Sharp decay estimates in a bioconvection model with quadratic degradation in bounded domains

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Abstract

The paper studies large time behavior of solutions to the Keller-Segel system with quadratic degradation in a liquid environment, as given by

$$\begin{cases} u_t + U \cdot \nabla u &= \Delta u - \nabla \cdot (u \nabla v) - \mu u^2, \qquad x \in \Omega, \ t > 0, \\ v_t + U \cdot \nabla v &= \Delta v - v + u, \qquad x \in \Omega, \ t > 0, \end{cases} \qquad (\star)$$

under Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^n$, where $n \geq 1$ is arbitrary.

It is shown that whenever $U : \Omega \times (0, \infty) \to \mathbb{R}^n$ is a bounded and sufficiently regular solenoidal vector field, any nontrivial global bounded solution of $(\star)$ approaches the trivial equilibrium at a rate which with respect to the norm in either of the spaces $L^1(\Omega)$ and $L^\infty(\Omega)$ can be controlled from above and below by appropriate multiples of $\frac{1}{t+1}$. This underlines that even up to this quantitative level of accuracy, the large time behavior in $(\star)$ is essentially independent not only of the particular fluid flow, but also of any effect originating from chemotactic cross-diffusion. The latter is in contrast to the corresponding Cauchy problem for which known results show that in the case $n = 2$ the presence of chemotaxis can significantly enhance biomixing by reducing the respective spatial L^1 norms of solutions (Kiselev/Ryzhik, *J. Math. Phys.*, 2012).

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1 Introduction

We consider nonnegative solutions of the boundary value problem

$$\begin{cases} u_t + U\cdot\nabla u &= \Delta u - \chi\nabla\cdot(u\nabla v) - \mu u^2, \qquad x\in\Omega,\ t>0,\\ v_t + U\cdot\nabla v &= \Delta v - v + u, \qquad x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & \qquad x\in\partial\Omega,\ t>0, \end{cases} \tag{1.1}$$

in a bounded domain $\Omega\subset\mathbb{R}^n$ with smooth boundary, where $n\ge 1$, where $\chi>0$ and μ are positive parameters, and where $U:\Omega\times(0,\infty)\to\mathbb{R}^n$ is a prescribed solenoidal vector field. Systems of this type arise in the macroscopic modeling of chemotactic migration under the influence of a liquid environment by transport through a given fluid, and in presence of quadratic degradation such as appearing in logistic-type cell kinetics. Here we focus on situations in which cell proliferation, in logistic models represented by linear production terms, can either be neglected on the considered time scales, or is absent in principle. A prototypical example for the latter arises in the context of coral broadcast spawning processes ([2], [6]) during which eggs release a chemical signal, with concentration denoted by $v=v(x,t)$, that attracts sperms, where both eggs and sperms jointly consitute a population with density $u=u(x,t)$, and where the transporting incompressible ocean flow is represented through its velocity field $U=U(x,t)$.

Already in the fluid-free case when $U\equiv 0$ a variety of previous results indicates quite a substantial effect of the cross-diffusive mechanism in (1.1), going far beyond well-established knowledge on the ability of the classical Keller-Segel system obtained on letting $\mu=0$, that is, of

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v),\\ v_t = \Delta v - v + u, \end{cases} \tag{1.2}$$

to generate singularities in the sense of finite-time blow-up of some solutions in two- and higher-dimensional settings ([5], [18]). Indeed, also in situations when $\mu>0$ in

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) - \mu u^2,\\ v_t = \Delta v - v + u, \end{cases} \tag{1.3}$$

and related systems, the destabilizing action of cross-diffusion may still enforce quite a complex solution behavior in comparison to the respective scalar absorptive parabolic equation, as indicated by numerical experiments ([11]) and rigorously confirmed by results on spontaneous emergence of large population densities at intermediate time scales ([20]; cf. also [19] and [8] for similar findings on associated parabolic-elliptic simplifications). In fact, even the drastic phenomenon of finite-time blow-up has been shown to be suppressed by the presence of quadratic degradation only when either $n\le 2$ ([10], [9]) or $n\ge 3$ and μ is suitably large ([16]; see also [14] for a precedent). The question how far such systems at all are globally solvable when $n\ge 3$ and $\mu>0$ is small has only been partially been answered so far by a statement on global existence of weak solutions, possibly unbounded but at least in the case $n=3$ eventually bounded and smooth and asymptotically decay in both components ([8]). Strong cross-diffusive effects become manifest also in an example of blow-up despite certain sub-quadratic but yet superlinear degradation terms in some appropriately high-dimensional chemotaxis systems ([17]).

In light of these premises, for the investigation of common large-scale qualitative features of solutions to (1.1) in general n-dimensional frameworks it seems adequate to explicitly resort to situations when solutions are globally regular. Upon a time shift if necessary this will in fact cover widely arbitrary solutions to (1.3) in all physically relevant cases $n \le 3$, but this will furthermore also capture more complex frameworks in which the fluid evolution itself is unknown, affected e.g. by the cell population, and governed by appropriate equations from fluid mechanics (cf. [1] for corresponding modeling aspects), at least in situations when the respectively obtained chemotaxis-fluid system is globally solvable by suitably regular functions ([12], [13]). Accordingly, the purpose of this work consists in describing the large time behavior of arbitrary global bounded solutions to (1.1) in bounded domains for any $n \ge 1$, thus ignoring the question under which particular assumptions on supposedly prescribed initial data $(u_0, v_0) \equiv (u(\cdot,0), v(\cdot,0))$ such solutions exist. Hence assuming to be given a sufficiently smooth vector field U and a nontrivial global bounded classical solution (u,v) of (1.1), we will more precisely focus on deriving optimal estimates for the decay rate of $u(\cdot,t)$ with respect to the norms both in $L^\infty(\Omega)$ and in $L^1(\Omega)$, bearing in mind the particular biological relevance of the latter as representing the total mass of the considered population.

Previous work in this direction addresses the Cauchy problem in $\Omega = \mathbb{R}^2$ for a simplified parabolic-ellitpic variant of (1.1) which can be rewritten in form of the scalar nonlocal parabolic equation

$$u_t + U \cdot \nabla u = \Delta u + \chi \nabla \cdot (u \nabla (\Delta)^{-1} u) - \mu u^q \tag{1.4}$$

with the additional parameter $q \ge 2$. For this problem with initial condition $u(\cdot,0) = u_0 \in L^1(\mathbb{R}^2)$, in the case $q > 2$ any sufficiently regular nonnegative global solution u is known to satisfy $\int_{\mathbb{R}^2} u(\cdot,t) \to m_\infty(\chi, u_0, U)$ as $t \to \infty$ with some $m_\infty(\chi, u_0, U) > 0$ fulfilling $m_\infty(\chi, u_0, U) \to 0$ as $\chi \to \infty$ ([6]). In the critical case $q = 2$, an influence of chemotaxis on the evolution of the total mass functional, which then decays to zero in both cases $\chi > 0$ and $\chi = 0$, has been shown to exist but to be of more subtle character, mainly relevant on finite time intervals ([7]).

Main results. It will turn out that in the presently considered framework of bounded domains, unlike in the latter Cauchy problem the solution behavior in (1.1) is essentially unaffected by chemotaxis at least on large time scales. Indeed, throughout the sequel assuming for simplicity that

$$U \in C^{1,0}(\bar\Omega \times [0,\infty); \mathbb{R}^n) \cap L^\infty(\Omega \times (0,\infty); \mathbb{R}^n) \quad \text{is such that} \quad \nabla \cdot U \equiv 0 \quad \text{in } \Omega \times (0,\infty), \tag{1.5}$$

we shall see that for any given nontrivial and sufficiently regular bounded solution of (1.1), with respect to the norms in either $X := L^1(\Omega)$ or in $L^\infty(\Omega)$ the quantity $\|u(\cdot,t)\|_X$ can be estimated from above and below by positive multiples, possibly depending on the solution e.g. through its norm in $L^\infty(\Omega \times (0,\infty))$, of $\frac{1}{t+1}$. More precisely, our main results read as follows.

Theorem 1.1 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, assume that $\mu > 0$ and that U satisfies (1.5), and suppose that $(u,v) \in (C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)))^2$ is a classical solution of (1.1) for which both u and v are nonnegative, and which is bounded in the sense that u belongs to $L^\infty(\Omega \times (0,\infty))$.*

i) There exists $C_1 > 0$ with the property that

$$\frac{1}{|\Omega|}\|u(\cdot,t)\|_{L^1(\Omega)} \le \|u(\cdot,t)\|_{L^\infty(\Omega)} \le \frac{C_1}{t+1} \qquad \text{for all } t > 0. \tag{1.6}$$

ii) If furthermore $u \not\equiv 0$, then one can find $C_2 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \ge \frac{1}{|\Omega|} \cdot \|u(\cdot,t)\|_{L^1(\Omega)} \ge \frac{C_2}{t+1} \qquad \text{for all } t > 0. \tag{1.7}$$

We remark that we do not pursue here the question how the constants appearing in the above statements depend on χ and μ, nor on the function U, thus leaving open whether chemotactic cross-diffusion possibly influences a fine structure in the large time asymptotics of solutions.
In corresponding chemotaxis-fluid systems in which the fluid evolution itself is affected by the presence of the other quantities e.g. through buoyant forces, the above results can directly be applied to solutions which are a priori known to enjoy the above regularity and boundedness properties; for two- and three-dimensional examples of situations when the latter in fact is guaranteed for all reasonably regular initial data we refer to [13] and [12]. However, Theorem 1.1 is actually more general by considering widely arbitrary fluid fields not necessarily receiving any feedback from the taxis components.

Plan of the paper. The main idea underlying approach is directly motivated by the result to be finally achieved: The goal pursued in our analysis consists in showing appropriate negligibility of the cross-diffusive action in (1.1) in comparison to the further mechanisms therein. After establishing a preliminary but fundamental decay information on solutions in $L^1(\Omega) \times L^1(\Omega)$ in Section 2, this will be accomplished in Section 3 on the basis of the latter by means of a series of arguments relying on the smoothing action of the heat semigroup in the second equation in (1.1). A first exploitation of the outcome thereby achieved will yield the estimate from Theorem 1.1 i) in Section 4, whereafter a second application thereof will show in Section 5 that also in the inequality

$$\frac{d}{dt}\int_\Omega \ln u \ge -\frac{\chi^2}{4}\int_\Omega |\nabla v|^2 - \mu\int_\Omega u, \qquad t > 0,$$

constituting the key step in or proof of Theorem 1.1 ii), the summand originating from the taxis term in (1.1) decays suitably fast so as to become asymptotically irrelevant.

2 Upper decay estimates for u and v in $L^1(\Omega)$

The following basic one-sided decay estimates for the spatial L^1 norms of both solution components can be gained in quite an elementary way, and similar observations have previously been made in [12, Lemma 5.1] already. Since they will be fundamental to our subsequent analysis, and since in particular they already underline the difference between the case of bounded Ω and the case $\Omega = \mathbb{R}^n$ in a quantitative manner, we include a short proof here.

Lemma 2.1 *Let (u, v) be a nonnegative global classical solution of (1.1). Then*

$$\int_\Omega u(\cdot,t) \le \frac{|\Omega|}{\mu} \cdot \frac{1}{t+\gamma} \qquad \text{for all } t > 0 \tag{2.1}$$

and

$$\int_\Omega v(\cdot,t) \le \frac{K}{t+2} \qquad \text{for all } t > 0, \tag{2.2}$$

where

$$\gamma := \frac{|\Omega|}{\mu \cdot \int_\Omega u(\cdot,0)} \tag{2.3}$$

and

$$K := \max\left\{ 2\int_\Omega v(\cdot,0)\,,\, 4\int_\Omega u(\cdot,0)\,,\, \frac{2|\Omega|}{\mu}\right\}. \tag{2.4}$$

Proof. We only need to consider the case when $u(\cdot,0) \not\equiv 0$, in which according to (1.1) and the Cauchy-Schwarz inequality,

$$\frac{d}{dt}\int_\Omega u = -\mu\int_\Omega u^2 \le -\frac{\mu}{|\Omega|}\left\{\int_\Omega u\right\}^2 \qquad \text{for all } t>0,$$

which on integration readily implies (2.1) with γ as in (2.3).
Since from (1.1) we moreover see that

$$\frac{d}{dt}\int_\Omega v = -\int_\Omega v + \int_\Omega u \qquad \text{for all } t>0,$$

we therefore obtain that

$$\frac{d}{dt}\int_\Omega v \le -\int_\Omega v + \frac{|\Omega|}{\mu(t+\gamma)} \qquad \text{for all } t>0. \tag{2.5}$$

Now with K as given by (2.4), $\overline{y}(t) := \frac{K}{t+2}$, $t \ge 0$, satisfies $\overline{y}(0) = \frac{K}{2} \ge \int_\Omega v(\cdot,0)$ by (2.4) and therefore

$$\begin{aligned}
\overline{y}'(t) + \overline{y}(t) - \frac{|\Omega|}{\mu(t+\gamma)} &= -\frac{K}{(t+2)^2} + \frac{K}{t+2} - \frac{|\Omega|}{\mu(t+\gamma)} \\
&= \frac{K}{t+2}\cdot\left\{1 - \frac{1}{t+2} - \frac{|\Omega|}{K\mu}\cdot\frac{t+2}{t+\gamma}\right\} \\
&\ge \frac{K}{t+2}\cdot\left\{1 - \frac{1}{2} - \frac{|\Omega|}{K\mu}\cdot\max\left\{\frac{2}{\gamma},1\right\}\right\} \\
&= \frac{K}{2(t+2)}\cdot\left\{1 - \frac{1}{K}\cdot\max\left\{4\int_\Omega u(\cdot,0)\,,\,\frac{2|\Omega|}{\mu}\right\}\right\} \\
&\ge 0 \qquad \text{for all } t>0
\end{aligned}$$

due to (2.3) and the second and third restrictions contained in (2.4). By an ODE comparison, we thus conclude from (2.5) that $\int_\Omega v(\cdot,t) \le \overline{y}(t)$ for all $t>0$, and that hence indeed (2.2) holds. □

3 Boundedness and decay properties of ∇v

A crucial step toward both parts of Theorem 1.1 will consist in adequately identifying the cross-diffusive term in (1.1) as asymptotically negligible relative to the diffusive action therein, which basically amounts to deriving appropriate quantitative bounds for the chemotactic gradient ∇v. This will be achieved in this section by firstly making use of the L^1 decay poperty of u from Lemma 2.1 in order to obtain decay of ∇v at an apparently optimal rate but in a yet unfavorable topology, and by secondly investing our assumption on boundedness of u to establish boundedness of v in certain higher norms but without any decay information. Interpolating these two extremal results will finally yield a decay result for ∇v in arbitrary L^p spaces at a rate which is probably far from optimal but sufficient for our purposes.

For what follows, let us recall that for $p \in (1,\infty)$, the realization $A = A_p$ of $-\Delta+1$ under homogeneous Neumann boundary conditions, that is, the operator defined by letting $A_p\varphi := -\Delta\varphi + \varphi$ for $\varphi \in D(A_p) := \{\varphi \in W^{2,p}(\Omega) \mid \frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \partial\Omega\}$, is sectorial in the space $L^p(\Omega)$, with its spectrum contained in the half-line $[1,\infty)$. Accordingly, A possesses closed and densely defined fractional powers A^β for all $\beta \in \mathbb{R}$, and A^β is bounded whenever $\beta < 0$ ([4, Theorem 1.4.2]).

Now the space $L^1(\Omega)$ is continuously embedded into suitable among the correspondingly obtained spaces $D(A^{-\beta})$, an explicit definition of which is actually not necessary and thus omitted here, keeping the focus rather on an associated embedding inequality:

Lemma 3.1 *Let $p > 1$ and $\beta > \frac{n(p-1)}{2p}$. Then there exists $C > 0$ such that*

$$\|A^{-\beta}\varphi\|_{L^p(\Omega)} \le C\|\varphi\|_{L^1(\Omega)} \qquad \textit{for all } \varphi \in L^1(\Omega). \tag{3.1}$$

Proof. Since $\beta > \frac{n(p-1)}{2p}$ implies that $p' := \frac{p}{p-1}$ satisfies $2\beta - \frac{n}{p'} > 0$, it follows from known embedding results ([4, Theorem 1.6.1]) that $D(A^\beta_{p'}) \hookrightarrow L^\infty(\Omega)$, whence there exists $c_1 > 0$ such that

$$\|\phi\|_{L^\infty(\Omega)} \le c_1\|A^\beta\phi\|_{L^{p'}(\Omega)} \qquad \text{for all } \phi \in D(A^\beta_{p'}). \tag{3.2}$$

Thus, given any $\varphi \in C_0^\infty(\Omega)$ and $\psi \in C_0^\infty(\Omega)$, using the self-adjointness of $A^{-\beta}$ in $L^2(\Omega)$ we can estimate

$$\Big|\int_\Omega A^{-\beta}\varphi\cdot\psi\Big| = \Big|\int_\Omega \varphi\cdot A^{-\beta}\psi\Big| \le \|\varphi\|_{L^1(\Omega)}\|A^{-\beta}\psi\|_{L^\infty(\Omega)} \le c_1\|\varphi\|_{L^1(\Omega)}\|\psi\|_{L^{p'}(\Omega)}.$$

Therefore,

$$\|A^{-\beta}\varphi\|_{L^p(\Omega)} = \sup_{\substack{\varphi\in C_0^\infty(\Omega)\\ \|\psi\|_{L^{p'}(\Omega)}\le 1}} \Big|\int_\Omega A^{-\beta}\varphi\cdot\psi\Big| \le c_1\|\varphi\|_{L^1(\Omega)},$$

as claimed. ∎

By appropriately making use of the latter in the course of an argument based on a variation-of-constants representation of v, we see that with respect to the norm in $L^p(\Omega)$ for suitably small $p > 1$, ∇v inherits the decay rate of the mass functional $\int_\Omega u$ from Lemma 2.1.

Lemma 3.2 *Let (u,v) be a nonnegative global classical solution of (1.1). Then for all $p \in (1,\frac{n}{n-1})$ one can find $C(p) > 0$ such that*

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(p)}{t} \qquad \textit{for all } t \ge 2. \tag{3.3}$$

PROOF. Since $\frac{n}{n-2(1-\alpha)} \to \frac{n}{n-1} > p$ as $\alpha \searrow \frac{1}{2}$, it is possible to fix $\alpha \in (\frac{1}{2}, 1)$ such that $p < \frac{n}{n-2(1-\alpha)}$, which means that

$$\alpha + \frac{n}{2}\Big(1 - \frac{1}{p}\Big) < 1. \tag{3.4}$$

We thereupon choose an arbitrary $\varepsilon \in (0, \alpha - \frac{1}{2})$ and pick $\beta > \frac{n(p-1)}{2p}$, so that since $D(A_p^{\frac{1}{2}+\varepsilon}) \hookrightarrow W^{1,p}(\Omega)$ ([4, Theorem 1.6.1]), employing a well-known interpolation argument ([3, Theorem 14.1]) we can find $c_1 > 0$ and $c_2 > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le c_1 \|A^{\frac{1}{2}+\varepsilon} v(\cdot,t)\|_{L^p(\Omega)} \le c_2 \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}^a \|A^{-\beta} v(\cdot,t)\|_{L^p(\Omega)}^{1-a} \qquad \text{for all } t > 0, \tag{3.5}$$

where

$$a := \frac{\frac{1}{2} + \varepsilon + \beta}{\alpha + \beta} \in (0,1).$$

Here the fact that $\beta > \frac{n(p-1)}{2p}$ enables us to invoke Lemma 3.1 and thereafter apply Lemma 2.1 to find $c_3 > 0$ and $c_4 > 0$ such that

$$\|A^{-\beta} v(\cdot,t)\|_{L^p(\Omega)} \le c_3 \|v(\cdot,t)\|_{L^1(\Omega)} \le \frac{c_3 c_4}{t} \qquad \text{for all } t > 0. \tag{3.6}$$

Now in order to derive (3.3), by means of a variation-of-constants representation of v we write

$$v(\cdot,t) = e^{-A} v(\cdot, t-1) + \int_{t-1}^t e^{-(t-s)A} u(\cdot,s)ds + \int_{t-1}^t e^{-(t-s)A} U(\cdot,s) \cdot \nabla v(\cdot,s)ds, \qquad t \ge 1,$$

and apply A^α to both sides to see that

$$\begin{aligned} \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} &\le \Big\| A^\alpha e^{-A} v(\cdot,t-1)\Big\|_{L^p(\Omega)} \\ &\quad + \int_{t-1}^t \Big\| A^\alpha e^{-(t-s)A} u(\cdot,s)\Big\|_{L^p(\Omega)} ds \\ &\quad + \int_t^{t-1} \Big\| A^\alpha e^{-(t-s)A} U(\cdot,s)\cdot\nabla v(\cdot,s)\Big\|_{L^p(\Omega)} ds \qquad \text{for all } t \ge 1. \end{aligned} \tag{3.7}$$

Here according to known smoothing properties of $(e^{-\tau A})_{\tau \ge 0}$ and Lemma 2.1, there exist $c_5 > 0$ and $c_6 > 0$ fulfilling

$$\Big\| A^\alpha e^{-A} v(\cdot,t-1)\Big\|_{L^p(\Omega)} \le c_5 \|v(\cdot,t-1)\|_{L^1(\Omega)} \le \frac{c_6}{t-1} \qquad \text{for all } t \ge 2, \tag{3.8}$$

and making use of Lemma 2.1 and (3.4), once more by a standard semigroup estimate we can find $c_7 > 0$ and $c_8 > 0$ such that

$$\begin{aligned} \int_{t-1}^t \Big\| A^\alpha e^{-(t-s)A} u(\cdot,s)\Big\|_{L^p(\Omega)} ds &\le c_7 \int_{t-1}^t (t-s)^{-\alpha - \frac{n}{2}(1-\frac{1}{p})} \|u(\cdot,s)\|_{L^1(\Omega)} ds \\ &\le c_8 \int_{t-1}^t (t-s)^{-\alpha-\frac{n}{2}(1-\frac{1}{p})} \cdot \frac{1}{s} ds \\ &\le c_8 \cdot \frac{1}{t-1} \cdot \int_{t-1}^t (t-s)^{-\alpha - \frac{n}{2}(1-\frac{1}{p})} ds \\ &= \frac{c_8}{1 - \alpha - \frac{n}{2}(1-\frac{1}{p})} \cdot \frac{1}{t-1} \qquad \text{for all } t \ge 2. \end{aligned} \tag{3.9}$$

To finally treat the last summand in (3.7) appropriately, let us introduce the numbers

$$M(T) := \sup_{t\in(1,T)} \Big\{ t \cdot \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \Big\}, \qquad T>2,$$

which are all finite due to our overall assumption that $v \in C^{2,1}(\bar\Omega \times (0,\infty))$. In terms of $M(T)$, by boundedness of U, (3.5) and (3.6), with some $c_9>0$ and $c_{10}>0$ the integral in question can be estimated according to

$$\begin{aligned}
\int_{t-1}^t \Big\| A^\alpha e^{-(t-s)A} U(\cdot,s)\cdot\nabla v(\cdot,s)\Big\|_{L^p(\Omega)} ds &\le c_9 \int_{t-1}^t (t-s)^{-\alpha} \Big\| U(\cdot,s)\cdot \nabla v(\cdot,s)\Big\|_{L^p(\Omega)} ds \\
&\le c_{10} \int_{t-1}^t (t-s)^{-\alpha} \|\nabla v(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le c_2 c_{10} \int_{t-1}^t (t-s)^{-\alpha} \cdot \Big\{\frac{M(T)}{s}\Big\}^a \cdot \Big\{\frac{c_3c_4}{s}\Big\}^{1-a} ds \\
&= c_2c_3c_4c_{10} M^a(T) \int_{t-1}^t (t-s)^{-\alpha}\cdot \frac{1}{s} ds \\
&\le c_2c_3c_4c_{10} M^a(T) \cdot \frac{1}{t-1} \cdot \int_{t-1}^t (t-s)^{-\alpha} ds \\
&= \frac{c_2c_3c_4c_{10}}{1-\alpha} M^a(T) \cdot \frac{1}{t-1} \qquad \text{for all } t\in[2,T].
\end{aligned}$$

Combined with (3.7)-(3.9), in view of the fact that $\frac{1}{t-1} \le \frac{2}{t}$ for all $t\ge 2$ this shows that there exists $c_{11}<0$ such that for each $T>2$,

$$t\cdot \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \le c_{11} + c_{11} M^a(T) \qquad \text{for all } t\in[2,T],$$

and that hence with the number

$$c_{12} := \max\Big\{ c_{11}\,,\, \sup_{t\in(1,2)} \Big\{ t\cdot \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}\Big\}\Big\},$$

finite again by the inclusion $v\in C^{2,1}(\bar\Omega\times(0,\infty))$ and the fact that $\alpha<1$, we have

$$M(T) \le c_{12} + c_{12} M^a(T) \qquad \text{for all } T>2.$$

As $a<1$, by an elementary argument this implies that

$$M(T) \le c_{13} := \max\Big\{ 1\,,\, (2c_{12})^{\frac{1}{1-a}} \Big\} \qquad \text{for all } T>2$$

and thereby proves (3.3), because e.g. once more by (3.5) and (3.6) this yields the inequality

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le c_2 \cdot \Big\{\frac{c_{13}}{t}\Big\}^a \cdot \Big\{\frac{c_3c_4}{t}\Big\}^{1-a}$$

for arbitrary $t\ge 1$. $\square$

We next modify the above argument but make use of different ingredients, in particular of the boundedness of u, to derive the following higher-order boundedness property of v.

Lemma 3.3 *Let (u,v) be a nonnegative global classical solution of (1.1) with the property that u is bounded in $\Omega\times(0,\infty)$. Then for all $p>1$ and each $\alpha\in(\frac{1}{2},1)$ there exists $C(p,\alpha)>0$ such that*

$$\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} \le C(\alpha,p) \qquad \textit{for all } t\ge 1. \tag{3.10}$$

PROOF. Following a variant of the strategy pursued in Lemma 3.2, we let

$$M(T) := \sup_{t\in(1,T)} \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}, \qquad T>2,$$

and note that since $\alpha<1$, the inclusion $v\in C^{2,1}(\bar\Omega\times(0,\infty))$ again warrants that $M(T)<\infty$ for all $T>2$.

To prepare an adequate estimation of $M(T)$ on the basis of a Duhamel formula associated with the second equation in (1.1), we once more invoke standard smoothing estimates for $(e^{-\tau A})_{\tau\ge 0}$ to find $c_1>0$ and $c_2>0$ such that

$$\left\|A^\alpha e^{-A} v(\cdot,t-1)\right\|_{L^p(\Omega)} \le c_1\|v(\cdot,t-1)\|_{L^1(\Omega)} \le c_2 \qquad \text{for all } t\ge 1, \tag{3.11}$$

for Lemma 2.1 in particular warrants that $(v(\cdot,t))_{t\ge0}$ is bounded in $L^1(\Omega)$. Next, as u is assumed to be bounded in $\Omega\times(0,\infty)$, there exist $c_3>0$ and $c_4>0$ such that

$$\begin{aligned}
\int_{t-1}^t \left\|A^\alpha e^{-(t-s)A}u(\cdot,s)\right\|_{L^p(\Omega)} ds &\le c_3\int_{t-1}^t (t-s)^{-\alpha}\|u(\cdot,s)\|_{L^p(\Omega)}ds \\
&\le c_3c_4\int_{t-1}^t (t-s)^{-\alpha}ds \\
&= \frac{c_3c_4}{1-\alpha} \qquad \text{for all } t\ge 1,
\end{aligned} \tag{3.12}$$

because $\alpha<1$. Moreover, once more fixing any $\varepsilon\in(0,\alpha-\frac{1}{2})$ and $\beta>\frac{n(p-1)}{2p}$ we may apply known embedding and interpolation estimates along with Lemma 3.1 to gain positive constants c_5,c_6,c_7 and c_8 such that with $a:=\frac{\frac{1}{2}+\varepsilon+\beta}{\alpha+\beta}\in(0,1)$ we have

$$\begin{aligned}
\|\nabla v(\cdot,t)\|_{L^p(\Omega)} &\le c_5\|A^{\frac{1}{2}+\varepsilon}v(\cdot,t)\|_{L^p(\Omega)} \\
&\le c_6\|A^\alpha v(\cdot,t)\|^a_{L^p(\Omega)}\|A^{-\beta}v(\cdot,t)\|^{1-a}_{L^p(\Omega)} \\
&\le c_7M^a(T)\|v(\cdot,t)\|^{1-a}_{L^1(\Omega)} \\
&\le c_8M^a(T) \qquad \text{for all } t\in[1,T],
\end{aligned} \tag{3.13}$$

again due to the fact that v belongs to $L^\infty((0,\infty);L^1(\Omega))$ by Lemma 2.1.

Now using (3.11)-(3.13), we can estimate

$$\begin{aligned}
\|A^\alpha v(\cdot,t)\|_{L^p(\Omega)} &\le \left\|A^\alpha e^{-A}v(\cdot,t-1)\right\|_{L^p(\Omega)} \\
&\quad + \int_{t-1}^t \left\|A^\alpha e^{-(t-s)A}u(\cdot,s)\right\|_{L^p(\Omega)}ds \\
&\quad + \int_t^{t-1} \left\|A^\alpha e^{-(t-s)A}U(\cdot,s)\cdot\nabla v(\cdot,s)\right\|_{L^p(\Omega)}ds \\
&\le c_2+\frac{c_3c_4}{1-\alpha}+c_8M^a(T) \qquad \text{for all } t\in[2,T],
\end{aligned}$$

so that with the evidently finite constant

$$c_9 := \max\left\{c_2 + \frac{c_3c_4}{1-\alpha}\,,\, c_8\,,\, \sup_{t\in(1,2)} \|A^\alpha v(\cdot,t)\|_{L^p(\Omega)}\right\}$$

we have

$$M(T) \le c_9 + c_9 M^a(T) \qquad \text{for all } T>1$$

and therefore

$$M(T) \le \max\left\{1\,,\,(2c_9)^{\frac{1}{1-a}}\right\} \qquad \text{for all } T>1,$$

which proves the lemma. $\square$

A straightforward interpolation shows that the above two lemmata imply decay of ∇v in Lebesgue spaces with high summability powers, but at rates slower than that in Lemma 3.2. The following statement on this will be applied to some large value of p and $\kappa := 0$ in proving the upper estimate claimed in Theorem 1.1 i), and to $p := 2$ with some $\kappa > \frac{1}{2}$ in Corollary 5.1 preparing the proof of the lower bound for $\int_\Omega u$ in Theorem 1.1 ii).

Lemma 3.4 *Let (u,v) be a nonnegative global classical solution of (1.1) such that u is bounded, and let $p>1$. Then for all $\kappa < \min\{1, \frac{n}{(n-1)p}\}$ there exists $C(p,\kappa)>0$ such that*

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \frac{C(p,\kappa)}{t^\kappa} \qquad \textit{for all } t\ge 2. \tag{3.14}$$

Proof. If $p < \frac{n}{n-1}$, the claim immediately results from Lemma 3.2. In the case $p \ge \frac{n}{n-1}$, our assumption ensures that $\kappa < \frac{n}{(n-1)p}$, so that we can fix $r \in [1, \frac{n}{n-1})$ such that still $\kappa < \frac{r}{p}$, whence writing

$$q := \frac{(1-\kappa)pr}{r-p\kappa},$$

we can easily verify that $q>p>r$, and that

$$\frac{\frac{1}{r}-\frac{1}{p}}{\frac{1}{r}-\frac{1}{q}} = 1-\kappa.$$

Therefore, the Hölder inequality says that

$$\|\nabla v(\cdot,t)\|_{L^p(\Omega)} \le \|\nabla v(\cdot,t)\|_{L^q(\Omega)}^{1-\kappa} \|\nabla v(\cdot,t)\|_{L^r(\Omega)}^{\kappa} \qquad \text{for all } t>0, \tag{3.15}$$

where picking any $\alpha \in (\frac{1}{2},1)$ we infer from the continuity of the embedding $D(A_q^\alpha) \hookrightarrow W^{1,q}(\Omega)$ ([4]) and from lemma 3.3 that

$$\|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1 \|A^\alpha v(\cdot,t)\|_{L^q(\Omega)} \le c_2 \qquad \text{for all } t\ge 2$$

with some $c_1>0$ and $c_2>0$. As moreover the inequality $r<\frac{n}{n-1}$ along with Lemma 3.2 yields $c_3>0$ fulfilling

$$\|\nabla v(\cdot,t)\|_{L^r(\Omega)} \le \frac{c_3}{t} \qquad \text{for all } t\ge 2,$$

from (3.15) we readily derive (3.14). $\square$

4 Upper bound for u in $L^\infty(\Omega)$. Proof of Theorem 1.1 i)

On the basis of a Duhamel formula now associated with the first equation in (1.1), knowing that cross-difusive gradient ∇v is bounded in $L^\infty((0,\infty);L^p(\Omega))$ for any finite $p>1$ we can now turn the L^1 decay information on u from Lemma 2.1 into a corresponding estimate in $L^\infty(\Omega)$.

Proof of Theorem 1.1 i). We fix an arbitrary $p>n$ and recall that then by standard regularization properties of the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ on Ω ([15]) one can pick $c_1>0$ and $c_2>0$ such that for all $\tau\in(0,1)$ we have

$$\|e^{\tau\Delta}\varphi\|_{L^\infty(\Omega)} \le c_1\tau^{-\frac{n}{2}}\|\varphi\|_{L^1(\Omega)} \qquad \text{for all } \varphi\in L^1(\Omega) \tag{4.1}$$

and

$$\|e^{\tau\Delta}\nabla\cdot\varphi\|_{L^\infty(\Omega)} \le c_2\tau^{-\frac{1}{2}-\frac{n}{2p}}\|\varphi\|_{L^p(\Omega)} \qquad \text{for all } \ \varphi\in C^1(\bar\Omega;\mathbb{R}^n) \text{ such that } \varphi\cdot\nu=0 \text{ on } \partial\Omega. \tag{4.2}$$

Now in order to estimate the numbers

$$M(T):=\sup_{t\in(0,T)}\Big\{(t+1)\cdot\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big\}, \qquad T>2,$$

we use that $\nabla\cdot U\equiv 0$ in representing $u(\cdot,t)$ according to

$$\begin{aligned} u(\cdot,t) &= e^{\Delta}u(\cdot,t-1)-\chi\int_{t-1}^t e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)ds \\ &\quad -\mu\int_{t-1}^t e^{(t-s)\Delta}u^2(\cdot,s)ds-\int_{t-1}^t e^{(t-s)\Delta}\nabla\cdot\Big(U(\cdot,s)u(\cdot,s)\Big)ds \qquad \text{for all } t\ge 1. \end{aligned}$$

Since $e^{(t-s)\Delta}u^2(\cdot,s)$ is nonnegative in Ω for all $t>0$ and $s\in(0,t)$ due to the maximum principle, by nonnegativity of u we therefore see that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)} \\ &\quad +\chi\int_{t-1}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \\ &\quad +\int_{t-1}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(U(\cdot,s)u(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \qquad \text{for all } t\ge 1, \end{aligned} \tag{4.3}$$

where combining (4.1) with Lemma 2.1 we can find $c_3>0$ such that

$$\|e^{\Delta}u(\cdot,t-1)\|_{L^\infty(\Omega)} \le c_1\|u(\cdot,t)\|_{L^1(\Omega)} \le \frac{c_3}{t-1} \le \frac{2c_3}{t} \qquad \text{for all } t\ge 2. \tag{4.4}$$

To relate the two rightmost integrals in (4.3) to $M(T)$, we first invoke (4.2) to obtain

$$\chi\int_{t-1}^t \Big\|e^{(t-s)\Delta}\nabla\cdot\Big(u(\cdot,s)\nabla v(\cdot,s)\Big)\Big\|_{L^\infty(\Omega)}ds \le c_2\chi\int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)}ds \qquad \text{for all } t\ge 1,$$

and then twice use the Hölder inequality to infer that again due to Lemma 2.1, and as a consequence of the boundedness of ∇v in $\Omega \times (1,2)$ and Lemma 3.4 when applied to $\kappa := 0$, with some $c_4 > 0$ and $c_5 > 0$ and $a := 1 - \frac{1}{2p}$ we have

$$\begin{aligned}
\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^p(\Omega)} &\le \|u(\cdot,s)\|_{L^{2p}(\Omega)} \|\nabla v(\cdot,s)\|_{L^{2p}(\Omega)} \\
&\le \|u(\cdot,s)\|_{L^\infty(\Omega)}^a \|u(\cdot,s)\|_{L^1(\Omega)}^{1-a} \|\nabla v(\cdot,s)\|_{L^{2p}(\Omega)} \\
&\le \left\{ \frac{M(T)}{s+1} \right\}^a \cdot \left\{ \frac{c_4}{s+1} \right\}^{1-a} \cdot c_5 \\
&= c_4^{1-a} c_5 M^a(T) \cdot \frac{1}{s+1} \qquad \text{for all } s \in (1,T)
\end{aligned}$$

and hence

$$\begin{aligned}
\chi \int_{t-1}^t \left\| e^{(t-s)\Delta} \nabla \cdot \Big(u(\cdot,s)\nabla v(\cdot,s) \Big) \right\|_{L^\infty(\Omega)} ds &\le c_2 c_4^{1-a} c_5 \chi M^a(T) \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \cdot \frac{1}{s+1} ds \\
&\le c_2 c_4^{1-a} c_5 \chi M^a(T) \cdot \frac{1}{t} \cdot \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} ds \\
&= \frac{c_2 c_4^{1-a} c_5 \chi}{\frac{1}{2} - \frac{n}{2p}} M^a(T) \cdot \frac{1}{t} \qquad \text{for all } t \in [2,T], \qquad (4.5)
\end{aligned}$$

because $p > n$.
Likewise, combining (4.2) with the boundedness of U we obtain $c_6 > 0$ such that

$$\begin{aligned}
\int_{t-1}^t \left\| e^{(t-s)\Delta} \nabla \cdot \Big(U(\cdot,s) u(\cdot,s) \Big) \right\|_{L^\infty(\Omega)} ds &\le c_2 \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \|U(\cdot,s)u(\cdot,s)\|_{L^p(\Omega)} ds \\
&\le c_6 \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \|u(\cdot,s)\|_{L^p(\Omega)} ds \qquad \text{for all } t \ge 1, \qquad (4.6)
\end{aligned}$$

where again by the Hölder inequality and Lemma 2.1, there exists $c_7 > 0$ such that

$$\begin{aligned}
\|u(\cdot,s)\|_{L^p(\Omega)} &\le \|u(\cdot,s)\|_{L^\infty(\Omega)}^b \|u(\cdot,s)\|_{L^1(\Omega)}^{1-b} \\
&\le \left\{ \frac{M(T)}{s+1} \right\}^b \cdot \left\{ \frac{c_7}{s+1} \right\}^{1-b} \\
&= c_7^{1-b} M^b(T) \cdot \frac{1}{s+1} \qquad \text{for all } s \in (1,T)
\end{aligned}$$

with $b := 1 - \frac{1}{p}$. Therefore, (4.6) implies that

$$\begin{aligned}
\int_{t-1}^t \left\| e^{(t-s)\Delta} \nabla \cdot \Big(U(\cdot,s) u(\cdot,s) \Big) \right\|_{L^\infty(\Omega)} ds &\le c_6 c_7^{1-b} M^b(T) \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} \cdot \frac{1}{s+1} ds \\
&\le c_6 c_7^{1-b} M^b(T) \cdot \frac{1}{t} \cdot \int_{t-1}^t (t-s)^{-\frac{1}{2}-\frac{n}{2p}} ds \\
&= \frac{c_6 c_7^{1-b}}{\frac{1}{2} - \frac{n}{2p}} M^b(T) \cdot \frac{1}{t} \qquad \text{for all } t \in [2,T],
\end{aligned}$$

so that summarizing (4.3), (4.4) and (4.5) and using Young's inequality yields $c_8 > 0$ and $c_9 > 0$ such that

$$\begin{aligned} t \cdot \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le c_8 + c_8 M^a(T) + c_8 M^b(T) \\ &\le c_9 + c_9 M^a(T) \qquad \text{for all } t \in [2,T], \end{aligned}$$

because $b < a$. Since u is bounded in $\Omega \times (0,2)$, this entails that for some $c_{10} > 0$ we have

$$M(T) \le c_{10} + c_{10} M^a(T) \qquad \text{for all } T > 2$$

and thus

$$M(T) \le \max\Big\{1\,,\,(2c_{10})^{\frac{1}{1-a}}\Big\} \qquad \text{for all } T > 2,$$

which readily yields (1.6), for $T > 2$ was arbitrary. $\square$

5 Lower bound for u in $L^1(\Omega)$. Proof of Theorem 1.1 ii)

In deriving the lower bound for $\int_\Omega u$ claimed in Theorem 1.1 ii), we will make essential use of the following consequence of Lemma 3.4 which strongly relies on the fact that the decay exponent κ appearing therein can be chosen favorably large at least in the particular case $p := 2$.

Corollary 5.1 *There exist $\lambda > 1$ and $C > 0$ such that*

$$\int_\Omega |\nabla v(\cdot,t)|^2 \le \frac{C}{t^\lambda} \qquad \textit{for all } t \ge 2. \tag{5.1}$$

Proof. This immediately results from an application of Lemma 3.4 to any $\kappa > \frac{1}{2}$ fulfilling $\kappa < \min\{1, \frac{n}{2(n-1)}\}$. $\square$

Now the fact that the function on the right of (5.1) is integrable over $t \in (2,\infty)$ enables us to make sure that the taxis term in (1.1) becomes asymptotically negligible in the framework of the following testing procedure.

Lemma 5.2 *There exists $C > 0$ such that*

$$\int_\Omega \ln u(\cdot,t) \ge -|\Omega| \ln(t+\gamma) - C \qquad \textit{for all } t \ge 2, \tag{5.2}$$

where $\gamma > 0$ is the constant defined in (2.3).

Proof. As u is positive in $\bar\Omega \times (0,\infty)$ according to the strong maximum principle, we may test the first equation in (1.1) against $\frac{1}{u}$ so as to see that

$$\begin{aligned} \frac{d}{dt}\int_\Omega \ln u &= \int_\Omega \frac{1}{u} u_t \\ &= \int_\Omega \frac{1}{u}\Delta u - \chi \int_\Omega \frac{1}{u}\nabla\cdot(u\nabla v) - \mu\int_\Omega u \\ &= \int_\Omega \frac{|\nabla u|^2}{u^2} - \chi\int_\Omega \frac{\nabla u}{u}\cdot\nabla v - \mu\int_\Omega u \qquad \text{for all } t > 0, \end{aligned} \tag{5.3}$$

where by Young's inequality,

$$-\chi \int_\Omega \frac{\nabla u}{u} \cdot \nabla v \geq -\int_\Omega \frac{|\nabla u|^2}{u^2} - \frac{\chi^2}{4} \int_\Omega |\nabla v|^2 \qquad \text{for all } t > 0. \tag{5.4}$$

Now from Lemma 2.1 we know that

$$\mu \int_\Omega u \leq \frac{|\Omega|}{t+\gamma} \qquad \text{for all } t > 0,$$

whereas Corollary 5.1 provides $\lambda > 1$ and $c_1 > 0$ satisfying

$$\frac{\chi^2}{4} \int_\Omega |\nabla v|^2 \leq \frac{c_1}{t^\lambda} \qquad \text{for all } t \geq 2.$$

From (5.3) and (5.4) we therefore obtain the inequality

$$\frac{d}{dt} \int_\Omega \ln u \geq -\frac{|\Omega|}{t+\gamma} - \frac{c_1}{t^\lambda} \qquad \text{for all } t \geq 2,$$

which on direct integration shows that

$$\begin{aligned}
\int_\Omega \ln u(\cdot,t) - \int_\Omega \ln u(\cdot,2) &\geq -|\Omega| \int_2^t \frac{ds}{s+\gamma} - c_1 \int_2^t \frac{ds}{s^\lambda} \\
&= -|\Omega| \ln(t+\gamma) + |\Omega| \ln(2+\gamma) - \frac{c_1}{2^{\lambda-1}(\lambda-1)} + \frac{c_1}{(\lambda-1)t^{\lambda-1}} \\
&\geq -|\Omega| \ln(t+\gamma) - \frac{c_1}{2^{\lambda-1}(\lambda-1)} \qquad \text{for all } t \geq 2.
\end{aligned}$$

As $\int_\Omega \ln u(\cdot,2)$ is finite by strict positivity of $u(\cdot,2)$ throughout $\bar{\Omega}$, this establishes (5.2). $\square$

Thanks to the precise information on the multiple of $\ln(t+\gamma)$ appearing in (5.2), upon a simple application of Jensen's inequality we can turn this into a lower estimate for $\int_\Omega u$ involving exactly the desired decay rate.

Lemma 5.3 *There exists $C > 0$ such that*

$$\int_\Omega u(\cdot,t) \geq \frac{C}{t+1} \qquad \textit{for all } t > 0. \tag{5.5}$$

PROOF. From Lemma 5.2 we know that with $\gamma > 0$ taken from (2.3), for some $c_1 > 0$ we have

$$\int_\Omega \ln u \geq -|\Omega| \ln(t+\gamma) - c_1 \qquad \text{for all } t \geq 2.$$

Since by Jensen's inequality we can estimate

$$\int_\Omega \ln u = |\Omega| \cdot \left\{ \frac{1}{|\Omega|} \int_\Omega \ln u \right\} \leq |\Omega| \cdot \ln \left\{ \frac{1}{|\Omega|} \int_\Omega u \right\} = |\Omega| \cdot \ln \left\{ \int_\Omega u \right\} - |\Omega| \ln |\Omega| \qquad \text{for all } t > 0,$$

this implies that

$$\begin{aligned}
\int_\Omega u &\geq |\Omega|\cdot e^{\frac{1}{|\Omega|}\int_\Omega \ln u} \\
&\geq |\Omega|\cdot e^{\frac{1}{|\Omega|}\cdot\Big\{-|\Omega|\ln(t+\gamma)-c_1\Big\}} \\
&= |\Omega| e^{-\frac{c_1}{|\Omega|}}\cdot\frac{1}{t+\gamma} \\
&\geq |\Omega| e^{-\frac{c_1}{|\Omega|}}\cdot\min\Big\{\frac{1}{\gamma},1\Big\}\cdot\frac{1}{t+1} \qquad \text{for all } t\geq 2.
\end{aligned}$$

Therefore, the proof is completed upon the observation that $\min_{t\in[0,2]}\Big\{(t+1)\int_\Omega u(\cdot,t)\Big\}$ must be positive by continuity of u and the fact that $u\not\equiv 0$. $\square$

We can thereby complete the proof of our main results.

Proof of Theorem 1.1 ii). For appropriately large $C>0$, the second inequality in (1.7) is precisely asserted by Lemma 5.3, whereas the first is obvious. $\square$

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