Slow growth of solutions of super-fast diffusion equations with unbounded initial data

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Abstract

We study positive solutions of the super-fast diffusion equation in the whole space with initial data which are unbounded as $|x| \to \infty$. We find an explicit dependence of the slow temporal growth rate of solutions on the initial spatial growth rate. A new class of self-similar solutions plays a significant role in our analysis.

Key words: super-fast diffusion, large time behaviour, self-similar solutions
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1 Introduction

Investigating mechanisms of mass flux plays an important role in the literature on nonlinear diffusion processes. In the specific context of the equation

$$v_t = \nabla \cdot (v^{m-1} \nabla v) \tag{1.1}$$

when posed in the entire space $\mathbb{R}^n$, by a large number of results quite a comprehensive understanding has been achieved with regard to phenomena related to mass *outflux*. For instance, in the case $m > 1$ when (1.1) becomes the porous medium equation, and also within the range $(n-2)_+/n < m < 1$ of fast diffusion, the large time behaviour of nonnegative solutions which decay sufficiently fast at spatial infinity, in the sense of having finite mass $\int_{\mathbb{R}^n} v$, is essentially determined by a particular element of a family of explicit self-similar solutions, the so-called Barenblatt solutions ([12], [18], [19]). For solutions of (1.1) with finite initial mass, the mass is conserved if $m \geq (n-2)/n$ when $n > 2$ and $m > 0$ when $n = 1, 2$ ([19]). If $n = 1$ and $-1 < m \leq 0$ then there is nonuniqueness and conservation of mass holds for the maximal solution ([8], [19]). In the remaining cases $n = 1, 2$ and $m \leq (n-2)_+$ or $n > 2$ and $m < (n-2)/n$, the mass is not conserved ([19]). In the borderline case $m = (n-2)/n$, $n > 2$, the large time behaviour of solutions with finite mass is more complicated than for $m > (n-2)/n$ because the solution does not evolve toward a single self-similar solution. The behaviour is different in an inner region and in an outer one ([14], [15]).

In the case $m < (n-2)/n$, mass flux toward spatial infinity occurs in an effective manner: If $0 < m < (n-2)/n$ then any positive solution emanating from initial data in $L^{n(1-m)/2}(\mathbb{R}^n)$ becomes extinct in finite time, and in the super-fast diffusion range $m < 0$ even instantaneous extinction occurs for such initial data in the sense that then no local-in-time solution exists ([6], cf. also [7] for the case $m = 0$). Beyond this, the literature has provided more detailed information on how the asymptotic behaviour near extinction depends on the initial spatial decay, again indicating an important role of self-similar solutions (see e.g. [2], [3], [4], [5], [10], [9], [13], [15], [19]).

In contrast to this, except for few results on essentially one-dimensional wave-like transport mechanisms ([1]; cf. also [23] and [24] for two recent examples involving non-constant wave speeds) only little seems known about processes of mass *influx* from infinity, here naively understood, rather than in a strict Fickian sense, as a transfer of high densities from far distant sites to e.g. regions located near the origin. In the present work we establish some results in this direction by exploring in detail how 'mass' initially concentrated at spatial infinity spreads over the entire space for the super-fast diffusion equation (1.1) with $m < 0$. More precisely, we shall be concerned with the Cauchy problem

$$\begin{cases} v_t = \nabla \cdot (v^{m-1} \nabla v), & x \in \mathbb{R}^n, \ t > 0, \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

in the strongly degenerate regime $m < 0$, assuming the initial data $v_0 \in C^0(\mathbb{R}^n)$ are positive, and such that

$$v_0(x) \to +\infty \qquad \text{as } |x| \to \infty \tag{1.3}$$

in an appropriate sense. Then solutions exist globally ([6]) and it is natural to expect that they tend to $+\infty$ everywhere in $\mathbb{R}^n$ in the large time limit. Our main objective now consists in investigating quantitatively the dependence of such growth phenomena on the particular asymptotics of the initial

data. Let us mention here that for $m < 0$ the diffusion is very fast where v is small but very slow where v is large. Therefore, it is natural to expect a slow growth process.

Main results: Decay estimates in a degenerate parabolic equation. In order to transfer the above situation to a convenient framework involving bounded functions decaying at spatial infinity, we substitute

$$u(x,t) := v^m(x,t), \qquad x \in \mathbb{R}^n, \ t > 0, \tag{1.4}$$

and then obtain formal equivalence of (1.2) to the Cauchy problem

$$\begin{cases} u_t = u^p \Delta u, & x \in \mathbb{R}^n, \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.5}$$

where $p := (m-1)/m = (|m|+1)/|m| > 1$ and $u_0 := v_0^m$ is positive and continuous in $\mathbb{R}^n$. Actually, most parts of our analysis also apply to the case $p = 1$ which does not stem from super-fast diffusion.

In view of known results on nonuniqueness of classical solutions to (1.2) ([17]), even in the framework of smooth positive solutions we cannot expect solutions of (1.5) to be uniquely determined. As a preliminary to our subsequent analysis, we shall therefore first make sure that after all, (1.5) possesses a *minimal* classical solution for any positive continuous and bounded initial data.

Proposition 1.1 *Let $p \geq 1$, and suppose that $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive. Then problem (1.5) possesses at least one global classical solution $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty))$ which satisfies*

$$0 < u(x,t) \leq \|u_0\|_{L^\infty(\mathbb{R}^n)} \qquad \text{for all } x \in \mathbb{R}^n \text{ and } t \geq 0. \tag{1.6}$$

Moreover, this solution is minimal in the sense that whenever $T \in (0,\infty]$ and $\widetilde{u} \in C^0(\mathbb{R}^n \times [0,T)) \cap C^{2,1}(\mathbb{R}^n \times (0,T))$ are such that $\widetilde{u}$ is positive and solves (1.5) classically in $\mathbb{R}^n \times (0,T)$, we necessarily have $u \leq \widetilde{u}$ in $\mathbb{R}^n \times (0,T)$.

Now if u_0 belongs to $L^{q_0}(\mathbb{R}^n)$ for some positive q_0 not necessarily exceeding the value 1, we can establish the following implications for the temporal decay of $\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)}$ for $q \in (q_0,\infty]$.

Theorem 1.2 *Let $p \geq 1$, and suppose that $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive and such that $u_0 \in L^{q_0}(\mathbb{R}^n)$ for some $q_0 > 0$. Then for any $q > q_0$ one can find $C = C(q) > 0$ with the property that the solution u of (1.5) from Proposition 1.1 satisfies*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-(1-\frac{q_0}{q})/(p+\frac{2q_0}{n})} \qquad \text{for all } t > 0. \tag{1.7}$$

Moreover, for any $\delta > 0$ there exists $\widehat{C}(\delta) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \widehat{C}(\delta)t^{-n/(np+2q_0)+\delta} \qquad \text{for all } t > 0. \tag{1.8}$$

In particular, if $u_0 \in \bigcap_{q_0 > 0} L^{q_0}(\mathbb{R}^n)$ then for any $\delta > 0$ one can find $\widetilde{C}(\delta) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \widetilde{C}(\delta)t^{-\frac{1}{p}+\delta} \qquad \text{for all } t > 0. \tag{1.9}$$

A natural next question appears to be how far the above one-sided estimates are optimal. Surprisingly, the above result on decay in $L^\infty(\mathbb{R}^n)$ for initial data with fast decay cannot be substantially improved in the sense that not even choosing $\delta = 0$ is possible in (1.9), however fast the initial data decay:

Proposition 1.3 *Let $p \geq 1$. Then for every positive $u_0 \in C^0(\mathbb{R}^n)$, any global positive classical u of (1.5) has the property that for any $R > 0$ we have*

$$\inf_{|x| < R} \left\{ t^{\frac{1}{p}} u(x, t) \right\} \to +\infty \qquad \text{as } t \to \infty.$$

But, indeed, also (1.7) and (1.8) are essentially sharp:

Theorem 1.4 *Let $p \geq 1$ and $q_0 > 0$. Then for every $q \in (q_0, \infty]$ there exists a positive function $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that $u_0 \in L^{q_0}(\mathbb{R}^n)$ and such that for any $\delta > 0$ one can find $C(\delta) > 0$ with the property that the solution u of (1.5) from Proposition 1.1 satisfies*

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \geq C(\delta) t^{-(1 - \frac{q_0}{q})/(p + \frac{2q_0}{n}) - \delta} \qquad \text{for all } t > 1. \tag{1.10}$$

In cases when the initial data satisfy pointwise decay estimates of algebraic type, we can even achieve more precise information on the respective large time behaviour. Fundamental for our analysis in this direction will be the observation that at least for $p > 1$, (1.5) possesses a two-parameter family of self-similar solutions with suitable spatial decay. As these solutions apparently have not yet been detected anywhere in the literature, let us describe them in the following separate statement.

Theorem 1.5 *Let $p > 1$, $\alpha \in (0, \frac{1}{p})$ and $A > 0$. Then the equation $u_t = u^p \Delta u$ possesses a radially symmetric positive classical self-similar solution $u_A^{(\alpha)}$ which can be written in the form*

$$u_A^{(\alpha)}(x, t) = t^{-\alpha} f\left(t^{-\beta} |x|\right), \qquad x \in \mathbb{R}^n, \ t > 0, \tag{1.11}$$

where $\beta := \frac{1 - p\alpha}{2} > 0$, and where $f : [0, \infty) \to (0, \infty)$ is the solution of the initial value problem

$$\begin{cases} f^p(\xi)\left(f''(\xi) + \frac{n-1}{\xi} f'(\xi)\right) + \beta \xi f'(\xi) + \alpha f(\xi) = 0, & \xi > 0, \\ f(0) = A, \quad f'(0) = 0. \end{cases} \tag{1.12}$$

Moreover, f satisfies
$$c(1 + \xi)^{-\frac{\alpha}{\beta}} \leq f(\xi) \leq C(1 + \xi)^{-\frac{\alpha}{\beta}} \qquad \text{for all } \xi \geq 0 \tag{1.13}$$

with appropriate positive constants c and C.

By means of two arguments based on parabolic comparison, in the upper estimate involving suitable members of the above self-similar family and in the lower estimate relying on certain compactly supported separated solutions, it is possible to give quite a comprehensive description of the temporal asymptotics in (1.5) for algebraically decaying initial data:

Theorem 1.6 *Let $p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be positive, and let u denote the solution of (1.5) constructed in Proposition 1.1.*

(i) If

$$u_0(x) \geq C_0(1+|x|)^{-\gamma} \qquad \text{for all } x \in \mathbb{R}^n \tag{1.14}$$

with some $C_0 > 0$ and $\gamma > 0$, then for any $q \in (\frac{n}{\gamma}, \infty]$ there exists $C > 0$ such that

$$\|u(\cdot, t)\|_{L^q(\Omega)} \geq Ct^{-(\gamma - \frac{n}{q})/(p\gamma + 2)} \qquad \text{for all } t > 1. \tag{1.15}$$

(ii) If $p > 1$ and there exist $\gamma > 0$ and $C_1 > 0$ such that

$$u_0(x) \leq C_1(1+|x|)^{-\gamma} \qquad \text{for all } x \in \mathbb{R}^n, \tag{1.16}$$

then for any $q \in (\frac{n}{\gamma}, \infty]$ one can find $C > 0$ with the property that

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq Ct^{-(\gamma - \frac{n}{q})/(p\gamma + 2)} \qquad \text{for all } t \geq 0. \tag{1.17}$$

Let us again underline that the above results exclusively address the respective minimal solution of (1.5) from Proposition 1.1. Restrictions of this type seem necessary for detecting a unique dependence of solution behavior on the initial data because of the quite drastic nature of the nonuniqueness phenomenon detected in [17], according to which it is namely possible to construct continua of solutions with substantially different, and actually widely arbitrary, behavior at spatial infinity.

Main results: Growth estimates for (1.2). Among the obvious translations of the above results to the original problem, let us highlight some implications for the respective maximal classical solutions $v := u^{1-p}$, $p = (m-1)/m$, of (1.2) obtained from Proposition 1.1. First, Theorem 1.2, Proposition 1.3 and Theorem 1.4 immediately yield the following.

Theorem 1.7 *Let $m < 0$, and let $v_0 \in C^0(\mathbb{R}^n)$ be positive and such that $v_0^m \in L^\infty(\mathbb{R}^n) \cap L^{q_0}(\mathbb{R}^n)$ for some $q_0 > 0$. Let v be the maximal solution of (1.2). Then for any $\delta > 0$ there exists $C(\delta) > 0$ such that*

$$\inf_{x \in \mathbb{R}^n} v(x, t) \geq C(\delta) t^{1/(1 - m - \frac{2q_0 m}{n}) - \delta} \qquad \text{for all } t > 0. \tag{1.18}$$

Moreover, if $u_0^m \in \bigcap_{q_0 > 0} L^{q_0}(\mathbb{R}^n)$ then for any $\delta > 0$ one can find $\widetilde{C}(\delta) > 0$ such that

$$\inf_{x \in \mathbb{R}^n} v(x, t) \geq C(\delta) t^{1/(1 - m - \frac{2q_0 m}{n}) - \delta} \qquad \text{for all } t > 0. \tag{1.19}$$

On the other hand, for every $q_0 > 0$ there exists a positive function $v_0 \in C^0(\mathbb{R}^n)$ such that $v_0^m \in L^\infty(\mathbb{R}^n) \cap L^{q_0}(\mathbb{R}^n)$ and such that for each $\delta > 0$ one can find $\widehat{C}(\delta) > 0$ with the property that

$$\inf_{x \in \mathbb{R}^n} v(x, t) \leq \widehat{C}(\delta) t^{1/(1 - m - \frac{2q_0 m}{n}) + \delta} \qquad \text{for all } t > 0. \tag{1.20}$$

Moreover, if $v_0 \in C^0(\mathbb{R}^n)$ is an arbitrary positive function such that $1/v_0 \in L^\infty(\mathbb{R}^n)$, then

$$t^{-\frac{1}{1-m}} v(\cdot, t) \to 0 \qquad \text{in } L^\infty_{loc}(\mathbb{R}^n) \qquad \text{as } t \to \infty. \tag{1.21}$$

Next, assuming algebraic growth of the initial data, we may rephrase Theorem 1.6 as follows.

Theorem 1.8 *Let $m < 0$ and $v_0 \in C^0(\mathbb{R}^n)$ be positive. Let v be the maximal solution of (1.2).*

(i) If
$$v_0(x) \le C_0(1 + |x|)^\theta \qquad \text{for all } x \in \mathbb{R}^n \tag{1.22}$$

with some $C_0 > 0$ and $\theta > 0$, then there exists $C > 0$ such that

$$\inf_{x \in \mathbb{R}^n} v(x,t) \le Ct^{\frac{\theta}{(1-m)\theta + 2}} \qquad \text{for all } t > 1. \tag{1.23}$$

(ii) If there exist $\theta > 0$ and $C_1 > 0$ such that

$$v_0(x) \ge C_1(1 + |x|)^\theta \qquad \text{for all } x \in \mathbb{R}^n \tag{1.24}$$

then

$$\inf_{x \in \mathbb{R}^n} v(x,t) \ge Ct^{\frac{\theta}{(1-m)\theta + 2}} \qquad \text{for all } t > 1. \tag{1.25}$$

with some $C > 0$.

Remark 1.9 *Theorem 1.8 describes explicitly how slow the growth process is and how it slows down as $m \to -\infty$. Namely, for $m < 0$ and $\theta > 0$ set*

$$\vartheta(\theta, m) := \frac{\theta}{(1-m)\theta + 2},$$

which is the exponent from (1.23), (1.25). Then for every fixed $\theta > 0$ we have that $\vartheta(\theta, m) \to 0$ as $m \to -\infty$, ϑ is increasing in both variables, and $0 < \vartheta(\theta, m) < 1$ for all $m < 0$ and $\theta > 0$.

The mechanism of mass influx from infinity is completely different for the linear heat equation. For example, for any positive even integer k, the solution of the problem

$$\begin{cases} u_t = u_{xx}, & x \in \mathbb{R}, \ t > 0, \\ u(x,0) = x^k, & x \in \mathbb{R}, \end{cases}.$$

is the heat polynomial

$$H_k(x,t) := \sum_{i=0}^{k/2} \frac{k!}{i!(k-2i)!} x^{k-2i} t^i,$$

and

$$\inf_{x \in \mathbb{R}} H_k(x,t) = \frac{k!}{(k/2)!} t^{k/2},$$

so the growth rate tends to infinity as $k \to \infty$ while $\vartheta(\theta, m) < 1/(1-m)$ for all $\theta > 0$.

The paper is organized as follows. We give a sketch of the proof of Proposition 1.1 in Section 2. The upper bounds from Theorem 1.2 are established in Section 3. We derive lower bounds used in the proofs of Proposition 1.3, Theorem 1.4 and Theorem 1.6 (i) in Section 4. The proof of Theorem 1.4 is finished in Section 5. Self-similar solutions are studied in Section 6 and they are used there to prove Theorem 1.6 (ii).

2 Global existence via approximation. Proof of Proposition 1.1

If $p > 1$ then Proposition 1.1 follows from [6]. Since we include also the case $p = 1$, we give a brief sketch of a proof which works for $p \geq 1$.

In order to construct solutions to (1.5), for $B_R := \{|x| < R\}$, $R > 0$ we consider the approximate problems

$$
\begin{cases}
u_{Rt} = u_R^p \Delta u_R, & x \in B_R, \ t > 0, \\
u_R(x,t) = 0, & x \in \partial B_R, \ t > 0, \\
u_R(x,0) = u_{0R}(x), & x \in B_R,
\end{cases} \tag{2.1}
$$

where $u_{0R} \in C^3(\bar{B}_R)$ satisfies $0 < u_{0R} < u_0$ in B_R and $u_{0R} = 0$ on ∂B_R as well as

$$
u_{0R} \nearrow u_0 \quad \text{in } \mathbb{R}^n \qquad \text{as } R \nearrow \infty. \tag{2.2}
$$

Moreover, for $\varepsilon \in (0,1)$ we consider

$$
\begin{cases}
u_{R\varepsilon t} = u_{R\varepsilon}^p \Delta u_{R\varepsilon}, & x \in B_R, \ t > 0, \\
u_{R\varepsilon}(x,t) = \varepsilon, & x \in \partial B_R, \ t > 0, \\
u_{R\varepsilon}(x,0) = u_{0R\varepsilon}(x), & x \in B_R,
\end{cases} \tag{2.3}
$$

where we have set

$$
u_{0R\varepsilon} := u_{0R} + \varepsilon.
$$

Lemma 2.1 *Let $p \geq 1$, and assume that $u_0 \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ is positive. Then with u_{0R} and $u_{0R\varepsilon}$ as above, problem (2.3) possesses for any $\varepsilon \in (0,1)$ a global classical solution $u_{R\varepsilon} \in C^0(\bar{B}_R \times [0,\infty)) \cap C^{2,1}(\bar{B}_R \times (0,\infty))$. As $\varepsilon \searrow 0$, we have that $u_{R\varepsilon} \searrow u_R$ where $u_R \in C^0(\bar{B}_R \times [0,\infty)) \cap C^{2,1}(B_R \times (0,\infty))$ is a positive classical solution of (2.1). Moreover, there exists a classical solution $u \in C^0(\mathbb{R}^n \times [0,\infty)) \cap C^{2,1}(\mathbb{R}^n \times (0,\infty))$ of (1.5) which is such that (1.6) holds, and that $u_R \nearrow u$ in $\mathbb{R}^n \times (0,\infty)$ as $R \nearrow \infty$.*

Proof.   Since all arguments are well-known, we may confine ourselves to sketching the main steps only and refer for details to [21], for example.

According to standard theory of quasilinear parabolic problems, each of these actually nondegenerate problems possesses a globally defined classical solution $u_{R\varepsilon}$ which satisfies

$$
\varepsilon \leq u_{R\varepsilon} \leq \|u_{0R}\|_{L^\infty(B_R)} + \varepsilon \qquad \text{in } B_R \times (0,\infty). \tag{2.4}
$$

By parabolic comparison it then follows that as $\varepsilon \searrow 0$ we have $u_{R\varepsilon} \searrow u_R$ in $B_R \times (0,\infty)$ with some limit function u_R. According to the interior positivity properties of u_{0R}, it can be seen by comparison that $\inf_{\varepsilon \in (0,1)} \inf_{(x,t) \in K \times [0,T]} u_{R\varepsilon}(x,t) > 0$ for any compact $K \subset B_R$ and every $T > 0$, which combined with parabolic Schauder estimates ([16]) shows that the convergence $u_{R\varepsilon} \to 0$ actually takes place in $C^0_{loc}(\bar{B}_R \times [0,\infty)) \cap C^{2,1}_{loc}(B_R \times [0,\infty))$, and that u_R is a classical solution of (2.1) which due to (2.4) satisfies

$$
0 < u_R \leq \|u_{0R}\|_{L^\infty(B_R)} \qquad \text{in } B_R \times (0,\infty). \tag{2.5}
$$

Now on the basis of this and the monotone approximation property (2.2), one more comparison argument asserts that $u_R \nearrow u$ in $\mathbb{R}^n \times (0,\infty)$ as $R \nearrow \infty$, where u is a limit function which according

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to (2.5) clearly satisfies (1.6). Again by means of parabolic regularity theory, the two-sided estimate in (1.6) guarantees that actually $u_R \to u$ in $C_{loc}^0(\mathbb{R}^n \times [0, \infty)) \cap C_{loc}^{2,1}(\mathbb{R}^n \times (0, \infty))$, and that hence u solves (1.5) classically. $\qquad \square$

PROOF of Proposition 1.1. Taking $u := \lim_{R\to\infty} u_R$ as provided by Lemma 2.1, in view of the latter we only need to show the claimed minimality property of u. Given $T \in (0, \infty]$ and a positive classical solution $\widetilde{u}$ of (1.5) in $\mathbb{R}^n \times (0, T)$ from the indicated class, however, by comparison we see that for every $R > 0$ we have $u_R < \widetilde{u}$ in $B_R \times [0, T)$ and that hence $u \le \widetilde{u}$ in $\mathbb{R}^n \times [0, T)$. $\qquad \square$

3 Upper decay estimates

3.1 Upper bounds for u in $L^q(\mathbb{R}^n)$ for $q < \infty$

The following elementary inequality will be used in Lemma 3.2.

Lemma 3.1 Let $\beta > 0$. Then there exists $\kappa(\beta) > 0$ such that

$$(a - b)_+^\beta \ge \kappa(\beta)a^\beta - b^\beta \qquad \text{for all } a \ge 0 \text{ and } b \ge 0. \tag{3.1}$$

PROOF. We take $c_1(\beta) > 0$ such that

$$(A + B)^\beta \le c_1(\beta)(A^\beta + B^\beta) \qquad \text{for all } A \ge 0 \text{ and } B \ge 0.$$

Then given $a \ge 0$ and $b \ge 0$, we apply this to $A := (a - b)_+$ and $B := b$ to obtain using the nonnegativity of β that

$$a^\beta \le \Big((a - b)_+ + b\Big)^\beta \le c_1(\beta)\Big((a - b)_+^\beta + b^\beta\Big).$$

Therefore, (3.1) holds if we let $\kappa(\beta) := 1/c_1(\beta)$, for instance. $\qquad \square$

Now the first main step toward our upper estimates for solutions is accomplished by means of a standard testing procedure applied to the approximate problems (2.3), followed by taking limits properly.

Lemma 3.2 Let $p \ge 1$ and $u_0 \in C^0(\mathbb{R}^n)$ be positive with

$$u_0 \in L^{q_0}(\mathbb{R}^n) \qquad \text{for some } q_0 > 0.$$

Then for any $q > q_0$ there exists $C(q) > 0$ such that for any $R > 0$, the solution u_R of (2.1) satisfies

$$\|u_R(\cdot, t)\|_{L^q(B_R)} \le C(q)\, t^{-(1 - \frac{q_0}{q})/(p + \frac{2q_0}{n})} \qquad \text{for all } t > 0. \tag{3.2}$$

PROOF. Observing that $u_{R\varepsilon}$ is smooth in $\bar{B}_R \times [0, \infty)$, for arbitrary $q > 0$ we may test (2.3) by $u_{R\varepsilon}^{q-1}$ and integrate by parts. Using that $u_{R\varepsilon} \ge \varepsilon$ and that $u_{R\varepsilon} = \varepsilon$ on ∂B_R, we thereby obtain

$$\frac{1}{q}\frac{d}{dt}\int_{B_R} u_{R\varepsilon}^q = \int_{B_R} u_{R\varepsilon}^{p+q-1}\Delta u_{R\varepsilon} = -(p + q - 1)\int_{B_R} u_{R\varepsilon}^{p+q-2}|\nabla u_{R\varepsilon}|^2 + \int_{\partial B_R} u_{R\varepsilon}^{p+q-1}\frac{\partial u_{R\varepsilon}}{\partial \nu}$$

$$\le -(p + q - 1)\int_{B_R} u_{R\varepsilon}^{p+q-2}|\nabla u_{R\varepsilon}|^2 = -\frac{4(p + q - 1)}{(p + q)^2}\int_{B_R}\Big|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}\Big|^2 \tag{3.3}$$

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for all $t > 0$. Since $p + q - 1$ is positive for any $q > 0$ due to the fact that $p \geq 1$, this implies that, in particular,

$$\|u_{R\varepsilon}(\cdot, t)\|_{L^{q_0}(B_R)} \leq \|u_{0R\varepsilon}\|_{L^{q_0}(B_R)} \qquad \text{for all } t > 0. \qquad (3.4)$$

In order to proceed, let us fix $q > q_0$ and note that then since $\frac{2q}{p+q} \leq 2$, an application of the Gagliardo-Nirenberg inequality provides $c_1 > 0$ such that

$$\|\varphi\|_{L^{\frac{2q}{p+q}}(\mathbb{R}^n)} \leq c_1 \|\nabla \varphi\|_{L^2(\mathbb{R}^n)}^a \|\varphi\|_{L^{\frac{2q_0}{p+q}}(\mathbb{R}^n)}^{1-a} \qquad \text{for all } \varphi \in W^{1,2}(\mathbb{R}^n), \qquad (3.5)$$

where $a \in (0, 1)$ is determined by the relation

$$-\frac{n(p+q)}{2q} = \left(1 - \frac{n}{2}\right)a - \frac{n(p+q)}{2q_0}(1 - a),$$

that is, where

$$a = \left(\frac{n(p+q)}{2}\left(\frac{1}{q_0} - \frac{1}{q}\right)\right)\left(1 - \frac{n}{2} + \frac{n(p+q)}{2q_0}\right)^{-1}. \qquad (3.6)$$

Now for fixed $t > 0$, we apply this to

$$\varphi(x) := \begin{cases} u_{R\varepsilon}^{\frac{p+q}{2}}(x, t) - \varepsilon^{\frac{p+q}{2}}, & x \in B_R, \\ 0, & x \in \mathbb{R}^n \setminus B_R, \end{cases}$$

which indeed defines a function φ belonging to $W^{1,2}(\mathbb{R}^n)$, because $u_{R\varepsilon}(\cdot, t) \in C^1(\bar{B}_R)$ satisfies $u_{R\varepsilon} \geq \varepsilon$ in B_R and $u_{R\varepsilon}|_{\partial B_R} = \varepsilon$.

Accordingly, (3.5) shows that

$$\left\|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\right\|_{L^{\frac{2q}{p+q}}(B_R)} \leq c_1 \left\|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t)\right\|_{L^2(B_R)}^a \left\|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\right\|_{L^{\frac{2q_0}{p+q}}(B_R)}^{1-a} \qquad \text{for all } t > 0, \quad (3.7)$$

where again using that $u_{R\varepsilon} \geq \varepsilon$ and recalling (3.4) we can estimate

$$\left\|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\right\|_{L^{\frac{2q_0}{p+q}}(B_R)}^{1-a} \leq \left\|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t)\right\|_{L^{\frac{2q_0}{p+q}}(B_R)}^{1-a} = \|u_{R\varepsilon}(\cdot, t)\|_{L^{q_0}(B_R)}^{\frac{p+q}{2q_0}(1-a)}$$

$$\leq \|u_{0R\varepsilon}\|_{L^{q_0}(B_R)}^{\frac{p+q}{2q_0}(1-a)} \qquad \text{for all } t > 0. \qquad (3.8)$$

Therefore, (3.7) entails that

$$\int_{B_R} \left|\nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t)\right|^2 \leq \left\{c_1 \|u_{0R\varepsilon}\|_{L^{q_0}(B_R)}^{\frac{p+q}{2q_0}(1-a)}\right\}^{-\frac{2}{a}} \left\|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\right\|_{L^{\frac{2q}{p+q}}(B_R)}^{\frac{2}{a}} \qquad \text{for all } t > 0. \qquad (3.9)$$

Here we apply Lemma 3.1 to see that with $\kappa(\cdot)$ as in (3.1) we have

$$\left\|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\right\|_{L^{\frac{2q}{p+q}}(B_R)}^{\frac{2}{a}} = \left\{\int_{B_R} \left|u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) - \varepsilon^{\frac{p+q}{2}}\right|^{\frac{2q}{p+q}}\right\}^{\frac{p+q}{qa}}$$

$$\geq \left\{\int_{B_R} \left(\kappa\left(\frac{2q}{p+q}\right)u_{R\varepsilon}^q(\cdot, t) - \varepsilon^q\right)\right\}^{\frac{p+q}{qa}} = \left\{\kappa\left(\frac{2q}{p+q}\right)\int_{B_R} u_{R\varepsilon}^q(\cdot, t) - |B_R|\varepsilon^q\right\}^{\frac{p+q}{qa}}$$

$$\geq \kappa\left(\frac{p+q}{qa}\right)\left(\kappa\left(\frac{2q}{p+q}\right)\right)^{\frac{p+q}{qa}}\left(\int_{B_R} u_{R\varepsilon}^q(\cdot, t)\right)^{\frac{p+q}{qa}} - \left(|B_R|\varepsilon^q\right)^{\frac{p+q}{qa}}$$

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for all $t > 0$. As a consequence, from (3.9) we obtain $c_2 > 0$ and $c_3(R) > 0$ such that writing $A_{R\varepsilon} := \|u_{0R\varepsilon}\|_{L^{q_0}(B_R)}^{-(p+q)(1-a)/(aq_0)}$ we have

$$\int_{B_R} \left| \nabla u_{R\varepsilon}^{\frac{p+q}{2}}(\cdot, t) \right|^2 \geq c_2 A_{R\varepsilon} \left(\int_{B_R} u_{R\varepsilon}^q(\cdot, t) \right)^{\frac{p+q}{qa}} - c_3(R) A_{R\varepsilon} \varepsilon^{\frac{p+q}{a}} \qquad \text{for all } t > 0,$$

so that (3.3) yields

$$\frac{d}{dt} \int_{B_R} u_{R\varepsilon}^q(\cdot, t) \leq -c_4 A_{R\varepsilon} \left(\int_{B_R} u_{R\varepsilon}^q(\cdot, t) \right)^{\frac{p+q}{qa}} + c_5(R) A_{R\varepsilon} \varepsilon^{\frac{p+q}{a}} \qquad \text{for all } t > 0,$$

where again since $p \geq 1$, both $c_4 := \frac{4q(p+q-1)}{(p+q)^2} c_2$ and $c_5(R) := \frac{4q(p+q-1)}{(p+q)^2} c_3(R)$ are positive. By an ODE comparison argument, this entails that

$$\int_{B_R} u_{R\varepsilon}^q(\cdot, t) \leq \overline{y}_{R\varepsilon}(t) \qquad \text{for all } t > 0, \tag{3.10}$$

where $\overline{y}_{R\varepsilon}$ denotes the solution if the initial-value problem

$$\begin{cases} \overline{y}_{R\varepsilon}'(t) = -c_4 A_{R\varepsilon} \overline{y}_{R\varepsilon}^{\frac{p+q}{qa}}(t) + c_5(R) A_{R\varepsilon} \varepsilon^{\frac{p+q}{a}}, & t > 0, \\ \overline{y}_{R\varepsilon}(0) = \int_{B_R} u_{0R\varepsilon}^q. \end{cases}$$

Now, since our construction of $(u_{0R\varepsilon})_{\varepsilon \in (0,1)}$ guarantees that

$$A_{R\varepsilon} \to A_R := \|u_{0R}\|_{L^{q_0}(B_R)}^{-(p+q)(1-a)/(aq_0)} \quad \text{and} \quad \int_{B_R} u_{0R\varepsilon}^q \to \int_{B_R} u_{0R}^q \qquad \text{as } \varepsilon \searrow 0,$$

it follows from standard results on continuous dependence that $\overline{y}_{R\varepsilon} \to \overline{y}_R$ in $C_{loc}^0([0,\infty))$ as $\varepsilon \searrow 0$, where

$$\begin{cases} \overline{y}_R'(t) = -c_4 A_R \overline{y}_R^{\frac{p+q}{qa}}(t), & t > 0, \\ \overline{y}_R(0) = \int_{B_R} u_{0R}^q. \end{cases}$$

Here an explicit integration shows that if we set $r := aq/(p+q-aq)$ then

$$\overline{y}_R(t) = \left(\overline{y}_R^{-\frac{1}{r}}(0) + \frac{c_4}{r} A_R t \right)^{-r} \leq c_6 (A_R t)^{-r} \qquad \text{for all } t > 0, \tag{3.11}$$

where $c_6 := (c_4/r)^{-r}$, and where thanks to (3.6),

$$\frac{1}{r} = \frac{p+q}{qa} - 1 = \frac{2q_0 + np}{n(q - q_0)} > 0$$

according to the fact that $q > q_0$. Therefore, using that $u_{0R} \leq u_0$ in estimating

$$A_R \geq A := \|u_0\|_{L^{q_0}(\mathbb{R}^n)}^{-(p+q)(1-a)/(aq_0)} \qquad \text{for all } R > 0,$$

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letting $\varepsilon \searrow 0$ in (3.10) we infer from (3.11) that for every $R > 0$ we have

$$\int_{B_R} u_R^r(\cdot, t) \leq c_6 (At)^{-(1 - \frac{q_0}{q})/(p + \frac{2q_0}{n})} \qquad \text{for all } t > 0,$$

which on taking the q-th root on both sides proves (3.2). $\qquad\square$

Taking $R \nearrow \infty$ and recalling Lemma 2.1, we can thereby already verify the first estimate in Theorem 1.2.

Corollary 3.3 *Let $p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n)$ be positive and such that*

$$u_0 \in L^{q_0}(\mathbb{R}^n) \qquad \text{for some } q_0 > 0.$$

Then for any $q > q_0$, the solution u of (1.5) obtained in Proposition 1.1 satisfies $u(\cdot, t) \in L^q(\mathbb{R}^n)$ for all $t > 0$, moreover, there exists $C(q) > 0$ such that

$$\|u(\cdot, t)\|_{L^q(\mathbb{R}^n)} \leq C(q) t^{-(1 - \frac{q_0}{q})/(p + \frac{2q_0}{n})} \qquad \text{for all } t > 0. \tag{3.12}$$

PROOF.   This is an evident consequence of Lemma 3.2 combined with (2.2) and the monotone convergence theorem. $\qquad\square$

3.2 Upper bounds for u in $L^\infty(\mathbb{R}^n)$. Proof of Theorem 1.2

The constant $C(q)$ inequality gained in Corollary 3.3 may depend on q; as Proposition 1.3 will reveal, in general we actually must have $C(q) \to \infty$ as $q \to \infty$. Accordingly, establishing the respective decay estimates from Theorem 1.2 which involve the norm in $L^\infty(\mathbb{R}^n)$ will, besides Lemma 3.2, require an additional ingredient. This role will be played by the following semi-convexity estimate which is a well-known feature of nonlinear diffusion equations of type (1.5). For its derivation, we may thus refer to the literature (see [1] or [22], for example).

Lemma 3.4 *Let $R > 0$. Then*

$$\frac{u_{Rt}(x, t)}{u_R(x, t)} \geq -\frac{1}{pt} \qquad \text{for all } x \in B_R \text{ and } t > 0. \tag{3.13}$$

In exploiting this, we shall need the following variant of the Gagliardo-Nirenberg inequality which involves small integrability powers, and in which the dependence of the constant on these powers is stressed.

Lemma 3.5 *Let $s > 2$ be such that $(n - 2)s \leq 2n$. Then there exists $C(s) > 0$ such that for any $\sigma \in (0, 2)$ we have*

$$\|\varphi\|_{L^s(\mathbb{R}^n)} \leq C(s) \|\nabla \varphi\|_{L^2(\mathbb{R}^n)}^a \|\varphi\|_{L^\sigma(\mathbb{R}^n)}^{1-a} \qquad \text{for all } \varphi \in W^{1,2}(\mathbb{R}^n) \cap L^\sigma(\mathbb{R}^n),$$

where

$$a = a(s, \sigma) := \left(\frac{n}{\sigma} - \frac{n}{s}\right) \left(1 - \frac{n}{2} + \frac{n}{\sigma}\right)^{-1} \in (0, 1].$$

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PROOF.   According to the standard Gagliardo-Nirenberg inequality ([11]), there exists $c_1(s) > 0$ such that

$$\|\varphi\|_{L^s(\mathbb{R}^n)} \leq c_1(s)\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^b\|\varphi\|_{L^2(\mathbb{R}^n)}^{1-b} \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n), \tag{3.14}$$

where

$$b = \frac{n}{2} - \frac{n}{s} \in (0,1]$$

due to our assumptions $s > 2$ and $(n-2)s \leq 2n$. Here we can apply the Hölder inequality to further interpolate

$$\|\varphi\|_{L^2(\mathbb{R}^n)}^{1-b} \leq \|\varphi\|_{L^s(\mathbb{R}^n)}^{(1-b)d}\|\varphi\|_{L^\sigma(\mathbb{R}^n)}^{(1-b)(1-d)} \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n) \tag{3.15}$$

with

$$d = d(s,\sigma) := \left(\frac{1}{\sigma} - \frac{1}{2}\right)\left(\frac{1}{\sigma} - \frac{1}{s}\right)^{-1}.$$

Combining (3.14) with (3.15) shows that

$$\|\varphi\|_{L^s(\mathbb{R}^n)}^{1-(1-b)d} \leq c_1(s)\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^b\|\varphi\|_{L^\sigma(\mathbb{R}^n)}^{(1-b)(1-d)} \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n),$$

so that since

$$\frac{b}{1-(1-b)d} = a \qquad \text{and} \qquad \frac{(1-b)(1-d)}{1-(1-b)d} = 1 - \frac{b}{1-(1-b)d} = 1 - a,$$

we obtain

$$\|\varphi\|_{L^s(\mathbb{R}^n)} \leq c_1^{\frac{a(s,\sigma)}{b(s)}}(s)\|\nabla\varphi\|_{L^2(\mathbb{R}^n)}^a\|\varphi\|_{L^\sigma(\mathbb{R}^n)}^{1-a} \qquad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n).$$

$\square$

With this tool at hand, we can derive the following consequence of Lemma 3.4 which will form the essential step in an iteration procedure to be performed in Lemma 3.7.

Lemma 3.6 *Let $s > 2$ be such that $(n-2)s < 2n$. Then there exists $C > 0$ such that for any $r \geq 1$, the solution u_R of (2.1) satisfies*

$$\|u_R(\cdot,t)\|_{L^{(p+r)s/2}(B_R)} \leq \left(\frac{Cr}{t}\right)^{\frac{a}{p+r}}\|u_R(\cdot,t)\|_{L^r(B_R)}^{1-\frac{pa}{p+r}} \qquad \text{for all } t > 1, \tag{3.16}$$

where

$$a = a(r) := \left(\frac{n(p+r)}{2r} - \frac{n}{s}\right)\left(1 - \frac{n}{2} + \frac{n(p+r)}{2r}\right)^{-1} \in (0,1). \tag{3.17}$$

PROOF.   We fix a sequence $(\chi_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{R})$ of cut-off functions satisfying $0 \leq \chi_k \leq 1$ and $\chi_k' \geq 0$ on $\mathbb{R}$ as well as $\chi_k \equiv 0$ in $(-\infty, 1/k)$ and $\chi_k \equiv 1$ in $(2/k, \infty)$. Then since u_R is continuous with $u_R|_{\partial B_R} = 0$, testing the inequality (3.13) by $\chi_k(u)u^r$ yields

$$(p+r-1)\int_{B_R}\chi_k(u_R)u_R^{p+r-2}|\nabla u_R|^2 + \int_{B_R}\chi_k'(u_R)u_R^{p+r-1}|\nabla u_R|^2 \leq \frac{1}{pt}\int_{B_R}\chi_k(u_R)u_R^r \quad \text{for all } t > 0.$$

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Since $\chi_k' \geq 0$, $\chi_k \leq 1$ and $p + r - 1 > 0$, this implies that

$$\int_{B_R} \chi_k(u_R) u_R^{p+r-2} |\nabla u_R|^2 \leq \frac{1}{p(p+r-1)t} \int_{B_R} \chi_k(u_R) u_R^r$$

$$\leq \frac{1}{p(p+r-1)t} \int_{B_R} u_R^r \qquad \text{for all } t > 0. \tag{3.18}$$

Thus, if we introduce

$$P_k(\xi) := \int_0^\xi \sqrt{\chi_k(\eta)} \eta^{(p+r-2)/2} d\eta, \qquad \xi \geq 0, \tag{3.19}$$

and extend $P_k(u_R(\cdot, t))$ by zero so as to become a function $P_k(u_R(\cdot, t)) \in C_0^\infty(\mathbb{R}^n)$, then (3.18) says that

$$\int_{\mathbb{R}^n} |\nabla P_k(u_R)|^2 \leq \frac{1}{p(p+r-1)t} \int_{B_R} u_R^r \qquad \text{for all } t > 0. \tag{3.20}$$

Here we apply the Gagliardo-Nirenberg inequality from Lemma 3.5 to obtain $c_1 > 0$ such that for any choice of $r > 0$, inter alia ensuring that

$$0 < \frac{2r}{p+r} < 2,$$

we have

$$\|P_k(u_R)\|_{L^s(\mathbb{R}^n)} \leq c_1 \|\nabla P_k(u_R)\|_{L^2(\mathbb{R}^n)}^a \|P_k(u_R)\|_{L^{2r/(p+r)}(\mathbb{R}^n)}^{1-a} \qquad \text{for all } t > 0 \tag{3.21}$$

with $a \in (0,1)$ determined in (3.17). Again since $0 \leq \chi_k \leq 1$, from (3.19) we see that $P_k(\xi) \leq 2\xi^{(p+r)/2}/(p+r)$ for all $\xi \geq 0$, so that

$$\|P_k(u_R)\|_{L^{2r/(p+r)}(\mathbb{R}^n)}^{1-a} \leq \left\{ \frac{2}{p+r} \left\| u_R^{(p+r)/2} \right\|_{L^{2r/(p+r)}(B_R)} \right\}^{1-a} = \left(\frac{2}{p+r} \right)^{1-a} \|u_R\|_{L^r(B_R)}^{(p+r)(1-a)/2}$$

for all $t > 0$. Accordingly, combining (3.21) with (3.20) shows that

$$\|P_k(u_R)\|_{L^s(\mathbb{R}^n)} \leq \left(\frac{2}{p+r} \right)^{1-a} \|\nabla P_K(u_R)\|_{L^2(\mathbb{R}^n)}^a \|u_R\|_{L^r(B_R)}^{(p+r)(1-a)/2}$$

$$\leq \left(\frac{2}{p+r} \right)^{1-a} \left(\frac{1}{p(p+r-1)t} \right)^{\frac{a}{2}} \left(\int_{B_R} u_R^r \right)^{\frac{a}{2}} \|u_R\|_{L^r(B_R)}^{(p+r)(1-a)/2}$$

$$= \left(\frac{2}{p+r} \right)^{1-a} \left(\frac{1}{p(p+r-1)t} \right)^{\frac{a}{2}} \|u_R\|_{L^r(B_R)}^{(p+r-pa)/2} \qquad \text{for all } t > 0. \tag{3.22}$$

Now since (3.19) along with our choice of χ_k ensures that for all $\xi < 0$ we have

$$P_k(\xi) \to \frac{2}{p+r} \xi^{\frac{p+r}{2}} \qquad \text{as } k \to \infty,$$

we may apply Fatou's lemma to conclude from (3.22) that

$$\frac{2}{p+r} \|u_R(\cdot, t)\|_{L^{(p+r)s/2}(B_R)}^{(p+r)/2} = \frac{2}{p+r} \left\| u_R^{(p+r)/2}(\cdot, t) \right\|_{L^s(B_R)} \leq \liminf_{k \to \infty} \|P_k(u_R(\cdot, t))\|_{L^s(\mathbb{R}^n)}$$

$$\leq \left(\frac{2}{p+r} \right)^{1-a} \left(\frac{1}{p(p+r-1)t} \right)^{\frac{a}{2}} \|u_R(\cdot, t)\|_{L^r(B_R)}^{(p+r-pa)/2}$$

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and hence

$$\|u_R(\cdot,t)\|_{L^{(p+r)s/2}(B_R)} \leq \left(\frac{p+r}{2}\right)^{\frac{2a}{p+r}} \left(\frac{1}{p(p+r-1)t}\right)^{\frac{a}{p+r}} \|u_R(\cdot,t)\|_{L^r(B_R)}^{(p+r-pa)/(p+r)} \qquad (3.23)$$

for all $t > 0$. Here, we can clearly find $c_2, c_3 > 1$ such that

$$\frac{p+r}{2} \leq c_2 r \quad \text{and} \quad \frac{1}{p(p+r-1)t} \leq \frac{c_3}{rt} \qquad \text{for all } r \geq 1,$$

so that (3.16) follows from (3.23) if we let $C := c_2^2 c_3$. $\qquad \square$

Now an estimate for u_R in $L^\infty(B_R)$ in the flavour of Theorem 1.2 can be achieved by means of a Moser-type iteration, making essential use of the dependence of the right-hand side of (3.16) on both r and t.

Lemma 3.7 *Let $p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n)$ be positive and such that $u_0 \in L^{q_0}(\mathbb{R}^n)$ for some $q_0 > 0$. Set $\nu := \frac{n}{np+2q_0}$. Then for any $\delta > 0$ there exists $C(\delta) > 0$ such that for any $R > 0$ we have*

$$\|u_R(\cdot,t)\|_{L^\infty(B_R)} \leq C(\delta) t^{-\nu+\delta} \qquad \text{for all } t > 0. \qquad (3.24)$$

PROOF.   Let us fix $c_1 > 0$ such that

$$\ln \xi \geq -c_1(1-\xi) \qquad \text{for all } \xi \in (1/2, 1) \qquad (3.25)$$

and pick $s > 2$ sufficiently close to 2 such that $(n-2)s < 2n$. Further, given $\delta > 0$ satisfying $\delta < \nu$ we can find $\delta' \in (0, \delta)$ such that $-\nu + \delta > -\nu(1-\delta')$ and then take $q > q_0$ large such that still

$$-\frac{q-q_0}{q}\nu(1-\delta') < -\nu + \delta. \qquad (3.26)$$

Next we choose a number $r_0 \geq 1$ large enough satisfying

$$r_0 \geq q \qquad (3.27)$$

and

$$\frac{p}{p+r_0} < \frac{1}{2} \qquad (3.28)$$

as well as

$$\frac{pc_1}{r_0} \sum_{i=0}^{\infty} \left(\frac{2}{s}\right)^i \leq \ln \frac{1}{1-\delta'} \qquad (3.29)$$

and recursively define

$$r_{k+1} := \frac{(p+r_k)s}{2}$$

for nonnegative integers k. Then, clearly, $(r_k)_{k \geq 0}$ is increasing with

$$r_k \geq r_0 \left(\frac{s}{2}\right)^k \qquad \text{for all } k \geq 0, \qquad (3.30)$$

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and there exists $c_2 > 0$ such that

$$r_k \leq c_2 s^k \qquad \text{for all } k \geq 0. \tag{3.31}$$

In particular, Lemma 3.6 applies to yield a constant $c_3 > 1$ such that writing

$$M_k(t) := \|u_R(\cdot, t)\|_{L^{r_k}(B_R)}, \qquad t > 0,$$

we have

$$M_{k+1}(t) \leq \left(\frac{c_3 r_k}{t}\right)^{\frac{a_k}{p+r_k}} M_k^{\theta_k}(t) \qquad \text{for all } t > 0 \text{ and } k \geq 0, \tag{3.32}$$

where

$$\theta_k := 1 - \frac{p a_k}{p + r_k} \equiv \frac{(1 - a_k)p + r_k}{p + r_k} \tag{3.33}$$

with

$$a_k := \left(\frac{n(p + r_k)}{2r_k} - \frac{n}{s}\right)\left(1 - \frac{n}{2} + \frac{n(p + r_k)}{2r_k}\right)^{-1} \in (0, 1) \tag{3.34}$$

for $k \geq 0$. Now by a straightforward induction, (3.32) implies that, for all $t > 0$ and $k \geq 0$, we have

$$M_{k+1}(t) \leq \left\{\left(\frac{c_3}{t}\right)^{\sum_{j=0}^{k} \frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i}\right\}\left\{\prod_{j=0}^{k} r_j^{\frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i}\right\} M_0^{\prod_{i=0}^{k} \theta_i}(t), \tag{3.35}$$

where we may use that (3.33) yields that $\theta_i \in (0, 1)$ for all $i \geq 0$ in estimating

$$\prod_{i=j+1}^{k} \theta_i \leq 1 \qquad \text{for all } k \geq 0 \text{ and } j \in \{0, \dots, k\}. \tag{3.36}$$

Along with (3.31), (3.30) and the fact that $a_j \leq 1$ and $r_j \geq r_0 \geq 1$ for all $j \geq 0$, this yields that

$$\ln\left\{\prod_{j=0}^{k} r_j^{\frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i}\right\} \leq \ln\left\{\prod_{j=0}^{k} r_j^{\frac{a_j}{p+r_j}}\right\} = \sum_{j=0}^{k} \frac{a_j}{p + r_j} \ln r_j \leq \sum_{j=0}^{k} \frac{1}{r_j} \ln r_j$$

$$\leq \sum_{j=0}^{k} \frac{1}{r_0 (\frac{s}{2})^j} \ln(c_2 s^j) = \frac{\ln c_2}{r_0} \sum_{j=0}^{k} \left(\frac{2}{s}\right)^j + \frac{\ln s}{r_0} \sum_{j=0}^{k} j\left(\frac{2}{s}\right)^j \qquad \text{for all } k \geq 0.$$

Since both $\sum_{j=0}^{\infty} (\frac{2}{s})^j$ and $\sum_{j=0}^{\infty} j(\frac{2}{s})^j$ converge thanks to the fact that $s > 2$, from this we infer the existence of $c_4 > 0$ such that

$$\prod_{j=0}^{k} r_j^{\frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i} \leq c_4 \qquad \text{for all } k \geq 0. \tag{3.37}$$

Next, (3.36) entails that since $c_3 > 1$ and $\theta_i > 0$ for all $i \geq 0$, for every $t \geq 1$ we can estimate

$$\left(\frac{c_3}{t}\right)^{\sum_{j=0}^{k} \frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i} \leq c_3^{\sum_{j=0}^{k} \frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i} \leq c_3^{\sum_{j=0}^{k} \frac{a_j}{p+r_j}} \qquad \text{for all } k \geq 0,$$

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whence again by using (3.30) and that $a_j \leq 1$ for all $j \geq 0$ we see that

$$\left(\frac{c_3}{t}\right)^{\sum_{j=0}^{k} \frac{a_j}{p+r_j} \prod_{i=j+1}^{k} \theta_i} \leq c_3^{\frac{1}{r_0} \sum_{j=0}^{k} (\frac{2}{s})^j} \leq c_5 := c_3^{\frac{1}{r_0} \sum_{j=0}^{\infty} (\frac{2}{s})^j} \qquad \text{for all } t \geq 1 \text{ and } k \geq 0. \qquad (3.38)$$

Finally, to control the rightmost factor in (3.35) we first apply Lemma 3.2 to find $c_6 > 1$ such that

$$M_0(t) \leq c_6 t^{-\frac{r_0-q_0}{r_0}\nu} \qquad \text{for all } t > 0,$$

which according to (3.27) implies that

$$M_0(t) \leq c_6 t^{-\frac{q-q_0}{q}\nu} \qquad \text{for all } t \geq 1. \qquad (3.39)$$

Now whenever $t \geq 1$ is such that the right-hand side herein is larger than 1, that is, when

$$1 \leq t < t_0 := c_6^{\frac{q}{\nu(q-q_0)}},$$

from (3.35), (3.37) and (3.38) we trivially infer that

$$M_{k+1}(t) \leq c_4 c_5 c_6 \leq c_4 c_5 c_6 t_0^{\nu-\delta} t^{-\nu+\delta}, \qquad (3.40)$$

again because $\theta_i \in (0, 1)$ for all $i \geq 0$. If, conversely, $t \geq t_0$ then we estimate

$$M_0^{\prod_{i=0}^{k} \theta_i}(t) \leq \left(c_6 t^{-\frac{q-q_0}{q}\nu}\right)^{\prod_{i=0}^{k} \theta_i}, \qquad (3.41)$$

where, since $\theta_i \geq 1 - \frac{p}{p+r_i}$ for all $i \geq 0$, we have

$$\theta_i \geq 1 - \frac{p}{p+r_0} > \frac{1}{2} \qquad \text{for all } i \geq 0$$

according to (3.34), (3.30) and (3.28), using (3.25) and once more (3.30) we know that

$$\ln\left\{\prod_{i=0}^{k} \theta_i\right\} = \sum_{i=0}^{k} \ln \theta_i \geq -c_1 \sum_{i=0}^{k} (1-\theta_i) \geq -c_1 \sum_{i=0}^{k} \frac{p}{p+r_i} \geq -pc_1 \sum_{i=0}^{k} \frac{1}{r_i}$$

$$\geq -\frac{pc_1}{r_0} \sum_{i=0}^{k} \left(\frac{2}{s}\right)^i \geq -\ln \frac{1}{1-\delta'} \qquad \text{for all } k \geq 0$$

as a consequence of (3.29). Therefore,

$$\prod_{i=0}^{k} \theta_i \geq 1 - \delta' \qquad \text{for all } k \geq 0,$$

so that (3.41) combined with (3.26) and the fact that $t_0 > 1$ guarantees that

$$M_0^{\prod_{i=0}^{k} \theta_i}(t) \leq \left(c_6 t^{-\frac{q-q_0}{q}\nu}\right)^{1-\delta'} \leq c_6^{1-\delta'} t^{-\nu+\delta} \qquad \text{for all } t \geq t_0.$$

In conjunction with (3.40) this establishes (3.24) upon an evident choice of $C(\delta)$. $\qquad\square$

In the limit $R \nearrow \infty$, this proves the estimate (1.8) in Theorem 1.2.

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Corollary 3.8 *Assume that $p \geq 1$, and that $u_0 \in C^0(\mathbb{R}^n)$ is positive fulfilling $u_0 \in L^{q_0}(\mathbb{R}^n)$ for some $q_0 > 0$. Then for any $\delta > 0$ one can find $C(\delta) > 0$ such that the solution u of (1.5) obtained in Proposition 1.1 satisfies*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\delta) t^{-\nu + \delta} \qquad \text{for all } t \geq 1.$$

PROOF.   The claim immediately results from Lemma 3.7 upon taking $R \nearrow \infty$.   □

On furthermore taking $q_0 \searrow 0$ herein, we can also deduce the estimate (1.9) on fast decay in $L^\infty(\mathbb{R}^n)$ for rapidly decreasing initial data.

Corollary 3.9 *Let $p \geq 1$ and $u_0 \in C^0(\mathbb{R}^n)$ be positive and such that $u_0 \in L^q(\mathbb{R}^n)$ for all $q > 0$. Then for any $\delta > 0$ there exists $C(\delta) > 0$ such that for the solution u of (1.5) from Proposition 1.1 we have*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C(\delta) t^{-\frac{1}{p} + \delta} \qquad \text{for all } t \geq 1. \tag{3.42}$$

PROOF.   Given $\delta > 0$, we fix $q_0 > 0$ small enough such that $-\nu < -\frac{1}{p} + \delta$, and then choose $\delta' > 0$ fulfilling $-\nu + \delta' \leq -\frac{1}{p} + \delta$. An application of Corollary 3.8 then yields (3.42).   □

Summarizing, we can thereby complete the proof of Theorem 1.2.

PROOF of Theorem 1.2.   We only need to collect the results provided by Corollary 3.3, Corollary 3.8 and Corollary 3.9.   □

4   Estimates from below. Optimality

In order to derive lower estimates for solutions of (1.5), given any such solution u, for $x \in \mathbb{R}^n$ and $t \geq 0$ we introduce

$$v(x, \tau) := (t + 1)^{\frac{1}{p}} u(x, t), \qquad \tau = \ln(t + 1), \tag{4.1}$$

whence, as can easily be verified, v is a positive classical solution of

$$\begin{cases} v_\tau = v^p \Delta v + \frac{1}{p} v, & x \in \mathbb{R}^n, \ t > 0, \\ v(x, 0) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

The following result can be obtained by a comparison argument from below with functions of the form $\underline{v}(x, \tau) := y_R(\tau)(\Theta_R(x) + \delta_R)$, where Θ_R denotes the principal eigenfunction of the Dirichlet Laplacian in B_R, and where the number $\delta_R > 0$ and the positive function y_R are chosen appropriately (cf. [22, Lemma 2.1]).

Lemma 4.1 *Let $p \geq 1$, and suppose that $u_0 \in C^0(\mathbb{R}^n)$ is positive and that u is a globally defined positive classical solution of (1.5). Then for every $R > 0$, the function v defined by (4.1) satisfies*

$$\inf_{x \in B_R} v(x, t) \to +\infty \qquad \text{as } t \to \infty.$$

PROOF of Proposition 1.3.   The claimed divergence property is an immediate consequence of Lemma 4.1 and (4.1).   □

4.1 Initial data with algebraic decay

Let us next focus on the particular case when the initial data essentially decay algebraically in space. The key step in our analysis of the corresponding solutions of (1.5) will consist in a comparison argument involving separated solutions of the PDE in (1.5) in suitably chosen balls. The spatial profiles arising therein are addressed in the following lemma asserting a favorable scaling property.

Lemma 4.2 *Let $p \geq 1$. For $R > 0$, let $w_R \in C^0(\bar{B}_R) \cap C^2(B_R)$ denote the positive solution of*

$$\begin{cases} -\Delta w_R = \frac{1}{p} w_R^{1-p}, & x \in B_R, \\ w_R = 0, & x \in \partial B_R. \end{cases} \tag{4.3}$$

Then for every $R > 0$ we have

$$w_R(x) = R^{\frac{2}{p}} w_1\left(\frac{x}{R}\right) \qquad \text{for all } x \in B_R. \tag{4.4}$$

PROOF. Since

$$\begin{aligned} \Delta\left\{R^{\frac{2}{p}} w_1\left(\frac{x}{R}\right)\right\} + \frac{1}{p}\left\{R^{\frac{2}{p}} w_1\left(\frac{x}{R}\right)\right\} &= R^{\frac{2}{p}-2}(\Delta w_1)\left(\frac{x}{R}\right) + \frac{1}{p} R^{\frac{2(1-p)}{p}} w_1^{1-p}\left(\frac{x}{R}\right) \\ &= R^{\frac{2}{p}-2}\left\{\Delta w_1 + \frac{1}{p} w_1^{1-p}\right\}\left(\frac{x}{R}\right) \qquad \text{for all } x \in B_R, \end{aligned}$$

it results from the properties defining w_1 that $B_R \ni x \mapsto R^{\frac{2}{p}} w_1(\frac{x}{R})$ solves (4.3). By uniqueness of positive solutions to (4.3) ([20]), (4.4) thus follows. $\square$

Now by an appropriate comparison argument from below we shall achieve the following quantitative implication of an algebraic lower decay estimate on u_0 for the size of v in balls with a certain time-dependent radius.

Lemma 4.3 *Let $p \geq 1$, and suppose that $u_0 \in C^0(\mathbb{R}^n)$ is such that there exist $\gamma > 0$ and $c_0 > 0$ fulfilling*

$$u_0(x) \geq C_0(1 + |x|)^{-\gamma} \qquad \text{for all } x \in \mathbb{R}^n. \tag{4.5}$$

Then there exists $C > 0$ such that for v as given by (4.1) we have

$$v(x, \tau) \geq C w_{R(\tau)}(x) \qquad \text{for all } x \in B_{R(\tau)} \text{ and } \tau > 0, \tag{4.6}$$

where $w_{R(\tau)}$ denotes the positive solution of (4.3) corresponding to $R(\tau) := e^{\tau/(p\gamma+2)}$.

PROOF. We fix an arbitrary $\tau_0 > 0$ and let

$$R := e^{\frac{\tau_0}{p\gamma+2}}. \tag{4.7}$$

Then writing $c_1 := \|w_1\|_{L^\infty(B_1)}$, we set

$$\delta := \frac{C_0}{2^\gamma c_1} R^{-\gamma-\frac{2}{p}} \tag{4.8}$$

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and define $y \in C^1([0, \infty))$ to be the solution of

$$\begin{cases} y'(\tau) = \frac{1}{p}y(\tau) - \frac{1}{p}y^{p+1}(\tau), & \tau > 0, \\ y(0) = \delta, \end{cases} \tag{4.9}$$

that is, we let

$$y(\tau) := \left\{ \delta^{-p}e^{-\tau} + 1 - e^{-\tau} \right\}^{-\frac{1}{p}} \qquad \text{for } \tau \ge 0. \tag{4.10}$$

We finally introduce

$$\underline{v}(x, \tau) := y(\tau)w_R(x), \qquad x \in \bar{B}_R, \ \tau \ge 0,$$

and first observe that clearly $\underline{v} = 0 < v$ on $\partial B_R \times (0, \infty)$. Moreover, since from Lemma 4.2 we know that

$$w_R(x) \le R^{\frac{2}{p}}\|w_1\|_{L^\infty(B_1)} = c_1 R^{\frac{2}{p}} \qquad \text{for all } x \in B_R,$$

and since (4.5) along with the fact that $R \ge 1$ implies that

$$u_0(x) \ge C_0(1 + |x|)^{-\gamma} \ge 2^{-\gamma}C_0 R^{-\gamma} \qquad \text{for all } x \in B_R,$$

thanks to (4.8) we obtain that

$$\frac{\underline{v}(x, 0)}{v(x, 0)} = \frac{\delta w_R(x)}{u_0(x)} \le \frac{2^\gamma c_1}{C_0} R^{\gamma + \frac{2}{p}}\delta = 1 \qquad \text{for all } x \in B_R.$$

As (4.9) entails that furthermore

$$\underline{v}_t - \underline{v}^p\Delta\underline{v} - \frac{1}{p}\underline{v} = y'w_R - y^{p+1}w_R^p\Delta w_R - \frac{1}{p}yw_R = \left\{ y' + \frac{1}{p}y^{p+1} - \frac{1}{p}y \right\}w_R = 0 \quad \text{in } B_R \times (0, \infty),$$

the comparison principle (cf. [20] for a version adequate for the present purpose) guarantees that

$$\underline{v} \le v \qquad \text{in } B_R \times (0, \infty). \tag{4.11}$$

Since from (4.10) we see that

$$y(\tau) \ge \left\{ \delta^{-p}e^{-\tau} + 1 \right\}^{-\frac{1}{p}} \qquad \text{for all } \tau > 0,$$

and that hence, by (4.7) and (4.8),

$$y(\tau_0) \ge \left\{ \delta^{-p}e^{-\tau_0} + 1 \right\}^{-\frac{1}{p}} = \left\{ \left(\frac{C_0}{2^\gamma c_1}\right)^{-p} R^{p\gamma + 2} R^{-(p\gamma + 2)} + 1 \right\}^{-\frac{1}{p}} = \left\{ \left(\frac{2^\gamma c_1}{C_0}\right)^p + 1 \right\}^{-\frac{1}{p}},$$

evaluating (4.11) at $\tau = \tau_0$ readily establishes (4.6). $\qquad \square$

Properly evaluating this lower estimate we obtain the claimed estimate from below for solutions which initially decay no faster than algebraically.

PROOF of Theorem 1.6 (i). According to Lemma 4.3 and (4.1), (1.14) implies that there exists $c_1 > 0$ such that

$$u(x,t) \geq c_1(t+1)^{-\frac{1}{p}} w_{R(\tau)}(x) \qquad \text{for all } x \in B_{R(\tau)} \text{ and } t > 0, \tag{4.12}$$

where $R(\tau) = e^{\frac{\tau}{p\gamma+2}}$ with $\tau = \ln(t+1)$ and $w_{R(\tau)}$ is taken from (4.3). As for the case $q = \infty$, in light of Lemma 4.2 this immediately implies that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \geq u(0,t) \geq c_1(t+1)^{-\frac{1}{p}} R^{\frac{2}{p}}(\tau) w_1(0) = c_1 w_1(0)(t+1)^{-\frac{\gamma}{p\gamma+2}} \qquad \text{for all } t \geq 0$$

and thereby proves (1.15) in this particular situation.

For finite q, an integration of (4.12) using the substitution $y = \frac{x}{R}$ shows that

$$
\begin{aligned}
\int_{\mathbb{R}^n} u^q(x,t)dx &\geq c_1^q(t+1)^{-\frac{q}{p}} \int_{B_{R(\tau)}} w_{R(\tau)}^q(x)dx \\
&= c_1^q(t+1)^{-\frac{q}{p}} \int_{B_{R(\tau)}} \left\{ R^{\frac{2}{p}}(\tau) w_1\left(\frac{x}{R}\right) \right\}^q dx \\
&= c_1^q(t+1)^{-\frac{q}{p}} R^{\frac{2q}{p}+n}(\tau) \int_{B_1} w_1(y)dy \\
&= c_1^q(t+1)^{-\frac{q\gamma-n}{p\gamma+2}} \qquad \text{for all } t > 0.
\end{aligned}
$$

Taking the q-th root here we readily arrive at (1.15) also in this case. $\qquad \square$

5 Optimality of L^q decay estimates. Proof of Theorem 1.4

Another application of Theorem 1.6 (i) to suitably chosen γ finally shows optimality of the estimates (1.7) and (1.8) as formulated in Theorem 1.4.

PROOF of Theorem 1.4. Since

$$\frac{q\gamma - n}{q(p\gamma+2)} \quad \rightarrow \quad \frac{n(q-q_0)}{q(np+2q_0)} \qquad \text{as } \gamma \searrow \frac{n}{q_0},$$

we see that given $q_0 > 0$, $q \in (q_0, \infty]$ and $\delta > 0$ we can fix $\gamma > n/q_0$ such that

$$\frac{q\gamma - n}{q(p\gamma+2)} < \gamma(q_0,q) + \delta.$$

We then choose any positive function $u_0 \in C^0(\mathbb{R}^n)$ with the property that

$$c_1(1+|x|)^{-\gamma} \leq u_0(x) \leq c_2(1+|x|)^{-\gamma} \qquad \text{for all } x \in \mathbb{R}^n \tag{5.1}$$

with certain positive constants c_1 and c_2. Since $\gamma > \frac{n}{q_0}$, the right inequality herein ensures that $u_0 \in L^{q_0}(\mathbb{R}^n)$, whereas the left allows for an application of Theorem 1.6 (i) which provides $c_3 > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^q(\Omega)} \geq c_3 t^{-\frac{\gamma-\frac{n}{q}}{p\gamma+2}} \qquad \text{for all } t > 1.$$

Thanks to (5.1), this proves (1.10). $\qquad \square$

6 A family of self-similar solutions of (1.5) with algebraic decay

In the following, given $p \geq 1, \alpha > 0, \beta > 0$ and $A > 0$ we let $f \equiv f_A \equiv f_A^{(\alpha,\beta)}$ denote the solution of the initial value problem (1.12) which we suppose to be defined as a positive classical solution in $[0, \xi_0)$ with maximally chosen $\xi_0 \in (0, \infty]$.

Lemma 6.1 *Let $p \geq 1$, $\alpha > 0$ and $\beta > 0$. Then for any choice of $A > 0$, f_A exists on all of $[0, \infty)$ and satisfies*

$$f_A(\xi) > 0 \quad and \quad f_A'(\xi) \leq 0 \qquad for \ all \ \xi \geq 0. \tag{6.1}$$

PROOF. Since $\alpha > 0$, it is clear that $f := f_A$ cannot attain a positive local minimum on $(0, \infty)$ and hence satisfies $f > 0$ as well as $f' \leq 0$ on $[0, \xi_0)$, whence we only need to show that $\xi_0 = \infty$. Thus assuming for contradiction that ξ_0 be finite, we accordingly obtain that $f(\xi) \searrow 0$ as $\xi \nearrow \xi_0$, and for $\xi \in (0, \xi_0)$ we may divide the ODE in (1.12) by $\xi f(\xi)$ to see that

$$\beta(\ln f)' = -\frac{\alpha}{\xi} - \xi^{-n} f^{p-1}(\xi^{n-1} f')' \qquad \text{for all } \xi \in (0, \xi_0),$$

and that hence with $\xi_1 := \xi_0/2$ we have

$$\beta \ln f(\xi) = \beta \ln f(\xi_1) - \alpha \ln \frac{\xi}{\xi_1} - \int_{\xi_1}^{\xi} \sigma^{-n} f^{p-1}(\sigma)(\sigma^{n-1} f'(\sigma))' d\sigma \qquad \text{for all } \xi \in (\xi_1, \xi_0). \tag{6.2}$$

Here an integration by parts shows that

$$-\int_{\xi_1}^{\xi} \sigma^{-n} f^{p-1}(\sigma)(\sigma^{n-1} f'(\sigma))' d\sigma = (p-1) \int_{\xi_1}^{\xi} \frac{1}{\sigma} f^{p-2}(\sigma) f'^2(\sigma) d\sigma - n \int_{\xi_1}^{\xi} \frac{1}{\sigma^2} f^{p-1}(\sigma) f'(\sigma) d\sigma$$

$$-\frac{1}{\xi} f^{p-1}(\xi) f'(\xi) + \frac{1}{\xi_1} f^{p-1}(\xi_1) f'(\xi_1) \geq -n \int_{\xi_1}^{\xi} \frac{1}{\sigma^2} f^{p-1}(\sigma) f'(\sigma) d\sigma - c_1 \qquad \text{for all } \xi \in (\xi_1, \xi_0) \tag{6.3}$$

with $c_1 := f^{p-1}(\xi_1) f'(\xi_1)/\xi_1 \geq 0$, because $p \geq 1$ and $f' \leq 0$ on $(0, \xi_0)$. Once more integrating by parts, since $f(\xi) \to 0$ as $\xi \nearrow \xi_0$ we obtain that

$$n \int_{\xi_1}^{\xi} \frac{1}{\sigma^2} f^{p-1}(\sigma) f'(\sigma) d\sigma = \frac{n}{p} \int_{\xi_1}^{\xi} \frac{1}{\sigma^2} (f^p)'(\sigma) d\sigma = \frac{2n}{p} \int_{\xi_1}^{\xi} \frac{1}{\sigma^3} f^p(\sigma) d\sigma + \frac{n}{p\xi^2} f^p(\xi) - \frac{n}{p\xi_1^2} f^p(\xi_1)$$

$$\leq \frac{2n}{p} \int_{\xi_1}^{\xi} \frac{1}{\sigma^3} f^p(\sigma) d\sigma + \frac{n}{p\xi^2} f^p(\xi) \quad \to \quad \frac{2n}{p} \int_{\xi_1}^{\xi_0} \frac{1}{\sigma^3} f^p(\sigma) d\sigma =: c_2 \qquad \text{as } \xi \nearrow \xi_0,$$

where the fact that $\xi_1 \in (0, \xi_0)$ clearly asserts that c_2 is finite. From (6.2) and (6.3) we therefore conclude that

$$\liminf_{\xi \nearrow \xi_0} \left\{ \beta \ln f(\xi) \right\} \geq \beta \ln f(\xi_1) - \alpha \ln \frac{\xi_0}{\xi_1} - c_1 - c_2,$$

which together with our assumption $\beta > 0$ contradicts the hypothesis that $f(\xi) \to 0$ as $\xi \nearrow \xi_0$ and thereby proves that actually $\xi_0 = \infty$, as claimed. $\square$

In the proofs of Lemma 6.4 and Lemma 6.5 we shall use the following:

Lemma 6.2 *Let $p > 1$, $\alpha > 0$, $\beta > 0$ and $A > 0$. Then the solution f_A of (1.12) satisfies*

$$\xi^{n-1}f'(\xi) + \frac{\beta}{p-1}\left(n + \frac{(p-1)\alpha}{\beta}\right)\int_0^\xi \sigma^{n-1}f^{1-p}(\sigma)d\sigma - \frac{\beta}{p-1}\xi^n f^{1-p}(\xi) = 0 \qquad \text{for all } \xi > 0. \quad (6.4)$$

PROOF.   We again let $f := f_A$ and rewrite (1.12) in the form

$$0 = (\xi^{n-1}f')' + \beta\xi^n f^{-p}f' + \alpha\xi^{n-1}f^{1-p} = (\xi^{n-1}f')' - \frac{\beta}{p-1}\xi^{n-1}\left\{\xi(f^{1-p})' - \frac{(p-1)\alpha}{\beta}f^{1-p}\right\},$$

whence computing

$$\xi^{\frac{(p-1)\alpha}{\beta}+1}\left(\xi^{-\frac{(p-1)\alpha}{\beta}}f^{1-p}\right)' = \xi(f^{1-p})' - \frac{(p-1)\alpha}{\beta}f^{1-p},$$

we obtain that

$$0 = (\xi^{n-1}f')' - \frac{\beta}{p-1}\xi^{n+\frac{(p-1)\alpha}{\beta}}\left(\xi^{-\frac{(p-1)\alpha}{\beta}}f^{1-p}\right)' \qquad \text{for all } \xi > 0.$$

Using that $f'(0) = 0$, by integration we achieve the identity

$$0 = \xi^{n-1}f'(\xi) - \frac{\beta}{p-1}\int_0^\xi \sigma^{n+\frac{(p-1)\alpha}{\beta}}\left(\sigma^{-\frac{(p-1)\alpha}{\beta}}f^{1-p}\right)'(\sigma)d\sigma \qquad \text{for all } \xi > 0, \quad (6.5)$$

where integrating by parts we find that

$$-\frac{\beta}{p-1}\int_0^\xi \sigma^{n+\frac{(p-1)\alpha}{\beta}}\left(\sigma^{-\frac{(p-1)\alpha}{\beta}}f^{1-p}\right)'(\sigma) = \frac{\beta}{p-1}\left(n + \frac{(p-1)\alpha}{\beta}\right)\int_0^\xi \sigma^{n-1}f^{1-p}(\sigma)d\sigma$$
$$-\frac{\beta}{p-1}\xi^n f^{1-p}(\xi) \qquad \text{for all } \xi > 0,$$

so that (6.4) is equivalent to (6.5).   □

6.1   A lower bound for $f_A^{(\alpha,\beta)}$

A preliminary estimate for f_A from below near $\xi = 0$ can be obtained by quite straightforward manipulations of (1.12).

Lemma 6.3 *Let $p \geq 1$, $\alpha > 0$ and $\beta > 0$. Then there exist $\xi_\star > 0$ and $K > 0$ such that for any $A > 0$, the solution f_A of (1.12) satisfies*

$$f_A(\xi) \geq \frac{A}{2} \qquad \text{for all } \xi \in [0, \xi_\star], \quad (6.6)$$

and that moreover

$$g_A(\xi) := \int_0^\xi \sigma^{n-1}f_A^{1-p}(\sigma)d\sigma, \qquad \xi \geq 0, \ A > 0, \quad (6.7)$$

has the property that

$$\xi_\star^{-n-\frac{(p-1)\alpha}{\beta}}g_A(\xi_\star) \leq KA^{1-p} \qquad \text{for all } A > 0. \quad (6.8)$$

PROOF.   Using that $f := f_A$ is nonincreasing by Lemma 6.1, from (1.12) we obtain that

$$f'' = -\frac{n-1}{\xi}f' - \beta\xi f' - \alpha f \geq -\alpha f \qquad \text{for all } \xi > 0,$$

and that hence $\frac{1}{2}(f'^2)' \geq -\frac{\alpha}{2}(f^2)'$ on $(0, \infty)$. On integration, this yields

$$\frac{1}{2}f'^2(\xi) \geq -\frac{\alpha}{2}f^2(\xi) + \frac{\alpha}{2}A^2 \qquad \text{for all } \xi > 0$$

and thus

$$f'(\xi) \geq -\sqrt{\alpha}\sqrt{A^2 - f^2(\xi)} \qquad \text{for all } \xi > 0.$$

By an ODE comparison, we thereby conclude that

$$f(\xi) \geq A\cos(\sqrt{\alpha}\xi) \qquad \text{for all } \xi \in \left(0, \frac{\pi}{2\sqrt{\alpha}}\right),$$

which means that if we let $\xi_\star := \frac{\pi}{3\sqrt{\alpha}}$, then

$$f(\xi) \geq A\cos\frac{\pi}{3} = \frac{A}{2} \qquad \text{for all } \xi \in (0, \xi_\star).$$

This establishes (6.6), and since $p \geq 1$ this moreover entails that

$$\int_0^{\xi_\star} \sigma^{n-1} f^{1-p}(\sigma)d\sigma \leq \frac{2^{p-1}\xi_\star^n}{n}A^{1-p},$$

which readily proves (6.8). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Making use of this information, on the basis of (6.4) we can make sure that in the case $p > 1$, a lower bound for the asymptotic behaviour of f_A is determined by the decay of positive solutions to the first-order equation $\beta\xi f' + \alpha f = 0$:

Lemma 6.4 *Let $p > 1$, and suppose that $\alpha > 0$ and $\beta > 0$. Then there exists $L > 0$ such that for every $A > 0$, the solution f_A of (1.12) satisfies*

$$f_A(\xi) \geq LA(1+\xi)^{-\frac{\alpha}{\beta}} \qquad \text{for all } \xi \geq 0. \tag{6.9}$$

PROOF.   Reformulating (6.4) in terms of $g(\xi) := \int_0^\xi \sigma^{n-1} f^{1-p}(\sigma)d\sigma$, $\xi \geq 0$, since $f' \leq 0$ on $[0, \infty)$ we obtain that

$$\left(n + \frac{(p-1)\alpha}{\beta}\right)g(\xi) - \xi^n f^{1-p}(\xi) \geq 0 \qquad \text{for all } \xi > 0 \tag{6.10}$$

and that hence

$$\xi g'(\xi) \leq \left(n + \frac{(p-1)\alpha}{\beta}\right)g(\xi) \qquad \text{for all } \xi > 0.$$

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By e.g. a comparison argument, from this we infer that with $\xi_\star > 0$ and $K > 0$ as provided by Lemma 6.3 we have

$$g(\xi) \leq g(\xi_\star)\left(\frac{\xi}{\xi_\star}\right)^{n+\frac{(p-1)\alpha}{\beta}} \leq KA^{1-p}\xi^{n+\frac{(p-1)\alpha}{\beta}} \qquad \text{for all } \xi > \xi_\star.$$

Going back to (6.10), we see that this yields information on f itself by implying the inequality

$$\xi^n f^{1-p}(\xi) \leq \left(n + \frac{(p-1)\alpha}{\beta}\right)g(\xi) \leq \left(n + \frac{(p-1)\alpha}{\beta}\right)KA^{1-p}\xi^{n+(p-1)\alpha/\beta} \qquad \text{for all } \xi > \xi_\star,$$

which is equivalent to

$$f(\xi) \geq \left(n + \frac{(p-1)\alpha}{\beta}\right)^{-\frac{1}{p-1}}K^{-\frac{1}{p-1}}\xi^{-\frac{\alpha}{\beta}} \qquad \text{for all } \xi > \xi_\star.$$

This shows that if $L > 0$ is sufficiently small then the estimate in (6.9) holds for all $\xi > \xi_\star$, whereas (6.6) guarantees that on diminishing L if necessary we can achieve the desired lower bound also for $\xi \in [0, \xi_\star]$. $\qquad \square$

6.2 An upper bound for $f_A^{(\alpha,\beta)}$

Again by means of (6.4), we can complete our description of the asymptotic decay of f_A for $p > 1$ as follows.

Lemma 6.5 *Let $p > 1, \alpha > 0, \beta > 0$ and $A > 0$. Then there exists $C > 0$ with the property that the solution f_A of (1.12) satisfies*

$$f_A(\xi) \leq C(1+\xi)^{-\frac{\alpha}{\beta}} \qquad \text{for all } \xi \geq 0. \tag{6.11}$$

PROOF.   Again abbreviating $f := f_A$ and $g(\xi) := \int_0^\xi \sigma^{n-1}f^{1-p}(\sigma)d\sigma$ for $\xi \geq 0$, from (6.4) we obtain

$$c_1\xi^{n-1}f'(\xi) + \left(n + \frac{(p-1)\alpha}{\beta}\right)g(\xi) - \xi g'(\xi) = 0 \qquad \text{for all } \xi > 0$$

with $c_1 := \frac{p-1}{\beta}$, that is,

$$g'(\xi) = \left(n + \frac{(p-1)\alpha}{\beta}\right)\frac{g(\xi)}{\xi} + c_1\xi^{n-2}f'(\xi) \qquad \text{for all } \xi > 0. \tag{6.12}$$

In order to prepare an integration thereof, let us first make sure that if we define

$$\xi_1 := \sqrt{2nc_1A^p}, \tag{6.13}$$

then

$$c_1\xi_1^{-2}f(\xi_1) \leq \frac{1}{2}\xi_1^{-n}g(\xi_1). \tag{6.14}$$

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Indeed, this follows from the observation that using $f(\sigma) \leq A$ for all $\sigma \geq 0$ by Lemma 6.1, we can estimate

$$2c_1\xi_1^{n-2}\frac{f(\xi_1)}{g(\xi_1)} = 2c_1\xi_1^{n-2}\frac{f(\xi_1)}{\int_0^{\xi_1}\sigma^{n-1}f^{1-p}(\sigma)d\sigma} \leq 2c_1\xi_1^{n-2}\frac{A}{A^{1-p}\int_0^{\xi_1}\sigma^{n-1}d\sigma} = \frac{2nc_1A^p}{\xi_1^2} = 1$$

thanks to (6.13).

We now invoke the variation-of-constants formula associated with the initial value problem for (6.12) starting from $\xi = \xi_1$ to see that

$$
\begin{aligned}
g(\xi) &= g(\xi_1)e^{\left(n+\frac{(p-1)\alpha}{\beta}\right)\int_{\xi_1}^{\xi}\frac{d\sigma}{\sigma}} + c_1\int_{\xi_1}^{\xi}e^{\left(n+\frac{(p-1)\alpha}{\beta}\right)\int_{\sigma}^{\xi}\frac{d\tau}{\tau}}\sigma^{n-2}f'(\sigma)d\sigma \\
&= g(\xi_1)\left(\frac{\xi}{\xi_1}\right)^{n+\frac{(p-1)\alpha}{\beta}} + c_1\xi^{n+\frac{(p-1)\alpha}{\beta}}\int_{\xi_1}^{\xi}\sigma^{-\frac{(p-1)\alpha}{\beta}-2}f'(\sigma)d\sigma \qquad \text{for all } \xi > \xi_1, \qquad (6.15)
\end{aligned}
$$

where an integration by parts shows that

$$
c_1\xi^{n+\frac{(p-1)\alpha}{\beta}}\int_{\xi_1}^{\xi}\sigma^{-\frac{(p-1)\alpha}{\beta}-2}f'(\sigma)d\sigma = \left(\frac{(p-1)\alpha}{\beta}+2\right)c_1\xi^{n+\frac{(p-1)\alpha}{\beta}}\int_{\xi_1}^{\xi}\sigma^{-\frac{(p+1)\alpha}{\beta}-3}f(\sigma)d\sigma
$$

$$
+ c_1\xi^{n-2}f(\xi) - c_1\xi^{n+\frac{(p-1)\alpha}{\beta}}\xi_1^{-\frac{(p-1)\alpha}{\beta}-2}f(\xi_1) \geq -c_1\xi^{n+\frac{(p-1)\alpha}{\beta}}\xi_1^{-\frac{(p-1)\alpha}{\beta}-2}f(\xi_1) \qquad \text{for all } \xi > \xi_1,
$$

because $p \geq 1$ and f is nonnegative. Using (6.14), from (6.15) we thus infer that

$$
g(\xi) \geq \left\{\xi_1^{-n}g(\xi_1) - c_1\xi_1^{-2}f(\xi_1)\right\}\xi_1^{-\frac{(p-1)\alpha}{\beta}}\xi^{n+\frac{(p-1)\alpha}{\beta}} \geq c_2\xi^{n+\frac{(p-1)\alpha}{\beta}} \qquad \text{for all } \xi > \xi_1 \qquad (6.16)
$$

with $c_2 := \frac{1}{2}\xi_1^{-n-(p-1)\alpha/\beta}$.

In order to derive (6.11) from this, we once again make use of the downward monotonicity of f in estimating

$$
g(\xi) = \int_0^{\xi}\sigma^{n-1}f^{1-p}(\sigma)d\sigma \leq \int_0^{\xi}\sigma^{n-1}f^{1-p}(\xi)d\sigma = \frac{\xi^n}{n}f^{1-p}(\xi) \qquad \text{for all } \xi > 0,
$$

which combined with (6.16) shows that

$$
f(\xi) \leq \left\{\frac{n}{\xi^n}c_2\xi^{n+\frac{(p-1)\alpha}{\beta}}\right\}^{-\frac{1}{p-1}} = (nc_2)^{-\frac{1}{p-1}}\xi^{-\frac{\alpha}{\beta}} \qquad \text{for all } \xi > \xi_1.
$$

Along with the boundedness of f on $[0, \xi_1]$, this proves (6.11). $\qquad\square$

6.3 Implications on the large time behaviour for solutions of (1.5) with algebraic spatial decay

Let us underline the outcome of the above construction:

Proposition 6.6 *Let $p > 1$ and $\alpha \in (0, \frac{1}{p})$, and set $\beta := \frac{1-p\alpha}{2}$. Then for any $A > 0$,*

$$u_A^{(\alpha)}(x,t) := t^{-\alpha} f_A^{(\alpha,\beta)}\left(t^{-\beta}|x|\right), \qquad x \in \mathbb{R}^n, \ t > 0,$$

defines a positive classical solution of the equation $u_t = u^p \Delta u$ in $\mathbb{R}^n \times (0, \infty)$.

PROOF. Writing $u := u_A^{(\alpha)}$ and $f := f_A^{(\alpha,\beta)}$ and abbreviating $\xi := t^{-\beta}|x|$ for $x \in \mathbb{R}^n$ and $t > 0$, we only need to compute $u_t(x,t) = -\alpha t^{-\alpha-1} f(\xi) - \beta t^{-\alpha-1}\xi f'(\xi)$ and

$$u^p(x,t)\Delta u(x,t) = t^{-(p+1)\alpha-2\beta} f^p(\xi)\left(f''(\xi) + \frac{n-1}{\xi}f'(\xi)\right)$$

to see that by the choice of β and (1.12) we have

$$u_t - u^p \Delta u = t^{-\alpha-1}\left\{ -\alpha f'(\xi) - \beta \xi f'(\xi) - f^p(\xi)\left(f''(\xi) + \frac{n-1}{\xi}f'(\xi)\right)\right\} = 0.$$

$\square$

PROOF of Theorem 1.5.   We only need to collect Proposition 6.6, Lemma 6.4 and Lemma 6.5. $\square$

We can thereby complete the proof of Theorem 1.6.

PROOF of Theorem 1.6 (ii).   Given $\gamma > 0$, we let $\alpha := \frac{\gamma}{p\gamma+2}$, so that clearly $\alpha \in (0, \frac{1}{p})$. Moreover, specifying $\beta := \frac{1-p\alpha}{2} > 0$, we then take $L > 0$ as provided by Lemma 6.4 and fix $A > 0$ suitably large fulfilling

$$LA \geq C_1. \tag{6.17}$$

Then with $u_A^{(\alpha)}$ taken from Proposition 6.6, the function $\overline{u}$ defined by

$$\overline{u}(x,t) := u_A^{(\alpha)}(x, t+1) \equiv (t+1)^{-\alpha} f_A^{(\alpha,\beta)}\left((t+1)^{-\beta}|x|\right), \qquad x \in \mathbb{R}^n, \ t \geq 0, \tag{6.18}$$

evidently satisfies $\overline{u}_t = \overline{u}^p \Delta \overline{u}$ in $\mathbb{R}^n \times (0, \infty)$ and

$$\overline{u}(x,0) = f_A^{(\alpha,\beta)}(|x|) \geq LA(1+|x|)^{-\frac{\alpha}{\beta}} \qquad \text{for all } x \in \mathbb{R}^n$$

by Lemma 6.4. Since according to our definitions of β and α we have $\frac{\alpha}{\beta} = \gamma$, in view of (6.17) and (1.16) this shows that $\overline{u}(x,0) \geq C_1(1+|x|)^{-\gamma} \geq u_0(x)$ for all $x \in \mathbb{R}^n$. By means of a comparison argument, we thus conclude that for any $R > 0$, the solution u_R of (2.1) satisfies $u_R \leq \overline{u}$ in $B_R \times [0, \infty)$, and that hence

$$u(x,t) \leq \overline{u}(x,t) \qquad \text{for all } x \in \mathbb{R}^n \text{ and } t \geq 0, \tag{6.19}$$

which by (6.18) and Lemma 6.1 in particular yields that

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} \leq \|\overline{u}(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = A(t+1)^{-\alpha} \qquad \text{for all } t \geq 0.$$

Again using the definition of α, we thereby immediately obtain (1.17) for $q = \infty$. Apart from that, once more writing $f := f_A^{(\alpha,\beta)}$ we see that (6.19) implies that whenever $q \in (\frac{n}{\gamma}, \infty)$, then

$$\begin{aligned}
\int_{\mathbb{R}^n} u^q(x,t)dx &\leq \int_{\mathbb{R}^n} \overline{u}^q(x,t)dx = (t+1)^{-q\alpha}\int_{\mathbb{R}^n} f^q\left((t+1)^{-\beta}|x|\right)dx \\
&= (t+1)^{-q\alpha}(t+1)^{n\beta}\int_{\mathbb{R}^n} f^q(|y|)dy \qquad \text{for all } t \geq 0. \tag{6.20}
\end{aligned}$$

Here, applying Lemma 6.5 we find $C_2 > 0$ such that $f(\xi) \leq C_2 \xi^{-\gamma}$ for all $\xi \geq 1$, so that our assumption $q > \frac{n}{\gamma}$ asserts that

$$\int_1^\infty \xi^{n-1} f^q(\xi) d\xi \leq C_2^q \int_1^\infty \xi^{n-1-q\gamma} d\xi = \frac{C_2^q}{n - q\gamma}$$

is finite. As furthermore, again by definition of β and α,

$$q\alpha - n\beta = q\alpha - \frac{n(1 - p\alpha)}{2} = \frac{q\gamma}{p\gamma + 2} - \frac{n}{2}\Big(1 - \frac{p\gamma}{p\gamma + 2}\Big) = \frac{q\gamma - n}{p\gamma + 2},$$

from (6.20) we thus infer the existence of $C_3 > 0$ such that

$$\int_{\mathbb{R}^n} u^q(x,t) dx \leq C_3 (t+1)^{-\frac{q\gamma - n}{p\gamma + 2}} \qquad \text{for all } t \geq 0,$$

which for these finite values of q is equivalent to (1.17). $\qquad \square$

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