A generalized solution concept for the Keller-Segel system with logarithmic sensitivity: Global solvability for large nonradial data

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Abstract

The chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (\frac{u}{v} \nabla v), \\ v_t = \Delta v - v + u, \end{cases}$$

is considered in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary, where $\chi > 0$.

An apparently novel type of generalized solution framework is introduced within which an extension of previously known ranges for the key parameter χ with regard to global solvability is achieved. In particular, it is shown that under the hypothesis that

$$\chi < \begin{cases} \infty & \text{if } n = 2, \\ \sqrt{8} & \text{if } n = 3, \\ \frac{n}{n-2} & \text{if } n \geq 4, \end{cases}$$

for all initial data satisfying suitable assumptions on regularity and positivity, an associated no-flux initial-boundary value problem admits a globally defined generalized solution. This solution inter alia has the property that

$$u \in L^1_{loc}(\overline{\Omega} \times [0, \infty)).$$

Key words: chemotaxis; logarithmic sensitivity; global existence; generalized solution
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1 Introduction

We consider the Keller-Segel system with logarithmic sensitivity, as given by the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot \left(\frac{u}{v} \nabla v \right), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \qquad x \in \Omega, \end{cases} \tag{1.1}$$

coupled to the parabolic problem

$$\begin{cases} v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x), \qquad x \in \Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary, χ is a positive parameter and the given initial data u_0 and v_0 satisfy suitable regularity and positivity assumptions.

This system can be viewed as a prototypical parabolic model for self-enhanced chemotactic migration processes in which cross-diffusion occurs in accordance with the Weber-Fechner law of stimulus perception ([9], [15]), and accordingly a considerable literature is concerned with its mathematical analysis. However, up to now it seems yet unclear to which extent the particular mechanism of taxis inhibition at large signal densities in (1.1) is sufficient to prevent phenomena of blow-up, known as the probably most striking qualitative feature of the classical Keller-Segel system: Indeed, in its fully parabolic version, as determined by the choice $\tau := 1$ in

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v), \\ \tau v_t = \Delta v - v + u, \end{cases} \tag{1.3}$$

the latter admits solutions blowing up in finite time for any choice of $\chi > 0$ whenever $n \geq 2$ ([8], [22]), and in the simplified parabolic-elliptic case obtained on choosing $\tau := 0$ it is even known that some radial solutions to an associated Cauchy problem in the whole plane collapse into a persistent Dirac-type singularity in the sense that a globally defined measure-valued solution exists which has a singular part beyond some finite time and asymptotically approaches a Dirac measure (cf. e.g. [19] or also [12]).

As opposed to this, the literature has identified various circumstances under which phenomena of this type are ruled out in (1.1)-(1.2): For instance, when $\chi < \chi_0(n)$ with some $\chi_0(2) > 1.015$ and $\chi_0(n) := \sqrt{\frac{2}{n}}$ for $n \geq 3$, global bounded classical solutions exist for all reasonably regular positive initial data ([11], [2], [4], [24], [13], [21]); in the corresponding parabolic-elliptic analogue, the same conclusion holds with $\chi_0(2) = \infty$ ([5]) and with $\chi_0(n) := \frac{2}{n-2}$ when $n \geq 3$ and the spatial setting is radially symmetric ([14], cf. also [6] for a related result addressing a variant with its second equation being $\tau v_t = \Delta v - v + u$ for small $\tau > 0$), whereas it is known that some exploding solutions exist if $n \geq 3$ and $\chi > \frac{2n}{n-2}$ ([14]). As for larger values of χ in the fully parabolic problem (1.1)-(1.2), in some cases at least certain global generalized solutions can be found which satisfy

$$u \in L^1_{loc}(\overline{\Omega} \times [0, \infty)) \tag{1.4}$$

and thereby indicate the absence of strong singularity formation of the flavor described above. Such constructions are possible in the context of natural weak solution concepts if

$$\chi < \sqrt{\frac{n+2}{3n-4}} \tag{1.5}$$

([21]) and within a slightly more generalized framework if merely

$$\chi < \sqrt{\frac{n}{n-2}} \tag{1.6}$$

but in addition the solutions are supposed to be radially symmetric ([17]). To the best of our knowledge, however, the question how far (1.5) is optimal with respect to the existence of not necessarily radial solutions fulfilling (1.4) is yet unsolved; in particular, it appears to be unknown whether in nonradial planar settings such solutions do exist also beyond the range $\chi < \sqrt{2}$ determined by (1.5).

Main results. The purpose of this work is to design a novel concept of generalized solvability which is yet suitably strong so as to require (1.4), but which on the other hand is mild enough so that it enables us to construct corresponding global solutions without any symmetry hypotheses and under conditions somewhat weaker than (1.5) and actually also than (1.6). More precisely, considering (1.1)-(1.2) under the assumptions that

$$\begin{cases} u_0 \in C^0(\overline{\Omega}) & \text{is such that } u_0 \ge 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \quad \text{and that} \\ v_0 \in W^{1,\infty}(\Omega) & \text{satisfies } v_0 > 0 \text{ in } \overline{\Omega}, \end{cases} \tag{1.7}$$

we can state our main results as follows.

Theorem 1.1. *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\chi > 0$ be such that*

$$\chi < \begin{cases} \infty & \text{if } n = 2, \\ \sqrt{8} & \text{if } n = 3, \\ \frac{n}{n-2} & \text{if } n \ge 4. \end{cases} \tag{1.8}$$

Then for any u_0 and v_0 fulfilling (1.7), the problem (1.1)-(1.2) has at least one global generalized solution (u, v) in the sense of Definition 2.4 below. In particular, this solution satisfies (1.4), and moreover we have

$$\int_\Omega u(\cdot, t) = \int_\Omega u_0 \qquad \text{for a.e. } t > 0. \tag{1.9}$$

Plan of the paper. A first substantial task will be related to the design of a suitable family of approximate versions of (1.1)-(1.2) in which, on the one hand, the crucial nonlinear interaction is regularized in such an effective manner that even global classical solutions exist, but which on the other hand retains, as far as possible, a fundamental dissipative property of (1.1)-(1.2). As is essentially well-known, namely, the functional $\int_\Omega u^p v^q$ enjoys certain quasi-entropy features along trajectories of (1.1)-(1.2), provided that the crucial positive parameter p therein satisfies $p < \frac{1}{\chi^2}$ and $q > 0$ is chosen adequately. After introducing the regularization (3.1) of (1.1)-(1.2) appropriate for our purposes, in Section 4 we will derive a rigorous counterpart of this entropy-like property for the corresponding approximate solutions. The main challenge now consists in taking appropriate advantage of accordingly implied a priori estimates obtained in Sections 5, 6 and 7, which inter alia seem far from sufficient to warrant L^1 bounds for the cross-diffusive flux $\chi\frac{u}{v}\nabla v$ especially in cases when χ is large and hence p needs to be chosen small.

In the preparatory Section 3, we will therefore resort to a solution framework involving certain sublinear powers of u rather than u itself, thus reminiscent of the celebrated concept of renormalized solutions ([3]). This idea has partially been adapted to the present context in [17] already, but in the present work we shall further weaken the requirements on solutions to a considerable extent: Namely, for the crucial first sub-problem (1.1) to be solved we shall only require that the *coupled* quantity u^pv^q, with certain positive p and q, satisfies a parabolic *inequality* associated with (1.1)-(1.2) in a weak form, and that moreover $\int_\Omega u(\cdot,t) \le \int_\Omega u_0$ for a.e. $t>0$; a key observation, to be made in Lemma 2.5, will reveal that if we furthermore assume the component v to fulfill (1.2) in a natural weak sense, then we indeed obtain a concept consistent with that of classical solvability in (1.1)-(1.2) for all suitably smooth functions.

As seen in Section 8 by means of appropriate compactness arguments, the previously gained estimates in fact enable us to construct a global solution within this framework.

2 A concept of generalized solvability

In specifying the subsequently pursued concept of weak solvability, we first require certain products u^pv^q to satisfy an inequality which can be viewed as generalizing a classical supersolution property of this quantity with regard to (1.1)-(1.2).

Definition 2.1. *Let $p\in(0,1)$ and $q\in(0,1)$, and suppose that u and v are measurable functions on $\Omega\times(0,\infty)$ such that $u>0$ and $v>0$ a.e. in $\Omega\times(0,\infty)$, that*

$$u^pv^q \in L^1_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \textit{and} \qquad u^{p+1}v^{q-1}\in L^1_{loc}(\overline{\Omega}\times[0,\infty)), \tag{2.1}$$

and that $\nabla u^{\frac{p}{2}}$ and $\nabla v^{\frac{q}{2}}$ belong to $L^1_{loc}(\Omega\times(0,\infty))$ and are such that

$$v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} \in L^2_{loc}(\overline{\Omega}\times[0,\infty)) \qquad \textit{and} \qquad u^{\frac{p}{2}}\nabla v^{\frac{q}{2}} \in L^2_{loc}(\overline{\Omega}\times[0,\infty)). \tag{2.2}$$

Then (u,v) will be called a global weak (p,q)-supersolution *of (1.1) if*

$$\begin{aligned}
-\int_0^\infty\int_\Omega u^pv^q\varphi_t - \int_\Omega u_0^pv_0^q\varphi(\cdot,0) \\
\ge\ & \frac{4(1-p)q-4q^2-p(1-p)^2\chi^2}{pq(p\chi+1-q)}\int_0^\infty\int_\Omega v^q|\nabla u^{\frac{p}{2}}|^2\varphi \\
&+\frac{4(p\chi+1-q)}{q}\int_0^\infty\int_\Omega \Big|u^{\frac{p}{2}}\nabla v^{\frac{q}{2}} - \frac{(1-p)\chi+2q}{2(p\chi+1-q)}v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\Big|^2\varphi \\
&-\frac{2p\chi}{q}\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^q\nabla u^{\frac{p}{2}}\cdot\nabla\varphi \\
&+\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega u^pv^q\Delta\varphi \\
&-q\int_0^\infty\int_\Omega u^pv^q\varphi + q\int_0^\infty\int_\Omega u^{p+1}v^{q-1}\varphi
\end{aligned} \tag{2.3}$$

for all nonnegative $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$ *such that* $\frac{\partial\varphi}{\partial\nu} = 0$ *on* $\partial\Omega \times (0,\infty)$ *and if moreover*

$$u^p v^q > 0 \qquad a.e. \ on \ \partial\Omega \times (0,\infty). \tag{2.4}$$

Remark 2.2. *(i) Observing that (2.1) in particular ensures that* $u^{\frac{p}{2}} v^{\frac{q}{2}} \in L^2_{loc}(\overline{\Omega} \times [0,\infty))$*, and that hence (2.1) and (2.2) warrant that*

$$u^{\frac{p}{2}} v^q \nabla u^{\frac{p}{2}} = (u^{\frac{p}{2}} v^{\frac{q}{2}}) \cdot (v^{\frac{q}{2}} \nabla u^{\frac{p}{2}}) \in L^1_{loc}(\overline{\Omega} \times [0,\infty))$$

and similarly $u^p v^{\frac{q}{2}} \nabla v^{\frac{q}{2}} \in L^1_{loc}(\overline{\Omega} \times [0,\infty))$*, it follows that under the above requirements all integrals in (2.3) are indeed well-defined.*
(ii) According to (2.1) *and* (2.2)*, for a.e.* $t > 0$*,* $u^{\frac{p}{2}}(\cdot,t) v^{\frac{q}{2}}(\cdot,t) \in W^{1,2}(\Omega)$ *so that* $u^{\frac{p}{2}} v^{\frac{q}{2}}(\cdot,t)\big|_{\partial\Omega} \in L^2(\partial\Omega)$ *exists in the sense of traces, giving meaning to the positivity requirement in* (2.4)*.*

Apart from that, we will require the second problem (1.2) to be satisfied in the following rather natural weak sense.

Definition 2.3. *A pair* (u,v) *of functions*

$$\begin{cases} u \in L^1_{loc}(\overline{\Omega} \times [0,\infty)), \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{2.5}$$

will be named a global weak solution *of (1.2) if*

$$-\int_0^\infty \int_\Omega v\varphi_t - \int_\Omega v_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi - \int_0^\infty \int_\Omega v\varphi + \int_0^\infty \int_\Omega u\varphi \tag{2.6}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$*.*

Following an approach already pursued in [23] in a considerably less involved related context, in order to complete our solution concept we will complement the above two requirements by merely postulating an upper bound for the mass functional $\int_\Omega u$ in terms of $\int_\Omega u_0$:

Definition 2.4. *A couple of nonnegative measurable functions* u *and* v *defined on* $\Omega \times (0,\infty)$ *will be said to be a* global generalized solution *of (1.1)-(1.2) if* (u,v) *is a global weak solution of (1.2) according to Definition 2.3, if there exist* $p \in (0,1)$ *and* $q \in (0,1)$ *such that* (u,v) *is a global weak* (p,q)*-supersolution of (1.1) in the sense of Definition 2.1, and if moreover*

$$\int_\Omega u(\cdot,t) \le \int_\Omega u_0 \qquad for \ a.e. \ t > 0. \tag{2.7}$$

This is indeed consistent with the concept of classical solvability in the following sense.

Lemma 2.5. *Let* $\chi > 0$*, and suppose that* $(u,v) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^2$ *is such that* (u,v) *is a global generalized solution of (1.1)-(1.2) in the sense of Definition 2.4. Then* (u,v) *satisfies (1.1)-(1.2) classically in* $\Omega \times (0,\infty)$*.*

Proof. By means of standard arguments relying on the assumed regularity properties of v, it can easily be verified that v solves (1.2) classically. According to the maximum principle, v hence is strictly positive in $\overline{\Omega}\times[0,\infty)$ and v^{q-1} is uniformly bounded in every set $\Omega\times[0,T)$ for $T\in(0,\infty)$. Positivity of v ensures that by (2.4) $u>0$ on a dense subset of $\partial\Omega\times(0,\infty)$ which moreover is open in $\partial\Omega\times(0,\infty)$ by continuity of u.

For arbitrary $\psi\in C^\infty(\overline{\Omega})$ with $\psi\geq 0$ and $\frac{\partial\psi}{\partial\nu}|_{\partial\Omega}=0$, testing (2.3) by $\varphi(x,t):=\psi(x)(1-\frac{1}{\varepsilon}t)_+$, $\varepsilon\in(0,1)$, which is permissible by Weierstrass' theorem, and invoking Lebesgue's dominated convergence theorem and continuity of $t\mapsto\int_\Omega u^p(\cdot,t)v^q(\cdot,t)$ at $t=0$ in taking $\varepsilon\searrow 0$ we readily achieve

$$\int_\Omega u^p(\cdot,0)v^q(\cdot,0)\psi\geq\int_\Omega u_0^pv_0^q\psi \qquad \text{for all } \psi\in C^\infty(\overline{\Omega}),\psi\geq 0,\frac{\partial\psi}{\partial\nu}|_{\partial\Omega}=0,$$

showing that $u^p(\cdot,0)v^q(\cdot,0)\geq u_0^pv_0^q$ throughout Ω. Because of $v(\cdot,0)=v_0>0$ and the monotonicity of $(\cdot)^{\frac{1}{p}}$ we obtain $u(\cdot,0)\geq u_0$ in Ω and from continuity of u and (2.7) we can conclude that $u(\cdot,0)=u_0$ in Ω.

In the first two integrals on the right of (2.3) straightforward computations yield

$$\begin{aligned}
&\frac{4(1-p)}{p}v^q|\nabla u^{\frac{p}{2}}|^2-\Big(\frac{4(1-p)\chi}{q}+8\Big)u^{\frac{p}{2}}v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\nabla v^{\frac{q}{2}}+\frac{4(p\chi+1-q)}{q}u^p|\nabla v^{\frac{q}{2}}|^2\\
&\quad=\frac{4(p\chi+1-q)}{q}\left\{u^p|\nabla v^{\frac{q}{2}}|^2-\frac{(1-p)\chi+2q}{p\chi+1-q}u^{\frac{p}{2}}v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\nabla v^{\frac{q}{2}}+\frac{(\chi-p\chi+2q)^2}{4(p\chi+1-q)^2}v^q|\nabla u^{\frac{p}{2}}|^2\right\}\\
&\quad\quad+\left\{\frac{4(1-p)}{p}-\frac{((1-p)\chi+2q)^2}{q(p\chi+1-q)}\right\}v^q|\nabla u^{\frac{p}{2}}|^2 \qquad (2.8)\\
&\quad=\frac{4(p\chi+1-q)}{q}\left|u^{\frac{p}{2}}\nabla v^{\frac{q}{2}}-\frac{(1-p)\chi+2q}{2(p\chi+1-q)}v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\right|^2+\frac{4(1-p)q-4q^2-p(1-p)^2\chi^2}{pq(p\chi+1-q)}v^q|\nabla u^{\frac{p}{2}}|^2,
\end{aligned}$$

since

$$\begin{aligned}
\frac{4(1-p)}{p}-\frac{((1-p)\chi+2q)^2}{q(p\chi+1-q)}&=\frac{4(1-p)q(p\chi+1-q)-((1-p)\chi+2q)^2p}{pq(p\chi+1-q)}\\
&=\frac{4(1-p)q-4q^2-p(1-p)^2\chi^2}{pq(p\chi+1-q)}.
\end{aligned}$$

In preparation of the following calculations we also note that for each positive function $w\in C^2(\overline{\Omega})$ and any $r>0$, we have the pointwise identities

$$\begin{aligned}
w^{\frac{r}{2}}\Delta w^{\frac{r}{2}}&=w^{\frac{r}{2}}\nabla\cdot\Big(\frac{r}{2}w^{\frac{r-2}{2}}\nabla w\Big)=w^{\frac{r}{2}}\Big(\frac{r(r-2)}{4}w^{\frac{r-4}{2}}|\nabla w|^2+\frac{r}{2}w^{\frac{r-2}{2}}\Delta w\Big)\\
&=\frac{r(r-2)}{4}w^{r-2}|\nabla w|^2+\frac{r}{2}w^{r-1}\Delta w=\frac{r-2}{r}|\nabla w^{\frac{r}{2}}|^2+\frac{r}{2}w^{r-1}\Delta w \qquad (2.9)
\end{aligned}$$

and

$$\Delta w^r=\nabla\cdot(rw^{r-1}\nabla w)=r(r-1)w^{r-2}|\nabla w|^2+rw^{r-1}\Delta w=\frac{4(r-1)}{r}|\nabla w^{\frac{r}{2}}|^2+rw^{r-1}\Delta w \qquad (2.10)$$

The positivity requirement on w in (2.9) and (2.10) prompts us to perform the following calculations only for test functions φ compactly supported in $\{u>0\}:=\{(x,t)\in\overline{\Omega}\times[0,\infty):\ u(x,t)>0\}$, ensuring strict positivity of u and boundedness of u^{p-1} on $\operatorname{supp}\varphi$.
Accordingly, for all nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times(0,\infty))$ with $\operatorname{supp}\varphi\subset\{u>0\}$ and $\frac{\partial\varphi}{\partial\nu}\big|_{\partial\Omega\times(0,\infty)}=0$, by (2.9) applied to u and p, an integration by parts in the integral in (2.3) containing $\nabla\varphi$ yields

$$\begin{aligned}
-\frac{2p\chi}{q}\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^q\nabla u^{\frac{p}{2}}\cdot\nabla\varphi &= \frac{2p\chi}{q}\int_0^\infty\int_\Omega v^q|\nabla u^{\frac{p}{2}}|^2\varphi+\frac{4p\chi}{q}\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^{\frac{q}{2}}\nabla v^{\frac{q}{2}}\cdot\nabla u^{\frac{p}{2}}\varphi\\
&\quad+\frac{2p\chi}{q}\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^q\Delta u^{\frac{p}{2}}\varphi-\frac{2p\chi}{q}\int_0^\infty\int_{\partial\Omega}u^{\frac{p}{2}}v^q\frac{\partial u^{\frac{p}{2}}}{\partial\nu}\varphi\\
&=\frac{2p\chi}{q}\int_0^\infty\int_\Omega v^q|\nabla u^{\frac{p}{2}}|^2\varphi+\frac{4p\chi}{q}\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^{\frac{q}{2}}\nabla v^{\frac{q}{2}}\cdot\nabla u^{\frac{p}{2}}\varphi\\
&\quad+\frac{2(p-2)\chi}{q}\int_0^\infty\int_\Omega v^q|\nabla u^{\frac{p}{2}}|^2\varphi+\frac{p^2\chi}{q}\int_0^\infty\int_\Omega u^{p-1}v^q\Delta u\varphi-\frac{2p\chi}{q}\int_0^\infty\int_{\partial\Omega}u^{\frac{p}{2}}v^q\frac{\partial u^{\frac{p}{2}}}{\partial\nu}\varphi,
\end{aligned}\tag{2.11}$$

whereas integrating by parts twice in the integral containing $\Delta\varphi$ in (2.3), by (2.10) applied to u,p and v,q, respectively, leads to

$$\begin{aligned}
\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega u^pv^q\Delta\varphi&=\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega v^q\Delta(u^p)\varphi+\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega u^p\Delta(v^q)\varphi\\
&\quad+2\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega 2u^{\frac{p}{2}}\nabla u^{\frac{p}{2}}\cdot 2v^{\frac{q}{2}}\nabla v^{\frac{q}{2}}\varphi\\
&\quad-\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_{\partial\Omega}2u^{\frac{p}{2}}\frac{\partial u^{\frac{p}{2}}}{\partial\nu}v^q\varphi-2\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_{\partial\Omega}u^pv^{\frac{q}{2}}\frac{\partial v^{\frac{q}{2}}}{\partial\nu}\varphi\\
&=\frac{4(p-1)}{p}\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega v^q|\nabla u^{\frac{p}{2}}|^2\varphi+\frac{4(q-1)}{q}\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega u^p|\nabla v^{\frac{q}{2}}|^2\varphi\\
&\quad+\Big(1-\frac{p\chi}{q}\Big)p\int_0^\infty\int_\Omega v^qu^{p-1}\Delta u\varphi+\Big(1-\frac{p\chi}{q}\Big)q\int_0^\infty\int_\Omega u^pv^{q-1}\Delta v\varphi\\
&\quad+8\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\cdot\nabla v^{\frac{q}{2}}\varphi\\
&\quad-\Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_{\partial\Omega}2u^{\frac{p}{2}}\frac{\partial u^{\frac{p}{2}}}{\partial\nu}v^q\varphi
\end{aligned}\tag{2.12}$$

for any such φ, for we already know that $\frac{\partial v}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$. If we combine (2.3) with (2.8), (2.11) and (2.12), we obtain

$$\begin{aligned}
\int_0^\infty\int_\Omega(u^pv^q)_t\varphi&\geq\Big\{\frac{4(1-p)}{p}+\frac{2p\chi}{q}+\frac{4(p-1)}{p}\Big(1-\frac{p\chi}{q}\Big)+\frac{2(p-2)\chi}{q}\Big\}\int_0^\infty\int_\Omega v^q|\nabla u^{\frac{p}{2}}|^2\varphi\\
&\quad+\Big\{\frac{4(p\chi+1-q)}{q}+\frac{4(q-1)}{q}\Big(1-\frac{p\chi}{q}\Big)\Big\}\int_0^\infty\int_\Omega u^p|\nabla v^{\frac{q}{2}}|^2\varphi\\
&\quad+\Big\{-\frac{4(1-p)\chi}{q}-8+\frac{4p\chi}{q}+8-\frac{8p\chi}{q}\Big\}\int_0^\infty\int_\Omega u^{\frac{p}{2}}v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\cdot\nabla v^{\frac{q}{2}}\varphi\\
&\quad+\Big\{\frac{p^2\chi}{q}+\Big(1-\frac{p\chi}{q}\Big)p\Big\}\int_0^\infty\int_\Omega v^qu^{p-1}\Delta u\varphi
\end{aligned}$$

$$
\begin{aligned}
&+\Big(1-\frac{p\chi}{q}\Big)q\int_0^\infty\int_\Omega u^pv^{q-1}\Delta v\varphi - q\int_0^\infty\int_\Omega u^pv^q\varphi + q\int_0^\infty\int_\Omega u^{p+1}v^{q-1}\varphi\\
&-\frac{2p\chi}{q}\int_0^\infty\int_{\partial\Omega} u^{\frac p2}v^q\frac{\partial u^{\frac p2}}{\partial\nu}\varphi - \Big(1-\frac{p\chi}{q}\Big)\int_0^\infty\int_{\partial\Omega} 2u^{\frac p2}\frac{\partial u^{\frac p2}}{\partial\nu}v^q\varphi\\
&=\int_0^\infty\int_\Omega\Big\{\frac{4p\chi}{q^2}u^p|\nabla v^{\frac q2}|^2-\frac{4\chi}{q}u^{\frac p2}v^{\frac q2}\nabla u^{\frac p2}\cdot\nabla v^{\frac q2}+pv^qu^{p-1}\Delta u-p\chi u^pv^{q-1}\Delta v\Big\}\varphi\\
&+q\int_0^\infty\int_\Omega u^pv^{q-1}\left\{\Delta v-v+u\right\}\varphi\\
&-2\int_0^\infty\int_{\partial\Omega}u^{\frac p2}\frac{\partial u^{\frac p2}}{\partial\nu}v^q\varphi
\end{aligned}
\tag{2.13}
$$

for every $\varphi\in C_0^\infty(\overline{\Omega}\times(0,\infty))$ satisfying $\varphi\ge 0$ throughout $\Omega\times(0,\infty)$ and $\frac{\partial\varphi}{\partial\nu}\big|_{\partial\Omega\times(0,\infty)}=0$ as well as $\operatorname{supp}\varphi\subset\{u>0\}$.

The observations that

$$
\begin{aligned}
pu^{p-1}v^q\Big(\Delta u-\chi\nabla\cdot(\frac uv\nabla v)\Big)&=pu^{p-1}v^q\Delta u-p\chi u^{p-1}v^{q-1}\nabla u\cdot\nabla v+p\chi u^pv^{q-2}|\nabla v|^2-p\chi u^pv^{q-1}\Delta v\\
&=pu^{p-1}v^q\Delta u-\frac{4\chi}{q}u^{\frac p2}v^{\frac q2}\nabla u^{\frac p2}\cdot\nabla v^{\frac q2}+\frac{4p\chi}{q^2}u^p|\nabla v^{\frac q2}|^2-p\chi u^pv^{q-1}\Delta v,
\end{aligned}
$$

and that v solves (1.2), now turn (2.13) into

$$
p\int_0^\infty\int_\Omega u^{p-1}v^qu_t\varphi\ge p\int_0^\infty\int_\Omega u^{p-1}v^q\left\{\Delta u-\chi\nabla\cdot\Big(\frac uv\nabla v\Big)\right\}\varphi-p\int_0^\infty\int_{\partial\Omega}u^{p-1}v^q\frac{\partial u}{\partial\nu}\varphi \tag{2.14}
$$

for all nonnegative $\varphi\in C_0^\infty(\overline{\Omega}\times(0,\infty))$ with $\operatorname{supp}\varphi\subset\{u>0\}$ and $\frac{\partial\varphi}{\partial\nu}\big|_{\partial\Omega\times(0,\infty)}=0$.

Specializing this to nonnegative $\varphi\in C_0^\infty(\Omega\times(0,\infty)\cap\{u>0\})$ by a Du Bois-Reymond lemma type argument we conclude

$$
u_t\ge\Delta u-\chi\nabla\cdot\Big(\frac uv\nabla v\Big)\qquad\text{in }\{u>0\}. \tag{2.15}
$$

Density of $\{u>0\}$ in $\Omega\times(0,\infty)$, obtained from the assumption that $u>0$ a.e., and continuity show that (2.15) actually holds on all of $\Omega\times(0,\infty)$.

We pick $t_0>0$ and some nonnegative $\psi\in C^1(\overline{\Omega})$ with $\frac{\partial\psi}{\partial\nu}\big|_{\partial\Omega}=0$ such that $\operatorname{supp}\psi\subset\{u(\cdot,t_0)>0\}:=\{x\in\overline{\Omega}:\ u(x,t_0)>0\}$. Then by continuity of u we can find some $\tau>0$ such that $\operatorname{supp}\psi\subset\cap_{t\in(t_0-\tau,t_0+\tau)}\{u(\cdot,t)>0\}$. Applying (2.14) to functions of the form $\varphi(x,t)=\zeta(t)\psi(x)$, $\zeta\in C_0^\infty((t_0-\tau,t_0+\tau))$ by once more invoking a Weierstrass type density argument and the Du Bois-Reymond lemma, we see that

$$
\int_\Omega u^{p-1}v^qu_t(\cdot,t)\psi\ge\int_\Omega u^{p-1}v^q\left\{\Delta u-\chi\nabla\cdot\Big(\frac uv\nabla v\Big)\right\}(\cdot,t)\psi-\int_{\partial\Omega}u^{p-1}v^q\frac{\partial u}{\partial\nu}(\cdot,t)\psi
$$

for every nonnegative $\psi\in C^1(\overline{\Omega})$ such that $\frac{\partial\psi}{\partial\nu}\big|_{\partial\Omega}=0$, $\operatorname{supp}\psi\subset\{u(\cdot,t_0)>0\}$ and for almost every $t\in(t_0-\tau,t_0+\tau)$ – and due to continuity especially for $t=t_0$. In particular inserting $\psi_\varepsilon(x):=(1-\frac1\varepsilon\operatorname{dist}(x,\partial\Omega))_+\cdot\psi$ and Lebesgue's theorem show that for every $t>0$, $\psi\in C^1(\overline{\Omega})$, $\psi\ge0$ with $\frac{\partial\psi}{\partial\nu}\big|_{\partial\Omega}=0$ and $\operatorname{supp}\psi\cap\partial\Omega\subset\{u(\cdot,t)>0\}$,

$$
\int_{\partial\Omega}u^{p-1}v^q\frac{\partial u}{\partial\nu}\psi\ge0. \tag{2.16}
$$

Since the integral only depends on $\psi\big|_{\partial\Omega}$ and not on the values of ψ inside Ω, it can be seen that (2.16) actually holds for any $t > 0$ and any nonnegative $\psi \in C^0(\overline{\Omega})$ such that $\operatorname{supp}\psi \cap \partial\Omega \subset \{u(\cdot,t) > 0\}$.
If $\frac{\partial u}{\partial\nu}(x_0,t_0) < 0$ for some $(x_0,t_0) \in \partial\Omega \times (0,\infty)$ with $u(x_0,t_0) > 0$, we pick $\psi_1 \in C^0(\partial\Omega)$ such that $\psi_1(x_0) > 0$, $\psi_1 \geq 0$, and $\operatorname{supp}\psi_1 \subset \{x \in \partial\Omega : \frac{\partial u}{\partial\nu}(x,t_0) < 0, u(x,t_0) > 0\} =: M$. Moreover, we let $d := \operatorname{dist}(\operatorname{supp}\psi_1, \partial\Omega \setminus M)$ (or $d = 1$ if $\partial\Omega \setminus M = \emptyset$) and let ψ_2 be the solution to $-\Delta\psi_2 = 0$ in Ω, $\psi_2 = -u^{1-p}(\cdot,t_0)v^{-q}(\cdot,t_0)\psi_1\frac{\partial u}{\partial\nu}$ on $\partial\Omega$. Then, thanks to the choice of $\operatorname{supp}\psi_1$, ψ_2 is nonnegative on the boundary and hence by the maximum principle in Ω. Defining $\psi_3(x) := \psi_2(x)(1 - \frac{2}{d}\operatorname{dist}(x,M))_+$ we obtain a nonnegative continuous function on $\overline{\Omega}$ whose support intersects the boundary only in $\{x \in \partial\Omega :\ u(x,t) > 0\}$ and which hence is a permissible test function in (2.16). We conclude that $0 \leq \int_{\partial\Omega} u^{p-1}v^q\frac{\partial u}{\partial\nu}\psi_3 = -\int_{\partial\Omega}|\frac{\partial u}{\partial\nu}|^2\psi_1$ and hence in particular $\frac{\partial u}{\partial\nu}(x_0,t_0) = 0$, which is a contradiction. We conclude that $\frac{\partial u}{\partial\nu} \geq 0$ on $\partial\Omega \times (0,\infty) \cap \{u > 0\}$ and hence, by continuity of $\frac{\partial u}{\partial\nu}$ and density of this set that $\frac{\partial u}{\partial\nu} \geq 0$ on $\partial\Omega \times (0,\infty)$.
Finally integrating (2.15) over $\Omega \times (0,t)$ and taking (2.7) into consideration, we see that

$$\int_\Omega u_0 \geq \int_\Omega u(\cdot,t) \geq \int_\Omega u_0 + \int_0^t\int_\Omega \Delta u - \chi\int_0^t\int_\Omega \nabla\cdot\left(\frac{u}{v}\nabla v\right) = \int_\Omega u_0 + \int_0^t\int_{\partial\Omega}\frac{\partial u}{\partial\nu}$$

by Gauss' theorem and $\frac{\partial v}{\partial\nu} = 0$, which firstly shows that $\frac{\partial u}{\partial\nu} = 0$ on $\partial\Omega \times (0,\infty)$ and secondly that (2.15) actually is an equality. □

3 Global smooth solutions to approximate problems

Now in order to approximate solutions by means of a convenient regularization of (1.1)-(1.2), for $\varepsilon \in (0,1)$ we consider

$$\begin{cases} u_{\varepsilon t} = \Delta u_\varepsilon - \chi\nabla\cdot\left(\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)v_\varepsilon}\nabla v_\varepsilon\right), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + u_\varepsilon, & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial\nu} = \frac{\partial v_\varepsilon}{\partial\nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{3.1}$$

and then first obtain the following.

Lemma 3.1. *For all $\varepsilon \in (0,1)$, the problem (3.1) admits a global classical solution $(u_\varepsilon, v_\varepsilon) \in (C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)))^2$ for which $u_\varepsilon > 0$ in $\overline{\Omega} \times (0,\infty)$ and $v_\varepsilon > 0$ in $\overline{\Omega} \times [0,\infty)$.*

Proof. The local existence of a solution can be obtained in a standard manner (cf. [1, Lemma 3.1] for a related general setting). Boundedness of the sensitivity term $\chi\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)v_\varepsilon}$, due to a strict positivity property of v_ε on $\overline{\Omega} \times (0,T)$ – to be made more precise in Lemma 3.3 below – allows for an iterative procedure converting boundedness information of $\|\nabla v_\varepsilon(\cdot,t)\|_{L^q(\Omega)}$ for some $q > 1$ into bounds for $\|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)}$ for $p \in (1, \frac{nq}{(n-q)_+})$ that in turn yield better estimates for ∇v_ε through application of semigroup estimates in the first and second equation, respectively. Finally, this serves to provide a uniform bound on u_ε on $\overline{\Omega} \times (0,T)$, in light of the extensibility criterion [1, (3.3)] thus ensuring global existence of the solution. Positivity of u_ε follows from a classical strong maximum principle. □

These approximate solutions clearly preserve mass:

Lemma 3.2. *Let $\varepsilon \in (0,1)$. Then*

$$\int_\Omega u_\varepsilon(\cdot,t) = \int_\Omega u_0 \qquad \text{for all } t > 0. \tag{3.2}$$

Proof. This directly results on integrating the first equation in (3.1). □

Moreover, the assumed positivity of v_0 enables us to control v_ε from below at least locally in time:

Lemma 3.3. *For each $\varepsilon \in (0,1)$, we have*

$$v_\varepsilon(x,t) \geq \Big(\inf_{x\in\Omega} v_0(x)\Big) \cdot e^{-t} \qquad \text{for all } x \in \Omega \text{ and } t > 0.$$

Proof. As u_ε is nonnegative, this is a straightforward consequence of a comparison argument applied to the second equation in (3.1). □

By means of well-known smoothing estimates of the heat semigroup, the mass conservation property (3.2) readily implies some basic regularity features of the second component.

Lemma 3.4. *Let $r \geq 1$ and $s \geq 1$ be such that $r < \frac{n}{n-2}$ and $s < \frac{n}{n-1}$. Then there exists $C > 0$ such that for each $\varepsilon \in (0,1)$,*

$$\int_\Omega v_\varepsilon^r(\cdot,t) \leq C \qquad \text{for all } t > 0 \tag{3.3}$$

and

$$\int_\Omega |\nabla v_\varepsilon(\cdot,t)|^s \leq C \qquad \text{for all } t > 0. \tag{3.4}$$

Proof. The representation of v_ε as

$$v_\varepsilon(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u_\varepsilon(\cdot,s)ds$$

makes it possible to apply well-known estimates for the Neumann heat-semigroup (cf. [20, Lemma 1.3]), which provide positive constants c_1, c_2, c_3 and c_4 such that

$$\|v_\varepsilon(\cdot,t)\|_{L^r(\Omega)} \leq c_1 \|v_0\|_{L^r(\Omega)} + c_2 \int_0^t (1 + (t-s)^{-\frac{n}{2}(1-\frac{1}{r})})e^{-(t-s)} \Big\|u_\varepsilon(\cdot,s)\Big\|_{L^1(\Omega)} ds$$

and

$$\|\nabla v_\varepsilon(\cdot,t)\|_{L^s(\Omega)} \leq c_3 \|v_0\|_{W^{1,\infty}(\Omega)} + c_4 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{s})})e^{-(t-s)} \Big\|u_\varepsilon(\cdot,s)\Big\|_{L^1(\Omega)} ds$$

for all $t > 0$ and all $\varepsilon \in (0,1)$, so that Lemma 3.2 and finiteness of $\int_0^\infty (1 + \tau^{-\frac{n}{2}(1-\frac{1}{r})})e^{-\tau}d\tau$ and $\int_0^\infty (1 + \tau^{-\frac{1}{2}-\frac{n}{2}(1-\frac{1}{s})})e^{-\tau}d\tau$ due to the conditions on r and s prove the lemma. □

4 A fundamental identity and first consequences thereof

Let us next formulate an identity which apparently reflects a fundamental structural propety of (1.1)-(1.2), as already used in a slightly modified form and for more restricted choices of χ in [21]. In Lemma 4.3 applied to $\varphi \equiv 1$, this will serve as a source for some essential a priori estimates for (3.1), whereas in Lemma 8.8 we will make use of the freedom to choose widely arbitrary test functions here in order to verify (2.3) for the limit couple (u,v) to be constructed in Lemma 8.1.

Lemma 4.1. *Let $p \in (0,1)$ and $q \in (0,1)$, and assume that $T>0$ and that $\varphi \in C^\infty(\overline{\Omega} \times [0,T])$ is such that $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,T)$. Then*

$$\begin{aligned}
&-\int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi_t + \int_\Omega u_\varepsilon^p(\cdot,T) v_\varepsilon^q(\cdot,T)\varphi(\cdot,T) - \int_\Omega u_0^p v_0^q \varphi(\cdot,0) \\
&\quad = \int_0^T \int_\Omega \frac{4(1-p)q - 4q^2 - p\frac{(1-p)^2\chi^2}{(1+\varepsilon u_\varepsilon)^2}}{pq(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q)} v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \varphi \\
&\qquad + \int_0^T \int_\Omega \frac{4}{q}\Big(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q\Big) \bigg| u_\varepsilon^{\frac{p}{2}} \nabla v_\varepsilon^{\frac{q}{2}} - \frac{\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon} + 2q}{2(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q)} v_\varepsilon^{\frac{q}{2}} \nabla u_\varepsilon^{\frac{p}{2}} \bigg|^2 \varphi \\
&\qquad + \int_0^T \int_\Omega \frac{2[(1-p)\varepsilon u_\varepsilon - p]\chi}{q(1+\varepsilon u_\varepsilon)^2} u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}} \cdot \nabla\varphi \\
&\qquad + \int_0^T \int_\Omega \Big(1 - \frac{p\chi}{q(1+\varepsilon u_\varepsilon)}\Big) u_\varepsilon^p v_\varepsilon^q \Delta\varphi \\
&\qquad - q\int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi + q\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1}\varphi \qquad \textit{for all } \varepsilon \in (0,1).
\end{aligned} \tag{4.1}$$

Proof. Using (3.1), we compute

$$\begin{aligned}
\int_\Omega \frac{\partial}{\partial t}(u_\varepsilon^p v_\varepsilon^q) \cdot \varphi &= -p\int_\Omega \nabla\Big(u_\varepsilon^{p-1} v_\varepsilon^q \varphi\Big) \cdot \Big(\nabla u_\varepsilon - \chi\frac{u_\varepsilon}{(1+\varepsilon u_\varepsilon)v_\varepsilon}\nabla v_\varepsilon\Big) \\
&\quad - q\int_\Omega \nabla\Big(u_\varepsilon^p v_\varepsilon^{q-1}\varphi\Big) \cdot \nabla v_\varepsilon - q\int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi + q\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1}\varphi \\
&= p(1-p)\int_\Omega u_\varepsilon^{p-2}|\nabla u_\varepsilon|^2 v_\varepsilon^q \varphi \\
&\quad - p(1-p)\chi\int_\Omega \frac{u_\varepsilon^{p-1}}{1+\varepsilon u_\varepsilon} v_\varepsilon^{q-1}(\nabla u_\varepsilon \cdot \nabla v_\varepsilon)\varphi - 2pq\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^{q-1}(\nabla u_\varepsilon \cdot \nabla v_\varepsilon)\varphi \\
&\quad + pq\chi\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon} v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2\varphi + q(1-q)\int_\Omega u_\varepsilon^p v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2\varphi \\
&\quad - p\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q \nabla u_\varepsilon \cdot \nabla\varphi + p\chi\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon} v_\varepsilon^{q-1}\nabla v_\varepsilon \cdot \nabla\varphi \\
&\quad - q\int_\Omega u_\varepsilon^p v_\varepsilon^{q-1}\nabla v_\varepsilon \cdot \nabla\varphi \\
&\quad - q\int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi + q\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1}\varphi
\end{aligned}$$

$$
\begin{aligned}
&= \frac{4(1-p)}{p}\int_\Omega v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2\varphi \\
&\quad -\frac{4(1-p)\chi}{q}\int_\Omega \frac{u_\varepsilon^{\frac{p}{2}}}{1+\varepsilon u_\varepsilon} v_\varepsilon^{\frac{q}{2}}(\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla v_\varepsilon^{\frac{q}{2}})\varphi - 8\int_\Omega u_\varepsilon^{\frac{p}{2}} v_\varepsilon^{\frac{q}{2}}(\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla v_\varepsilon^{\frac{q}{2}})\varphi \\
&\quad +\frac{4p\chi}{q}\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon}|\nabla v_\varepsilon^{\frac{q}{2}}|^2\varphi + \frac{4(1-q)}{q}\int_\Omega u_\varepsilon^p |\nabla v_\varepsilon^{\frac{q}{2}}|^2\varphi \\
&\quad -2\int_\Omega u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla\varphi \\
&\quad +\frac{p\chi}{q}\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon^q\cdot\nabla\varphi - \int_\Omega u_\varepsilon^p \nabla v_\varepsilon^q\cdot\nabla\varphi \\
&\quad -q\int_\Omega u_\varepsilon^p v_\varepsilon^q\varphi + q\int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1}\varphi \qquad \text{for all } t>0. \qquad (4.2)
\end{aligned}
$$

Here a straightforward rearrangement in the first five integrands on the right along the lines of (2.8) shows that

$$
\begin{aligned}
&\frac{4(1-p)}{p} v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2 - \frac{4(1-p)\chi}{q}\frac{u_\varepsilon^{\frac{p}{2}}}{1+\varepsilon u_\varepsilon} v_\varepsilon^{\frac{q}{2}}(\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla v_\varepsilon^{\frac{q}{2}}) - 8u_\varepsilon^{\frac{p}{2}} v_\varepsilon^{\frac{q}{2}}(\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla v_\varepsilon^{\frac{q}{2}}) \\
&\qquad +\frac{4p\chi}{q}\frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon}|\nabla v_\varepsilon^{\frac{q}{2}}|^2 + \frac{4(1-q)}{q} u_\varepsilon^p |\nabla v_\varepsilon^{\frac{q}{2}}|^2 \\
&= \frac{4}{q}\left(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q\right)\left(u_\varepsilon^p|\nabla v_\varepsilon^{\frac{q}{2}}|^2 - \frac{\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}+2q}{\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q} u_\varepsilon^{\frac{p}{2}} v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla v_\varepsilon^{\frac{q}{2}} + \frac{1}{4}\left(\frac{\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon}+2q}{\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q}\right)^2 v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2\right) \\
&\qquad + \frac{4(1-p)q - 4q^2 - p\frac{(1-p)^2\chi^2}{(1+\varepsilon u_\varepsilon)^2}}{pq(\frac{p\chi}{1+\varepsilon u_\varepsilon}+1-q)} v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \qquad \text{in } \Omega\times(0,\infty). \qquad (4.3)
\end{aligned}
$$

Moreover, in two of the three summands in (4.2) which contain $\nabla\varphi$ we once more integrate by parts to see that

$$
\begin{aligned}
&-2\int_\Omega u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla\varphi + \frac{p\chi}{q}\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon}\nabla v_\varepsilon^q\cdot\nabla\varphi - \int_\Omega u_\varepsilon^p\nabla v_\varepsilon^q\cdot\nabla\varphi \\
&\quad = -2\int_\Omega u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla\varphi \\
&\qquad -\frac{p^2\chi}{q}\int_\Omega \frac{u_\varepsilon^{p-1}}{1+\varepsilon u_\varepsilon} v_\varepsilon^q \nabla u_\varepsilon\cdot\nabla\varphi + \frac{p\chi\varepsilon}{q}\int_\Omega \frac{u_\varepsilon^p}{(1+\varepsilon u_\varepsilon)^2} v_\varepsilon^q\nabla u_\varepsilon\cdot\nabla\varphi - \frac{p\chi}{q}\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon} v_\varepsilon^q\Delta\varphi \\
&\qquad +p\int_\Omega u_\varepsilon^{p-1} v_\varepsilon^q\nabla u_\varepsilon\cdot\nabla\varphi + \int_\Omega u_\varepsilon^p v_\varepsilon^q\Delta\varphi \\
&\quad = -2\int_\Omega u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla\varphi \\
&\qquad -\frac{2p\chi}{q}\int_\Omega \frac{u_\varepsilon^{\frac{p}{2}}}{1+\varepsilon u_\varepsilon} v_\varepsilon^q\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla\varphi + \frac{2\chi\varepsilon}{q}\int_\Omega \frac{u_\varepsilon^{\frac{p}{2}+1}}{(1+\varepsilon u_\varepsilon)^2} v_\varepsilon^q\nabla u_\varepsilon^{\frac{p}{2}}\cdot\nabla\varphi - \frac{p\chi}{q}\int_\Omega \frac{u_\varepsilon^p}{1+\varepsilon u_\varepsilon} v_\varepsilon^q\Delta\varphi
\end{aligned}
$$

$$+2\int_\Omega u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}} \cdot \nabla\varphi + \int_\Omega u_\varepsilon^p v_\varepsilon^q \Delta\varphi$$

$$= \int_\Omega \frac{2[(1-p)\varepsilon u_\varepsilon - p]\chi}{q(1+\varepsilon u_\varepsilon)^2} u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}} \cdot \nabla\varphi + \int_\Omega \Big(1 - \frac{p\chi}{q(1+\varepsilon u_\varepsilon)}\Big) u_\varepsilon^p v_\varepsilon^q \Delta\varphi \qquad \text{in } (0,T)$$

thanks to the assumption that $\frac{\partial\varphi}{\partial\nu} = 0$ on $\partial\Omega \times (0,T)$. Combining this with (4.2) and (4.3) establishes (4.1). □

An elementary but crucial observation now identifies a condition on the relationship between the exponents p and q which ensure positivity of the coefficient appearing in the first summand on the right-hand side in (4.1).

Lemma 4.2. *Given $\chi > 0$ and $p \in (0,1)$ such that $p < \frac{1}{\chi^2}$, let $q_+(p) \in (0,1)$ and $q_-(p) \in (0, q_+(p))$ be defined by*

$$q_\pm(p) := \frac{1-p}{2} \cdot \Big(1 \pm \sqrt{1 - p\chi^2}\Big). \tag{4.4}$$

Then for any choice of $q \in (q_-(p), q_+(p))$ and all $\varepsilon \in (0,1)$,

$$\frac{4(1-p)q - 4q^2 - p\frac{(1-p)^2\chi^2}{(1+\varepsilon u_\varepsilon)^2}}{pq(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q)} \geq \frac{4(1-p)q - 4q^2 - p(1-p)^2\chi^2}{pq(p\chi + 1 - q)} > 0 \qquad \text{in } \Omega \times (0,\infty). \tag{4.5}$$

Proof. We use that $1 + \varepsilon u_\varepsilon \geq 1$ to firstly obtain

$$4(1-p)q - 4q^2 - \frac{p(1-p)^2\chi^2}{(1+\varepsilon u_\varepsilon)^2} \geq 4(1-p)q - 4q^2 - p(1-p)^2\chi^2 \qquad \text{in } \Omega \times (0,\infty).$$

Since here our hypothesis $q \in (q_-(p), q_+(p))$ guarantees that

$$4(1-p)q - 4q^2 - p(1-p)^2\chi^2 = -4 \cdot \Big(q - q_+(p)\Big) \cdot \Big(q - q_-(p)\Big) > 0,$$

we may once again estimate $1 + \varepsilon u_\varepsilon \geq 1$ to infer that indeed both inequalities in (4.5) hold. □

As a consequence, for p and q as in Lemma 4.2 we can readily derive the following from Lemma 4.1 when combined with the pointwise lower estimate for v_ε in Lemma 3.3.

Lemma 4.3. *Let $p \in (0,1)$ be such that $p < \frac{1}{\chi^2}$, and let $q \in (q_-(p), q_+(p))$ with $q_\pm(p)$ taken from (4.4). Then for each $T > 0$ there exists $C(p,q,T) > 0$ fulfilling*

$$\int_0^T \int_\Omega v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \leq C(p,q,T) \tag{4.6}$$

and

$$\int_0^T \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \leq C(p,q,T) \tag{4.7}$$

as well as

$$\int_0^T \int_\Omega \frac{4}{q} \left(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q\right) \left| u_\varepsilon^{\frac{p}{2}} \nabla v_\varepsilon^{\frac{q}{2}} - \frac{\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon} + 2q}{2(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q)} v_\varepsilon^{\frac{q}{2}} \nabla u_\varepsilon^{\frac{p}{2}} \right|^2 \leq C(p,q,T) \tag{4.8}$$

and

$$\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} \le C(p,q,T) \tag{4.9}$$

for all $\varepsilon \in (0,1)$.

Proof. According to Lemma 4.2, our assumption $q \in (q_-(p), q_+(p))$ ensures that with some $c_1 > 0$ we have

$$\frac{4(1-p)q - 4q^2 - p\frac{(1-p)^2\chi^2}{(1+\varepsilon u_\varepsilon)^2}}{pq(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q)} \ge c_1$$

for all $\varepsilon \in (0,1)$, whence applying Lemma 4.1 to $\varphi \equiv 1$ shows that

$$\begin{aligned}
c_1 \int_0^T \int_\Omega v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \\
&+ \int_0^T \int_\Omega \frac{4}{q} \left(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q\right) \left| u_\varepsilon^{\frac{p}{2}} \nabla v_\varepsilon^{\frac{q}{2}} - \frac{\frac{(1-p)\chi}{1+\varepsilon u_\varepsilon} + 2q}{2(\frac{p\chi}{1+\varepsilon u_\varepsilon} + 1 - q)} v_\varepsilon^{\frac{q}{2}} \nabla u_\varepsilon^{\frac{p}{2}} \right|^2 \varphi \\
&+ q \int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} \\
\le & \int_\Omega u_\varepsilon^p(\cdot,T) v_\varepsilon^q(\cdot,T) - \int_\Omega u_0^p v_0^q + q \int_0^T \int_\Omega u_\varepsilon^p v_\varepsilon^q \qquad \text{for all } \varepsilon \in (0,1).
\end{aligned} \tag{4.10}$$

Now by the Hölder inequality,

$$\int_\Omega u_\varepsilon^p v_\varepsilon^q \le \left\{ \int_\Omega u_\varepsilon \right\}^p \cdot \left\{ \int_\Omega v_\varepsilon^{\frac{q}{1-p}} \right\}^{1-p} \qquad \text{for all } t > 0,$$

so that since

$$\frac{q}{1-p} < \frac{q_+(p)}{1-p} = \frac{1+\sqrt{1-p\chi^2}}{2} < 1 < \frac{n}{n-2},$$

we may combine (3.2) with Lemma 3.4 to find $c_3 > 0$ fulfilling

$$\int_\Omega u_\varepsilon^p v_\varepsilon^q \le c_3 \qquad \text{for all } t > 0$$

whenever $\varepsilon \in (0,1)$. The estimates in (4.6), (4.8) and (4.9) therefore result from (4.10), whereupon (4.7) is a consequence of (4.6) and the fact that Lemma 3.3 along with (1.7) says that given $T > 0$ we can find $c_2 > 0$ such that

$$v_\varepsilon(x,t) \ge c_2 \qquad \text{for all } x \in \Omega,\ t \in (0,T) \text{ and } \varepsilon \in (0,1).$$

□

5 A further consequence: A bound for u_ε in L^r for some $r > 1$

Now in view of the desired integrability feature in (1.4), a crucial step in our analysis will consist in deriving a spatio-temporal equi-integrability property of u_ε. This will result from bounds therefor in some reflexive L^r spaces, to be obtained by an interpolation between (4.9) and (3.3). The following statement identifies the minimal possible choice of an integrability exponent arising in the course of this argument (cf. (5.6) below), and will thereby form the core of our requirement (1.8) on χ.

Lemma 5.1. *Let $\chi > 0$, and for $p \in (0, \min\{1, \frac{1}{\chi^2}\})$ let $q_\pm(p)$ be as in (4.4). Then*

$$\inf_{\substack{p\in(0,1),\ p<\frac{1}{\chi^2}\\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = \begin{cases} 1 & \textit{if } \chi \le 1, \\ \chi & \textit{if } \chi \in (1,2), \\ 1+\frac{\chi^2}{4} & \textit{if } \chi \ge 2. \end{cases} \tag{5.1}$$

Proof. By an evident monotonicity property,

$$\begin{aligned} \inf_{\substack{p\in(0,1),\ p<\frac{1}{\chi^2}\\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} &= \inf_{p\in(0,1),\ p<\frac{1}{\chi^2}} \frac{1-q_+(p)}{p} \\ &= \inf_{p\in(0,1),\ p<\frac{1}{\chi^2}} \frac{1-\frac{1-p}{2}(1+\sqrt{1-p\chi^2})}{p} \\ &= \inf_{p\in(0,1),\ p<\frac{1}{\chi^2}} \frac{1+p-(1-p)\sqrt{1-p\chi^2})}{2p} =: I(\chi) \end{aligned} \tag{5.2}$$

for any $\chi > 0$. Since $(1-p)\sqrt{1-p\chi^2} < 1-p$ and thus $\frac{1+p-(1-p)\sqrt{1-p\chi^2}}{2p} > 1$ for all $p \in (0, \min\{1, \frac{1}{\chi^2}\})$, and since on the other hand for $\chi \le 1$ we have

$$I(\chi) \le \liminf_{p\nearrow 1} \frac{1+p-(1-p)\sqrt{1-p\chi^2}}{2p} = 1,$$

this firstly implies that $I(\chi) = 1$ for any such χ.
In the case $\chi > 1$, having in mind the substitution $\xi = \sqrt{1-p\chi^2}$ in (5.2), we note that

$$\begin{aligned} \rho(\xi) &:= \frac{1+\frac{1-\xi^2}{\chi^2} - (1-\frac{1-\xi^2}{\chi^2})\cdot\xi}{2\cdot\frac{1-\xi^2}{\chi^2}} \\ &= \frac{1}{2}\cdot\Big\{\frac{\chi^2}{1-\xi^2}\cdot(1-\xi)+1+\xi\Big\} \\ &= \frac{1}{2}\cdot\Big\{\frac{\chi^2}{1+\xi}+1+\xi\Big\}, \qquad \xi \in [0,1), \end{aligned}$$

satisfies

$$\rho'(\xi) = -\frac{\chi^2}{2(1+\xi)^2} + \frac{1}{2} \qquad \text{for all } \xi \in (0,1),$$

so that ρ' attains a zero at $\xi = \chi - 1 \in (0,1)$ if and only if $\chi \in (1,2)$, while $\rho' \le 0$ throughout $(0,1)$ if $\chi \ge 2$. Therefore, $\inf_{\xi\in[0,1)} \rho(\xi) = \rho(\chi-1) = \chi$ if $\chi \in (1,2)$, whereas $\inf_{\xi\in[0,1)} \rho(\xi) = \lim_{\xi\nearrow 1} \rho(\xi) = 1 + \frac{\chi^2}{4}$ if $\chi \ge 2$. In conjunction with (5.2), these observations verify (5.1). □

Now under the assumptions on χ from Theorem 1.1, the announced interpolation argument indeed bears fruit of the desired flavour.

Lemma 5.2. *Suppose that $\chi > 0$ is such that (1.8) holds. Then there exists $r > 1$ such that for any $T > 0$ one can find $C(T) > 0$ with the property that*

$$\int_0^T \int_\Omega u_\varepsilon^r \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{5.3}$$

Proof. As a consequence of Lemma 5.1, our assumption on χ warrants that we can pick $p \in (0, \min\{1, \frac{1}{\chi^2}\})$ and $q \in (q_-(p), q_+(p))$ such that

$$\frac{1-q}{p} < \frac{n}{n-2}. \tag{5.4}$$

Indeed, if $n = 2$ this is obvious, while if $n \ge 4$ this is immediate from (5.1), because then due to the fact that $\frac{n}{n-2} \le 2$, the hypothesis (1.8) in particular requires that $\chi < 2$, so that in both cases $\chi \le 1$ and $\chi > 1$, (5.1) shows that the assumption $\chi < \frac{n}{n-2}$ implies that

$$\inf_{\substack{p\in(0,1),\ p<\frac{1}{\chi^2} \\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = \max\{1,\chi\} < \frac{n}{n-2}.$$

If $n = 3$, in the case $\chi < 2$ we similarly obtain that

$$\inf_{\substack{p\in(0,1),\ p<\frac{1}{\chi^2} \\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = \max\{1,\chi\} < 2 < 3 = \frac{n}{n-2},$$

whereas when $\chi \ge 2$ we use our restriction $\chi < \sqrt{8}$ to infer from (5.1) that

$$\inf_{\substack{p\in(0,1),\ p<\frac{1}{\chi^2} \\ q\in(q_-(p),q_+(p))}} \frac{1-q}{p} = 1 + \frac{\chi^2}{4} < 3$$

and that thus (5.4) can be achieved also in this case.

Henceforth keeping p and q fixed such that (5.4) holds, e.g. by means of a continuity argument we can pick $r > 1$ sufficiently close to 1 such that still $p + 1 - r > 0$ and

$$\frac{(1-q)r}{p+1-r} < \frac{n}{n-2}. \tag{5.5}$$

Then using Young's inequality, for $T > 0$ we can estimate

$$\int_0^T \int_\Omega u_\varepsilon^r = \int_0^T \int_\Omega \Big(u_\varepsilon^{p+1} v_\varepsilon^{q-1}\Big)^{\frac{r}{p+1}} \cdot v_\varepsilon^{\frac{(1-q)r}{p+1}}$$

$$\leq \int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} + \int_0^T \int_\Omega v_\varepsilon^{\frac{(1-q)r}{p+1-r}} \qquad \text{for all } \varepsilon \in (0,1), \tag{5.6}$$

so that (5.3) results on using (4.9) and applying (3.3) together with (5.5). □

6 A weighted L^2 bound for ∇v_ε

In order to complement (4.7) by an analogous L^2 estimate for ∇v_ε merely involving v_ε but not u_ε as a weight function, independently from the above we apply a standard testing technique to the second equation in (3.1) with the following outcome.

Lemma 6.1. *For all $q \in (0,1)$ and any $T > 0$ one can find $C(T) > 0$ such that*

$$\int_0^T \int_\Omega |\nabla v_\varepsilon^{\frac{q}{2}}|^2 \leq C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{6.1}$$

Proof. Thanks to the positivity of v_ε, we may use v_ε^{q-1} as a test function in the second equation in (3.1) to see that

$$\begin{aligned}
\frac{1}{q}\frac{d}{dt}\int_\Omega v_\varepsilon^q &= (1-q)\int_\Omega v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^q + \int_\Omega u_\varepsilon v_\varepsilon^{q-1} \\
&\geq (1-q)\int_\Omega v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 - \int_\Omega v_\varepsilon^q \qquad \text{for all } t > 0,
\end{aligned} \tag{6.2}$$

where according to Lemma 3.4 and the fact that $q < 1 < \frac{n}{n-2}$, we can find $c_1 > 0$ such that

$$\int_\Omega v_\varepsilon^q \leq c_1 \qquad \text{for all } t > 0.$$

On integration, we thus obtain from (6.2) that

$$\begin{aligned}
\frac{4(1-q)}{q^2}\int_0^T \int_\Omega |\nabla v_\varepsilon^{\frac{q}{2}}|^2 &= (1-q)\int_0^T \int_\Omega v_\varepsilon^{q-2}|\nabla v_\varepsilon|^2 \\
&\leq \frac{1}{q}\int_\Omega v_\varepsilon^q(\cdot,T) + \int_0^T \int_\Omega v_\varepsilon^q \\
&\leq \frac{c_1}{q} + Tc_1
\end{aligned}$$

for all $\varepsilon \in (0,1)$. □

7 Time regularity

As a final preparation for our limit procedure, we establish some regularity features of the time derivatives in (3.1), beginning with a conveniently transformed version of the first solution component.

Lemma 7.1. *Assume (1.8), and let $p \in (0,1)$ be such that $p < \frac{1}{\chi^2}$. Then for all $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^T \Big\| \partial_t \Big(u_\varepsilon(\cdot,t) + 1 \Big)^{\frac{p}{2}} \Big\|_{(W_0^{1,\infty}(\Omega))^\star} dt \le C(T) \qquad \textit{for all } \varepsilon \in (0,1). \tag{7.1}$$

Proof. We fix $\psi \in C_0^\infty(\Omega)$ such that $\|\psi\|_{W^{1,\infty}(\Omega)} \le 1$ and use (3.1) and Young's inequality as well as the trivial estimate $u_\varepsilon + 1 \ge 1$ to see that

$$\begin{aligned}
\left| \int_\Omega \partial_t (u_\varepsilon+1)^{\frac{p}{2}} \psi \right| &= \Bigg| \frac{p(2-p)}{4} \int_\Omega (u_\varepsilon+1)^{\frac{p-4}{2}} |\nabla u_\varepsilon|^2 \psi - \frac{p}{2} \int_\Omega (u_\varepsilon+1)^{\frac{p-2}{2}} \nabla u_\varepsilon \cdot \nabla\psi \\
&\quad - \frac{p(2-p)\chi}{4} \int_\Omega \frac{u_\varepsilon (u_\varepsilon+1)^{\frac{p-4}{2}}}{(1+\varepsilon u_\varepsilon) v_\varepsilon} (\nabla u_\varepsilon \cdot \nabla v_\varepsilon) \psi + \frac{p\chi}{2} \int_\Omega \frac{u_\varepsilon (u_\varepsilon+1)^{\frac{p-2}{2}}}{(1+\varepsilon u_\varepsilon) v_\varepsilon} \nabla v_\varepsilon \cdot \nabla\psi \Bigg| \\
&\le \frac{p(2-p)}{4} \int_\Omega (u_\varepsilon+1)^{\frac{p-4}{2}} |\nabla u_\varepsilon|^2 + \frac{p}{2} \int_\Omega (u_\varepsilon+1)^{\frac{p-2}{2}} |\nabla u_\varepsilon| \\
&\quad + \frac{p(2-p)\chi}{4} \int_\Omega (u_\varepsilon+1)^{\frac{p-2}{2}} |\nabla u_\varepsilon| \cdot \frac{|\nabla v_\varepsilon|}{v_\varepsilon} + \frac{p\chi}{2} \int_\Omega (u_\varepsilon+1)^{\frac{p}{2}} \frac{|\nabla v_\varepsilon|}{v_\varepsilon} \\
&\le \frac{p(2-p)}{4} \int_\Omega (u_\varepsilon+1)^{p-2} |\nabla u_\varepsilon|^2 + \frac{p}{4} \int_\Omega (u_\varepsilon+1)^{p-2} |\nabla u_\varepsilon|^2 + \frac{p|\Omega|}{4} \\
&\quad + \frac{p(2-p)\chi}{8} \int_\Omega (u_\varepsilon+1)^{p-2} |\nabla u_\varepsilon|^2 + \frac{p(2-p)\chi}{8} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \\
&\quad + \frac{p\chi}{4} \int_\Omega (u_\varepsilon+1)^p + \frac{p\chi}{4} \int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1).
\end{aligned}$$

Since Lemma 3.3 provides $c_1 > 0$ such that $v_\varepsilon \ge c_1$ in $\Omega \times (0,T)$ and hence

$$\int_\Omega \frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}(\cdot,t) \le 9 c_1^{-\frac{2}{3}} \int_\Omega |\nabla v_\varepsilon^{\frac{1}{3}}(\cdot,t)|^2 \qquad \text{for all } t \in (0,T) \text{ and } \varepsilon \in (0,1),$$

and since

$$\int_\Omega (u_\varepsilon(\cdot,t)+1)^p \le \int_\Omega u_0 + |\Omega| \qquad \text{for all } t>0 \text{ and } \varepsilon \in (0,1)$$

by (3.2), we thus infer that there exists $c_2 > 0$ such that for all $\varepsilon \in (0,1)$,

$$\begin{aligned}
\Big\| \partial_t \Big(u_\varepsilon(\cdot,t)+1 \Big)^{\frac{p}{2}} \Big\|_{(W_0^{1,\infty}(\Omega))^\star} &= \sup_{\substack{\psi \in C_0^\infty(\Omega) \\ \|\psi\|_{W^{1,\infty}(\Omega)} \le 1}} \left| \int_\Omega \partial_t \Big(u_\varepsilon(\cdot,t)+1 \Big)^{\frac{p}{2}} \psi \right| \\
&\le c_2 \cdot \left\{ \int_\Omega |\nabla u_\varepsilon^{\frac{p}{2}}(\cdot,t)|^2 + \int_\Omega |\nabla v_\varepsilon^{\frac{1}{3}}(\cdot,t)|^2 + 1 \right\} \qquad \text{for all } t \in (0,T).
\end{aligned}$$

Thanks to the outcomes of Lemma 4.3 and Lemma 6.1, an integration over $t \in (0,T)$ therefore yields (7.1). $\square$

As for the second component, we can directly address the quantity $v_{\varepsilon t}$.

Lemma 7.2. *Let $\chi > 0$. Then there exists $C > 0$ such that whenever $\varepsilon \in (0,1)$,*

$$\|v_{\varepsilon t}(\cdot,t)\|_{(W_0^{1,\infty}(\Omega))^\star} dt \le C \qquad \textit{for all } t > 0. \tag{7.2}$$

Proof. We again fix $\psi \in C_0^\infty(\Omega)$ fulfilling $\|\psi\|_{W^{1,\infty}(\Omega)} \le 1$, and using (3.1) we find that

$$\begin{aligned} \left| \int_\Omega v_{\varepsilon t}\psi \right| &= \left| - \int_\Omega \nabla v_\varepsilon \cdot \nabla\psi - \int_\Omega v_\varepsilon \psi + \int_\Omega u_\varepsilon \psi \right| \\ &\le \int_\Omega |\nabla v_\varepsilon| + \int_\Omega v_\varepsilon + \int_\Omega u_\varepsilon \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1). \end{aligned}$$

Therefore,

$$\|v_{\varepsilon t}(\cdot,t)\|_{(W_0^{1,\infty}(\Omega))^\star} \le \sup_{\tau>0} \left\{ \int_\Omega |\nabla v_\varepsilon(\cdot,\tau)| + \int_\Omega v_\varepsilon(\cdot,\tau) + \int_\Omega u_\varepsilon(\cdot,\tau) \right\} \qquad \text{for all } t > 0 \text{ and } \varepsilon \in (0,1),$$

so that (7.2) results from Lemma 3.4 and (3.2). ☐

8 Construction of limit functions. Proof of Theorem 1.1

Collecting the above estimates, by means of a straightforward extraction procedure we can pass to the limit $\varepsilon \searrow 0$ in the following sense.

Lemma 8.1. *Suppose that (1.8) holds, and let $p \in (0,1)$ and $q \in (0,1)$ be such that $p < \frac{1}{\chi^2}$ and $q \in (q_-(p), q_+(p))$. Then there exist $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ and functions u and v defined on $\Omega \times (0,\infty)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $u \ge 0$ and $v \ge 0$ a.e. in $\Omega \times (0,\infty)$, and that*

$$\begin{aligned} & u_\varepsilon \to u && \textit{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ and a.e. in } \Omega \times (0,\infty), && (8.1) \\ & \nabla u_\varepsilon^{\frac{p}{2}} \rightharpoonup \nabla u^{\frac{p}{2}} && \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)), && (8.2) \\ & v_\varepsilon \to v && \textit{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \textit{ and a.e. in } \Omega \times (0,\infty), && (8.3) \\ & \nabla v_\varepsilon \rightharpoonup \nabla v && \textit{in } L^1_{loc}(\overline{\Omega} \times [0,\infty)) \qquad \textit{and} && (8.4) \\ & \nabla v_\varepsilon^{\frac{q}{2}} \rightharpoonup \nabla v^{\frac{q}{2}} && \textit{in } L^2_{loc}(\overline{\Omega} \times [0,\infty)) && (8.5) \end{aligned}$$

as $\varepsilon = \varepsilon_j \searrow 0$, and

$$\int_\Omega u(\cdot,t) = \int_\Omega u_0 \qquad \textit{for a.e. } t > 0. \tag{8.6}$$

Proof. We fix $p \in (0,1)$ such that $p < \frac{1}{\chi^2}$ and combine Lemma 4.3 with (3.2) and Lemma 7.1 to see that

$$\Big((u_\varepsilon + 1)^{\frac{p}{2}} \Big)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}([0,\infty); W^{1,2}(\Omega))$$

and that

$$\Big(\partial_t (u_\varepsilon + 1)^{\frac{p}{2}} \Big)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^1_{loc}([0,\infty); (W_0^{1,\infty}(\Omega))^\star).$$

Apart from that, Lemma 3.4 and Lemma 7.2 show that there exists $r > 1$ such that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^r_{loc}([0,\infty); W^{1,r}(\Omega))$$

and

$$(v_{\varepsilon t})_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^\infty((0,\infty); (W_0^{1,\infty}(\Omega))^\star).$$

Therefore, by means of two applications of an Aubin-Lions lemma [16, Cor. 8.4] we can find $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, that $u_\varepsilon^{\frac{p}{2}} \to u^{\frac{p}{2}}$ in $L^2_{loc}(\overline{\Omega}\times[0,\infty))$ and a.e. in $\Omega\times(0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, and that (8.2), (8.3) and (8.4) hold with some nonnegative functions u and v defined on $\Omega \times (0,\infty)$. Since thus also $u_\varepsilon \to u$ a.e. in $\Omega \times (0,\infty)$ as $\varepsilon = \varepsilon_j \searrow 0$, making use of the equi-integrability property implied by Lemma 5.2 we may invoke the Vitali convergence theorem to infer that in fact also (8.1) holds, whereupon (8.6) becomes a consequence of (3.2). The additional convergence statement in (8.5) finally results from Lemma 6.1 and (8.3) in a straightforward manner. □

Our next aim is to make sure that the functions u and v we have just constructed form a generalized solution of (1.1)-(1.2). We begin with the second equation.

Lemma 8.2. *If (1.8) holds, then the pair (u,v) obtained in Lemma 8.1 is a global weak solution of (1.2) in the sense of Definition 2.3.*

Proof. From (8.1), (8.3) and (8.4) we immediately see that the regularity properties in (2.5) hold, and that moreover for arbitrary $\varphi \in C_0^\infty(\overline{\Omega} \times [0,\infty))$, in the identity

$$-\int_0^\infty \int_\Omega v_\varepsilon \varphi_t - \int_\Omega v_0 \varphi(\cdot,0) = -\int_0^\infty \int_\Omega \nabla v_\varepsilon \cdot \nabla\varphi - \int_0^\infty \int_\Omega v_\varepsilon\varphi + \int_0^\infty \int_\Omega u_\varepsilon \varphi,$$

valid for all $\varepsilon \in (0,1)$ due to (3.1), we may let $\varepsilon = \varepsilon_j \searrow 0$ in each integral separately to readily verify (2.6). □

Another important part of Definition 2.4 are positivity requirements, which will be established in Lemma 8.6. The following technical lemmas prepare the main argument therein, where we will derive a differential inequality for $\int_\Omega \ln u_\varepsilon$, and where further exploting the latter will in particular require some ε-independent lower bound for this functional at some suitable initial value, despite the fact that (1.7) does not guarantee finiteness of $\int_\Omega \ln u_0$. An appropriate replacement, to be provided by Lemma 8.5, is entailed by the comparison-type Lemma 8.3 in combination with a differential inequality, the derivation of which rests on Lemma 8.4.

Lemma 8.3. *Let $a > 0$, $b > 0$ and $T > 0$, and let $y\colon (0,T) \to \mathbb{R}$ be a continuously differentiable function satisfying*

$$y'(t) \le -ay^2(t) + b \qquad \text{for all } t \in (0,T) \text{ at which } y(t) > 0.$$

Then

$$y(t) \le \sqrt{\frac{b}{a}}\coth(\sqrt{ab}t) \qquad \text{for all } t \in (0,T).$$

Proof. Let $\eta > 0$. Then $M_\eta := \left\{t \in (0,T) : y(t) > \sqrt{\frac{b}{a}}+\eta\right\}$ (is either empty or) can be written as union of its connected components, i.e. $M_\eta = \bigcup_{k\in\mathbb{N}} I_k$ with disjoint open intervals I_k. If we consider any nonempty I_k with $\inf I_k \neq 0$, by continuity $y(\inf I_k) = \sqrt{\frac{b}{a}}+\eta$ and hence $y'(\inf I_k) \le -2\sqrt{ab}\eta - a\eta^2 < 0$, contradicting the definition of $\inf I_k$ as infimum of a set where $y > \sqrt{\frac{b}{a}} + \eta$. Hence there is $t_\eta \in [0,T)$ such that $y \le \sqrt{\frac{b}{a}} + \eta$ in (t_η, T) and that $y > \sqrt{\frac{b}{a}} + \eta$ in $(0, t_\eta)$ so that $b - ay^2$ is negative in $(0, t_\eta)$ and for $t_0 \in (0, t_\eta)$ and $t \in (t_0, t_\eta)$ we find that

$$t-t_0 \le \int_{t_0}^t \frac{y'(s)}{b-ay^2(s)}ds = \frac{1}{\sqrt{ab}} \int_{\sqrt{\frac{a}{b}}y(t_0)}^{\sqrt{\frac{a}{b}}y(t)} \frac{1}{1-z^2}dz = \frac{1}{\sqrt{ab}} \left\{ \operatorname{arcoth}\left(\sqrt{\frac{a}{b}}y(t)\right) - \operatorname{arcoth}\left(\sqrt{\frac{a}{b}}y(t_0)\right)\right\},$$

leading to

$$\sqrt{ab}(t-t_0) + \operatorname{arcoth}\left(\sqrt{\frac{a}{b}}y(t_0)\right) \le \operatorname{arcoth}\left(\sqrt{\frac{a}{b}}y(t)\right)$$

and hence to

$$y(t) \le \sqrt{\frac{b}{a}} \coth\left(\sqrt{ab}(t-t_0) + \operatorname{arcoth}\left(\sqrt{\frac{a}{b}}y(t_0)\right)\right) \le \sqrt{\frac{b}{a}} \coth\left(\sqrt{ab}(t-t_0)\right).$$

Using that $t_0 \in (0,t)$ and $\eta > 0$ were arbitrary, we conclude that $y(t) \le \max\left\{\sqrt{\frac{b}{a}}\coth(\sqrt{ab}t), \sqrt{\frac{b}{a}}\right\} = \sqrt{\frac{b}{a}}\coth(\sqrt{ab}t)$. □

The following statement essentially goes back to an observation made in [18].

Lemma 8.4. *Let $\eta > 0$. Then there exists $C > 0$ such that every positive function $\varphi \in C^1(\overline{\Omega})$ fulfilling*

$$\left|\{x \in \Omega;\ \varphi(x) > \delta\}\right| > \eta$$

for some $\delta > 0$ satisfies

$$\int_\Omega \frac{|\nabla\varphi|^2}{\varphi^2} \ge C \cdot \left\{\int_\Omega \ln\frac{\delta}{\varphi}\right\}^2 \quad \text{or} \quad \int_\Omega \ln\frac{\delta}{\varphi} < 0.$$

Proof. This directly follows from the inequality provided by [18, Lemma 4.3] upon squaring. A requirement on convexity of the domain, as additionally made there in order to allow for an application of the Poincaré inequality from [10, Cor 9.1.4] in the proof, can actually be removed by replacing the latter with Lemma 9.1. □

We can now pass to our derivation of lower bounds for $\int_\Omega \ln u_\varepsilon$ in the following form.

Lemma 8.5. *There exists $T > 0$ such that for every $t \in (0,T)$,*

$$\inf_{\varepsilon\in(0,1)} \int_\Omega \ln u_\varepsilon(\cdot,t) > -\infty.$$

Proof. We let $M_\varepsilon(t) := \sup_{\tau\in[0,t]} \|u_\varepsilon(\cdot,\tau)\|_{L^\infty(\Omega)}$ for $t \in (0,\infty)$ and $\varepsilon \in (0,1)$, and pick $p > n$. From known L^p-L^q estimates for the Neumann heat semigroup [20, Lemma 1.3 iii) and ii)] we obtain $c_1 > 0$ and $c_2 > 0$ such that

$$\begin{aligned} \|\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} &\le c_1 \|v_0\|_{W^{1,\infty}(\Omega)} + c_2 \int_0^t (1 + (t-s)^{-\frac{1}{2}}) e^{-(t-s)} \|u_\varepsilon\|_{L^\infty(\Omega)} \, ds \\ &\le c_3 \cdot (1 + M_\varepsilon(t)) \qquad \text{for all } t \in (0,\infty), \end{aligned}$$

where $c_3 = \max\left\{ c_1 \|v_0\|_{W^{1,\infty}(\Omega)}, c_2 \int_0^\infty (1+\tau^{-\frac{1}{2}}) e^{-\tau} d\tau \right\}$. Invoking further semigroup estimates (in the form of [20, Lemma 1.3 iv)]) we find $c_4 > 0$ such that

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty(\Omega)} &\le \|u_0\|_{L^\infty(\Omega)} + \chi c_4 \int_0^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2p}}) \left\| \frac{u_\varepsilon}{v_\varepsilon(\cdot,s)(1+\varepsilon u_\varepsilon(\cdot,s))} \nabla v_\varepsilon(\cdot,s) \right\|_{L^p(\Omega)} ds \\ &\le \|u_0\|_{L^\infty(\Omega)} + \frac{\chi c_4}{\inf v_0} e^t \cdot c_3 (1 + M_\varepsilon(t)) \cdot \|u_0\|_{L^1(\Omega)}^{\frac{1}{p}} (M_\varepsilon(t))^{\frac{p-1}{p}} \int_0^t (1 + \tau^{-\frac{1}{2}-\frac{n}{2p}}) d\tau \end{aligned}$$

for all $t \in (0,\infty)$ and all $\varepsilon \in (0,1)$, because $\left\| \frac{u_\varepsilon(\cdot,t)}{1+\varepsilon u_\varepsilon(\cdot,t)} \right\|_{L^p(\Omega)} \le \|u_\varepsilon(\cdot,t)\|_{L^p(\Omega)} \le \|u_0\|_{L^1(\Omega)}^{\frac{1}{p}} \|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)}^{\frac{p-1}{p}}$ for all $t \in (0,\infty)$, so that in conclusion we can find $c_5 > 0$ fulfilling

$$M_\varepsilon(t) \le \|u_0\|_{L^\infty(\Omega)} + c_5 (t + t^{\frac{1}{2}-\frac{n}{2p}}) e^t (1 + M_\varepsilon(t)) (M_\varepsilon(t))^{\frac{p-1}{p}} \qquad \text{for all } t \in (0,\infty),$$

and hence by the fact that for all $a, b \in [0,\infty)$, $\gamma \in (0,1)$

$$\sup\{x \in [0,\infty);\ x \le a + bx^\gamma\} \le \frac{a}{1-\gamma} + b^{\frac{1}{1-\gamma}}$$

we can achieve that

$$M_\varepsilon(t) \le p \|u_0\|_{L^\infty(\Omega)} + \left(c_5 (t + t^{\frac{1}{2}-\frac{n}{2p}}) e^t (1 + M_\varepsilon(t)) \right)^p$$

If we let $T_\varepsilon := \sup\left\{ t \in (0,\infty) : M_\varepsilon(t) \le p \|u_0\|_{L^\infty(\Omega)} + 1 \right\}$, then certainly

$$T_\varepsilon > \min\left\{ 1, \frac{1}{2} \left(c_5 \cdot 2e (2 + p \|u_0\|_{L^\infty(\Omega)})^{-\frac{1}{\frac{1}{2}-\frac{n}{2p}}} \right) \right\} =: T.$$

In conclusion, this means that for all $\varepsilon \in (0,1)$,

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le p \|u_0\|_{L^\infty(\Omega)} + 1 =: M \quad \text{and} \quad \|\nabla v_\varepsilon(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_3(1+M) \qquad \text{for all } t \in (0,T).$$

In particular with $\delta := \frac{1}{2|\Omega|} \int_\Omega u_0$ and $\eta := \frac{1}{2M} \int_\Omega u_0$ we have $|\{u_\varepsilon(\cdot,t) \ge \delta\}| \ge \eta$ for every $t \in (0,T)$ and each $\varepsilon \in (0,1)$, because

$$\int_\Omega u_0 = \int_\Omega u(\cdot,t) = \int_{\{u(\cdot,t)\ge\delta\}} u(\cdot,t) + \int_{\{u(\cdot,t)<\delta\}} u(\cdot,t) \le M|\{u(\cdot,t) \ge \delta\}| + |\Omega|\delta = M|\{u(\cdot,t) \ge \delta\}| + \frac{1}{2} \int_\Omega u_0$$

and therefore

$$\eta = \frac{1}{2M}\int_\Omega u_0 \le |\{u(\cdot,t)\ge\delta\}|.$$

From Lemma 8.4 we hence obtain $c_6>0$ such that

$$\begin{aligned}
\frac{d}{dt}\left(\int_\Omega \ln\frac{\delta}{u_\varepsilon(\cdot,t)}\right) &= -\int_\Omega \frac{|\nabla u_\varepsilon(\cdot,t)|^2}{u_\varepsilon^2(\cdot,t)} + \chi\int_\Omega \frac{1}{(1+\varepsilon u_\varepsilon)u_\varepsilon(\cdot,t)v_\varepsilon(\cdot,t)}\nabla u_\varepsilon(\cdot,t)\cdot\nabla v_\varepsilon(\cdot,t)\\
&\le -\frac12\int_\Omega \frac{|\nabla u_\varepsilon(\cdot,t)|^2}{u_\varepsilon^2(\cdot,t)} + \frac{\chi^2}{2}\int_\Omega\frac{|\nabla v_\varepsilon(\cdot,t)|^2}{v_\varepsilon^2(\cdot,t)}\\
&\le -\frac{c_6}{2}\left(\int_\Omega \ln\frac{\delta}{u_\varepsilon(\cdot,t)}\right)^2 + \frac{\chi^2 e^{2T}c_3^2(1+M)^2}{2(\inf v_0)^2}
\end{aligned}$$

for every $t\in(0,T)$ at which $\int_\Omega \ln\frac{\delta}{u_\varepsilon(\cdot,t)}>0$. Lemma 8.3 hence proves the claim. □

This enables us to verify the positivity requirements from Definition 2.4 without any assumptions on the initial data beyond (1.7).

Lemma 8.6. *The functions u, v obtained in Lemma 8.1 satisfy $v>0$, $u>0$ a.e. in $\Omega\times(0,\infty)$ and $u^pv^q>0$ a.e. on $\partial\Omega\times(0,\infty)$.*

Proof. According to Lemma 3.3 and (8.3), $\operatorname{essinf}_{x\in\Omega} v(x,t) > \inf v_0 e^{-t}$ for any $t>0$, and (8.2) and (8.5) together with (8.1) or (8.3), respectively, show that u and v can be evaluated on $\partial\Omega\times(0,\infty)$ in the sense of traces. For the proof of the remaining positivity properties $u>0$ a.e. in $\Omega\times(0,\infty)$ and $u>0$ a.e. on $\partial\Omega\times(0,\infty)$, we intend to prove

$$\ln u \in L^2_{loc}((0,\infty);W^{1,2}(\Omega)), \tag{8.7}$$

which entails $\ln u\in L^2_{loc}(\overline\Omega\times(0,\infty))$ and, due to the embedding $W^{1,2}(\Omega)\hookrightarrow L^2(\partial\Omega)$, also $\ln u\in L^2_{loc}(\partial\Omega\times(0,\infty))$, proving positivity of u a.e. in the respective sets. By Lemmata 3.3 and 3.2,

$$\begin{aligned}
\frac{d}{dt}\left[-\int_\Omega \ln u_\varepsilon - \chi^2\int_\Omega \ln v_\varepsilon\right] &= -\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} + \chi\int_\Omega\frac{1}{u_\varepsilon v_\varepsilon(1+\varepsilon u_\varepsilon)}\nabla u_\varepsilon\cdot\nabla v_\varepsilon - \chi^2\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2}\\
&\quad + \chi^2|\Omega| - \chi^2\int_\Omega\frac{u_\varepsilon}{v_\varepsilon}\\
&\le -\frac12\int_\Omega\frac{|\nabla u_\varepsilon|^2}{u_\varepsilon^2} - \frac{\chi^2}{2}\int_\Omega\frac{|\nabla v_\varepsilon|^2}{v_\varepsilon^2} + \chi^2|\Omega| + \frac{\chi^2}{\inf v_0}e^t\int_\Omega u_0
\end{aligned}$$

for any $t>0$. Now picking $\tau>0$, according to Lemma 8.5 we can find $\tau_0\in(0,\tau)$ and $c_1>0$ such that

$$\inf_{\varepsilon\in(0,1)}\int_\Omega \ln u_\varepsilon(\cdot,\tau_0) > -\infty. \tag{8.8}$$

For any fixed $T>\tau$ we then obtain

$$-\int_\Omega \ln u_\varepsilon(\cdot,t) + \frac12\int_{\tau_0}^t\int_\Omega |\nabla\ln u_\varepsilon|^2 \le -\int_\Omega \ln u_\varepsilon(\cdot,\tau_0) + \chi^2\int_\Omega \ln\frac{v_\varepsilon(\cdot,t)}{v_\varepsilon(\cdot,\tau_0)} + \chi^2|\Omega|T + \frac{\chi^2}{\inf v_0}e^T\int_\Omega u_0 \tag{8.9}$$

for $t \in (\tau,T)$, where according to (8.8) and by Lemma 3.3 and (3.3) the right-hand side is bounded independently of ε. We invoke the Poincaré inequality to find $c_1 > 0$ such that $\|\varphi\|_{W^{1,2}(\Omega)} \le c_1(\|\nabla\varphi\|_{L^2(\Omega)} + \|\varphi\|_{L^1(\Omega)})$ for all $\varphi \in W^{1,2}(\Omega)$, and since the elementary estimate $|\ln s| \le 2s - \ln s$ valid for all $s > 0$, Lemma 3.2 and (8.9) provide $c_2 > 0$ such that $\int_\Omega |\ln u_\varepsilon(\cdot,t)| \le c_2$ for all $t \in (\tau,T)$, we conclude that for every $\tau > 0$ and $T > \tau$ there exists $c_3 > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$\|\ln u_\varepsilon\|_{L^2((\tau,T);W^{1,2}(\Omega))} \le c_3,$$

which by a weak compactness argument immediately results in (8.7). □

In order to demonstrate that (u,v) satisfies (2.3), we prepare the following.

Lemma 8.7. *Let $(\phi_\varepsilon)_{\varepsilon\in(0,1)} \subset C^0([0,\infty)) \cap L^\infty((0,\infty))$ be such that*

$$\sup_{\varepsilon\in(0,1)} \|\phi_\varepsilon\|_{L^\infty((0,\infty))} < \infty \tag{8.10}$$

and that there exists $\phi \in C^0([0,\infty))$ such that

$$\phi_\varepsilon \to \phi \quad \text{in } L^\infty_{loc}([0,\infty)) \qquad \text{as } \varepsilon \searrow 0. \tag{8.11}$$

Then given $\chi > 0$ such that (1.8) holds and taking u, v and $(\varepsilon_j)_{j\in\mathbb{N}} \subset (0,1)$ from Lemma 8.1, for all $p \in (0, \min\{1, \frac{1}{\chi^2}\})$, any $q \in (q_-(p), q_+(p))$ and each $T > 0$ we have

$$\phi_\varepsilon(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} \rightharpoonup \phi(u)v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} \quad \text{in } L^2(\Omega\times(0,T)) \qquad \text{as } \varepsilon = \varepsilon_j \searrow 0. \tag{8.12}$$

Proof. Let $T > 0$. Then since from (8.10) we know that there exists $c_1 > 0$ such that

$$|\phi_\varepsilon(s)| \le c_1 \qquad \text{for all } s > 0 \text{ and } \varepsilon \in (0,1) \tag{8.13}$$

and hence

$$\int_0^T \int_\Omega \left|\phi_\varepsilon(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}}\right|^2 \le c_1 \int_0^T\int_\Omega v_\varepsilon^q |\nabla u_\varepsilon^{\frac{p}{2}}|^2 \qquad \text{for all } \varepsilon \in (0,1),$$

writing $w_\varepsilon := \phi_\varepsilon(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}}$, $\varepsilon \in (0,1)$, we infer from (4.6) that $(w_\varepsilon)_{\varepsilon\in(0,1)}$ is bounded in $L^2(\Omega\times(0,T))$ and hence relatively compact therein with respect to the weak topology. According to a standard argument, in order to verify (8.12) it is thus sufficient to make sure that whenever $(\varepsilon_{j_k})_{k\in\mathbb{N}}$ is a subsequence of $(\varepsilon_j)_{j\in\mathbb{N}}$ with the property that

$$w_\varepsilon \rightharpoonup w \quad \text{in } L^2(\Omega\times(0,T)) \qquad \text{as } \varepsilon = \varepsilon_{j_k} \searrow 0 \tag{8.14}$$

with some $w \in L^2(\Omega\times(0,T))$, we have $w = \phi(u)v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}$ a.e. in $\Omega\times(0,T)$. To achieve this, we note that due to $q < 1$ and (8.3), $v_\varepsilon^{\frac{q}{2}} \to v^{\frac{q}{2}}$ in $L^2(\Omega\times(0,T))$. By (8.13) and the pointwise convergence asserted in (8.1) together with Lebesgue's dominated convergence theorem, we therefore even have $\phi_\varepsilon(u_\varepsilon)v_\varepsilon^{\frac{q}{2}} \to \phi(u)v^{\frac{q}{2}}$ in $L^2(\Omega\times(0,T))$. Furthermore taking into account (8.2), we see that

$$\phi_\varepsilon(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} \rightharpoonup \phi(u)v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} \qquad \text{in } L^1(\Omega\times(0,T)),$$

which ensures $w = \phi(u)v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}$ a.e. in $\Omega\times(0,T)$. □

We can now make sure that indeed the obtained pair (u,v) has all the properties required in Definition 2.1.

Lemma 8.8. *Suppose that (1.8) holds, and let $p \in (0,1)$ and $q \in (0,1)$ be such that $p < \frac{1}{\chi^2}$ and $q \in (q_-(p), q_+(p))$. Then the functions u and v constructed in Lemma 8.1 form a global weak (p,q)-supersolution of (1.1) in the framework of Definition 2.1.*

Proof. Writing

$$\phi_\varepsilon^{(1)}(s) := \frac{\frac{(1-p)\chi}{1+\varepsilon s} + 2q}{\sqrt{\frac{pq\chi}{1+\varepsilon s} + q(1-q)}}$$

and

$$\phi_\varepsilon^{(2)}(s) := \frac{2}{\sqrt{q}} \cdot \sqrt{\frac{p\chi}{1+\varepsilon s} + 1 - q} \tag{8.15}$$

as well as

$$\phi_\varepsilon^{(3)}(s) := \sqrt{\frac{4(1-p)q - 4q^2 - p\frac{(1-p)^2\chi^2}{(1+\varepsilon s)^2}}{pq(\frac{p\chi}{1+\varepsilon s} + 1 - q)}}$$

and

$$\phi_\varepsilon^{(4)}(s) := \frac{2[(1-p)\varepsilon s - p]\chi}{q(1+\varepsilon s)^2}$$

for $\varepsilon \in (0,1)$ and $s \geq 0$, we first observe that $\phi_\varepsilon^{(k)}$ is well-defined for $k \in \{1,2,3,4\}$ with

$$\begin{aligned}
\phi_\varepsilon^{(1)} &\to c_1 := \frac{(1-p)\chi + 2q}{\sqrt{pq\chi - q(1-q)}}, \\
\phi_\varepsilon^{(2)} &\to c_2 := \frac{2\sqrt{p\chi + 1 - q}}{\sqrt{q}}, \\
\phi_\varepsilon^{(3)} &\to c_3 := \sqrt{\frac{4(1-p)q - 4q^2 - p(1-p)^2\chi^2}{p[pq\chi - q(1-q)]}} \qquad \text{and} \\
\phi_\varepsilon^{(4)} &\to c_4 := -\frac{2p\chi}{q}
\end{aligned} \tag{8.16}$$

in $L^\infty_{loc}([0,\infty))$ as $\varepsilon \searrow 0$. Using these auxiliary functions, given $T > 0$ we now invoke Lemma 4.3 to fix $c_5 > 0$ and $c_6 > 0$ such that

$$\int_0^T \int_\Omega \Big| \phi_\varepsilon^{(1)}(u_\varepsilon) v_\varepsilon^{\frac{q}{2}} \nabla u_\varepsilon^{\frac{p}{2}} - \phi_\varepsilon^{(2)}(u_\varepsilon) u_\varepsilon^{\frac{p}{2}} \nabla v_\varepsilon^{\frac{q}{2}} \Big|^2 \leq c_5 \tag{8.17}$$

and

$$\int_0^T \int_\Omega u_\varepsilon^{p+1} v_\varepsilon^{q-1} \leq c_6 \tag{8.18}$$

for all $\varepsilon \in (0,1)$.
Next, since evidently $(\phi_\varepsilon^{(1)})_{\varepsilon\in(0,1)}$, $(\phi_\varepsilon^{(3)})_{\varepsilon\in(0,1)}$ and $(\phi_\varepsilon^{(4)})_{\varepsilon\in(0,1)}$ are bounded in $L^\infty((0,\infty))$, three applications of Lemma 8.7 on the basis of (8.16) show that

$$\phi_\varepsilon^{(1)}(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} \rightharpoonup c_1 v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} \qquad \text{in } L^2(\Omega\times(0,T)) \tag{8.19}$$

and

$$\phi_\varepsilon^{(3)}(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} \rightharpoonup c_3 v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} \qquad \text{in } L^2(\Omega\times(0,T)) \tag{8.20}$$

as well as

$$\phi_\varepsilon^{(4)}(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} \rightharpoonup c_4 v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} \qquad \text{in } L^2(\Omega\times(0,T)) \tag{8.21}$$

as $\varepsilon = \varepsilon_j \searrow 0$. We moreover observe that (8.1) implies that $c_2 \ge \phi_\varepsilon^{(2)}(u_\varepsilon) \to c_2$ a.e. in $\Omega\times(0,T)$ as $\varepsilon=\varepsilon_j\searrow 0$, so that since

$$\int_0^T\int_\Omega \Big|\phi_\varepsilon^{(2)}(u_\varepsilon)u_\varepsilon^{\frac{p}{2}}\Big|^{\frac{2}{p}} \le c_2^{\frac{2}{p}}\int_0^T\int_\Omega u_\varepsilon = c_2^{\frac{2}{p}}T\int_\Omega u_0 \qquad \text{for all } \varepsilon\in(0,1)$$

by (3.2), the Vitali convergence theorem ensures that $\phi_\varepsilon^{(2)}(u_\varepsilon)u_\varepsilon^{\frac{p}{2}} \to c_2 u^{\frac{p}{2}}$ in $L^2(\Omega\times(0,T))$ as $\varepsilon=\varepsilon_j\searrow 0$ due to the fact that $\frac{2}{p}>2$. Since $\nabla v_\varepsilon^{\frac{q}{2}} \rightharpoonup \nabla v^{\frac{q}{2}}$ in $L^2(\Omega\times(0,T))$ as $\varepsilon=\varepsilon_j\searrow 0$ according to Lemma 8.1, we thus infer that

$$\phi_\varepsilon^{(2)}(u_\varepsilon)u_\varepsilon^{\frac{p}{2}}\nabla v_\varepsilon^{\frac{q}{2}} \rightharpoonup c_2 u^{\frac{p}{2}}\nabla v^{\frac{q}{2}} \qquad \text{in } L^1(\Omega\times(0,T))$$

as $\varepsilon=\varepsilon_j\searrow 0$. But since $(\phi_\varepsilon^{(1)}(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} - \phi_\varepsilon^{(2)}(u_\varepsilon)u_\varepsilon^{\frac{p}{2}}\nabla v_\varepsilon^{\frac{q}{2}})_{\varepsilon\in(0,1)}$ is relatively compact with respect to the weak topology in $L^2(\Omega\times(0,T))$ by (8.17), together with (8.19) the latter guarantees that in fact

$$\phi_\varepsilon^{(2)}(u_\varepsilon)u_\varepsilon^{\frac{p}{2}}\nabla v_\varepsilon^{\frac{q}{2}} \rightharpoonup c_2 u^{\frac{p}{2}}\nabla v^{\frac{q}{2}} \qquad \text{in } L^2(\Omega\times(0,T) \tag{8.22}$$

as $\varepsilon=\varepsilon_j\searrow 0$. Along with e.g. (8.20), this particularly asserts the regularity requirements in (2.2), whereas those in (2.1) result from (8.18) and the fact that by Young's inequality, (3.2) and Lemma 3.4 there exists $c_7>0$ such that

$$\int_0^T\int_\Omega \Big|u_\varepsilon^p v_\varepsilon^q\Big|^{\frac{1}{p+q}} \le \int_0^T\int_\Omega u_\varepsilon + \int_0^T\int_\Omega v_\varepsilon \le c_7 \qquad \text{for all } \varepsilon\in(0,1)$$

and hence, as $\varepsilon=\varepsilon_j\searrow 0$,

$$u_\varepsilon^p v_\varepsilon^q \to u^p v^q \qquad \text{in } L^1(\Omega\times(0,T)) \tag{8.23}$$

thanks to Lemma 8.1 and again the Vitali convergence theorem, because $p+q<p+q_+(p)<1$.

Positivity properties of u, v and u^pv^q, and the validity of (2.6) are ensured by Lemma 8.6 and Lemma 8.2, respectively.

Now for the verification of (2.3) we let $0\le\varphi\in C_0^\infty(\overline{\Omega}\times[0,\infty))$ be such that $\frac{\partial\varphi}{\partial\nu}=0$ on $\partial\Omega\times(0,\infty)$ and fix $T>0$ such that $\varphi\equiv 0$ in $\Omega\times(T,\infty)$. Then an application of Lemma 4.1 shows that

$$\int_0^T\int_\Omega \Big|\phi_\varepsilon^{(3)}(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}}\Big|^2\varphi + \int_0^T\int_\Omega \Big|\phi_\varepsilon^{(1)}(u_\varepsilon)v_\varepsilon^{\frac{q}{2}}\nabla u_\varepsilon^{\frac{p}{2}} - \phi_\varepsilon^{(2)}(u_\varepsilon)u_\varepsilon^{\frac{p}{2}}\nabla v_\varepsilon^{\frac{q}{2}}\Big|^2\varphi + q\int_0^T\int_\Omega u_\varepsilon^{p+1}v_\varepsilon^{q-1}\varphi$$

$$
\begin{aligned}
= & -\int_0^T\int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi_t - \int_\Omega u_0^p v_0^q \varphi(\cdot,0) \\
& -\int_\Omega \phi_\varepsilon^{(4)}(u_\varepsilon) u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}} \cdot \nabla\varphi - \int_0^T\int_\Omega \Big(1-\frac{p\chi}{q(1+\varepsilon u_\varepsilon)}\Big) u_\varepsilon^p v_\varepsilon^q \Delta\varphi \\
& +q\int_0^T\int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi \qquad \text{for all } \varepsilon\in(0,1),
\end{aligned}
\tag{8.24}
$$

where again employing (8.23) we see that

$$
-\int_0^T\int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi_t \to -\int_0^T\int_\Omega u^p v^q \varphi_t \tag{8.25}
$$

and

$$
q\int_0^T\int_\Omega u_\varepsilon^p v_\varepsilon^q \varphi \to q\int_0^T\int_\Omega u^p v^q \varphi \tag{8.26}
$$

as well as

$$
-\int_0^T\int_\Omega \Big(1-\frac{p\chi}{q(1+\varepsilon u_\varepsilon)}\Big) u_\varepsilon^p v_\varepsilon^q \Delta\varphi \to -\Big(1-\frac{p\chi}{q}\Big)\int_0^T\int_\Omega u^p v^q \Delta\varphi \tag{8.27}
$$

as $\varepsilon=\varepsilon_j \searrow 0$, the derivation of the latter additionally relying on an application of the dominated convergence theorem. Since (8.23) together with Lemma 8.1 clearly warrants that $u_\varepsilon^{\frac{p}{2}} v_\varepsilon^{\frac{q}{2}} \to u^{\frac{p}{2}} v^{\frac{q}{2}}$ in $L^2(\Omega\times(0,T))$ and hence

$$
\begin{aligned}
-\int_0^T\int_\Omega \phi_\varepsilon^{(4)}(u_\varepsilon) u_\varepsilon^{\frac{p}{2}} v_\varepsilon^q \nabla u_\varepsilon^{\frac{p}{2}} \cdot \nabla\varphi &= -\int_0^T\int_\Omega \Big(u_\varepsilon^{\frac{p}{2}} v_\varepsilon^{\frac{q}{2}}\Big)\cdot\Big(\phi_\varepsilon^{(4)}(u_\varepsilon) v_\varepsilon^{\frac{q}{2}} \nabla u_\varepsilon^{\frac{p}{2}}\Big)\cdot\nabla\varphi \\
&\to -\int_0^T\int_\Omega \Big(u^{\frac{p}{2}} v^{\frac{q}{2}}\Big)\cdot\Big(c_4 v^{\frac{q}{2}} \nabla u^{\frac{p}{2}}\Big)\cdot\nabla\varphi \\
&= -c_4\int_0^T\int_\Omega u^{\frac{p}{2}} v^q \nabla u^{\frac{p}{2}} \cdot\nabla\varphi
\end{aligned}
$$

as $\varepsilon=\varepsilon_j \searrow 0$, in view of Fatou's lemma and a standard argument based on lower semicontinuity of the norm in $L^2(\Omega\times(0,T))$ with respect to weak convergence it follows from (8.24), (8.20), (8.19), (8.22) and (8.25)-(8.27) that

$$
\begin{aligned}
&\int_0^T\int_\Omega \Big|c_3 v^{\frac{q}{2}}\nabla u^{\frac{p}{2}}\Big|^2\varphi + \int_0^T\int_\Omega \Big|c_1 v^{\frac{q}{2}}\nabla u^{\frac{p}{2}} - c_2 u^{\frac{p}{2}}\nabla v^{\frac{q}{2}}\Big|^2\varphi + q\int_0^T\int_\Omega u^{p+1}v^{q-1}\varphi \\
&\qquad \le -\int_0^T\int_\Omega u^p v^q\varphi_t - \int_\Omega u_0^p v_0^q\varphi(\cdot,0) \\
&\qquad\quad -c_4\int_0^T\int_\Omega u^{\frac{p}{2}} v^q\nabla u^{\frac{p}{2}}\cdot\nabla\varphi - \Big(1-\frac{p\chi}{q}\Big)\int_0^T\int_\Omega u^p v^q\Delta\varphi \\
&\qquad\quad +q\int_0^T\int_\Omega u^p v^q\varphi,
\end{aligned}
$$

which is equivalent to (2.3) and thus completes the proof.

□

Our main result thereby becomes evident.

PROOF of Theorem 1.1. We only need to combine Lemma 8.2 with Lemma 8.8. □

9 Appendix: A Poincaré inequality in non-convex domains

Our proof of Lemma 8.5 relies on Lemma 8.4, which in its original formulation in [18, Lemma 4.2] requires convexity of the domain due to the version of Poincaré's inequality ([10, Corollary 9.1.4]) used. In this appendix we state this Poincaré inequality without any such convexity condition, and since we could not find any reference to this in the literature, we briefly outline an argument. Here and in the following, by u_X we denote the average $\frac{1}{|X|}\int_X u(x)dx$ for $u \in L^1(\Omega)$ and any measurable set $X \subset \Omega$ with positive measure.

Lemma 9.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded with smooth boundary, and let $\delta > 0$ and $p \in [1,\infty)$. Then there exists $C = C(\Omega,\delta,p)$ with the property that for all $u \in W^{1,p}(\Omega)$,*

$$\left(\int_\Omega |u-u_B|^p\right)^{\frac{1}{p}} \le C(\Omega,\delta,p)\left(\int_\Omega |Du|^p\right)^{\frac{1}{p}}$$

holds for any measurable set $B \subset \Omega$ with $|B| = \delta$.

A derivation of this can be based on the following.

Lemma 9.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\delta > 0$. Then there exists $C > 0$ such that for every measurable set $B \subset \Omega$ with $|B| = \delta$, and for each $u \in W^{1,1}(\Omega)$ we have*

$$|u(x)-u_B| \le C\int_\Omega \frac{|Du(z)|}{|x-z|^{n-1}}dz \tag{9.1}$$

for almost every $x \in \Omega$.

Proof. We follow the proof of [7, Theorem 10], where (9.1) is shown for $B = \Omega$, and indicate necessary changes. With B_0 being a certain ball in Ω, defined as in the proof of [7, Theorem 10], in [7, (14)] it is shown that there is $c_1 > 0$ such that for every $u \in W^{1,1}(\Omega)$ and almost every $x \in \Omega$ the estimate

$$|u(x)-u_{B_0}| \le c_1\int_\Omega \frac{|\nabla u(z)|}{|x-z|^{n-1}}dz \tag{9.2}$$

holds, whose proof relies on a Poincaré inequality on balls that takes into account the dependence of the constant on the radius and on the existence of a chain of balls connecting B_0 with x that allows for certain estimates independently of x (cf. [7, p. 119]). Whereas the first summand in the right-hand side of

$$|u(x)-u_B| \le |u(x)-u_{B_0}| + |u_{B_0}-u_B| \tag{9.3}$$

is immediately covered by (9.2), as to the second we observe that, again by (9.2),

$$|u_{B_0}-u_B| \le \frac{1}{|B|}\int_B |u_{B_0}-u(y)|dy$$

$$\leq \frac{c_1}{|B|}\int_B\int_\Omega \frac{|\nabla u(z)|}{|y-z|^{n-1}}dzdy$$
$$\leq \frac{c_1}{|B|}\int_\Omega |\nabla u(z)|\int_B \frac{1}{|y-z|^{n-1}}dydz. \tag{9.4}$$

In estimating $\int_B \frac{1}{|y-z|^{n-1}}dy$ we employ the fact that with some $c_2 = c(n)$, for all $z \in \mathbb{R}^n$ and any measurable $E \subset \mathbb{R}^n$, $\int_E \frac{dy}{|y-z|^{n-1}} \leq c_2|E|^{\frac{1}{n}}$ ([7, (13)]), because if $\tilde{E}$ is a ball centered in z with $|E| = |\tilde{E}|$ and radius $R = \tilde{c}_2(n)|E|^{\frac{1}{n}}$,

$$\int_E \frac{dy}{|y-z|^{n-1}} \leq \int_{\tilde{E}} \frac{dy}{|y-z|^{n-1}} = \int_0^R r^{n-1}\frac{1}{r^{n-1}}dr = c_2|\tilde{E}|^{\frac{1}{n}}.$$

We moreover use that $1 \leq \frac{(\operatorname{diam}\Omega)^{n-1}}{|x-z|^{n-1}}$ for any $x, z \in \Omega$. With these observations, (9.4) turns into

$$|u_{B_0} - u_B| \leq \frac{c_1c_2}{|B|}|B|^{\frac{1}{n}}\int_\Omega |\nabla u(z)|dz \leq c_1c_2|B|^{\frac{1}{n}-1}(\operatorname{diam}\Omega)^{n-1}\int_\Omega \frac{|\nabla u(z)|}{|x-z|^{n-1}}dz. \tag{9.5}$$

Noting that $|B|^{\frac{1}{n}-1} = \delta^{\frac{1}{n}-1}$ and combining (9.2) and (9.5) with (9.3) proves (9.1). □

Proof of Lemma 9.1. For convex domains, this is exactly Corollary 9.1.4 of [10], which follows from [10, Lemma 9.1.3] and [10, Lemma 9.1.2], the latter of which (a continuity property of the Riesz potential operator) poses no convexity condition on Ω. As replacement of the former, in the case of general Ω we now rather rely on Lemma 9.2. □

Remark 9.3. *In Lemma 9.2 (and hence in Lemma 9.1), for the domain it is actually sufficient to be (bounded and) a John domain, instead of having smooth boundary. In particular, any bounded domain satisfying the interior cone condition is admissible in these lemmata. For details concerning this, we once more refer the reader to [7].*

Remark 9.4. *With Lemma 9.1, it is also possible to remove the convexity condition on the domain in [18].*

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