{m+4}dr\Big)^{\frac{m+3}{m+4}}, \end{aligned}$$

whence we obtain that

$$J_6(t) \le c_{11} \cdot \left\{ 1 + \left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \right\} \qquad \text{for all } t \in (0, T_{max}) \tag{5.21}$$

with $c_{11} := \max \left\{ \frac{(n-1)c_2\chi\mu^3 R^{n+2}}{n^3(n+2)c_3^2}, \frac{2(n-1)c_2\chi\mu^3}{n^3 c_2^3} \cdot \frac{R_1^{n+2}}{n} \right\}$.

In summary, (5.10), (5.16), (5.17), (5.18), (5.11), (5.12), (5.15) and (5.21) combined with (5.9) show that

$$\begin{aligned}
\frac{1}{c_1^3} \int_0^{R_0} r^{n-1} u_r^{m+4} dr \quad \le \quad & c_7 \cdot \left\{ 1 + \left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \right\} \cdot \left(\int_0^{R_0} r^{n-1} z_+^{m+4} dr \right)^{\frac{1}{m+4}} \\
& + c_8 L^{m+1} + c_9 L^{m+3} \\
& + c_{10} \cdot \left\{ 1 + \left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \right\} \\
& + c_4 + \frac{1}{2c_1^3} \int_0^{R_0} r^{n-1} u_r^{m+4} dr \\
& + c_5 L^{m+4} + \frac{c_6}{m+4} \int_0^{R_0} r^{n-1} u_r^{m+4} dr \\
& + c_{11} \left\{ 1 + \left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \right\} \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Since $L \ge 1$, this means that if $m_\star$ is sufficiently large such that

$$\frac{c_6}{m_\star + 4} \le \frac{1}{4c_1^3}, \tag{5.22}$$

and if $m \ge m_\star$, then

$$\begin{aligned}
\frac{1}{4c_1^3} \int_0^{R_0} r^{n-1} u_r^{m+4} dr \quad \le \quad & c_7 \cdot \left\{ 1 + \left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \right\} \cdot \left(\int_0^{R_0} r^{n-1} z_+^{m+4} dr \right)^{\frac{1}{m+4}} \\
& + c_{12} \left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \\
& + c_{13} L^{m+4} \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{5.23}$$

where $c_{12} := c_{10} + c_{11}$ and $c_{13} := c_8 + c_9 + c_{10} + c_4 + c_5 + c_{11}$.

Now in order to prove (5.2) for some suitably large $C > 0$ independent of $t \in (0, T_{max})$, we fix any such t and first consider the case when there exists a sequence of even numbers $m = m_j \ge m_\star, j \in \mathbb{N}$, such that $m_j \to \infty$ as $j \to \infty$ and

$$\left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} \le L^{m+4} \qquad \text{for all } m \in (m_j)_{j \in \mathbb{N}}. \tag{5.24}$$

Then taking $j \to \infty$ here, we directly obtain that

$$\|u_r(\cdot,t)\|_{L^\infty((0,R_0))} = \lim_{j\to\infty} \left(\int_0^{R_0} r^{n-1} u_r^{m_j+4}(r,t)dr \right)^{\frac{1}{m_j+4}} = \lim_{j\to\infty} L^{\frac{m_j+4}{m_j+3}} = L. \tag{5.25}$$

If conversely, such a sequence does not exist, then we can find $m_{\star\star} \geq m_\star$ such that for all even $m \geq m_{\star\star}$,

$$\left(\int_0^{R_0} r^{n-1} u_r^{m+4} dr \right)^{\frac{m+3}{m+4}} > L^{m+4}.$$

Using that $L \geq 1$, we thus infer from (5.23) that for any such m we have

$$\begin{aligned}\frac{1}{4c_1^3} \int_0^{R_0} r^{n-1} u_r^{m+4}(r,t)dr \quad < \quad & 2c_7 \left(\int_0^{R_0} r^{n-1} u_r^{m+4}(r,t)dr \right)^{\frac{m+3}{m+4}} \cdot \left(\int_0^{R_0} r^{n-1} z_+^{m+4}(r,t)dr \right)^{\frac{1}{m+4}} \\ & + c_{12} \left(\int_0^{R_0} r^{n-1} u_r^{m+4}(r,t)dr \right)^{\frac{m+3}{m+4}} + c_{13} \left(\int_0^{R_0} r^{n-1} u_r^{m+4}(r,t)dr \right)^{\frac{m+3}{m+4}}\end{aligned}$$

and hence

$$\frac{1}{4c_1^3} \left(\int_0^{R_0} r^{n-1} u_r^{m+4}(r,t)dr \right)^{\frac{1}{m+4}} \leq 2c_7 \left(\int_0^{R_0} r^{n-1} z_+^{m+4}(r,t)dr \right)^{\frac{1}{m+4}} + c_{12} + c_{13}.$$

In the limit $m \to \infty$, we therefore conclude that in this case,

$$\frac{1}{4c_1^3} \cdot \|u_r(\cdot,t)\|_{L^\infty((0,R_0))} \leq 2c_7 \cdot \|z_+(\cdot,t)\|_{L^\infty((0,R_0))} + c_{12} + c_{13}. \tag{5.26}$$

Since c_7, c_{12} and c_{13} as well as L are independent of $t \in (0, T_{max})$, (5.25) and (5.26) establish (5.2). $\square$

5.1.2 Estimating $|u_r|$ near the boundary

For fixed $t \in (0, T_{max})$, the above lemma in particular implies an upper bound for u_r in terms of $\|z_+\|_{L^\infty((0,t)\times(0,R_0))}$ on the lateral boundary line $r = R_0$ of the parabolic cylinder $(R_0, R) \times (0, t)$. This will enable us to apply a comparison argument to derive an estimate from above for u_r in this region on the basis of (2.10) to achieve the following.

Lemma 5.2 *Assume that $T_{max} < \infty$, but that $\sup_{(r,t)\in(0,R)\times(0,T_{max})} u(r,t) < \infty$. Then with $R_0 \in (0, R)$ taken from Lemma 5.1, we can find $C > 0$ such that*

$$\|u_r(\cdot,t)\|_{L^\infty((R_0,R))} \leq C \cdot \Big(1 + \|z_+\|_{L^\infty((0,R_0)\times(0,t))} \Big) \qquad \textit{for all } t \in (0, T_{max}). \tag{5.27}$$

PROOF. According to Lemma 5.1, we can pick $c_1 > 0$ such that

$$u_r(R_0, t) \leq c_1 \cdot \Big(1 + \|z_+(\cdot,t)\|_{L^\infty((0,R_0))} \Big) \qquad \text{for all } t \in (0, T_{max}),$$

which in particular implies that given any $t_0 \in (0, T_{max})$ we have

$$u_r(R_0, t) \le D_1(t_0) := c_1 \cdot \Big(1 + \|z_+\|_{L^\infty((0,R_0)\times(0,t_0))}\Big) \qquad \text{for all } t \in (0, t_0). \tag{5.28}$$

Let us next use our hypothesis and recall Lemma 3.2 to pick $c_2 > 0$ and $c_3 > 0$ fulfilling

$$c_2 \le u(r,t) \le c_3 \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}), \tag{5.29}$$

and apply Lemma 2.5 to find $c_4 > 0$ and $c_5 > 0$ such that

$$|v_r(r,t)| \le c_4 r \quad \text{and} \quad |v_{rr}(r,t)| \le c_5 \qquad \text{for all } r \in (0, R) \text{ and } t \in (0, T_{max}). \tag{5.30}$$

Therefore, the coefficient functions $\tilde{A}_3$ and $\tilde{A}_4$ in (2.12) can be estimated according to

$$\tilde{A}_3(r,t) \le c_6 := \frac{n-1}{R_0} + \frac{2\chi}{c_2^2} + \chi c_5 + \chi \cdot c_4^2 R^2 \cdot c_5 + (n-1)\chi \cdot c_4^3 R^2 \tag{5.31}$$

and

$$\tilde{A}_4(r,t) \le C_7 \;\; := \;\; 3c_3 + \frac{n-1}{R_0^2} \cdot c_2 + 3\chi\mu \cdot c_3 \cdot c_4 R \cdot c_5 + 3\chi \cdot c_3^2 \cdot c_4 R \cdot c_5 \tag{5.32}$$

$$+(n-1)\chi \cdot c_3 \cdot c_4^3 R + 3(n-1) \cdot \cdot c_3 \cdot c_4^2 R \cdot c_5 \tag{5.33}$$

for all $r \in (0, R)$ and $t \in (0, T_{max})$. We now fix $\alpha > 0$ large such that

$$\alpha > c_6 + \frac{c_7}{D} \tag{5.34}$$

and, given $t_0 \in (0, T_{max})$, define

$$\overline{\varphi}(r,t) := D\, e^{\alpha t} \qquad \text{for } r \in [R_0, R] \text{ and } t \in [0, t_0],$$

where

$$D := \max\Big\{D_1(t_0)\,,\, \sup_{r\in(R_0,R)} u_{0r}(r)\Big\} + 1. \tag{5.35}$$

Then (5.28) asserts that

$$u_r(R_0, t) \le \overline{\varphi}(R_0, t) \qquad \text{for all } t \in (0, t_0),$$

whereas clearly

$$u_r(R, t) = \overline{\varphi}(R, t) = 0 \qquad \text{for all } t \in (0, t_0)$$

and

$$u_r(r, 0) = u_{0r}(r) < D = \overline{\varphi}(r, 0) \qquad \text{for all } r \in [R_0, R].$$

Moreover, since $\overline{\varphi}$ is positive and $\overline{\varphi}_r = \overline{\varphi}_{rr} \equiv 0$, we may use (5.31), (5.32) and (5.34) to see that with $\mathcal{Q}$ as in (2.11) we have

$$\begin{aligned}
\mathcal{Q}\overline{\varphi} &= \overline{\varphi}_t - \tilde{A}_3(r,t)\overline{\varphi} - \tilde{A}_4(r,t) \\
&= \alpha D e^{\alpha t} - \tilde{A}_3(r,t) \cdot D e^{\alpha t} - \tilde{A}_4(r,t) \\
&\geq (\alpha - c_6) \cdot D e^{\alpha t} - c_7 \\
&\geq (\alpha - c_6) D - c_7 \\
&> 0 \qquad \text{for all } r \in (R_0, R) \text{ and } t \in (0, t_0).
\end{aligned}$$

As $\mathcal{Q}u_r \equiv 0$ due to Lemma 2.3, by comparison we conclude that $u_r \leq \overline{\varphi}$ in $(R_0,R) \times (0,t_0)$, which in view of (5.35) and (5.28) readily entails (5.27). $\square$

5.1.3 A bound for $|u_r|$ in the entire domain

Let us summarize the outcome of Lemma 5.1 and Lemma 5.2:

Corollary 5.3 *If $T_{max} < \infty$ but $\sup_{(r,t)\in(0,R)\times(0,T_{max})} u(r,t) < \infty$, then there exists $C > 0$ such that*

$$\|u_r(\cdot,t)\|_{L^\infty((0,R))} \leq C \cdot \Big(1 + \|z_+\|_{L^\infty((0,R)\times(0,t))}\Big) \qquad \text{for all } t \in (0, T_{max}).$$

PROOF. We only need to combine Lemma 5.1 with Lemma 5.2. $\square$

5.2 A nonlocal parabolic inequality for z

In order to bound z from above, let us first identify a linear inhomogeneous parabolic equation satisfied by this function.

Lemma 5.4 *The function $z = \frac{u_t}{u}$ satisfies*

$$z_t = B_1(r,t)z_{rr} + B_{21}(r,t)z_r + \frac{B_{22}(r,t)}{r} z_r + B_3(r,t)z + B_4(r,t) \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}), \tag{5.36}$$

where

$$\begin{cases}
B_1(r,t) := \dfrac{u^3}{\sqrt{u^2+u_r^2}^3}, \\
B_{21}(r,t) := 2\dfrac{u^2 u_r}{\sqrt{u^2+u_r^2}^3} - 3\dfrac{u^3 u_r u_{rr}}{\sqrt{u^2+u_r^2}^5} + 4\dfrac{u_r^3}{\sqrt{u^2+u_r^2}^3} - 3\dfrac{u_r^5}{\sqrt{u^2+u_r^2}^5} - \chi\dfrac{v_r}{\sqrt{1+v_r^2}}, \\
B_{22}(r,t) := (n-1)\dfrac{u^3}{\sqrt{u^2+u_r^2}^3}, \\
B_3(r,t) := \chi\dfrac{u}{\sqrt{1+v_r^2}^3} \qquad \text{and} \\
B_4(r,t) := -3\chi\dfrac{u(\mu-u)u_r v_r}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5} + 3\chi^2\dfrac{u(\mu-u)v_r^2}{(1+v_r^2)^3} + \chi\dfrac{u_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^3} - \chi^2\dfrac{u_r v_r}{(1+v_r^2)^2} \\
\qquad +3\chi\cdot\frac{n-1}{r}\cdot\frac{u u_r v_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5} - 3\chi^2\cdot\frac{n-1}{r}\cdot\frac{u v_r^3}{(1+v_r^2)^3}
\end{cases} \tag{5.37}$$

for $r \in (0,R)$ and $t \in (0,T_{max})$.

PROOF. We divide (2.4) by u and differentiate each term on the right-hand side of the resulting identity separately. Using that $u_t = uz$ and hence $u_{rt} = uz_r + u_r z$ and $u_{rrt} = uz_{rr} + 2u_r z_r + u_{rr} z$, we first obtain

$$\begin{aligned}
\left(\frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3}\right)_t &= \frac{u^2 u_{rrt}}{\sqrt{u^2+u_r^2}^3} + 2\frac{u u_t u_{rr}}{\sqrt{u^2+u_r^2}^3} - \frac{3}{2}\cdot\frac{u^2 u_{rr}\cdot(2uu_t + 2u_r u_{rt})}{\sqrt{u^2+u_r^2}^5} \\
&= \frac{u^3}{\sqrt{u^2+u_r^2}^3}\cdot z_{rr} + 2\frac{u^2 u_r}{\sqrt{u^2+u_r^2}^3}\cdot z_r + \frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3}\cdot z + 2\frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3}\cdot z \\
&\quad -3\frac{u^4 u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot z - 3\frac{u^3 u_r u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot z_r - 3\frac{u^2 u_r^2 u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot z.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3}\cdot z + 2\frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3}\cdot z - 3\frac{u^4 u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot z - 3\frac{u^2 u_r^2 u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot z \\
&\qquad = \frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^5}\cdot\Big\{(u^2+u_r^2) + 2(u^2+u_r^2) - 3u^2 - 3u_r^2\Big\} \\
&\qquad = 0,
\end{aligned}$$

this yields

$$\left(\frac{u^2 u_{rr}}{\sqrt{u^2+u_r^2}^3}\right)_t = \frac{u^3}{\sqrt{u^2+u_r^2}^3}\cdot z_{rr} + \left\{2\frac{u^2 u_r}{\sqrt{u^2+u_r^2}^3} - 3\frac{u^3 u_r u_{rr}}{\sqrt{u^2+u_r^2}^5}\right\}\cdot z_r \tag{5.38}$$

for all $r \in (0,R)$ and $t \in (0,T_{max})$.
Next,

$$\begin{aligned}
\left(\frac{u_r^4}{u\sqrt{u^2+u_r^2}^3}\right)_t &= 4\frac{u_r^3 u_{rt}}{u\sqrt{u^2+u_r^2}^3} - \frac{u_r^4 u_t}{u^2\sqrt{u^2+u_r^2}^3} - \frac{3}{2}\cdot\frac{u_r^4\cdot(2uu_t + 2u_r u_{rt})}{u\sqrt{u^2+u_r^2}^5} \\
&= 4\frac{u_r^3}{\sqrt{u^2+u_r^2}^3}\cdot z_r + 4\frac{u_r^4}{u\sqrt{u^2+u_r^2}^3}\cdot z - \frac{u_r^4}{u\sqrt{u^2+u_r^2}^3}\cdot z \\
&\quad -3\frac{u u_r^4}{\sqrt{u^2+u_r^2}^5}\cdot z - 3\frac{u_r^5}{\sqrt{u^2+u_r^2}^5}\cdot z_r - 3\frac{u_r^6}{u\sqrt{u^2+u_r^2}^5}\cdot z,
\end{aligned}$$

where again the zero-order terms have a vanishing sum in the sense that

$$\begin{aligned}
&4\frac{u_r^4}{u\sqrt{u^2+u_r^2}^3}\cdot z - \frac{u_r^4}{u\sqrt{u^2+u_r^2}^3}\cdot z - 3\frac{u u_r^4}{\sqrt{u^2+u_r^2}^5}\cdot z - 3\frac{u_r^6}{u\sqrt{u^2+u_r^2}^5}\cdot z \\
&\qquad = \frac{u_r^4}{u\sqrt{u^2+u_r^2}^5}\cdot\Big\{4(u^2+u_r^2) - (u^2+u_r^2) - 3u^2 - 3u_r^2\Big\}\cdot z \\
&\qquad = 0,
\end{aligned}$$

so that

$$\Big(\frac{u_r^4}{u\sqrt{u^2+u_r^2}^3}\Big)_t = \Big\{4\frac{u_r^3}{\sqrt{u^2+u_r^2}^3} - 3\frac{u_r^5}{\sqrt{u^2+u_r^2}^5}\Big\}\cdot z_r \tag{5.39}$$

for all $r \in (0,R)$ and $t \in (0,T_{max})$.
Likewise,

$$\begin{aligned}
\Big(\frac{u_r}{\sqrt{u^2+u_r^2}}\Big)_t &= \frac{u_{rt}}{\sqrt{u^2+u_r^2}} - \frac{1}{2}\cdot\frac{u_r\cdot(2uu_t+2u_ru_{rt})}{\sqrt{u^2+u_r^2}^3} \\
&= \frac{u}{\sqrt{u^2+u_r^2}}\cdot z_r + \frac{u_r}{\sqrt{u^2+u_r^2}}\cdot z - \frac{u^2u_r}{\sqrt{u^2+u_r^2}^3}\cdot z - \frac{uu_r^2}{\sqrt{u^2+u_r^2}^3}\cdot z_r - \frac{u_r^3}{\sqrt{u^2+u_r^2}^3}\cdot z \\
&= \frac{u}{\sqrt{u^2+u_r^2}^3}\cdot\Big\{(u^2+u_r^2)-u_r^2\Big\}\cdot z_r \\
&\quad +\frac{u_r}{\sqrt{u^2+u_r^2}^3}\cdot\Big\{(u^2+u_r^2)-u^2-u_r^2\Big\}\cdot z \\
&= \frac{u^3}{\sqrt{u^2+u_r^2}^3}\cdot z_r,
\end{aligned}$$

whence

$$\Big(\frac{n-1}{r}\frac{u_r}{\sqrt{u^2+u_r^2}}\Big)_t = \frac{n-1}{r}\cdot\frac{u^3}{\sqrt{u^2+u_r^2}^3}\cdot z_r \tag{5.40}$$

for $r \in (0,R)$ and $t \in (0,T_{max})$.
As for the respective terms originating from the rightmost three summands in (2.4), we make use of (2.16) to express v_{rt} conveniently. We thereby compute

$$\begin{aligned}
\Big(-\chi\frac{\mu-u}{\sqrt{1+v_r^2}^3}\Big)_t &= \chi\frac{u_t}{\sqrt{1+v_r^2}^3} + \frac{3}{2}\chi\frac{\mu-u}{\sqrt{1+v_r^2}^5}\cdot 2v_rv_{rt} \\
&= \chi\frac{u}{\sqrt{1+v_r^2}^3}\cdot z + 3\chi\frac{(\mu-u)v_r}{\sqrt{1+v_r^2}^5}\cdot\Big\{-\frac{uu_r}{\sqrt{u^2+u_r^2}}+\chi\frac{uv_r}{\sqrt{1+v_r^2}}\Big\} \\
&= \chi\frac{u}{\sqrt{1+v_r^2}^3}\cdot z - 3\chi\frac{u(\mu-u)u_rv_r}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5} + 3\chi^2\frac{u(\mu-u)v_r^2}{(1+v_r^2)^3}
\end{aligned} \tag{5.41}$$

and

$$\begin{aligned}
\Big(-\chi\frac{u_rv_r}{u\sqrt{1+v_r^2}}\Big)_t &= -\chi\frac{u_{rt}v_r}{u\sqrt{1+v_r^2}} - \chi\frac{u_rv_{rt}}{u\sqrt{1+v_r^2}} + \chi\frac{u_rv_ru_t}{u^2\sqrt{1+v_r^2}} + \frac{1}{2}\chi\frac{u_rv_r}{u\sqrt{1+v_r^2}^3}\cdot 2v_rv_{rt} \\
&= -\chi\frac{v_r}{\sqrt{1+v_r^2}}\cdot z_r - \chi\frac{u_rv_r}{u\sqrt{1+v_r^2}}\cdot z \\
&\quad -\chi\frac{u_r}{u\sqrt{1+v_r^2}}\cdot\Big\{-\frac{uu_r}{\sqrt{u^2+u_r^2}}+\chi\frac{uv_r}{\sqrt{1+v_r^2}}\Big\} \\
&\quad +\chi\frac{u_rv_r}{u\sqrt{1+v_r^2}}\cdot z + \chi\frac{u_rv_r^2}{u\sqrt{1+v_r^2}^3}\cdot\Big\{-\frac{uu_r}{\sqrt{u^2+u_r^2}}+\chi\frac{uv_r}{\sqrt{1+v_r^2}}\Big\}
\end{aligned}$$

$$
\begin{aligned}
&= -\chi\frac{v_r}{\sqrt{1+v_r^2}}\cdot z_r+\chi\frac{u_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}}-\chi^2\frac{u_rv_r}{(1+v_r^2)^2} \\
&= -\chi\frac{v_r}{\sqrt{1+v_r^2}}\cdot z_r+\chi\frac{u_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^3}-\chi^2\frac{u_rv_r}{(1+v_r^2)^2}, \qquad (5.42)
\end{aligned}
$$

and observe that

$$
\begin{aligned}
\left(\frac{v_r^3}{\sqrt{1+v_r^2}^3}\right)_t &= \left\{\frac{3v_r^2}{\sqrt{1+v_r^2}^3}-\frac{3}{2}\cdot\frac{v_r^3\cdot 2v_r}{\sqrt{1+v_r^2}^5}\right\}\cdot v_{rt} \\
&= -3\frac{uu_rv_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5}+3\chi\frac{uv_r^3}{(1+v_r^2)^3}
\end{aligned}
$$

to obtain

$$
\left(-\chi\cdot\frac{n-1}{r}\cdot\frac{v_r^3}{\sqrt{1+v_r^2}^3}\right)_t = 3\chi\cdot\frac{n-1}{r}\cdot\frac{uu_rv_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5}-3\chi^2\cdot\frac{n-1}{r}\cdot\frac{uv_r^3}{(1+v_r^2)^3} \qquad (5.43)
$$

for $r\in(0,R)$ and $t\in(0,T_{max})$. In light of (5.38)-(5.43), (2.4) easily yields (5.36) with B_1,B_{21},B_{22},B_3 and B_4 as in (5.37). $\Box$

On suitably estimating the inhomogeneous term B_4 herein by means of Corollary 5.3, we can develop (5.36) into a nonlocal parabolic inequality for z as follows.

Lemma 5.5 *Suppose that $T_{max}<\infty$, but that $\sup_{(r,t)\in(0,R)\times(0,T_{max})}u(r,t)<\infty$. Then there exist a constant $d>0$ and continuous functions b_1,b_{21},b_{22} and b_3 on $[0,R]\times[0,T_{max})$ with the properties that b_1 and b_{22} are nonnegative, that b_3 is bounded on $(0,R)\times(0,T_{max})$, and such that $z=\frac{u_t}{u}$ satisfies*

$$
z_t(r,t)\le b_1(r,t)z_{rr}+b_{21}(r,t)z_r+\frac{b_{22}(r,t)}{r}z_r+b_3(r,t)z+d\cdot\Big(1+\|z_+\|_{L^\infty((0,R)\times(0,t))}\Big) \qquad (5.44)
$$

for all $r\in(0,R)$ and $t\in(0,T_{max})$.

Proof. With B_1,B_{21},B_{22},B_3 and B_4 taken from Lemma 5.4, we let $b_1:=B_1, b_{21}:=B_{21}, b_{22}:=B_{22}$ and $b_3:=B_3$. Then from (5.37) we immediately obtain that b_1,b_{21},b_{22} and b_3 are continuous in $[0,R]\times[0,T_{max})$, and that $b_1\ge 0$ and $b_{22}\ge 0$. Since our boundedness assumption on u ensures that b_3 is bounded, it remains to control the inhomogeneity B_4 in (5.36) adequately. To this end, we once more use our hypothesis along with Lemma 2.5 to pick positive constants c_1,c_2 and c_3 such that

$$
u(r,t)\le c_1, \quad |v_r(r,t)|\le c_2r \quad \text{and} \quad |v_{rr}(r,t)|\le c_3 \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}). \qquad (5.45)
$$

Then in (5.37) we can estimate

$$
\begin{aligned}
-3\chi\frac{u(\mu-u)u_rv_r}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5} &\le 3\chi\cdot c_1(\mu+c_1)\cdot c_2R\cdot\frac{|u_r|}{\sqrt{u^2+u_r^2}} \\
&\le 3c_1(\mu+c_1)c_2\chi R
\end{aligned}
$$

and

$$3\chi^2 \frac{u(\mu-u)v_r v_{rr}}{(1+v_r^2)^3} \le 3\chi^2 c_1(\mu+c_1)\cdot c_2 R\cdot c_3$$
$$\le 3c_1(\mu+c_1)c_2c_3\chi^2 R$$

as well as

$$3\chi\cdot\frac{n-1}{r}\cdot\frac{uu_rv_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^5} \le 3\chi\cdot\frac{n-1}{r}\cdot c_1\cdot c_2^2r^2\cdot\frac{|u_r|}{\sqrt{u^2+u_r^2}}$$
$$\le 3(n-1)c_1c_2^2\chi R$$

and

$$-3\chi^2\cdot\frac{n-1}{r}\cdot\frac{uv_r^3}{(1+v_r^2)^3} \le 3\chi^2\cdot\frac{n-1}{r}\cdot c_1\cdot c_2^3r^3$$
$$\le 3(n-1)c_1c_2^3\chi^2R^2$$

for all $r\in(0,R)$ and $t\in(0,T_{max})$. In the third and fourth summands in the definition (5.37) of B_4, however, apparently we can only estimate

$$\chi\frac{u_r^2}{\sqrt{u^2+u_r^2}\cdot\sqrt{1+v_r^2}^3} \le \chi\frac{u_r^2}{\sqrt{u^2+u_r^2}} \le \chi|u_r|$$

and

$$-\chi^2\frac{u_rv_r}{(1+v_r^2)^2} \le \chi^2|u_r|$$

for all $r\in(0,R)$ and $t\in(0,T_{max})$, with the possibly unbounded factors $|u_r|$ remaining. Fortunately, applying Corollary 5.3 yields $c_4>0$ such that

$$|u_r(r,t)| \le c_4\cdot\Big(1+\|z_+\|_{L^\infty((0,R)\times(0,t))}\Big) \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}).$$

Therefore, (5.44) results from (5.36) if we choose $d>0$ conveniently large. $\Box$

5.3 Boundedness of z from above

Apparently, nonlocal parabolic inequalities of type (5.44) do not allow for general comparison priciples. After all, the fact that here the memory term enjoys a certain linear boundedness property with respect to z_+ enables us to follow a maximum principle-type reasoning to establish an essentially exponential upper bound for z and thereby obtain the following.

Lemma 5.6 *Assume that $T_{max}<\infty$ and $\sup_{(r,t)\in(0,R)\times(0,T_{max})}u(r,t)<\infty$. Then there exists $C>0$ such that $z=\frac{u_t}{u}$ satisfies*

$$z(r,t)\le C \qquad \textit{for all } r\in(0,R) \textit{ and } t\in(0,T_{max}). \tag{5.46}$$

Proof. We let b_1, b_{21}, b_{22}, b_3 and d be as provided by Lemma 5.5, so that by boundedness of b_3 we can find $c_1 > 0$ such that

$$b_3(r,t) \le c_1 \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}). \tag{5.47}$$

We than fix $\alpha > 0$ large enough fulfilling

$$\alpha > c_1 + d \tag{5.48}$$

and let

$$\varphi(r,t) := e^{-\alpha t} z(r,t) - dt \qquad \text{for } r \in [0,R] \text{ and } t \in [0,T_{max}).$$

Then according to Lemma 5.5,

$$\begin{aligned}
\varphi_t &= e^{-\alpha t}(z_t - \alpha z) - d \\
&\le e^{-\alpha t} \cdot \Big\{ b_1(r,t) z_{rr} + b_{21}(r,t) z_r + \frac{b_{22}}{r} z_r + b_3(r,t) z + d\|z_+\|_{L^\infty((0,R)\times(0,t))} + d - \alpha z \Big\} - d \\
&= b_1(r,t)\varphi_{rr} + b_{21}(r,t)\varphi_r + \frac{b_{22}(r,t)}{r}\varphi_r + \Big(b_3(r,t) - \alpha\Big) \cdot \Big(\varphi + dt\Big) \\
&\quad + de^{-\alpha t}\|z_+\|_{L^\infty((0,R)\times(0,t))} + de^{-\alpha t} - d \qquad \text{for all } r \in (0,R) \text{ and } t \in (0,T_{max}),
\end{aligned} \tag{5.49}$$

and since $z_r = (\frac{u_t}{u})_r = \frac{u_{rt}}{u} - \frac{u_r u_t}{u^2}$ in $[0,R] \times [0,T_{max})$, the fact that $u_r(0,t) = u_r(R,t) = 0$ for all $t \in (0,T_{max})$ entails that

$$\varphi_r(0,t) = \varphi_r(R,t) = 0 \qquad \text{for all } t \in (0,T_{max}). \tag{5.50}$$

Now if for some $T \in (0,T_{max})$, the value $S := \sup_{(r,t)\in(0,R)\times(0,T)} \varphi(r,t)$ was positive and attained at some point $(r_0,t_0) \in [0,R] \times [0,T]$ with $t_0 > 0$, then necessarily

$$\varphi_t(r_0,t_0) \ge 0, \tag{5.51}$$

and (5.50) ensures that in both cases $r_0 \in (0,R)$ and $r_0 \in \{0,R\}$ we moreover must have

$$\varphi_r(r_0,t_0) = 0 \qquad \text{and} \varphi_{rr}(r_0,t_0) \le 0. \tag{5.52}$$

We claim that these properties imply that

$$0 \le \Big(b_3(r_0,t_0) - \alpha\Big) \cdot \Big(\varphi(r_0,t_0) + dt_0\Big) + de^{-\alpha t_0}\|z_+\|_{L^\infty((0,R)\times(0,t_0))} + de^{-\alpha t_0} - d. \tag{5.53}$$

Indeed, in the case $r_0 \in (0,R)$ we may directly apply (5.49) to easily deduce this from (5.51) and (5.52). When $r_0 = R$, by continuity of $\varphi, \varphi_t, \varphi_r$ and φ_{rr} in $[0,R] \times (0,T_{max})$ it is clear that (5.49) actually remains valid at (r_0,t_0), so that (5.53) follows from the same argument. If $r_0 = 0$, however, we make use of the favorable sign of the singular term $\frac{b_{22}}{r}$ in (5.49) by first choosing, once more relying on the extremal property of $\varphi(r_0,t_0)$, a sequence $(r_j)_{j\in\mathbb{N}} \subset (0,R)$ such that $r_j \searrow 0$ as $j \to \infty$ and

$$\varphi_r(r_j,t_0) \le 0 \qquad \text{for all } j \in \mathbb{N},$$

and then evaluating (5.49) at $r = r_j$ to see that

$$\begin{array}{rcl} \varphi_t(r_j,t_0) & \le & b_1(r_j,t_0)\varphi_{rr}(r_j,t_0) + b_{21}(r_j,t_0)\varphi_r(r_j,t_0) + \Big(b_3(r_j,t_0)-\alpha\Big)\cdot\Big(\varphi(r_j,t_0)+dt_0\Big) \\ & & +de^{-\alpha t_0}\|z_+\|_{((0,R)\times(0,t_0))} + de^{-\alpha t_0} - d \end{array}$$

for all $j \in \mathbb{N}$. Again by continuity of $\varphi, \varphi_t, \varphi_r$ and φ_{rr}, we may take $j \to \infty$ to conclude that

$$\begin{array}{rcl} \varphi_t(0,t_0) & \le & b_1(0,t_0)\varphi_{rr}(0,t_0) + b_{21}(0,t_0)\varphi_r(0,t_0) + \Big(b_3(0,t_0)-\alpha\Big)\cdot\Big(\varphi(0,t_0)+dt_0\Big) \\ & & +de^{-\alpha t_0}\|z_+\|_{((0,R)\times(0,t_0))} + de^{-\alpha t_0} - d, \end{array}$$

whereupon one more application of (5.51) and (5.52) yields (5.53) also in this case.

Now observing that $e^{-\alpha t_0} \le 1$ and using that $S = \varphi(r_0,t_0)$ is positive, in view of (5.47) we obtain from (5.53) that

$$0 \le (c_1-\alpha)\cdot\Big(\varphi(r_0,t_0)+dt_0\Big) + de^{-\alpha t_0}\|z_+\|_{L^\infty((0,R)\times(0,t_0))}. \qquad (5.54)$$

Here we rewrite $z = e^{\alpha t}\varphi + dte^{\alpha t}$ and use that if f and g are two functions on a set $D \subset \mathbb{R}^N$, $N \ge 1$, then both $(f+g)_+$ and $\sup_D\{f+g\} \le \sup_D f + \sup_D g$. We thereby obtain that

$$\begin{array}{rcl} de^{-\alpha t_0}\|z_+\|_{L^\infty((0,R)\times(0,t_0))} & = & de^{-\alpha t_0}\cdot \sup\limits_{(r,s)\in(0,R)\times(0,t_0)} \Big\{e^{\alpha s}\varphi(r,s) + dse^{\alpha s}\Big\} \\ & \le & de^{\alpha t_0}\cdot \sup\limits_{(r,s)\in(0,R)\times(0,t_0)} \Big\{e^{\alpha s}\varphi_+(r,s) + dse^{\alpha s}\Big\} \\ & \le & de^{\alpha t_0}\cdot \bigg\{ \sup\limits_{(r,s)\in(0,R)\times(0,t_0)} \Big\{e^{\alpha s}\varphi_+(r,s)\Big\} + \sup\limits_{s\in(0,t_0)} \Big\{dse^{\alpha s}\Big\}\bigg\} \\ & \le & de^{\alpha t_0}\cdot \sup\limits_{(r,s)\in(0,R)\times(0,t_0)} \Big\{e^{\alpha s}\varphi_+(r,s)\Big\} + d^2t_0. \end{array}$$

Since from the definition of S we know that

$$\sup_{(r,s)\in(0,R)\times(0,t_0)} \Big\{e^{\alpha s}\varphi_+(r,s)\Big\} \le e^{\alpha t_0}\cdot \sup_{(r,s)\in(0,R)\times(0,t_0)} \varphi_+(r,s) = e^{\alpha t_0}\cdot\varphi(r_0,t_0),$$

this entails that

$$de^{-\alpha t_0}\|z_+\|_{L^\infty((0,R)\times(0,t_0))} \le de^{-\alpha t_0}\|z_+\|_{L^\infty((0,R)\times(0,t_0))} \le d\varphi(r_0,t_0) + d^2t_0,$$

so that (5.54) yields the inequality

$$\begin{array}{rcl} 0 & \le & (c_1-\alpha)\cdot\Big(\varphi(r_0,t_0)+dt_0\Big) + d\varphi(r_0,t_0) + dt_0 \\ & = & (c_1-\alpha+d)\cdot\Big(\varphi(r_0,t_0)+dt_0\Big). \end{array}$$

In light of our restriction (5.48) on α, however, this contradicts the positivity of $\varphi(r_0,t_0)$ and thereby proves that actually φ cannot attain a positive maximum over any such region $[0,R]\times[0,T]$, $T \in (0,T_{max})$, at a positive time t_0. This means that in fact

$$\varphi(r,t) \le \|\varphi_+(\cdot,0)\|_{L^\infty((0,R))} = \|z_+(\cdot,0)\|_{L^\infty((0,R))} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max})$$

and hence

$$\begin{aligned} z(r,t) &= e^{\alpha t}\Big(\varphi(r,t)+dt\Big) \\ &\le e^{\alpha T_{max}}\cdot\Big\{\|z_+(\cdot,0)\|_{L^\infty((0,R))}+dT_{max}\Big\} \qquad \text{for all } r\in(0,R) \text{ and } t\in(0,T_{max}), \end{aligned}$$

which establishes (5.46). □

5.4 Boundedness of u implies extensibility. Proof of Theorem 1.1

Combining the latter lemma with Corollary 5.3 now directly yields the desired bound for u_r.

Corollary 5.7 *If $T_{max}<\infty$ but $\sup_{(r,t)\in(0,R)\times(0,T_{max})} u(r,t)<\infty$, then there exists $C>0$ such that*

$$\|u_r(\cdot,t)\|_{L^\infty((0,R))}\le C \qquad \textit{for all } t\in(0,T_{max}).$$

Proof. Thanks to the upper estimate for z obtained in Lemma 5.6, this is an immediate consequence of Corollary 5.3. □

We can thereby readily verify our main statement on local existence and extensibility.

Proof of Theorem 1.1. In view of the local existence result established in Lemma 2.1, we only need to verify (1.6). Indeed, if (1.6) was false, then for some solution the respective maximal existence time would satisfy $T_{max}<\infty$ but $\limsup_{t\nearrow T_{max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}<\infty$. Then, however, Corollary 5.7 would apply to assert that also $\limsup_{t\nearrow T_{max}}\|\nabla u(\cdot,t)\|_{L^\infty(\Omega)}$ would be finite. Along with the lower bound for u provided by Lemma 3.2, this would contradict the extensibility criterion (2.2) in Lemma 2.1. □

6 Boundedness for small χ. Proof of Theorem 1.2

In light of the extensibility criterion provided by Theorem 1.1, in order to prove both global existence and boundedness of a solution it is sufficient to derive an a priori estimate for (u,v) in $(L^\infty(\Omega\times(0,T)))^2$ which does not explicitly depend on $T<T_{max}\le\infty$. As a preparation for the proof of this in Theorem 1.2 below, let us state the following elementary inequality.

Lemma 6.1 *Let $p\ge 1$. Then*

$$\int_\Omega u^{p-1}|\nabla u|\le\int_\Omega\frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}+\int_\Omega u^p \qquad \textit{for all } t\in(0,T_{max}). \tag{6.1}$$

Proof. By means of Young's inequality, we see that

$$\int_\Omega u^{p-1}|\nabla u|\le\frac12\int_\Omega\frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}+\frac12\int_\Omega u^{p-1}\sqrt{u^2+|\nabla u|^2} \qquad \text{for all } t\in(0,T_{max}), \tag{6.2}$$

where using the elementary inequality $\sqrt{X+Y}\le\sqrt{X}+\sqrt{Y}$, valid for all $X\ge0$ and $Y\ge0$, we can estimate

$$\frac12\int_\Omega u^{p-1}\sqrt{u^2+|\nabla u|^2}\le\frac12\int_\Omega u^p+\frac12\int_\Omega u^{p-1}|\nabla u|.$$

Therefore, (6.1) results from (6.2). $\square$

We are now in the position to make sure that if either $n \geq 2, \chi < 1$ and u_0 is an arbitrary function satisfying (1.4), or $n = 1$, $\chi > 0$ and $\int_\Omega u_0 < m_c$ with m_c as in (1.9), then the solution of (1.3) in fact is global and remains bounded:

PROOF of Theorem 1.2. We let $p_k := 2^k$ and, given $T \in (0, T_{max})$, introduce

$$M_k := \sup_{t \in (0,T)} \int_\Omega u^{p_k}(x,t)dx \tag{6.3}$$

for nonnegative integers k. Then clearly M_k is well-defined for any such T and k, and in order to control M_k appropriately, we fix $k \geq 1$ and multiply the first equation in (1.3) by pu^{p-1} for $p := p_k$ to see upon integrating by parts that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} &= p(p-1)\chi \int_\Omega \frac{u^{p-1}\nabla u \cdot \nabla v}{\sqrt{1+|\nabla v|^2}} \\ &\leq p(p-1)\chi \int_\Omega u^{p-1}|\nabla u| \cdot \frac{|\nabla v|}{\sqrt{1+|\nabla v|^2}} \end{aligned} \tag{6.4}$$

for all $t \in (0, T_{max})$. Here in the multi-dimensional case, in which no evident uniform a priori bound for $|\nabla v|$ seems available, we use the trivial pointwise inequality $\frac{|\nabla v|}{\sqrt{1+|\nabla v|^2}} \leq 1$ to obtain

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} \leq p(p-1)\chi\int_\Omega u^{p-1}|\nabla u| \quad \text{for all } t \in (0,T_{max}) \qquad \text{if } n \geq 2. \tag{6.5}$$

In the one-dimensional setting, however, from (2.17) and (1.5) we know that $|\nabla v| = |v_r| \leq m$ throughout $\Omega \times (0, T_{max})$, whence by monotonicity of $0 \leq \xi \mapsto \frac{\xi}{\sqrt{1+\xi^2}}$ we infer from (6.4) that

$$\frac{d}{dt}\int_\Omega u^p + p(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} \leq p(p-1)\chi \cdot \frac{m}{\sqrt{1+m^2}} \cdot \int_\Omega u^{p-1}|\nabla u| \quad \text{for all } t \in (0,T_{max}) \quad \text{if } n = 1. \tag{6.6}$$

In both (6.5) and (6.6) we now apply Lemma 6.1 to estimate

$$p(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} \geq p(p-1)\int_\Omega u^{p-1}|\nabla u| - p(p-1)\int_\Omega u^p$$

and thus obtain on writing

$$\Lambda := \begin{cases} \frac{m}{\sqrt{1+m^2}} & \text{if } n = 1, \\ 1 & \text{if } n \geq 2, \end{cases} \tag{6.7}$$

and adding $\int_\Omega u^p$ on both sides of (6.5) and (6.6), respectively, that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + p(p-1)(1-\chi\Lambda)\int_\Omega u^{p-1}|\nabla u| &\leq \Big\{p(p-1)+1\Big\} \cdot \int_\Omega u^p \\ &\leq p^2 \int_\Omega u^p \qquad \text{for all } t \in (0,T_{max}). \end{aligned} \tag{6.8}$$

We next invoke the Gagliardo-Nirenberg inequality ([32]) to find $c_1 > 0$ such that with $a := \frac{n}{n+1}$ we have

$$\|\varphi\|_{L^1(\Omega)} \le c_1 \|\nabla\varphi\|_{L^1(\Omega)}^a \|\varphi\|_{L^{\frac{1}{2}}(\Omega)}^{1-a} + c_1 \|\varphi\|_{L^{\frac{1}{2}}(\Omega)} \qquad \text{for all } \varphi \in W^{1,1}(\Omega),$$

and thereby obtain that

$$\int_\Omega u^p \le c_1 \Big(\int_\Omega |\nabla u^p| \Big)^a \cdot \Big(\int_\Omega u^{\frac{p}{2}} \Big)^{2(1-a)} \qquad \text{for all } t \in (0, T_{max}).$$

Since our specification of $p = p_k = 2^k$ allows us to use the definition (6.3) of M_{k-1} in estimating

$$\int_\Omega u^{\frac{p}{2}} \le M_{k-1} \qquad \text{for all } t \in (0,T),$$

this implies that

$$\int_\Omega u^p \le c_1 \Big(\int_\Omega |\nabla u^p| \Big)^a \cdot M_{k-1}^{2(1-a)} + c_1 M_{k-1}^2 \qquad \text{for all } t \in (0,T).$$

Thanks to the fact that our assumptions ensure that $\chi\Lambda < 1$, another application of Young's inequality therefore provides $c_2 > 0$ fulfilling

$$p^2 \int_\Omega u^p \le (p-1)(1-\chi\Lambda) \int_\Omega |\nabla u^p| + c_2 p^{\frac{2}{1-a}} M_{k-1}^2 + c_1 p^2 M_{k-1}^2 \qquad \text{for all } t \in (0,T),$$

from which due to the evident fact that $p^2 \le p^{\frac{2}{1-a}}$ we obtain that

$$p^2 \int_\Omega u^p \le p(p-1)(1-\chi\Lambda) \int_\Omega u^{p-1} |\nabla u| + c_3 p^{\frac{2}{1-a}} M_{k-1}^2 \qquad \text{for all } t \in (0,T)$$

with $c_3 := c_1 + c_2$. Therefore, (6.8) entails the autonomous ODI

$$\frac{d}{dt} \int_\Omega u^p + \int_\Omega u^p \le c_3 p^{\frac{2}{1-a}} M_{k-1}^2, \qquad t \in (0,T),$$

for $(0,T) \ni t \mapsto \int_\Omega u^p(x,t)dx$, which upon a comparison argument implies that

$$M_k \le \max\Big\{ \int_\Omega u_0^{p_k} \, , \, c_3 p_k^{\frac{2}{1-a}} M_{k-1}^2 \Big\} \qquad \text{for all } k \ge 1. \tag{6.9}$$

Now if there exists a sequence $(k_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $k_j \to \infty$ as $j \to \infty$ and

$$M_{k_j} \le \int_\Omega u_0^{p_{k_j}} \qquad \text{for all } j \in \mathbb{N}, \tag{6.10}$$

we may take the p_{k_j}-th root on both sides here to see that according to the definition (6.3) of M_{k_j} we have

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^{p_{k_j}}(\Omega)} \le \|u_0\|_{L^{p_{k_j}}(\Omega)},$$

which on letting $j \to \infty$ implies that

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} \tag{6.11}$$

in this case.
Conversely, if no such sequence exists, then (6.9) means that with some suitably large $k_0 \in \mathbb{N}$ we have

$$M_k \le c_3 p_k^{\frac{2}{1-a}} M_{k-1} \qquad \text{for all } k \ge k_0.$$

Since $p_k^{\frac{2}{1-a}} = (2^{\frac{2}{1-a}})^k$, it is easy to see that this entails the existence of a number $b > 1$ independent of T which satisfies

$$M_k \le b^k M_{k-1} \qquad \text{for all } k \ge 1. \tag{6.12}$$

By a straightforward induction, this warrants that

$$M_k \le b^{\sum_{j=0}^k j\cdot 2^{k-j}} \cdot M_0^{2^k} \qquad \text{for all } k \ge 1,$$

where by an elementary computation,

$$\begin{aligned}
\sum_{j=0}^k j\cdot 2^{k-j} &= 2^{k-1}\cdot \sum_{j=0}^k j\cdot \Big(\frac{1}{2}\Big)^{j-1} \\
&= 2^{k-1}\cdot \frac{k\cdot(\frac{1}{2})^{k+1} - (k+1)\cdot(\frac{1}{2})^k + 1}{(\frac{1}{2})^2} \\
&= k - 2(k+1) + 2^{k+1} \\
&\le 2^{k+1} \qquad \text{for all } k \ge 1.
\end{aligned}$$

Thus,

$$M_k \le B^{2^{k+1}} \cdot M_0^{2^k} \qquad \text{for all } k \ge 1,$$

by (6.3) implying that

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^{p_k}(\Omega)} = M_k^{\frac{1}{2^k}} \le b^2 M_0 \qquad \text{for all } k \ge 1. \tag{6.13}$$

Now as by the evident mass conservation property in (1.3) we have $\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx$ for all $t \in (0,T_{max})$ and hence $M_0 = \|u_0\|_{L^1(\Omega)}$, taking $k \to \infty$ in (6.13) shows that in this second case,

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \le b^2\|u_0\|_{L^1(\Omega)}. \tag{6.14}$$

Since all expressions on the right-hand sides of (6.11) and (6.14) do not depend on $T \in (0,T_{max})$, and since boundedness of u clearly implies boundedness of v by standard elliptic estimates, the proof is complete. $\square$

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