Global bounded solutions in a two-dimensional quasilinear Keller-Segel system with exponentially decaying diffusivity and subcritical sensitivity

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Abstract

The quasilinear chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \qquad (\star)$$

is considered under homogeneous Neumann boundary conditions in a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary.

It is shown that if D and S are sufficiently smooth nonnegative functions on $[0, \infty)$ satisfying

$$K_1 e^{-\beta^- s} \le D(s) \le K_2 e^{-\beta^+ s} \qquad \text{for all } s \ge 0$$

with some $K_1 > 0, K_2 > 0$, $\beta^+ > 0$ and $\beta^- \ge \beta^+$, then whenever S satisfies the condition of subcritical growth relative to D given by

$$\frac{S(s)}{D(s)} \le K_3 s^\alpha \qquad \text{for all } s \ge 0$$

with some $K_3 > 0$ and $\alpha \in (0, 1)$, for all suitably regular nonnegative initial data the corresponding initial-boundary value problem for $(\star)$ possesses a global classical solution for which the component u is bounded in $\Omega \times (0, \infty)$.

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1 Introduction

In a bounded domain $\Omega \subset \mathbb{R}^2$, with prescribed nonnegative initial data u_0 and v_0 we consider the quasilinear parabolic system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

which arises in the modeling of chemotactic migration in populations of cells that partially orient their movement toward increasing concentrations of chemical secreted by themselves. In this work we will focus on cases when the diffusivity D and the chemotactic sensitivity S in (1.1) remain significantly below the respective coefficient functions in the classical Keller-Segel system ([9]), as obtained on letting

$$D(s) = 1 \quad \text{and} \quad S(s) = s \qquad \text{for } s \geq 0, \tag{1.2}$$

thereby following refined modeling approaches which account for the finite volume of cells. In [12], for instance, the authors propose to choose D and S in dependence of a supposedly given function Q for which $Q(u)$ measures the probability that a cell, when localized at a position (x,t) with population density $u(x,t)$, may find space in some neighboring region; in terms of this parameter function, an accordingly modified random walk approach suggests the precise functional relationships determined by

$$D(s) = Q(s) - sQ'(s) \quad \text{and} \quad S(s) = sQ(s), \qquad s \geq 0, \tag{1.3}$$

with the choice $Q \equiv 1$ corresponding to (1.2).

In comparison to the case determined by (1.2), previous results indicate that dampening the growth of S relative to D may substantially inhibit the well-known tendency of the original Keller-Segel system to spontaneously generate extreme structures in the sense of singularity formation within finite time. Indeed, such exploding solutions are known to exist for D and S as in (1.2) if either $n \geq 3$ ([18]), or $n = 2$ and $\int_\Omega u_0 > 8\pi$ ([7], [10]), but any unboundedness phenomenon of this type is ruled out for all reasonably regular initial data whenever $n \geq 2$ and

$$\frac{S(s)}{D(s)} \leq Cs^{\frac{2}{n} - \varepsilon} \qquad \text{for all } s \geq 0 \tag{1.4}$$

with some $\varepsilon > 0$ and $C > 0$, provided that in addition to this, D decays at most algebraically in the sense that

$$\liminf_{s \to \infty} \Big(s^p D(s) \Big) > 0 \tag{1.5}$$

for some $p > 0$ ([15], cf. also [13] for some preceding partial results in this direction). That herein the condition (1.4) is essentially optimal is indicated by a corresponding result on nonexistence of global bounded solutions, asserting blow-up, either in finite or infinite time, of some solutions under the single condition

$$\liminf_{s \to \infty} \frac{s \left(\frac{S}{D}\right)'(s)}{\left(\frac{S}{D}\right)(s)} > \frac{2}{n}, \tag{1.6}$$

for instance, without the additional requirement (1.5) ([17]).

In cases when D decays at a rate faster than algebraic, besides the latter blow-up result the literature apparently provides no further information concerning the question how far the growth of $s^{\frac{2}{n}}$ remains critical for the increase of $\frac{S(s)}{D(s)}$ with respect to the occurrence of blow-up. In fact, the derivation of time-independent L^∞ bounds for solutions to (1.1) when $D \not\equiv const.$ has up to now mainly been based on Moser-type iteration procedures which at their core make essential use of (1.5) in order to adequately exploit the dissipative action of diffusion ([15]); in the case of the diffusivity decaying faster than in an algebraic way, however, both Moser-type iterations ([2]) as well as De-Giorgi-type methods could so far only be used to establish global existence results, without any further boundedness information ([4]). For instance, it can thereby shown that whenever S decays sufficiently fast in the sense that $S(s) \le Cs^{-n+1-\varepsilon}$ for all $s > 0$ with some $\varepsilon > 0$ and $C > 0$, then irrespective of the decay of D, the mere assumption that $\sup_{s>0} \frac{S(s)}{sD(s)}$ be finite, evidently weaker than (1.4) if $n \ge 3$, is sufficient to warrant global existence of solutions ([5, Theorem 1.6]), thereby complementing some finite-time blowup results in cases when D decays at a rate faster than algebraic, as available in [5, Theorem 1.1].

An alternative approach based on a Moser iteration for the function e^u, rather than for u itself, has recently been developed for the case when D decays exponentially ([19]). Using this, both a global existence result as well as a quantitative upper bound for the possible growth of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ could be established under an assumption on the growth of $\frac{S}{D}$ which inter alia allows the latter quantity to increase at a suitably small exponential rate, and thereby also includes cases in which (1.6) holds and hence some unbounded solutions are known to exist. Correspondingly, the upper bound for $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ obtained in [19] is not time-independent but rather grows logarithmically as $t \to \infty$, and accordingly it seems open whether solutions to a system of type (1.1) with exponentially decreasing diffusivity remain bounded for suitably slow growth of S relative to D.

Main results. It is the intention of the present work to introduce a method of proving global boundedness of solutions to the spatially two-dimensional version of (1.1) with D decreasing exponentially and S satisfying the corresponding optimal relative growth condition (1.4). To make this more precise, let us henceforth assume that $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and require the regularity hypotheses that

$$\begin{cases} u_0 \in W^{1,\vartheta}(\Omega) \text{ for some } \vartheta > 2 \quad \text{with } u_0 > 0 \text{ in } \bar\Omega \qquad \text{and} \\ v_0 \in W^{1,\vartheta}(\Omega) \text{ for some } \vartheta > 2 \quad \text{with } v_0 \ge 0 \text{ in } \Omega, \end{cases} \tag{1.7}$$

and that with some $\iota > 0$,

$$\begin{cases} D \in C^{1+\iota}([0,\infty)) \quad \text{is positive and} \\ S \in C^{1+\iota}([0,\infty)) \quad \text{is nonnegative with } S(0) = 0. \end{cases} \tag{1.8}$$

Moreover, we shall suppose that there exist constants $\beta^+ > 0, \beta^- \ge \beta^+$, $K_1 > 0$ and $K_2 > 0$ such that

$$K_1 e^{-\beta^- s} \le D(s) \le K_2 e^{-\beta^+ s} \qquad \text{for all } s \ge 0, \tag{1.9}$$

and that

$$\frac{S(s)}{D(s)} \le K_3 s^\alpha \qquad \text{for all } s \ge 0 \tag{1.10}$$

with some $K_3 > 0$ and $\alpha \in (0,1)$. Then our main results can be stated as follows.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary. Assume that D and S comply with (1.8), and that there exist positive constants K_1, K_2 and K_3 such that (1.9) and (1.10) are valid with some $\beta^+ > 0, \beta^- \geq \beta^+$ and*

$$\alpha \in (0,1).$$

Then for any u_0 and v_0 satisfying (1.7), the problem (1.1) possesses a uniquely determined global classical solution (u,v) with

$$\begin{cases} u \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)), \\ v \in C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\vartheta}(\Omega)), \end{cases} \tag{1.11}$$

such that both u and v are nonnegative in $\Omega \times (0,\infty)$. Moreover, this solution is bounded in the sense that there exists $C > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq C \qquad \text{for all } t > 0. \tag{1.12}$$

For the particular version of (1.1) obtained through choosing D and S as in (1.3) with

$$Q(s) := e^{-\beta s}, \qquad s \geq 0,$$

for $\beta > 0$, that is, for the volume-filling chemotaxis system

$$\begin{cases} u_t = \nabla \cdot \Big((1+\beta u) e^{-\beta u} \nabla u \Big) - \nabla \cdot (u e^{-\beta u} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.13}$$

this will imply the following boundedness result.

Theorem 1.2 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary and $\beta > 0$, and suppose that (u_0, v_0) is such that (1.7) holds. Then (1.13) possesses a unique global classical solution (u,v) fulfilling (1.11) which is bounded in $\Omega \times (0,\infty)$ in the sense that (1.12) holds.*

Thereby asserting boundedness in the two-dimensional chemotaxis system (1.13) with volume-filling effect described by any exponentially decaying jump probability function Q, Theorem 1.2 complements know results for algebraically decreasing counterparts, which e.g. in the specific case $Q(s) = (1+s)^{-\gamma}$, $s \geq 0, \gamma > 0$, are known to enforce unboundedness of some solutions ([6, Theorem 4.2 (ii)]).

Plan of the paper. After collecting some preliminary material in Section 2, we shall first exploit the well-known natural energy inequality (3.1) associated with (1.1) in order to obtain a time-independent bound for the superlinear functional $\int_\Omega u^{2-\alpha}$ (Lemma 3.3). Making use of bound on ∇v in a sub-quadratic L^q space thereby implied (Lemma 4.3), in Lemma 4.4 we will establish an estimate for $\int_\Omega e^{\beta^+ u}$ by tracking the time evolution of $y(t) := \int_\Omega \Psi(u(\cdot,t))$ with Ψ denoting a second primitive of $\frac{1}{D}$, where as an essential ingredient we will employ the two-dimensional Moser-Trudinger inequality in deriving from the respective diffusive contribution a suitable nonlinear absorptive summand in a corresponding ODI for y. Using this L^{β^+} estimate for e^u as a starting point, in Section 5 we will pursue a Moser-type iterative procedure for e^u in order to derive boundedness of u and thereby prove both Theorem 1.1 and Theorem 1.2.

2 Preliminaries

The following basic statement on local existence and extensibility of classical solutions can be obtained in a straightforward manner by adapting arguments well-established in the context of semilinear and quasilinear chemotaxis systems, so that we may omit giving details here and rather refer to the reasonings in [4, Theorem 2.1] and also in [8], or alternatively to the general result in [1], for instance.

Lemma 2.1 *Suppose that D and S satisfy (1.8) and that u_0 and v_0 fulfill (1.7). Then there exist $T_{max} \in (0,\infty]$ and a unique couple of nonnegative functions*

$$\begin{cases} u \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})), \\ v \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,\vartheta}(\Omega)) \end{cases}$$

such that (u,v) is a classical solution of (1.1) in $\Omega \times (0,T_{max})$, and such that we have the alternative

$$\text{either } T_{max} = \infty, \qquad \text{or} \qquad \limsup_{t \nearrow T_{max}} \Big(\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\vartheta}(\Omega)} \Big) = \infty. \tag{2.1}$$

A first basic property of this solution is immediate.

Lemma 2.2 *The solution of (1.1) satisfies*

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t \in (0,T_{max}). \tag{2.2}$$

Proof. This directly follows on integrating the first equation in (1.1) over $x \in \Omega$. □

In deriving further regularity properties from this and subsequently obtained boundedness features of u, we shall apply the following variant of [8, Lemma 4.1] to the second solution component v in several situations.

Lemma 2.3 *Let $p \ge 1$ and $q \ge p$ be such that*

$$\begin{cases} q < \frac{2p}{2-p} & \text{if } p \le 2, \\ q \le \infty & \text{if } p > 2. \end{cases} \tag{2.3}$$

Then there exists $C > 0$ such that whenever $T \in (0,\infty]$, $f \in L^\infty((0,T); L^p(\Omega))$ and $w \in C^0(\bar\Omega \times [0,T)) \cap C^{2,1}(\bar\Omega \times (0,T))$ are such that

$$\begin{cases} w_t = \Delta w - w + f(x,t), & x \in \Omega, \ t \in (0,T), \\ \frac{\partial w}{\partial \nu} = 0, & x \in \partial\Omega, \ t \in (0,T), \end{cases} \tag{2.4}$$

we have

$$\|w(\cdot,t)\|_{W^{1,q}(\Omega)} \le C \cdot (1 + t^{-\frac{3}{2}+\frac{1}{q}}) \|w(\cdot,0)\|_{L^1(\Omega)} + C\|f\|_{L^\infty((0,T);L^p(\Omega))} \qquad \text{for all } t \in (0,T). \tag{2.5}$$

Proof. According to a variation-of-constants representation of w, by known smoothing estimates for the Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ in Ω ([16]), we can find $c_1>0$ such that

$$\begin{aligned}
\|w(\cdot,t)\|_{W^{1,q}(\Omega)} &= \Big\| e^{t(\Delta-1)}w(\cdot,0)+\int_0^t e^{(t-s)(\Delta-1)}f(\cdot,s)ds\Big\|_{W^{1,q}(\Omega)} \\
&\le c_1e^{-t}\cdot(1+t^{-\frac{1}{2}-(1-\frac{1}{q})})\|w(\cdot,0)\|_{L^1(\Omega)} \\
&\quad +c_1\int_0^t e^{-(t-s)}\cdot\Big\{1+(t-s)^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}\Big\}\|f(\cdot,s)\|_{L^p(\Omega)}ds \qquad \text{for all } t\in(0,T).
\end{aligned}$$

Since $e^{-t}\le 1$ for all $t>0$, this entails that

$$\begin{aligned}
\|w(\cdot,t)\|_{W^{1,q}(\Omega)} &\le c_1\cdot(1+t^{-\frac{3}{2}+\frac{1}{q}})\|w(\cdot,0)\|_{L^1(\Omega)} \\
&\quad +c_1\|f\|_{L^\infty((0,T);L^p(\Omega))}\int_0^t e^{-(t-s)}\cdot\Big\{1+(t-s)^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}\Big\}ds \\
&\le c_1\cdot(1+t^{-\frac{3}{2}+\frac{1}{q}})\|w(\cdot,0)\|_{L^1(\Omega)} \\
&\quad +c_1c_2\|f\|_{L^\infty((0,T);L^p(\Omega))} \qquad \text{for all } t\in(0,T)
\end{aligned}$$

with $c_2:=\int_0^\infty e^{-\sigma}\cdot\{1+\sigma^{-\frac{1}{2}-(\frac{1}{p}-\frac{1}{q})}\}d\sigma$ being finite as a consequence of (2.3). $\Box$

As announced, the mass conservation property (2.2) can thereby be seen to imply some first regularity property of v.

Corollary 2.4 *Let $r\ge 2$. Then there exists $C>0$ such that*

$$\|v(\cdot,t)\|_{L^r(\Omega)}\le C \qquad \textit{for all } t\in(0,T_{max}). \tag{2.6}$$

Proof. We let $q:=\frac{2r}{r+2}\ge 1$ and note that then $W^{1,q}(\Omega)\hookrightarrow L^r(\Omega)$. Since u belongs to $L^\infty((0,T_{max});L^1(\Omega))$ by (2.2), in view of the boundary-value sub-problem of (1.1) satisfied by v we may invoke Lemma 2.3 to see that writing $\tau:=\min\{1,\frac{1}{2}T_{max}\}$ we have

$$\|v(\cdot,t)\|_{L^r(\Omega)}\le c_1 \qquad \text{for all } t\in(\tau,T_{max})$$

with some $c_1>0$. As v is continuous and hence bounded in $\bar\Omega\times[0,\tau]$, this implies (2.6). $\Box$

For later use, let us recall the following consequence of the compactness of the embedding $W^{1,2}(\Omega)\hookrightarrow L^2(\Omega)$ and an associated Ehrling-type lemma ([19, Lemma 2.5]).

Lemma 2.5 *There exists $C_E>0$ with the property that for any $q\ge 1$ we have*

$$\|\varphi\|^2_{W^{1,2}(\Omega)}\le C_E\cdot\Big\{\|\nabla\varphi\|^2_{L^2(\Omega)}+C_E\|\varphi\|^2_{L^q(\Omega)}\Big\} \qquad \textit{for all } \varphi\in W^{1,2}(\Omega).$$

We shall moreover need the following version of the two-dimensional Gagliardo-Nirenberg inequality, with particular emphasis on the independence of the constant appearing therein on the involved parameters within an appropriate range (cf. also [19, Lemma 2.6]).

Lemma 2.6 *Let $q^\star, p_\star$ and $p^\star$ be positive constants such that $q^\star \le p_\star < p^\star$. Then there exists $C > 0$ such that for any choice of $q \in [1, q^\star]$ and $p \in [p_\star, p^\star]$ we have*

$$\|\varphi\|_{L^p(\Omega)}^p \le C\|\nabla\varphi\|_{L^2(\Omega)}^{p-q}\|\varphi\|_{L^q(\Omega)}^q + C\|\varphi\|_{L^q(\Omega)}^p \qquad \textit{for all } \varphi \in W^{1,2}(\Omega). \tag{2.7}$$

PROOF. We first employ the Gagliardo-Nirenberg inequality to fix $c_1 \ge 1$ such that

$$\|\varphi\|_{L^{p^\star}(\Omega)}^{p^\star} \le c_1\|\varphi\|_{W^{1,2}(\Omega)}^{p^\star-q^\star}\|\varphi\|_{L^{q^\star}(\Omega)}^{q^\star} \qquad \text{for all } \varphi \in W^{1,2}(\Omega), \tag{2.8}$$

and using Lemma 2.5 we can find $c_2 \ge 1$ such that whenever $q \in [1, q^\star]$,

$$\|\varphi\|_{W^{1,2}(\Omega)}^2 \le c_2 \cdot \Big\{\|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^q(\Omega)}^2\Big\} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{2.9}$$

Now if $q \in [1, q^\star]$ and $p \in [p_\star, p^\star]$ are such that $q < p$, then the Hölder inequality says that

$$\|\varphi\|_{L^p(\Omega)} \le \|\varphi\|_{L^{p^\star}(\Omega)}^a\|\varphi\|_{L^q(\Omega)}^{1-a} \quad \text{and} \quad \|\varphi\|_{L^{q^\star}(\Omega)} \le \|\varphi\|_{L^p(\Omega)}^b\|\varphi\|_{L^q(\Omega)}^{1-b} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

where

$$a := \frac{p^\star(p-q)}{p(p^\star-q)} \in (0,1] \qquad \text{and} \qquad b := \frac{p(q^\star-q)}{q^\star(p-q)} \in (0,1].$$

Therefore, (2.8) entails that

$$\begin{aligned}
\|\varphi\|_{L^p(\Omega)} &\le \left\{c_1^{\frac{1}{p^\star}}\|\varphi\|_{W^{1,2}(\Omega)}^{\frac{p^\star-q^\star}{p^\star}} \cdot \Big(\|\varphi\|_{L^p(\Omega)}^b\|\varphi\|_{L^q(\Omega)}^{1-b}\Big)^{\frac{q^\star}{p^\star}}\right\}^a \cdot \|\varphi\|_{L^q(\Omega)}^{1-a} \\
&= c_1^{\frac{a}{p^\star}}\|\varphi\|_{W^{1,2}(\Omega)}^{\frac{p^\star-q^\star}{p^\star}a}\|\varphi\|_{L^p(\Omega)}^{\frac{q^\star}{p^\star}ab}\|\varphi\|_{L^q(\Omega)}^{\frac{q^\star}{p^\star}a(1-b)+1-a} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),
\end{aligned}$$

that is,

$$\|\varphi\|_{L^p(\Omega)}^p \le c_1^{\frac{pa}{p^\star\cdot(1-\frac{q^\star}{p^\star}ab)}} \cdot \|\varphi\|_{W^{1,2}(\Omega)}^{\frac{p(p^\star-q^\star)a}{p^\star(1-\frac{q^\star}{p^\star}ab)}}\|\varphi\|_{L^q(\Omega)}^{\frac{p[\frac{q^\star}{p^\star}a(1-b)+1-a]}{1-\frac{q^\star}{p^\star}ab}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Since herein

$$1 - \frac{q^\star}{p^\star}ab = \frac{p^\star-q^\star}{p^\star-q}$$

and hence

$$\frac{p(p^\star-q^\star)a}{p^\star\Big(1-\frac{q^\star}{p^\star}ab\Big)} = p - q$$

as well as

$$\frac{p\Big[\frac{q^\star}{p^\star}a(1-b)+1-a\Big]}{1-\frac{q^\star}{p^\star}ab} = q$$

and

$$\frac{pa}{p^\star\Big(1-\frac{q^\star}{p^\star}ab\Big)}=\frac{p-q}{p^\star-q^\star}\le\frac{p^\star}{p^\star-q^\star},$$

it follows from our restriction $c_1\ge 1$ that with $c_3:=c_1^{\frac{p^\star}{p^\star-q^\star}}$ we have

$$\|\varphi\|_{L^p(\Omega)}^p\le c_3\|\varphi\|_{W^{1,2}(\Omega)}^{p-q}\|\varphi\|_{L^q(\Omega)}^q \qquad \text{for all } \varphi\in W^{1,2}(\Omega).$$

As a consquence of (2.9) and the validity of the elementary inequality $(A+B)^\kappa\le 2^\kappa(A^\kappa+B^\kappa)$ for all $A\ge 0, B\ge 0$ and $\kappa>0$, we thus conclude that

$$\begin{aligned}
\|\varphi\|_{L^p(\Omega)}^p &\le c_3\cdot c_2^{\frac{p-q}{2}}\cdot\Big\{\|\nabla\varphi\|_{L^2(\Omega)}^2+\|\varphi\|_{L^q(\Omega)}^2\Big\}^{\frac{p-q}{2}}\cdot\|\varphi\|_{L^q(\Omega)}^q\\
&\le c_2^{\frac{p-q}{2}}c_3\cdot 2^{\frac{p-q}{2}}\cdot\Big\{\|\nabla\varphi\|_{L^2(\Omega)}^{p-q}+\|\varphi\|_{L^q(\Omega)}^{p-q}\Big\}\cdot\|\varphi\|_{L^q(\Omega)}^q\\
&\le (2c_2)^{\frac{p^\star}{2}}c_3\Big\{\|\nabla\varphi\|_{L^2(\Omega)}^{p-q}\|\varphi\|_{L^q(\Omega)}^q+\|\varphi\|_{L^q(\Omega)}^p\Big\} \qquad \text{for all } \varphi\in W^{1,2}(\Omega),
\end{aligned}$$

because $c_2\ge 1$. This proves (2.7) whenever $q<p$, whereupon the corresponding estimate in the borderline case $q=p$ can easily be established upon an appropriate limit procedure. □

3 Estimates implied by the natural energy inequality

The natural energy inequality (3.1), known as a powerful tool in the derivation of various solution properties in quasilinear chemotaxis systems of the considered structure, including the occurrence of blow-up ([11], [5], [6], [17], [18], [10]), will also constitute the starting point of our analysis of regularity beyond the information in (2.2) and Corollary 2.4.

Lemma 3.1 *We have*

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t),v(\cdot,t))\le-\mathcal{D}(u(\cdot,t),v(\cdot,t)) \qquad \textit{for all } t\in(0,T_{max}), \tag{3.1}$$

where we have set

$$\mathcal{F}(\varphi,\psi):=\frac12\int_\Omega|\nabla\psi|^2+\frac12\int_\Omega\psi^2-\int_\Omega\varphi\psi+\int_\Omega\Phi(\varphi) \tag{3.2}$$

for $\psi\in W^{1,2}(\Omega)$ and positive functions $\varphi\in C^0(\bar\Omega)$, and

$$\mathcal{D}(\varphi,\psi):=\int_\Omega\bigg|\frac{D(\varphi)}{\sqrt{S(\varphi)}}\nabla\varphi-\sqrt{S(\varphi)}\nabla\psi\bigg|^2+\int_\Omega\Big|\Delta\psi-\psi+\varphi\Big|^2$$

for $\psi\in W^{1,2}(\Omega)$ and positive $\varphi\in C^0(\bar\Omega)\cap W^{1,2}(\Omega)$, with

$$\Phi(s):=\int_1^s\int_1^\sigma\frac{D(\xi)}{S(\xi)}d\xi, \qquad s>0. \tag{3.3}$$

PROOF. Using that u and v are smooth in $\bar{\Omega} \times (0, T_{max})$ and that $u > 0$ in $\bar{\Omega} \times [0, T_{max})$ according to the strong maximum principle and our assumption that u_0 be positive in $\bar{\Omega}$, one can easily verify (3.1) by means of a straightforward computation ([17]). $\Box$

Now the assumption $\alpha < 1$ in (1.10) becomes essential in the following observation which is elementary but of fundamental importance in further exploiting (3.1).

Lemma 3.2 *If (1.10) holds with some $\alpha \in (0,1)$ and $K_3 > 0$, then the function Φ defined in (3.3) satisfies*

$$\Phi(s) \geq \frac{1}{(1-\alpha)(2-\alpha)K_3} \cdot s^{2-\alpha} - \frac{1}{(1-\alpha)K_3} \cdot s \qquad \text{for all } s > 0. \tag{3.4}$$

PROOF. By (1.10), in both cases $s \in (0,1]$ and $s > 1$ we can estimate

$$\begin{aligned} \Phi(s) &\geq \int_1^s \int_1^\sigma \frac{1}{K_3 \xi^\alpha} d\xi d\sigma \\ &= \frac{1}{(1-\alpha)K_3} \int_1^s (\sigma^{1-\alpha} - 1) d\sigma \\ &= \frac{1}{(1-\alpha)(2-\alpha)K_3} \cdot (s^{2-\alpha} - 1) - \frac{1}{(1-\alpha)K_3} \cdot (s-1) \\ &= \frac{1}{(1-\alpha)(2-\alpha)K_3} \cdot s^{2-\alpha} - \frac{1}{(1-\alpha)K_3} \cdot s + \frac{1}{(2-\alpha)K_3}. \end{aligned}$$

As $2 - \alpha$ is positive, this immediately yields (3.4). $\Box$

In view of the latter, namely, the energy inequality (3.1) can be used to improve the boundedness statement contained in (2.2) as follows.

Lemma 3.3 *Under assumptions of Theorem 1.1 there exists $C > 0$ such that*

$$\int_\Omega u^{2-\alpha}(\cdot, t) \leq C \qquad \text{for all } t \in (0, T_{max}). \tag{3.5}$$

PROOF. Form Corollary 2.4 we obtain $c_1 > 0$ such that

$$\int_\Omega v^{\frac{2-\alpha}{1-\alpha}} \leq c_1 \qquad \text{for all } t \in (0, T_{max}).$$

Since Young's inequality provides $c_2 > 0$ such that

$$\int_\Omega uv \leq \frac{1}{2(1-\alpha)(2-\alpha)K_3} \int_\Omega u^{2-\alpha} + c_2 \int_\Omega v^{\frac{2-\alpha}{1-\alpha}} \qquad \text{for all } t \in (0, T_{max}),$$

by means of Lemma 3.2 and (2.2) we thus infer that with $\mathcal{F}$ as in (3.2) and $m := \int_\Omega u_0$ we have

$$\begin{aligned} \mathcal{F}(u,v) &\geq -\int_\Omega uv + \int_\Omega \Phi(u) \\ &\geq -\int_\Omega uv + \frac{1}{(1-\alpha)(2-\alpha)K_3} \int_\Omega u^{2-\alpha} - \frac{1}{(1-\alpha)K_3} \int_\Omega u \end{aligned}$$

$$
\begin{aligned}
&= -\int_\Omega uv + \frac{1}{(1-\alpha)(2-\alpha)K_3}\int_\Omega u^{2-\alpha} - \frac{m}{(1-\alpha)K_3} \\
&\ge \frac{1}{2(1-\alpha)(2-\alpha)K_3}\int_\Omega u^{2-\alpha} - c_2\int_\Omega v^{\frac{2-\alpha}{1-\alpha}} - \frac{m}{(1-\alpha)K_3} \\
&\ge c_3\int_\Omega u^{2-\alpha} - c_4 \qquad \text{for all } t\in(0,T_{max})
\end{aligned}
$$

with $c_3 := \frac{1}{2(1-\alpha)(2-\alpha)K_3}$ and $c_4 := c_1c_2 + \frac{m}{(1-\alpha)K_3}$. Therefore, (3.1) implies that

$$
\begin{aligned}
c_3\int_\Omega u^{2-\alpha}(\cdot,t) - c_4 &\le \mathcal{F}(u(\cdot,t),v(\cdot,t)) \\
&\le \mathcal{F}(u_0,v_0) \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}
$$

and that hence

$$
\int_\Omega u^{2-\alpha}(\cdot,t) \le \frac{1}{c_3}\cdot\Big\{c_4 + \mathcal{F}(u_0,v_0)\Big\} \qquad \text{for all } t\in(0,T_{max}),
$$

as claimed. □

4 Boundedness of $e^{\beta^+ u}$

Our next goal is to further improve our knowledge on regularity of u, but unlike previous approaches concerned with at most algebraically decaying diffusivities, our subsequent arguments will aim at deriving estimates for integrals involving certain exponentials of u, rather than powers. To prepare our analysis in this direction, for arbitrary $\gamma\ge 0$ let us introduce $\Psi_\gamma:[0,\infty)\to\mathbb{R}$ by defining

$$
\Psi_\gamma(s) := \int_0^s\int_0^\sigma \frac{e^{\gamma\xi}}{D(\xi)}d\xi d\sigma, \qquad s\ge 0, \tag{4.1}
$$

and first state two elementary features thereof, as immediately implied by (1.9).

Lemma 4.1 *Let $\gamma\ge 0$. Then*

$$
\Psi_\gamma(s) \ge \frac{1}{K_2(\gamma+\beta^+)^2}\cdot e^{(\gamma+\beta^+)s} - \frac{1}{K_2(\gamma+\beta^+)}\cdot s - \frac{1}{K_2(\gamma+\beta^+)^2} \qquad \textit{for all } s\ge 0 \tag{4.2}
$$

and

$$
\Psi_\gamma(s) \le \frac{1}{K_1(\gamma+\beta^-)^2}\cdot e^{(\gamma+\beta^-)s} \qquad \textit{for all } s\ge 0. \tag{4.3}
$$

Proof. Using the right inequality in (1.9), we directly obtain that

$$
\Psi_\gamma(s) \ge \frac{1}{K_2}\int_0^s\int_0^\sigma e^{(\gamma+\beta^+)\xi}d\xi d\sigma \qquad \text{for all } s\ge 0,
$$

from which (4.2) directly results after integration. Likewise, the left inequality in (1.9) entails that

$$
\begin{aligned}
\Psi_\gamma(s) &\le \frac{1}{K_1}\int_0^s\int_0^\sigma e^{(\gamma+\beta^-)\xi}d\xi d\sigma \\
&= \frac{1}{K_1(\gamma+\beta^-)^2}\cdot e^{(\gamma+\beta^-)s} - \frac{1}{K_1(\gamma+\beta^-)}\cdot s - \frac{1}{K_1(\gamma+\beta^-)^2} \qquad \text{for all } s\ge 0,
\end{aligned}
$$

and thereby implies (4.3) on dropping the two rightmost summands. □

As a consequence of (1.10), we obtain the following basic property of $\int_\Omega \Psi_\gamma(u)$ when evaluated along trajectories of (1.1).

Lemma 4.2 *Let $\gamma \geq 0$. Then under assumptions of Theorem 1.1,*

$$\frac{d}{dt}\int_\Omega \Psi_\gamma(u) + \frac{1}{2}\int_\Omega e^{\gamma u}|\nabla u|^2 \leq \frac{K_3^2}{2}\int_\Omega u^{2\alpha}e^{\gamma u}|\nabla v|^2 \qquad \textit{for all } t \in (0, T_{max}). \tag{4.4}$$

PROOF. Since $\Psi''_\gamma(s) = \frac{e^{\gamma s}}{D(s)}$ for all $s \geq 0$, on straightforward computation using the first equation in (1.1) we obtain

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \Psi_\gamma(u) &= \int_\Omega \Psi'_\gamma(u)\nabla\cdot\Big\{D(u)\nabla u - S(u)\nabla v\Big\} \\
&= -\int_\Omega \Psi''_\gamma(u)D(u)|\nabla u|^2 + \int_\Omega \Psi''_\gamma(u)S(u)\nabla u\cdot\nabla v \qquad (4.5) \\
&= -\int_\Omega e^{\gamma u}|\nabla u|^2 + \int_\Omega \frac{S(u)}{D(u)}e^{\gamma u}\nabla u\cdot\nabla v \qquad \text{for all } t \in (0, T_{max}). \qquad (4.6)
\end{aligned}$$

As here by Young's inequality and (1.10) we can estimate

$$\begin{aligned}
\int_\Omega \frac{S(u)}{D(u)}e^{\gamma u}\nabla u\cdot\nabla v &\leq \frac{1}{2}\int_\Omega e^{\gamma u}|\nabla u|^2 + \frac{1}{2}\int_\Omega \Big(\frac{S(u)}{D(u)}\Big)^2 e^{\gamma u}|\nabla v|^2 \\
&\leq \frac{1}{2}\int_\Omega e^{\gamma u}|\nabla u|^2 + \frac{K_3^2}{2}\int_\Omega u^{2\alpha}e^{\gamma u}|\nabla v|^2 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned}$$

from (4.5) we immediately obtain (4.4). □

The following consequence of Lemma 3.3 and Lemma 2.3 will enable us to appropriately estimate the integral on the right-hand side of (4.4) in the first step of our iterative argument to be performed below.

Lemma 4.3 *Under assumptions of Theorem 1.1 there exist $q > 2$ and $C > 0$ such that*

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \leq C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.7}$$

PROOF. We take $\vartheta > 2$ from the hypothesis (1.7) and fix any $q > 2$ such that $q \leq \vartheta$ and

$$q < \frac{2(2-\alpha)}{\alpha},$$

which is possible because $\frac{2(2-\alpha)}{\alpha} > 2$ due to our assumption that $\alpha < 1$. Now since Lemma 3.3 yields $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^{2-\alpha}(\Omega)} \leq c_1 \qquad \text{for all } t \in (0, T_{max}),$$

on an application of Lemma 2.3 to $p := 2-\alpha$ we infer that with some $c_2 > 0$ we have

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,q}(\Omega)} &\leq c_2(1 + t^{-\frac{3}{2}+\frac{1}{q}})\|v_0\|_{L^1(\Omega)} + c_2\|u\|_{L^\infty((0,T_{max});L^{2-\alpha}(\Omega))} \\
&\leq c_2(1 + t^{-\frac{3}{2}+\frac{1}{q}})\|v_0\|_{L^1(\Omega)} + c_1c_2 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Again writing $\tau := \min\{1, \frac{1}{2}T_{max}\}$, we see that this implies the inequality

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le 2c_2\tau^{-\frac{3}{2}+\frac{1}{q}}\|v_0\|_{L^1(\Omega)} + c_1c_2 \qquad \text{for all } t \in (\tau, T_{max}),$$

whereupon recalling that $v \in L^\infty((0,\tau); W^{1,\vartheta}(\Omega))$ by continuity, we conclude. □

We shall next see that indeed the latter information can be used to suitably estimate the integral on the right of (4.4) in terms of the dissipative integral on the left in the case $\gamma = 0$. However, in order to turn (4.4) into an autonomous ODI for the functional $\int_\Omega \Psi_\gamma(u)$ *containing an appropriate absorbing term*, an adequate control of the Dirichlet integral $\int_\Omega |\nabla u|^2$ from below will be required. In the presently considered two-dimensional context, this can be achieved by means of the Moser-Trudinger inequality, as demonstrated in the following lemma.

Lemma 4.4 *Under assumptions of Theorem 1.1 there exists $C > 0$ such that*

$$\int_\Omega e^{\beta^+ u(\cdot,t)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.8}$$

Proof. We apply Lemma 4.2 to $\gamma := 0$ to see that with Ψ_γ as correspondingly given by (4.1) we have

$$\frac{d}{dt}\int_\Omega \Psi_\gamma(u) + \frac{1}{2}\int_\Omega |\nabla u|^2 \le c_1 \int_\Omega u^{2\alpha}|\nabla v|^2 \qquad \text{for all } t \in (0, T_{max}) \tag{4.9}$$

with $c_1 := \frac{K_3^2}{2}$. In order to estimate the right-hand side herein, we invoke Lemma 4.3 to find $q > 2$ and $c_2 > 0$ such that

$$\int_\Omega |\nabla v|^q \le c_2 \qquad \text{for all } t \in (0, T_{max}),$$

and hence by the Hölder inequality we obtain

$$\begin{aligned} c_1 \int_\Omega u^{2\alpha}|\nabla v|^2 &\le c_1 \bigg\{ \int_\Omega u^{\frac{2q\alpha}{q-2}} \bigg\}^{\frac{q-2}{q}} \bigg\{ \int_\Omega |\nabla v|^q \bigg\}^{\frac{2}{q}} \\ &= c_1 c_2^{\frac{2}{q}} \|u\|^{2\alpha}_{L^{\frac{2q\alpha}{q-2}}(\Omega)} \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Now since $W^{1,2}(\Omega) \hookrightarrow L^{\frac{2q\alpha}{q-2}}(\Omega)$, there exists $c_3 > 0$ such that

$$\|u\|_{L^{\frac{2q\alpha}{q-2}}(\Omega)} \le c_3 \|u\|_{W^{1,2}(\Omega)} \qquad \text{for all } t \in (0, T_{max}),$$

so that with $C_E > 0$ as given by Lemma 2.5, in view of (2.2) we can estimate

$$\begin{aligned} \|u\|^{2\alpha}_{L^{\frac{2q\alpha}{q-2}}(\Omega)} &\le c_3^{2\alpha} \cdot \Big\{ C_E \|\nabla u\|^2_{L^2(\Omega)} + C_E \|u\|^2_{L^1(\Omega)} \Big\}^\alpha \\ &\le c_3^{2\alpha} C_E^\alpha \cdot \bigg\{ \int_\Omega |\nabla u|^2 + m^2 \bigg\}^\alpha \qquad \text{for all } t \in (0, T_{max}) \end{aligned}$$

with $m := \int_\Omega u_0$. Thus, using that $\alpha < 1$ in employing Young's inequality we infer that there exists $c_4 > 0$ such that

$$\begin{aligned} c_1 \int_\Omega u^{2\alpha} |\nabla v|^2 &\le c_1 c_2^{\frac{2}{q}} c_3^{2\alpha} C_E^\alpha \cdot \left\{ \int_\Omega |\nabla u|^2 + m^2 \right\}^\alpha \\ &\le \frac{1}{4} \cdot \left\{ \int_\Omega |\nabla u|^2 + m^2 \right\} + c_4 \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

and that therefore (4.9) implies the inequality

$$\frac{d}{dt} \int_\Omega \Psi_\gamma(u) + \frac{1}{4} \int_\Omega |\nabla u|^2 \le c_5 \qquad \text{for all } t \in (0, T_{max}) \tag{4.10}$$

with $c_5 := \frac{1}{4} m^2 + c_4$.
We now make use of the fact that writing $c_6 := \frac{1}{K_1 (\beta^-)^2}$, by Lemma 4.1 with $\gamma = 0$ we have

$$\int_\Omega \Psi_\gamma(u) \le c_6 \int_\Omega e^{\beta^- u} \qquad \text{for all } t \in (0, T_{max}),$$

and recall that according to the Moser-Trudinger inequality ([3]) there exists $c_7 > 0$ fulfilling

$$\int_\Omega e^{|\varphi|} \le c_7 e^{c_7 \|\varphi\|^2_{W^{1,2}(\Omega)}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Therefore, again by Lemma 2.5 and (2.2),

$$\begin{aligned} \int_\Omega \Psi_\gamma(u) &\le c_6 \int_\Omega e^{\beta^- u} \\ &\le c_6 c_7 \exp \left\{ (\beta^-)^2 c_7 \|u\|^2_{W^{1,2}(\Omega)} \right\} \\ &\le c_6 c_7 \exp \left\{ (\beta^-)^2 c_7 C_E \int_\Omega |\nabla u|^2 + (\beta^-)^2 c_7 C_E m^2 \right\} \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

which entails that

$$\begin{aligned} \int_\Omega |\nabla u|^2 &\ge \frac{1}{(\beta^-)^2 c_7 C_E} \ln \left\{ \frac{1}{c_6 c_7} \int_\Omega \Psi_\gamma(u) \right\} - m^2 \\ &= \frac{1}{(\beta^-)^2 c_7 C_E} \ln \left\{ \int_\Omega \Psi_\gamma(u) \right\} - \frac{1}{(\beta^-)^2 c_7 C_E} \ln(c_6 c_7) - m^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Consequently, from (4.10) we obtain that

$$\frac{d}{dt} \Psi_\gamma(u) + c_8 \ln \left\{ \int_\Omega \Psi_\gamma(u) \right\} \le c_9 \qquad \text{for all } t \in (0, T_{max})$$

with $c_8 := \frac{1}{4(\beta^-)^2 c_7 C_E}$ and $c_9 := c_5 + \frac{1}{4(\beta^-)^2 c_7 C_E} \ln(c_6 c_7) + \frac{1}{4} m^2$. As a result of an ODE comparison, we thus conlcude that

$$\int_\Omega \Psi_\gamma(u) \le c_{10} := \max \left\{ \int_\Omega \Psi_\gamma(u_0)\, ,\, e^{\frac{c_9}{c_8}} \right\} \qquad \text{for all } t \in (0, T_{max}),$$

and that hence, as a consequence of the lower estimate for Ψ_γ provided by Lemma 4.1, we have

$$\begin{aligned}\int_\Omega e^{\beta^+ u} &\le K_2(\beta^+)^2 \int_\Omega \Psi_\gamma(u) + \beta^+ \int_\Omega u + |\Omega| \\ &\le K_2(\beta^+)^2 c_{10} + \beta^+ \int_\Omega u + |\Omega| \qquad \text{for all } t \in (0, T_{max}).\end{aligned}$$

Once more by means of (2.2), this finally establishes (4.8). □

5 A Moser-type iteration for e^u. Proof of Theorems 1.1 and 1.2

The goal of this section is to achieve boundedness of u in $\Omega \times (0, T_{max})$ by employing a Moser-type recursive argument to e^u, as a starting point using the L^{β^+} bound for this quantity asserted by Lemma 4.4. To accomplish this, let us first draw an immediate consequence of the latter concerning our knowledge about the regularity of v.

Lemma 5.1 *Under assumptions of Theorem 1.1, for all $\tau \in (0, T_{max})$ there exists $C(\tau) > 0$ such that*

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le C(\tau) \qquad \textit{for all } t \in (0, T_{max}).$$

PROOF. We fix $p > 2$ and then obtain as a particular consequence of Lemma 4.4 that $u \in L^\infty((0, T_{max}); L^p(\Omega))$. Therefore, the claim readily results upon an application of Lemma 2.3 to $q := \infty$. □

With the information provided by Lemma 4.4 and Lemma 5.1 at hand, we are now in the position to pursue the announced recursive reasoning. As a final preparation for this, let us briefly prove a comparison principle for sub- and supersolutions of difference equations involving nonlinearities with a favorable monotonicity property.

Lemma 5.2 *Let $F : \mathbb{N}_0 \times [0, \infty) \to [0, \infty)$ be such that $F(k, \cdot)$ is nondecreasing for all $k \in \mathbb{N}_0$, and suppose that $(\underline{M}_k)_{k \in \mathbb{N}_0} \subset [0, \infty)$ and $(\overline{M}_k)_{k \in \mathbb{N}_0} \subset [0, \infty)$ are such that*

$$\underline{M}_k \le F(k, \underline{M}_{k-1}) \quad \textit{and} \quad \overline{M}_k \ge F(k, \overline{M}_{k-1}) \qquad \textit{for all } k \in \mathbb{N} \tag{5.1}$$

as well as $\underline{M}_0 \le \overline{M}_0$. Then

$$\underline{M}_k \le \overline{M}_k \qquad \textit{for all } k \in \mathbb{N}_0. \tag{5.2}$$

PROOF. By hypothesis, $S := \{k \in \mathbb{N} \mid \underline{M}_j \le \overline{M}_j \text{ for all } j \in \{0, ..., k-1\}\}$ is not empty, and unless $S = \mathbb{N}$, $k_0 := \max S$ is a well-defined number in $\mathbb{N}$ satisfying $\underline{M}_{k_0 - 1} \le \overline{M}_{k_0 - 1}$ and $\underline{M}_{k_0} > \overline{M}_{k_0}$. According to (5.1) and the assumed monotonicity property of F, this entails that

$$\overline{M}_{k_0} < \underline{M}_{k_0} \le F(k_0, \underline{M}_{k_0 - 1}) \le F(k_0, \overline{M}_{k_0 - 1}) \le \overline{M}_{k_0},$$

which is absurd. We therefore must have $S = \mathbb{N}$, meaning that indeed (5.2) holds. □

We can now proceed to derive boundedness of e^u, and hence of u, by refining an iterative argument of Moser-type that has been applied to a slightly different framework in [19]. In its basic design, our reasoning follows a procedure detailed in the appendix of [15], but since the present setting apparently does not allow for a direct application of the latter, we include a full proof for completeness.

Lemma 5.3 *Assume that all the assumptions of Theorem 1.1 are satisfied. Then there exists $C > 0$ with the property that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{5.3}$$

Proof. We let $\beta_k := 2^k\beta^+$ for $k \in \mathbb{N}_0$ and $\tau := \min\{1, \frac{1}{2}T_{max}\}$, and given $T \in (\tau, T_{max})$ we introduce the numbers

$$M_k(T) := \max\left\{1\,,\ \sup_{t\in(\tau,T)} \int_\Omega e^{\beta_k u(\cdot,t)}\right\}, \qquad k \in \mathbb{N}_0,$$

which are all finite thanks to the continuity of u in $\bar\Omega \times [0, T_{max})$. In order to derive suitable information on $M_k(T)$ for $k \in \mathbb{N}$, we let $\gamma \equiv \gamma_k := \beta_k - \beta^+$ and then obtain from Lemma 4.1 and (2.2) that

$$\begin{aligned}\int_\Omega e^{\beta_k u(\cdot,t)} &\le K_2\beta_k^2 \int_\Omega \Psi_\gamma(u(\cdot,t)) + \beta_k \int_\Omega u(\cdot,t) + |\Omega| \\ &\le K_2\beta_k^2 \int_\Omega \Psi_\gamma(u(\cdot,t)) + \beta_k \int_\Omega u_0 + |\Omega| \qquad \text{for all } t \in (0, T_{max}),\end{aligned} \tag{5.4}$$

whence estimating $M_k(T)$ essentlially amounts to finding appropriate bounds for $\int_\Omega \Psi_\gamma(u)$ in $(0,T)$. To achieve this, according to Lemma 5.1 we fix $c_1 > 0$ such that

$$|\nabla v(x,t)| \le c_1 \qquad \text{for all } x \in \Omega \text{ and } t \in (\tau, T_{max}),$$

so that using (4.4) we find that

$$\frac{d}{dt}\int_\Omega \Psi_\gamma(u) + \frac{1}{2}\int_\Omega e^{(\beta_k-\beta^+)u}|\nabla u|^2 \le \frac{K_3^2c_1^2}{2}\int_\Omega u^{2\alpha}e^{(\beta_k-\beta^+)u} \qquad \text{for all } t \in (\tau, T_{max}). \tag{5.5}$$

To proceed from this, let us fix $c_2 > 0$ satisfying

$$\frac{K_3^2c_1^2}{2}\xi^{2\alpha} \le c_2 e^{\frac{\beta^+}{2}\xi} \qquad \text{for all } \xi \ge 0,$$

whence from (5.5) and the evident fact that $\beta_k - \beta^+ \le \beta_k$ we obtain that

$$\beta_k^2\frac{d}{dt}\int_\Omega \Psi_\gamma(u) + 2\int_\Omega \left|\nabla e^{\frac{\beta_k-\beta^+}{2}u}\right|^2 \le c_2\beta_k^2\int_\Omega e^{(\beta_k-\frac{\beta^+}{2})u} \qquad \text{for all } t \in (\tau, T_{max}). \tag{5.6}$$

Now an application of Lemma 2.6 to $p \equiv p_k := \frac{2\beta_k-\beta^+}{\beta_k-\beta^+}$ and $q \equiv q_k := \frac{\beta_k}{\beta_k-\beta^+}$ yields $c_3 > 0$ such that for all $k \in \mathbb{N}$ we have

$$\begin{aligned}\int_\Omega e^{(\beta_k-\frac{\beta^+}{2})u} &= \left\|e^{\frac{\beta_k-\beta^+}{2}u}\right\|_{L^{\frac{2\beta_k-\beta^+}{\beta_k-\beta^+}}(\Omega)}^{\frac{2\beta_k-\beta^+}{\beta_k-\beta^+}} \\ &\le c_3\left\|\nabla e^{\frac{\beta_k-\beta^+}{2}u}\right\|_{L^2(\Omega)}\left\|e^{\frac{\beta_k-\beta^+}{2}u}\right\|_{L^{\frac{\beta_k}{\beta_k-\beta^+}}(\Omega)}^{\frac{\beta_k}{\beta_k-\beta^+}} + c_3\left\|e^{\frac{\beta_k-\beta^+}{2}u}\right\|_{L^{\frac{\beta_k}{\beta_k-\beta^+}}(\Omega)}^{\frac{2\beta_k-\beta^+}{\beta_k-\beta^+}}\end{aligned} \tag{5.7}$$

for all $t \in (0, T_{max})$, because

$$1 = \lim_{j\to\infty} q_j \le q_k \le q_1 = 2 = \lim_{j\to\infty} p_j \le p_k \le p_1 = 3 \qquad \text{for all } k \in \mathbb{N}.$$

As

$$\left\| e^{\frac{\beta_k-\beta^+}{2}u} \right\|_{L^{\frac{\beta_k}{\beta_k-\beta^+}}(\Omega)}^{\frac{\beta_k}{\beta_k-\beta^+}} = \int_\Omega e^{\frac{\beta_k}{2}u} = \int_\Omega e^{\beta_{k-1}u} \le M_{k-1}(T) \qquad \text{for all } t \in (\tau, T), \tag{5.8}$$

from (5.7) we thus infer upon employing Young's inequality that

$$\begin{aligned} c_2\beta_k^2 \int_\Omega e^{(\beta_k - \frac{\beta^+}{2})u} &\le c_2c_3\beta_k^2 M_{k-1}(T) \cdot \left\| \nabla e^{\frac{\beta_k-\beta^+}{2}u} \right\|_{L^2(\Omega)} + c_2c_3\beta_k^2 M_{k-1}^{\frac{2\beta_k-\beta^+}{\beta_k}}(T) \\ &\le \int_\Omega \left| \nabla e^{\frac{\beta_k-\beta^+}{2}u} \right|^2 + \frac{c_2^2c_3^2\beta_k^4}{4} M_{k-1}^2(T) + c_2c_3\beta_k^2 M_{k-1}^{\frac{2\beta_k-\beta^+}{\beta_k}}(T) \end{aligned} \tag{5.9}$$

for all $t \in (\tau, T)$. Since

$$\beta_k^2 \le \frac{1}{4(\beta^+)^2}\beta_k^4 \qquad \text{for all } k \in \mathbb{N},$$

and since $\frac{2\beta_k-\beta^+}{\beta_k} < 2$ together with the fact that $M_{k-1}(T) \ge 1$ implies that

$$M_{k-1}^{\frac{2\beta_k-\beta^+}{\beta_k}}(T) \le M_{k-1}^2(T) \qquad \text{for all } k \in \mathbb{N},$$

this shows that if we abbreviate $c_4 := \frac{c_2^2c_3^2}{4} + \frac{c_2c_3}{4(\beta^+)^2}$, then (5.6) along with (5.9) implies that

$$\beta_k^2 \frac{d}{dt} \int_\Omega \Psi_\gamma(u) + \int_\Omega \left| \nabla e^{\frac{\beta_k-\beta^+}{2}u} \right|^2 \le c_4\beta_k^4 M_{k-1}^2(T) \qquad \text{for all } t \in (\tau, T). \tag{5.10}$$

We now recall that, again by Lemma 4.1 and our definition of γ_k, and due to the fact that $\beta^- \ge \beta^+$, we have

$$\begin{aligned} \int_\Omega \Psi_\gamma(u) &\le \frac{1}{K_1(\beta_k + \beta^- - \beta^+)^2} \int_\Omega e^{(\beta_k+\beta^--\beta^+)u} \\ &\le \frac{1}{K_1\beta_k^2} \int_\Omega e^{(\beta_k+\beta^--\beta^+)u} \qquad \text{for all } t \in (0, T_{max}), \end{aligned} \tag{5.11}$$

where we once more apply Lemma 2.6, this time with $p \equiv \tilde{p}_k := \frac{2(\beta_k+\beta^--\beta^+)}{\beta_k-\beta^+}$ and $q \equiv \tilde{q}_k := \frac{\beta_k}{\beta_k-\beta^+}$, to find $c_5 \ge 1$ such that for all $k \in \mathbb{N}$ and any $t \in (0, T_{max})$,

$$\begin{aligned} \int_\Omega e^{(\beta_k+\beta^--\beta^+)u} &= \left\| e^{\frac{\beta_k-\beta^+}{2}u} \right\|_{L^{\frac{2(\beta_k+\beta^--\beta^+)}{\beta_k-\beta^+}}(\Omega)}^{\frac{2(\beta_k+\beta^--\beta^+)}{\beta_k-\beta^+}} \\ &\le c_5 \left\| \nabla e^{\frac{\beta_k-\beta^+}{2}u} \right\|_{L^2(\Omega)}^{\frac{\beta_k+2\beta^--2\beta^+}{\beta_k-\beta^+}} \left\| e^{\frac{\beta_k-\beta^+}{2}u} \right\|_{L^{\frac{\beta_k}{\beta_k-\beta^+}}(\Omega)}^{\frac{\beta_k}{\beta_k-\beta^+}} + c_5 \left\| e^{\frac{\beta_k-\beta^+}{2}u} \right\|_{L^{\frac{\beta_k}{\beta_k-\beta^+}}(\Omega)}^{\frac{2(\beta_k+\beta^--\beta^+)}{\beta_k-\beta^+}}, \end{aligned}$$

for

$$1=\lim_{j\to\infty}\tilde q_j\le\tilde q_k\le\tilde q_1=2=\lim_{j\to\infty}\tilde p_j\le\tilde p_k\le\tilde p_1=2+\frac{2\beta^-}{\beta^+}\qquad\text{for all }k\in\mathbb{N}.$$

Again by (5.8), this entails that

$$\int_\Omega e^{(\beta_k+\beta^--\beta^+)u}\le c_5M_{k-1}(T)\cdot\Big\|\nabla e^{\frac{\beta_k-\beta^+}{2}u}\Big\|_{L^2(\Omega)}^{\frac{\beta_k+2\beta^--2\beta^+}{\beta_k-\beta^+}}+c_5M_{k-1}^{\frac{2(\beta_k+\beta^--\beta^+)}{\beta_k}}(T)\qquad\text{for all }t\in(\tau,T),$$

and that hence

$$\begin{aligned}\int_\Omega\Big|\nabla e^{\frac{\beta_k-\beta^+}{2}u}\Big|^2&\ge\left\{\frac{1}{c_5M_{k-1}(T)}\cdot\int_\Omega e^{(\beta_k+\beta^--\beta^+)u}-M_{k-1}^{\frac{\beta_k+2\beta^--2\beta^+}{\beta_k}}(T)\right\}_+^{\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}\\&\ge2^{-\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}\cdot\left\{\frac{1}{c_5M_{k-1}(T)}\cdot\int_\Omega e^{(\beta_k+\beta^--\beta^+)u}\right\}^{\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}\\&\quad-M_{k-1}^{\frac{2(\beta_k-\beta^+)}{\beta_k}}(T)\qquad\text{for all }t\in(\tau,T),\end{aligned}\tag{5.12}$$

because $(a-b)_+^\kappa\ge2^{-\kappa}a^\kappa-b^\kappa$ for all $a\ge0,b\ge0$ and $\kappa>0$. Since clearly

$$2\ge\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}\ge\frac{2(\beta_1-\beta^+)}{\beta_1+2\beta^--2\beta^+}=\frac{\beta^+}{\beta^-}\qquad\text{for all }k\in\mathbb{N}\tag{5.13}$$

and thus

$$2^{-\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}\ge\frac14,\quad c_5^{-\frac{2(\beta_k-\beta^+)}{\beta_k}}\ge c_5^{-2}\quad\text{and}\quad K_1^{\frac{2(\beta_k-\beta^+)}{\beta_k}}\ge c_6:=\min\{1\,,\,K_1^2\}\qquad\text{for all }k\in\mathbb{N}$$

due to our restriction $c_5\ge1$, and since furthermore

$$M_{k-1}^{\frac{2(\beta_k-\beta^+)}{\beta_k}}(T)\le M_{k-1}^2(T)$$

thanks to the fact that $M_{k-1}(T)\ge1$, combining (5.12) with (5.11) shows that

$$\begin{aligned}\int_\Omega\Big|\nabla e^{\frac{\beta_k-\beta^+}{2}u}\Big|^2&\ge\frac{1}{4c_5^2}M_{k-1}^{-\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}(T)\cdot\left\{\int_\Omega e^{(\beta_k+\beta^--\beta^+)u}\right\}^{\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}-M_{k-1}^2(T)\\&\ge\frac{1}{4c_5^2}M_{k-1}^{-\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}(T)\cdot\left\{K_1\beta_k^2\int_\Omega\Psi_\gamma(u)\right\}^{\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}-M_{k-1}^2(T)\\&\ge\frac{c_6}{4c_5^2}M_{k-1}^{-\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}(T)\cdot\left\{\beta_k^2\int_\Omega\Psi_\gamma(u)\right\}^{\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^--2\beta^+}}-M_{k-1}^2(T)\qquad\text{for all }t\in(\tau,T).\end{aligned}$$

Therefore, (5.10) implies that $y_k(t):=\beta_k^2\int_\Omega\Psi_\gamma(u(\cdot,t))$, $t\in[\tau,T)$, satisfies the autonomous ODI

$$\begin{aligned}y'(t)+\frac{c_6}{4c_5^2}M_{k-1}^{-\frac{2(\beta_k-\beta)}{\beta_k+2\beta^--2\beta^+}}(T)\cdot y^{\frac{2(\beta_k-\beta)}{\beta_k+2\beta^--2\beta^+}}(t)&\le c_4\beta_k^4M_{k-1}^2(T)+M_{k-1}^2(T)\\&\le c_7\beta_k^4M_{k-1}^2(T)\qquad\text{for all }t\in(\tau,T)\end{aligned}\tag{5.14}$$

with $c_7 := c_4 + \frac{1}{(2\beta^+)^4}$, because $1 \le \frac{\beta_k^4}{(2\beta^+)^4}$ for all $k \in \mathbb{N}$. Now by an ODE comparison, (5.14) warrants that for all $t \in (\tau, T)$

$$\begin{aligned} y(t) &\le \max\left\{ y(\tau)\,,\, \left\{ \frac{4c_5^2 c_7}{c_6} \beta_k^4 M_{k-1}^{2+\frac{2(\beta_k-\beta^+)}{\beta_k+2\beta^- -2\beta^+}}(T) \right\}^{\frac{\beta_k+2\beta^- -2\beta^+}{2(\beta_k-\beta^+)}} \right\} \\ &= \max\left\{ y(\tau)\,,\, \left\{ \frac{4c_5^2 c_7}{c_6} \right\}^{\frac{\beta_k+2\beta^- -2\beta^+}{2(\beta_k-\beta^+)}} \beta_k^{4\cdot\frac{\beta_k+2\beta^- -2\beta^+}{2(\beta_k-\beta^+)}} M_{k-1}^{\frac{2\beta_k+2\beta^- -3\beta^+}{\beta_k-\beta^+}}(T) \right\}, \end{aligned} \tag{5.15}$$

where recalling (5.13) we can estimate

$$\left\{ \frac{4c_5^2 c_7}{c_6} \right\}^{\frac{\beta_k+2\beta^- -2\beta^+}{2(\beta_k-\beta^+)}} \le c_8 := \max\left\{ 1\,,\, \left\{ \frac{4c_5^2 c_7}{c_6} \right\}^{\frac{\beta^-}{\beta^+}} \right\} \qquad \text{for all } k \in \mathbb{N}$$

and

$$\beta_k^{4\cdot\frac{\beta_k+2\beta^- -2\beta^+}{2(\beta_k-\beta^+)}} \le c_9 \beta_k^8 \qquad \text{for all } k \in \mathbb{N}$$

with $c_9 := \max\{1\,,\, \frac{1}{(2\beta^+)^8}\}$, noting that the latter is immediate from (5.13) when $\beta_k \ge 1$, whereas in the case $\beta_k < 1$ it is sufficient to observe that $1 \le \frac{\beta_k^8}{(2\beta^+)^8}$ by definition of β_k. Since furthermore $\beta_k - \beta^+ \ge \frac{\beta_k}{2}$ implies that

$$\frac{2\beta_k + 2\beta^- - 3\beta^+}{\beta_k - \beta^+} = 2 + \frac{2\beta^- - \beta^+}{\beta_k - \beta^+} \le 2 + \frac{2\beta^-}{\beta_k - \beta^+} \le 4 + \frac{4\beta^-}{\beta_k} = 2 + c_{10} \cdot 2^{-k}$$

with $c_{10} := \frac{4\beta^-}{\beta^+}$, and hence

$$M_{k-1}^{\frac{2\beta_k+2\beta^- -3\beta^+}{\beta_k-\beta^+}}(T) \le M_{k-1}^{2+c_{10}\cdot 2^{-k}}(T)$$

for all $k \in \mathbb{N}$, (5.15) along with (5.11) entails that

$$y(t) \le \max\left\{ 1\,,\, \frac{1}{K_1} \int_\Omega e^{(\beta_k+\beta^- -\beta^+)u(\cdot,\tau)}\,,\, c_8 c_9 \beta_k^8 M_{k-1}^{2+c_{10}\cdot 2^{-k}}(T) \right\} \qquad \text{for all } t \in (\tau, T)$$

and that therefore, by (5.4),

$$M_k(T) \le \max\left\{ 1\,,\, K_2\,,\, \frac{K_2}{K_1} \int_\Omega e^{(\beta_k+\beta^- -\beta^+)u(\cdot,\tau)}\,,\, c_8 c_9 K_2 \beta_k^8 M_{k-1}^{2+c_{10}\cdot 2^{-k}}(T) \right\} + \beta_k \int_\Omega u_0 + |\Omega|$$

for all $T \in (\tau, T_{max})$. Since

$$K_2 \le \frac{K_2}{(2\beta^+)^8} \beta_k^8, \quad \beta_k \int_\Omega u_0 \le \frac{\int_\Omega u_0}{(2\beta^+)^7} \beta_k^8 \quad \text{and} \quad |\Omega| \le \frac{|\Omega|}{(2\beta^+)^8} \beta_k^8$$

as well as $M_{k-1}(T) \geq 1$ for all $T \in (\tau, T_{max})$ and $k \in \mathbb{N}$, this can easily be seen to imply

$$M_k(T) \leq \max \left\{ 1\,,\, \frac{K_2}{K_1} \int_\Omega e^{(\beta_k+\beta^- - \beta^+)u(\cdot,\tau)}\,,\, c_{11}\beta_k^8 M_{k-1}^{2+c_{10}\cdot 2^{-k}}(T) \right\} \qquad \text{for all } T \in (\tau, T_{max}) \tag{5.16}$$

with e.g. $c_{11} := c_8 c_9 K_2 + \frac{K_2}{(2\beta^+)^8} + \frac{\int_\Omega u_0}{(2\beta^+)^7} + \frac{|\Omega|}{(2\beta^+)^8}$. According to Lemma 5.2 when applied to

$$F(k,\xi) := \max \left\{ 1\,,\, \frac{K_2}{K_1} \int_\Omega e^{(\beta_k+\beta^- - \beta^+)u(\cdot,\tau)}\,,\, c_{11}\beta_k^8 \xi^{2+c_{10}\cdot 2^{-k}} \right\}, \qquad k \in \mathbb{N}_0,\ \xi \geq 0,$$

this means that if we recursively define $(\overline{M}_k)_{k\in\mathbb{N}_0} \subset [0,\infty)$ by letting

$$\overline{M}_0 := \max \left\{ 1\,,\, \sup_{t\in(\tau,T_{max})} \int_\Omega e^{\beta^+ u(\cdot,t)} \right\}$$

and

$$\overline{M}_k := \max \left\{ 1\,,\, \frac{K_2}{K_1} \int_\Omega e^{(\beta_k+\beta^- - \beta^+)u(\cdot,\tau)}\,,\, c_{11}\beta_k^8 \overline{M}_{k-1}^{2+c_{10}\cdot 2^{-k}} \right\}, \qquad k \in \mathbb{N},$$

then since $\overline{M}_0$ and hence also all numbers $\overline{M}_k, k \in \mathbb{N}$, are finite by Lemma 4.4 with $\overline{M}_0 \geq M_0(T)$ for all $T \in (\tau, T_{max})$, we have

$$M_k(T) \leq \overline{M}_k \qquad \text{for all } k \in \mathbb{N}_0 \tag{5.17}$$

for any such T. Therefore, the remaining part of the conclusion is now straightforward: If, for some $T \in (\tau, T_{max})$, there exist infinitely many $k \in \mathbb{N}$ such that $\overline{M}_k \leq \max\left\{1\,,\, \int_\Omega e^{(\beta_k+\beta^- - \beta^+)u(\cdot,\tau)}\right\}$, then by (5.17)

$$\begin{aligned}
\|e^{u(\cdot,t)}\|_{L^\infty(\Omega)} &= \liminf_{k\to\infty} \left\{ \int_\Omega e^{\beta_k u(\cdot,t)} \right\}^{\frac{1}{\beta_k}} \\
&\leq \liminf_{k\to\infty} M_k^{\frac{1}{\beta_k}}(T) \\
&\leq \liminf_{k\to\infty} \overline{M}_k^{\frac{1}{\beta_k}} \\
&\leq \max \left\{ 1\,,\, \liminf_{k\to\infty} \left\{ \int_\Omega e^{(\beta_k+\beta^- - \beta^+)u(\cdot,\tau)} \right\}^{\frac{1}{\beta_k}} \right\} \\
&= \|e^{u(\cdot,\tau)}\|_{L^\infty(\Omega)} \qquad \text{for all } t \in (\tau, T).
\end{aligned} \tag{5.18}$$

Otherwise, (5.16) entails the existence of $b > 1$ such that

$$\overline{M}_k \leq b^k \overline{M}_{k-1}^{2+c_{10}\cdot 2^{-k}} \qquad \text{for all } k \in \mathbb{N},$$

which by a well-known inductive argument (see [19, Lemma 4.3] for a corresponding statement precisely covering the present situation) shows that with $c_{12} := e^{\frac{c_{10}}{2}}$ we have

$$\overline{M}_k \leq b^{k+c_{12}\cdot 2^{k+1}} \cdot \overline{M}_0^{c_{12}\cdot 2^k} \qquad \text{for all } k \in \mathbb{N}$$

and hence, again together with (5.17), implies that in this case,

$$\begin{aligned}\|e^{u(\cdot,t)}\|_{L^\infty(\Omega)} &\le \liminf_{k\to\infty}\left\{b^{k+c_{12}\cdot 2^{k+1}}\cdot \overline{M}_0^{c_{12}\cdot 2^k}\right\}^{\frac{1}{2^k\beta+}} \\ &= b^{\frac{2c_{12}}{\beta+}}\overline{M}_0^{\frac{c_{12}}{\beta+}} \qquad \text{for all } t\in(\tau,T). \end{aligned} \tag{5.19}$$

Therefore, e^u and thus also u is bounded in $\Omega\times(\tau,T_{max})$. Once more by continuity of u in $\bar\Omega\times[0,\tau]$, this establishes (5.3). □

Without any further difficulties, we can thereby proceed to the proof of our main result on global boundedness in (1.1).

PROOF of Theorem 1.1. In view of Lemma 2.1, both the statement of global existence and the claimed boundedness property are immediate consequences of Lemma 5.3 when combined with Lemma 5.1. □

From this, boundedness in the volume-filling chemotaxis model (1.13) follows as a straightforward consequence:

PROOF of Theorem 1.2. Writing $D(s) := (1+\beta s)e^{-\beta s}$ and $S(s) := se^{-\beta s}$ for $s\ge 0$, it is evident that if we fix any $\varepsilon\in(0,\beta)$, then we can find $c_1>0$ such that

$$e^{-\beta s}\le D(s)\le c_1 e^{-(\beta-\varepsilon)s} \qquad \text{for all } s\ge 0.$$

Since furthermore

$$\frac{S(s)}{D(s)} = \frac{s}{1+\beta s} \qquad \text{for all } s\ge 0$$

and hence, for arbitrary $\alpha\in(0,1)$,

$$\frac{S(s)}{D(s)}\le \min\left\{s\,,\,\frac{1}{\beta}\right\}\le \max\left\{1\,,\,\frac{1}{\beta}\right\}\cdot s^\alpha \qquad \text{for all } s\ge 0,$$

Theorem 1.1 directly applies so as to yield the desired conclusion. □

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References

[1] H. AMANN, *Dynamic theory of quasilinear parabolic systems III. Global existence.* Math. Z. **202**, 219-250 (1989)

[2] CALVEZ, V., CARILLO, J.A.: *Volume effects in the Keller-Segel model: energy estimates preventing blow-up.* J. Math. Pures Appl. **86**, 155-175 (2006)

[3] CHANG, S.Y.A., YANG, P.C.: *Conformal deformation of metrics on S^2.* J. Differential Geometry **27**, 259-296 (1988)

[4] Cieślak, T.: *Quasilinear nonuniformly parabolic system modelling chemotaxis.* J. Math. Anal. Appl. **326**, 1410-1426 (2007)

[5] Cieślak, T., Stinner, C.: *Finite-time blowup and global-in-time unbounded solutions to a parabolic-parabolic quasilinear Keller-Segel system in higher dimensions.* J. Differential Equations **252** (10), 5832-5851 (2012)

[6] Cieślak, T., Stinner, C.: *New critical exponents in a fully parabolic quasilinear Keller-Segel and applications to volume filling models.* J. Differential Equations **258** (6), 2080-2113 (2015)

[7] Herrero, M. A., Velázquez, J. J. L.: *A blow-up mechanism for a chemotaxis model.* Ann. Scuola Normale Superiore Pisa Cl. Sci. **24**, 633-683 (1997)

[8] Horstmann, D., Winkler, M.: *Boundedness vs. blow-up in a chemotaxis system.* J. Differential Equations **215** (1), 52-107 (2005)

[9] Keller, E.F., Segel, L.A.: *Initiation of slime mold aggregation viewed as an instability.* J. Theor. Biol. **26**, 399-415 (1970)

[10] Mizoguchi, N., Winkler, M.: *Blow-up in the two-dimensional parabolic Keller-Segel system.* Preprint

[11] Nagai, T., Senba, T., Yoshida, K.: *Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis.* Funkc. Ekvacioj, Ser. Int. **40**, 411-433 (1997)

[12] Painter, K.J., Hillen, T.: *Volume-filling and quorum-sensing in models for chemosensitive movement.* Can. Appl. Math. Q. **10**, 501-543 (2002)

[13] Senba, T., Suzuki, T.: *A quasi-linear system of chemotaxis.* Abstr. Appl. Anal. 2006, 1-21 (2006)

[14] Tao, Y., Winkler, M.: *A chemotaxis-haptotaxis model: the roles of porous medium diffusion and logistic source.* SIAM J. Math. Anal. **43** (2), 685-704 (2011)

[15] Tao, Y., Winkler, M.: *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity.* J. Differential Equations **252** (1), 692-715 (2012)

[16] Winkler, M.: *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model.* J. Differential Equations **248**, 2889-2905 (2010)

[17] Winkler, M.: *Does a 'volume-filling effect' always prevent chemotactic collapse?* Math. Meth. Appl. Sci. **33**, 12-24 (2010)

[18] Winkler, M: *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system.* J. Math. Pures Appl. **100**, 748-767 (2013), `arXiv:1112.4156v1`

[19] Winkler, M.: *Global existence and slow grow-up in a quasilinear Keller-Segel system with exponentially decaying diffusivity.* Preprint

[20] Wrzosek, D.: *Global attractor for a chemotaxis model with prevention of overcrowding.* Nonlinear Anal. TMA **59**, 1293-1310 (2004)