Critical mass for infinite-time aggregation in a chemotaxis model with indirect signal production

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Abstract

We study the Neumann initial-boundary problem for the chemotaxis system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \mu(t) + w, & x \in \Omega,\ t > 0, \\ \tau w_t + \delta w = u, & x \in \Omega,\ t > 0, \end{cases} \qquad (\star)$$

in the unit disk $\Omega := B_1(0) \subset \mathbb{R}^2$, where $\delta \geq 0$ and $\tau > 0$ are given parameters and $\mu(t) := \fint_\Omega w(x,t)dx$, $t > 0$.

It is shown that this problem exhibits a novel type of critical mass phenomenon with regard to the formation of singularities, which drastically differs from the well-known threshold property of the classical Keller-Segel system, as obtained upon formally taking $\tau \to 0$, in that it refers to blow-up in infinite time rather than in finite time:
Specifically, it is first proved that for any sufficiently regular nonnegative initial data u_0 and w_0, $(\star)$ possesses a unique global classical solution. In particular, this shows that in sharp contrast to classical Keller-Segel-type systems reflecting immediate signal secretion by the cells themselves, the indirect mechanism of signal production in $(\star)$ entirely rules out any occurrence of blow-up in finite time.
However, within the framework of radially symmetric solutions it is next proved that

- whenever $\delta > 0$ and $\int_\Omega u_0 < 8\pi\delta$, the solution remains uniformly bounded, whereas
- for any choice of $\delta \geq 0$ and $m > 8\pi\delta$, one can find initial data such that $\int_\Omega u_0 = m$, and such that for the corresponding solution we have

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \qquad \text{as } t \to \infty.$$

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1 Introduction

A chemotaxis model with indirect signal production. Chemotaxis, the biased movement of cells along concentration gradients of a chemical signal, is known to play a significant role in numerous biological circumstances such as bacterial aggregation, spatial pattern formation, embryonic morphogenesis, cell sorting, immune response, wounding healing, tumor-induced angiogenesis, and also tumor invasion (see [35], [20], [28], [11], [1], [10], [6] and [7], for instance). The renowned Keller-Segel model (cf. (1.4) below), describing the collective behavior of cells in response to a signal produced by the cells themselves, has been well-studied with regard to biological implications, but beyond this, during the last decades quite a thorough comprehension of its mathematical features has grown in various directions ([35], [13], [2]).

In contrast to this well-understood paradigmatic case, the theoretical understanding is much less developed in situations when a chemotactic cue is not released by the cells themselves. Typical examples for such mechanisms include cases when the signal is not produced at all, such as in oxygenotaxis processes of swimming aerobic bacteria which preferably move toward higher concentrations of externally provided oxygen as their nutrient ([36]), and also cases in which signal production occurs within more complex processes, possibly involving chemical reactions or even cascades thereof, such as e.g. in the glycolysis reaction ([9], [29]; cf. also [23] and [5] for further extensions of chemotaxis models involving additional couplings.

It is the purpose of the present work to achieve some insight into possible features of chemotaxis models accounting for the latter type of more complex signal production mechanisms. Specifically, we shall be concerned with the prototypical parabolic-elliptic-ODE system

$$\begin{cases} u_t = \Delta u - \nabla\cdot(u\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v - \mu(t) + w, & x\in\Omega,\ t>0,\\ \tau w_t + \delta w = u, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0(x), \quad w(x,0)=w_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

in the unit disk $\Omega := B_1 := B_1(0) \subset \mathbb{R}^2$, where $\delta \geq 0$ and $\tau > 0$ are fixed parameters and

$$\mu(t) := \fint_\Omega w(x,t)dx, \qquad t>0. \tag{1.2}$$

In a concrete biological framework, this model arises as a simplification of the chemotaxis model recently proposed by Strohm, Tyson and Powell in [32] to describe the spread and aggregative behavior of the Mountain Pine Beetle (MPB) in a forest habitat considered negligibly thin in its vertical dimension. Their model involves three variables: the density of flying MPB, denoted by u, the density of nesting MPB, represented by w, and the concentration v of beetle pheromone, the latter being secreted only by those MBP which are nested in trees. Besides random diffusive motion, the flying MPB can partially orient their movement according to concentration gradients of MPB pheromone. Once MPB nest they do not move any longer, thus meaning that apart from the increase of w through transition from the flying to the nested state, the only further quantity relevant to their evolution remains their death rate δ. For more details on the physical background, we refer the reader to [32, Section 2].

From a mathematical point of view, (1.1) can be viewed as a variant of the Keller-Segel model associated with the system

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - \widetilde{\mu} + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where $\widetilde{\mu} := \fint_\Omega u \equiv \fint_\Omega u_0$. which can formally be obtained from (1.1) upon taking $\tau \searrow 0$. In the case when Ω coincides with the entire space $\mathbb{R}^2$ or $\mathbb{R}^3$, the corresponding limit system of the latter arises in the modeling of self-gravitating particles ([4]), and furthermore it was introduced in [16] as a simplification of the well-known classical Keller-Segel model ([17]) of chemotaxis, the original version of which being

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0. \end{cases} \tag{1.4}$$

Here the hypotheses justifying the reduction of (1.4) to (1.3), namely the physically meaningful assumptions that chemicals diffuse much faster than cells, and that the particular signal substance in question degrades sufficiently slowly, have been used in various related contexts and are also part of the simplification of the original model in [32] to (1.1) (cf. also the review paper [14]).

Let us emphasize here the evident difference between (1.1) for $\tau > 0$ on the one hand and the two-component Keller-Segel systems (1.3) and (1.4) on the other: In both of the latter, the quantity u *directly* produces the quantity v governing its cross-diffusion, whereas the corresponding signal production in (1.1) occurs in an *indirect* process, with first u producing the third quantity w, and with the latter being exclusively responsible for the release of v.

Blow-up and critical mass phenomena. It is known that chemotactic cross-diffusion terms, constituting the apparently most characteristic model ingredient in all systems (1.1), (1.3) and (1.4), may have a strong destabilizing potential and even enforce the formation of singularities. Correspondingly, a striking feature of both Keller-Segel systems (1.3) and (1.4) appears to be the occurrence of some solutions blowing up in finite time, which is commonly viewed as mathematically expressing numerous processes of spontaneous cell aggregation which can be observed in experiments (see [13] and also [2] for a survey). Indeed, in the spatially two-dimensional framework considered here, the appearance of such explosion phenomena is closely related to the initially present total mass $\int_\Omega u_0$ of cells. For instance, it was shown in [16] and [3] that in the spatially radial setting, the system (1.3) possesses some solutions which blow up in finite time provided that this mass $\int_\Omega u_0$ is large enough, whereas solutions remain bounded whenever $\int_\Omega u_0$ is small; as a precise value distinguishing the respective mass regimes either allowing for or suppressing explosions the *critical mass* $m_c = 8\pi$ could be identified (cf. [3], [26] and [30] for (1.3) and closely related variants thereof).

As for the fully parabolic chemotaxis system (1.4), an analogous critical mass phenomenon is known to occur, the respective threshold value again being $m_c = 8\pi$ in the radially symmetric situation. For corresponding results on boundedness in the subcritical regime we refer to [25]; some quite particular blow-up solutions with $\int_\Omega u_0 > 8\pi$ have been detected in [12], whereas recently in [22] it was shown that such a singularity formation indeed occurs within a considerably large set of supercritical-mass initial data, which can even be viewed generic in an appropriate sense.

In the nonradial setting, corresponding critical mass phenomena seem to be present, with a reduced value of $m_c = 4\pi$. For parabolic-elliptic Keller-Segel systems, rigorous proofs for this can be found in, or easily adapted from [26] and [25]; in the parabolic-parabolic case, only a respective boundedness result is available in the case $\int_\Omega u_0 < 4\pi$ ([25]), whereas the occurrence of any nonradial finite-time blow-up solution to (1.4) appears to be a challenging open problem (cf. [15] for a partial result on unboundedness).

Let us mention that in the spatially one-dimensional versions of both (1.3) and (1.4), all solutions emanating from conveniently smooth initial data are global in time and remain uniformly bounded ([27]), while in three- or higher-dimensional balls, for arbitrarily small values of $m > 0$ one can find smooth initial data fulfilling $\int_\Omega u_0 = m$, for which the corresponding solution will blow up in finite time (see [24] for a parabolic-elliptic and [37] for the fully parabolic case). A critical mass phenomenon thus occurs only in the two-dimensional situation.

Main results. A novel type of critical mass phenomenon. It is the purpose of the present paper to rigorously investigate the qualitative features of the system (1.1) with regard to its original intention to model processes of aggregation. Here our focus will be on the question in how far the indirect signal production mechanism in (1.1) can enforce singularity formation in the first solution component u. Our main results in this direction show that actually also (1.1) exhibits a type of critical mass phenomenon, but that the latter appears to be novel in the context of chemotaxis problems: Surprisingly, namely, unlike that for (1.3) and (1.4), the mass threshold property we shall identify here will refer to blow-up *in infinite time* rather than in finite time.

Indeed, by deriving energy-type estimates through rather straightforward testing procedures we can first show that for all reasonably regular initial data with arbitrary mass $\int_\Omega u_0$, (1.1) is globally classically solvable:

Proposition 1.1 *Let $\delta \ge 0$ and $\tau > 0$, and suppose that $u_0 \in C^0(\bar\Omega)$ and $w_0 \in C^1(\overline{\Omega})$ are nonnegative. Then there exists a unique triple (u, v, w) of nonnegative functions*

$$
\begin{aligned}
&u \in C^0(\overline{\Omega} \times [0,\infty)) \cap C^{2,1}(\overline{\Omega} \times (0,\infty)),\\
&v \in C^{2,0}(\overline{\Omega} \times [0,\infty)),\\
&w \in C^{0,1}(\overline{\Omega} \times [0,\infty)),
\end{aligned}
$$

which solves (1.1) in the classical sense.

We shall next detect the number

$$m_c := 8\pi\delta$$

to be critical with regard to boundedness of radial solutions. The first part of this characterization is contained in the following.

Theorem 1.2 *Let $\delta > 0$ and $\tau > 0$, and suppose that $u_0 \in C^0(\overline{\Omega})$ and $w_0 \in C^1(\overline{\Omega})$ are radially symmetric and nonnegative, and that $m := \int_\Omega u_0$ satisfies*

$$m < 8\pi\delta.$$

Then the solution of (1.1) is bounded in $\Omega \times (0,\infty)$*; that is, there exists a constant* $C>0$ *such that*

$$u(x,t) \le C, \quad v(x,t) \le C \quad \text{and} \quad w(x,t) \le C \qquad \text{for all } x \in \Omega \text{ and } t>0.$$

Secondly, the above picture is completed by our final statement: In fact, for any $m > 8\pi\delta$ we shall derive an essentially explicit condition on the radial initial data u_0 and w_0 which under the assumption $\int_\Omega u_0 = m$ ensures that in the large time limit, the solution diverges exponentially in its first component when measured in $L^\infty(\Omega)$:

Theorem 1.3 *Let* $\delta \ge 0$ *and* $\tau > 0$*. Then for any*

$$m > 8\pi\delta$$

there exist $R \in (0,1)$ *and* $\alpha > 0$ *such that for each* $\eta > 0$ *one can find positive constants* $\Gamma_u(m,\eta), \gamma(m,\eta)$ *and* $\Gamma_w(m,\eta)$ *with the property that for all radially symmetric nonnegative functions* $u_0 \in C^0(\overline{\Omega})$ *and* $w_0 \in C^1(\overline{\Omega})$ *satisfying*

$$\int_\Omega u_0 = m > 8\pi\delta \tag{1.5}$$

and

$$⨍_{B_r} u_0 \ge \Gamma_u(m,\eta) \qquad \text{for all } r \in (0,R) \tag{1.6}$$

and

$$⨍_{B_1\setminus B_r} u_0 \le \gamma \qquad \text{for all } r \in (R,1) \tag{1.7}$$

as well as

$$⨍_{B_r} w_0 \ge ⨍_{B_1} w_0 + \Gamma_w(m,\eta) \qquad \text{for all } r \in (0,R) \tag{1.8}$$

and

$$⨍_{B_1\setminus B_r} w_0 \le ⨍_{B_1} w_0 - \eta \qquad \text{for all } r \in (R,1), \tag{1.9}$$

the corresponding solution (u,v,w) *of (1.1) is unbounded in the sense that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \ge ce^{\alpha t} \qquad \text{for all } t>0$$

with some $c = c(m,\eta,\delta,\tau,\|w_0\|_{L^1(\Omega)}) > 0$*.*

As a particular consequence, this provides some quantitative information on the damping role of the death rate δ in (1.1). For instance, it follows from Theorem 1.2 that for any given initial data (u_0,w_0) the associated solution will remain bounded whenever $\delta > 0$ is suitably large. On the other hand, if δ vanishes then unbounded solutions can be found for arbitrarily small values of the initial mass $\int_\Omega u_0$.

Moreover, the criticality of $m_c = 8\pi\delta$, as thus detected to predict the possibility or impossibility of aggregation phenomena in (1.1) for positive values of τ and δ, appears to be consistent with the above mass threshold properties of (1.3): Indeed, in the limit case $\tau = 0$, in which in (1.1) clearly any initial condition on w becomes obsolete, we will have $w \equiv \frac{1}{\delta}u$. Hence, upon substituting $\tilde{u} := \frac{1}{\delta}u$ we see that we may assume that $\delta = 1$, and that (1.1) reduces to (1.3), having critical mass $m_c = 8\pi = 8\pi\delta$; the fact that m_c is then related to finite-time blow-up, rather than to inifinite-time aggregation, may be viewed as a consequence of the lacking relaxation mechanism reflected in the ODE for w in (1.1) when $\tau > 0$. In summary, varying τ over the interval $[0, \infty)$ does not change the *value* of the critical mass, but it significantly affects its precise *role* when passing from positive τ to the case $\tau = 0$.

Main ideas underlying our approach. Let us briefly outline the methods we pursue in the derivation of Theorem 1.2 and Theorem 1.3. Our approach to both of these will be based on a transformation reducing (1.1) to an initial-boundary value problem for a scalar degenerate parabolic equation. Though well-established in related contexts, this transformation results in an equation which, unlike the corresponding situation in the standard Keller-Segel system (1.3) ([16]), now contains a nonlinear production term that is nonlocal in time. More precisely, we shall see that the mass distribution function U associated with a given radial solution $u = u(r,t)$ of (1.1), that is, the function defined by

$$U(\xi, t) := \int_0^{\sqrt{\xi}} ru(r,t)dr, \qquad \xi \in [0,1],\ t \geq 0,$$

satisfies the single equation

$$U_t = 4\xi U_{\xi\xi} + \frac{2}{\tau}\left\{ \int_0^t e^{-\frac{\delta}{\tau}(t-s)} \Big(U(\xi, s) - \frac{m}{2\pi}\xi \Big)\, ds \right\} \cdot U_\xi + 2\Big(W_0(\xi) - K_0\xi \Big) \cdot e^{-\frac{\delta}{\tau}t} U_\xi, \tag{1.10}$$

for $\xi \in (0,1)$ and $t > 0$, where $W_0(\xi) := \int_0^{\sqrt{\xi}} rw_0(r)dr$, $\xi \in [0,1]$, and $K_0 := W_0(1)$ (cf. Lemma 4.1). Clearly, u is bounded if and only if the spatial gradient U_ξ is bounded. Fortunately, the corresponding parabolic operator allows for a comparison principle (Lemma 4.2), and thus enables us to focus our subsequent analysis on the construction of appropriate super- and subsolutions.

Based on such a comparison argument, under the subcriticality assumption $m < 8\pi\delta$ from Theorem 1.2 we shall first obtain an estimate of the form $U(\xi, t) \leq C\xi$ for all $(\xi, t) \in (0,1) \times (0, \infty)$ and some $C > 0$ (Lemma 5.2). This means that given $\varepsilon > 0$, adjusting $r_0 \in (0,1)$ suitably we can achieve that the mass which the original solution accumulates in the ball $B_{r_0}(0)$ satisfies $\int_{B_{r_0}(0)} u(x,t)dx < \varepsilon$ for all $t > 0$. In conjunction with a corresponding ε-regularity result (Section 5.3) this will yield the desired boundedness property of such solutions.

In the case $m > 8\pi\delta$ addressed in Theorem 1.3, we will construct subsolutions exhibiting gradient grow-up at the origin; that is, we shall find a family of adequate subsolutions $\underline{U}$ to (1.10) with the properties $\underline{U}(0,t) = 0$ for all $t > 0$ and $\underline{U}_\xi(0,t) \to +\infty$ as $t \to \infty$. Proving Theorem 1.3 then amounts to finding sufficient conditions for u_0 and w_0 ensuring that $U(\xi, 0) \geq \underline{U}(\xi, 0)$ for all $\xi \in (0,1)$.
We find it worthwhile to underline here that the structure near the origin of the latter comparison functions, to be explicitly constructed and analyzed in detail in Section 6, will be given by

$$\underline{U}(\xi, t) := \frac{a(t)\xi}{b(t) + \xi}, \qquad \xi \in [0, \xi_0),\ t \geq 0, \tag{1.11}$$

with $b(t) = b_0 e^{-\alpha t}$, $t \geq 0$, and appropriately chosen $a \in C^1([0,\infty))$, $\xi_0 \in (0,1)$, $b_0 > 0$ and $\alpha > 0$. The idea for this construction originates from standard knowledge on equilibria for the classical parabolic-elliptic Keller-Segel system obtained from (1.3) in the limit case $\Omega = \mathbb{R}^2$. Indeed, choosing $a \equiv 4$ and

$b \equiv const.$ in (1.11) one would rediscover a well-known family of explicit radial steady states for the corrseponding version of (1.3) ([19]).

2 Local existence

The following basic result on local existence of solutions to (1.1) can be proved by adapting approaches that are well-established in the context of parabolic-elliptic models for taxis mechanisms involving both cross-diffusion terms and ODE dynamics (cf. [34], [21], [18] and [8], for instance). Here we note that our assumption that w_0 belong to $C^1(\overline{\Omega})$ enables us to use standard elliptic Schauder theory to gain appropriate knowledge on the spatial regularity of v. Indeed, expressing w via the formula

$$w(x,t) = w_0(x)e^{-\frac{\delta}{\tau}t} + \frac{1}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}u(x,s)ds, \qquad x \in \Omega,\ t > 0, \tag{2.1}$$

we see that $v(\cdot,t)$ actually solves the Poisson equation with a temporally nonlocal inhomogeneity which thanks to the inclusion $w_0 \in C^1(\overline{\Omega})$ will be Hölder continuous in $\overline{\Omega}$ provided that $u(\cdot,t)$ is sufficiently regular, where the latter can be guaranteed by standard arguments involving appropriate smoothing properties of the Neumann heat semigroup in Ω.

Lemma 2.1 *Let $\delta \ge 0$, and suppose that $u_0 \in C^0(\bar{\Omega})$ and $w_0 \in C^1(\bar{\Omega})$ are nonnegative. Then there exist $T_{max} \in (0,\infty]$ and uniquely determined nonnegative functions*

$$\begin{aligned} &u \in C^0(\overline{\Omega} \times [0,T_{max})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{max})),\\ &v \in C^{2,0}(\overline{\Omega} \times [0,T_{max})),\\ &w \in C^{0,1}(\overline{\Omega} \times [0,T_{max})), \end{aligned}$$

which solve (1.1) classically in $\Omega \times (0,T_{max})$ and which are such that

$$\text{if } T_{max} < \infty, \text{ then } \|u(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \qquad \text{as } t \nearrow T_{max}. \tag{2.2}$$

The following identities describing the evolution of the total masses of the first and third components in (1.1) can easily be checked.

Lemma 2.2 *Let $\delta \ge 0$. Then the solution (u,v,w) of (1.1) satisfies*

$$\int_\Omega u(\cdot,t) = m := \int_\Omega u_0 \qquad \text{for all } t \in (0,T_{max}), \tag{2.3}$$

and for all $t \in (0,T_{max})$ we have

$$\begin{aligned} \int_\Omega w(\cdot,t) &= e^{-\frac{\delta}{\tau}t}\int_\Omega w_0 + \frac{m}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}ds \\ &= \begin{cases} e^{-\frac{\delta}{\tau}t}\int_\Omega w_0 + \frac{m}{\delta}(1-e^{-\frac{\delta}{\tau}t}) & \text{if } \delta > 0, \\ e^{-\frac{\delta}{\tau}t}\int_\Omega w_0 + \frac{m}{\tau}\cdot t & \text{if } \delta = 0, \end{cases} \end{aligned} \tag{2.4}$$

Proof. Integrating the first equation in (1.1) with respect to $x \in \Omega$, we see that $\frac{d}{dt}\int_\Omega u \equiv 0$, which immediately yields (2.3). Using this, we only need to integrate (2.1) in space to obtain (2.4). □

Based on (2.4) we can now explicitly rewrite the degradation term $\mu(t)$ in the second equation in (1.1).

Corollary 2.3 *Let $\delta \ge 0$. Then the function μ defined in (1.2) is given by*

$$\mu(t) = \frac{1}{\pi} e^{-\frac{\delta}{\tau}t} \int_\Omega w_0 + \frac{m}{\pi\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \qquad \textit{for all } t \in (0, T_{max}), \tag{2.5}$$

where $m := \int_\Omega u_0$.

3 Global existence

The following basic statement on the time evolution of the functional $\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1}$ will be the starting point for our derivation of bounds for u, and also for w, in spaces of the form $L^\infty((0, T_{max}); L^p(\Omega))$ with $p > 1$. Besides in Lemma 3.2, it will be referred to in Lemma 5.4 below.

Lemma 3.1 *Let $\delta \ge 0$. Then for all $p > 1$, the solution of (1.1) satisfies*

$$\frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1}\right\} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \delta\int_\Omega w^{p+1} \le \frac{p-1}{p}\int_\Omega u^p w + \int_\Omega u w^p \tag{3.1}$$

for all $t \in (0, T_{max})$.

Proof. We multiply the first equation in (1.1) by u^{p-1} and integrate by parts using the identity $\Delta v = \mu(t) - w$ to find that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 &= (p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v \\
&= -\frac{p-1}{p}\int_\Omega u^p \Delta v \\
&= -\frac{p-1}{p}\int_\Omega u^p \Big(\mu(t) - w\Big) \\
&\le \frac{p-1}{p}\int_\Omega u^p w \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{3.2}$$

because $\mu(t) \ge 0$ by Corollary 2.3. On the other hand, multiplying the third equation in (1.1) by w^p and integrating with respect to $x \in \Omega$ we see that

$$\frac{\tau}{p+1}\frac{d}{dt}\int_\Omega w^{p+1} + \delta\int_\Omega w^{p+1} = \int_\Omega u w^p \qquad \text{for all } t \in (0, T_{max}).$$

Adding this to (3.2) proves (3.1). □

Further estimating the terms on the right of (3.1) shows that the functional in question actually satisfies the following autonomous differential inequality.

Lemma 3.2 *Let $\delta \geq 0$. Then for any $p > 1$ there exists $C(p) > 0$ such that the solution of (1.1) satisfies*

$$\frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1}\right\} \leq C(p)\cdot\left\{\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1}\right\} \qquad \textit{for all } t \in (0, T_{max}). \quad (3.3)$$

Proof. Let us first invoke the Gagliardo-Nirenberg inequality to fix $c_1 > 0$ such that

$$\begin{aligned}
\int_\Omega \varphi^{p+1} &= \|\varphi^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\leq c_1\|\nabla\varphi^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \cdot \|\varphi^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + c_1\|\varphi^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&= c_1\|\nabla\varphi^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \cdot \|\varphi\|_{L^1(\Omega)} + c_1\|\varphi\|_{L^1(\Omega)}^{p+1} \qquad \text{for all nonnegative } \varphi \in W^{1,2}(\Omega). \quad (3.4)
\end{aligned}$$

We now let $\varepsilon := \frac{2(p-1)}{mc_1p^2}$ and use the Young inequality to estimate the two terms on the right of (3.1) according to

$$\frac{p-1}{p}\int_\Omega u^p w + \int_\Omega uw^p \leq 2\varepsilon\int_\Omega u^{p+1} + (\varepsilon^{-p} + \varepsilon^{-\frac{1}{p}})\int_\Omega w^{p+1} \qquad \text{for all } t \in (0, T_{max}). \quad (3.5)$$

Here since $\|u\|_{L^1(\Omega)} = \int_\Omega u = m$ for all $t \in (0, T_{max})$ due to Lemma 2.2, by the Hölder inequality and (3.4) we obtain

$$\begin{aligned}
2\varepsilon\int_\Omega u^{p+1} &\leq 2\varepsilon c_1 m\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + 2\varepsilon c_1 m\left(\int_\Omega u\right)^p \\
&\leq 2\varepsilon c_1 m\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + 2\varepsilon c_1 m\cdot|\Omega|^{\frac{p-1}{p}}\int_\Omega u^p \\
&= \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{4(p-1)}{p^2}\cdot|\Omega|^{\frac{p-1}{p}}\int_\Omega u^p \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Inserting this into (3.5) and recalling (3.1) proves (3.3). □

We are now in the position to assert our global existence result for (1.1).

Proof of Proposition 1.1. For any given $T \in (0, T_{max})$, the ODI (3.3) yields

$$\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1} \leq c_1(p,T) \qquad \text{for all } t \in (0,T)$$

with $c_1(p,T) := \left(\frac{1}{p}\int_\Omega u_0^p + \frac{\tau}{p+1}\int_\Omega w_0^{p+1}\right)\cdot e^{C(p)\cdot T}$, where $C(p) > 0$ is as defined by Lemma 3.2. Since $\tau > 0$, this immediately yields

$$\int_\Omega u^p \leq pc_2(p,T) \qquad \text{for all } t \in (0,T) \quad (3.6)$$

and

$$\int_\Omega w^{p+1} \leq c_2(p,T) \qquad \text{for all } t \in (0,T)$$

with $c_2(p,T) := \max\{p, \frac{p+1}{\tau}\} \cdot c_1(p,T)$. From the latter and standard elliptic regularity theory we obtain a bound for v in all spaces $L^\infty((0,T);W^{2,p}(\Omega))$ for any $p \in (1,\infty)$, whence in particular there exists $c_3(p,T) > 0$ such that

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le c_3(p,T) \qquad \text{for all } t \in (0,T).$$

Along with (3.6), this ensures that Lemma 4.1 in [33] becomes applicable so as to assert via a Moser-type iteration that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_4(p,T) \qquad \text{for all } t \in (0,T)$$

holds for some $c_4(p,T) > 0$. Finally, Proposition 1.1 is an evident consequence of this and the extensibility criterion in Lemma 2.1. □

4 Radial solutions. A comparison principle

Throughout the sequel we shall assume that the initial data u_0 and w_0, and hence clearly also all components of the solution (u,v,w), are radially symmetric with respect to the spatial origin, and unless stated otherwise we fix

$$m := \int_\Omega u_0. \tag{4.1}$$

Then without danger of confusion we may and will switch to the usual radial notation and write $u = u(r,t)$ for $r = |x| \in [0,1]$ whenever this appears convenient.

Lemma 4.1 *Suppose that $\delta \ge 0$, and given a radial solution (u,v,w) of (1.1), let*

$$U(\xi,t) := \int_0^{\sqrt{\xi}} r u(r,t)dr, \qquad \xi \in [0,1],\ t \ge 0. \tag{4.2}$$

Then

$$U(0,t) = 0 \quad \textit{as well as} \quad U(1,t) = \frac{m}{2\pi} \qquad \textit{for all } t \ge 0 \tag{4.3}$$

and

$$U_\xi(\xi,t) \ge 0 \qquad \textit{for all } \xi \in (0,1) \textit{ and } t > 0. \tag{4.4}$$

Moreover,

$$\mathcal{P}U(\xi,t) = 0 \qquad \textit{for all } \xi \in (0,1) \textit{ and } t > 0, \tag{4.5}$$

where the operator $\mathcal{P}$ is defined by setting

$$\mathcal{P}\widetilde{U}(\xi,t) := \widetilde{U}_t - 4\xi\widetilde{U}_{\xi\xi} - \frac{2}{\tau}\left\{\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\Big(\widetilde{U}(\xi,s) - \frac{m}{2\pi}\xi\Big)\,ds\right\} \cdot \widetilde{U}_\xi - 2\Big(W_0(\xi) - K_0\xi\Big) \cdot e^{-\frac{\delta}{\tau}t}\widetilde{U}_\xi \tag{4.6}$$

for $\xi \in (0,1), t > 0$ and $\widetilde{U} \in C^1((0,1)\times(0,\infty)) \cap C^0((0,\infty);W^{2,\infty}((0,1)))$, with

$$W_0(\xi) := \int_0^{\sqrt{\xi}} r w_0(r)dr, \quad \xi \in [0,1], \qquad \textit{and} \qquad K_0 := W_0(1) = \int_0^1 r w_0(r)dr. \tag{4.7}$$

PROOF. The boundary properties in (4.3) are immediate from (4.2) and (2.3), whereas the monotonicity statement in (4.4) is equivalent to the nonnegativity of u. Moreover, upon differentiation in (1.1) we see that for $\xi \in (0,1)$ and $t > 0$,

$$\begin{aligned} U_t(\xi,t) &= \int_0^{\sqrt{\xi}} r \cdot \Big\{ \frac{1}{r}(ru_r)_r - \frac{1}{r}(ruv_r)_r \Big\} dr \\ &= \sqrt{\xi} u_r(\sqrt{\xi},t) - \sqrt{\xi} u(\sqrt{\xi},t) v_r(\sqrt{\xi},t), \end{aligned}$$

where by (4.2) we have

$$u(\sqrt{\xi},t) = 2U_\xi(\xi,t) \qquad \text{and} \qquad u_r(\sqrt{\xi},t) = 4\sqrt{\xi} U_{\xi\xi}(\xi,t).$$

Since the second equation in (1.1) implies that

$$rv_r(r,t) = -\int_0^r \rho w(\rho,t) d\rho + \frac{\mu(t) r^2}{2} \qquad \text{for all } r \in (0,1) \text{ and } t > 0,$$

we thus obtain

$$U_t = 4\xi U_{\xi\xi} + 2U_\xi W - \mu(t) \xi U_\xi \qquad \text{for all } \xi \in (0,1) \text{ and } t > 0, \tag{4.8}$$

with $W(\xi,t) := \int_0^{\sqrt{\xi}} rw(r,t) dr$, $\xi \in [0,1], t \ge 0$. Now by (4.7) and (2.1),

$$W(\xi,t) = W_0(\xi) e^{-\frac{\delta}{\tau}t} + \frac{1}{\tau} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} U(\xi,s) ds \qquad \text{for all } \xi \in (0,1) \text{ and } t > 0,$$

whereas

$$\mu(t) = 2K_0 e^{-\frac{\delta}{\tau}t} + \frac{m}{\pi\tau} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \qquad \text{for all } t > 0$$

according to (2.5) and (4.7). Therefore,

$$\begin{aligned} 2U_\xi W - \mu(t)\xi U_\xi &= \frac{2}{\tau} \Big\{ \int_0^t e^{-\frac{\delta}{\tau}(t-s)} U(\xi,s) ds \Big\} \cdot U_\xi(\xi,t) + 2W_0(\xi) e^{-\frac{\delta}{\tau}t} U_\xi(\xi,t) \\ &\quad - 2K_0 \xi \cdot e^{-\frac{\delta}{\tau}t} U_\xi(\xi,t) - \Big\{ \int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \frac{m}{\pi\tau} \xi ds \Big\} \cdot U_\xi(\xi,t) \end{aligned}$$

for all $\xi \in (0,1)$ and $t > 0$, which along with (4.8) proves (4.5). □

Fortunately, the parabolic operator $\mathcal{P}$ introduced above falls into a class of operators allowing for a comparison principle. To see this, for functions A, B and D to be specified below, let us consider

$$\mathcal{Q}\widetilde{U}(\xi,t) := \widetilde{U}_t(\xi,t) - A(\xi,t)\widetilde{U}_{\xi\xi}(\xi,t) - \Big\{ B(\xi,t) + \int_0^t D(\xi,t,s) \widetilde{U}(\xi,s) ds \Big\} \cdot \widetilde{U}_\xi(\xi,t), \quad \xi \in (0,1),\ t \in (t_0,T), \tag{4.9}$$

for $0 \le t_0 < T$ and sufficiently regular $\widetilde{U} : (0,1) \times (0,T) \to \mathbb{R}$. Then assuming, besides parabolicity, that the memory term has a favorable sign, we can indeed derive the following comparison principle for spatially nondecreasing functions.

Lemma 4.2 *Let $t_0 \geq 0$ and $T > t_0$, and suppose that $A \in C^0((0,1)\times(t_0,T))$, $B \in C^0((0,1)\times(t_0,T))$ and $D \in C^0([0,1]\times[0,T]\times[0,T])$ satisfy*

$$A \geq 0 \quad \text{in } (0,1)\times(t_0,T) \qquad \text{and} \qquad D \geq 0 \quad \text{in } [0,1]\times[0,T]\times[0,T]. \tag{4.10}$$

Moreover, assume that $\underline{U}$ and $\overline{U}$ are nonnegative functions belonging to

$$C^0([0,1]\times[0,T]) \cap C^1((0,1)\times(t_0,T)) \cap C^0((t_0,T);W^{2,\infty}((0,1))), \tag{4.11}$$

which are such that

$$0 \leq \underline{U}_\xi(\xi,t) \leq M \qquad \text{for all } \xi \in (0,1) \text{ and } t \in (t_0,T) \tag{4.12}$$

with some $M > 0$, and such that with $\mathcal{Q}$ as defined in (4.9) we have

$$\mathcal{Q}\underline{U}(\cdot,t) \leq \mathcal{Q}\overline{U}(\cdot,t) \qquad \text{a.e. in } (0,1) \text{ for all } t \in (t_0,T). \tag{4.13}$$

Then if

$$\underline{U}(\xi,t) \leq \overline{U}(\xi,t) \qquad \text{for all } \xi \in [0,1] \text{ and } t \in [0,t_0] \tag{4.14}$$

as well as

$$\underline{U}(0,t) \leq \overline{U}(0,t) \quad \text{for all } t \in [t_0,T] \qquad \text{and} \qquad \underline{U}(1,t) \leq \overline{U}(1,t) \quad \text{for all } t \in [t_0,T], \tag{4.15}$$

we have the global ordering property

$$\underline{U}(\xi,t) \leq \overline{U}(\xi,t) \quad \text{for all } \xi \in [0,1] \text{ and } t \in [0,T]. \tag{4.16}$$

PROOF. We let $c_1 := \|D\|_{L^\infty((0,1)\times(0,T)\times(0,T))}$ and $\alpha := \sqrt{c_1 M}$ with M as in (4.12), and for arbitrary $\varepsilon_0 > 0$ we let

$$\varepsilon(t) := \varepsilon_0 e^{\alpha t}, \qquad t \geq 0, \tag{4.17}$$

and

$$d(\xi,t) := \underline{U}(\xi,t) - \overline{U}(\xi,t) - \varepsilon(t) \qquad \text{for } \xi \in [0,1] \text{ and } t \in [0,T].$$

Then d is continuous in $[0,1]\times[0,T]$ with

$$d(\xi,t) \leq -\varepsilon_0 e^{\alpha t} < 0 \qquad \text{for all } \xi \in [0,1] \text{ and } t \in [0,t_0]$$

by (4.14) and

$$d(\xi,t) \leq -\varepsilon_0 e^{\alpha t} < 0 \qquad \text{for } \xi \in \{0,1\} \text{ and } t \in [t_0,T]$$

according to (4.15). Thus,

$$t_\star := \sup\Big\{ t \in (0,T) \ \Big|\ d < 0 \text{ in } [0,1]\times[0,t] \Big\}$$

is well-defined and satisfies $t_\star \in (t_0, T]$, and if we had $t_\star < T$, then there would exist $\xi_\star \in (0,1)$ such that

$$d(\xi_\star, t_\star) = \max_{\xi \in [0,1]} d(\xi, t_\star) = 0, \tag{4.18}$$

whence evidently

$$d_t(\xi_\star, t_\star) \geq 0 \qquad \text{and} \qquad d_\xi(\xi_\star, t_\star) = 0, \tag{4.19}$$

because $d \in C^1((0,1) \times (t_0, T))$ by (4.11). Now by (4.13) we know that there exists a null set $N \subset (0,1)$ such that $d_{\xi\xi}(\xi, t_\star)$ exists for all $\xi \in (0,1) \setminus N$ and

$$\begin{aligned} d_t(\xi, t_\star) \quad &\leq \quad A(\xi, t_\star) d_{\xi\xi}(\xi, t_\star) + \left\{ B(\xi, t_\star) + \int_0^{t_\star} d(\xi, t_\star, s) \overline{U}(\xi, s) ds \right\} \cdot d_\xi(\xi, t_\star) \\ &\quad + \underline{U}_\xi(\xi, t_\star) \cdot \int_0^{t_\star} D(\xi, t_\star, s) d(\xi, s) ds \\ &\quad + \underline{U}_\xi(\xi, t_\star) \cdot \int_0^{t_\star} D(\xi, t_\star, s) \cdot \varepsilon(s) ds - \varepsilon'(t_\star) \qquad \text{for all } \xi \in (0,1) \setminus N. \end{aligned} \tag{4.20}$$

In order to make appropriate use of (4.19) and the maximality property in (4.18), we observe that (4.18) necessarily implies that there exists $(\xi_j)_{j \in \mathbb{N}} \subset (0,1) \setminus N$ such that $\xi_j \to \xi_\star$ as $j \to \infty$ and

$$d_{\xi\xi}(\xi_j, t_\star) \leq 0 \qquad \text{for all } j \in \mathbb{N},$$

for otherwise we would have $\mathrm{essliminf}_{\xi \to \xi_\star} \mathrm{d}_{\xi\xi}(\xi, \mathrm{t}_\star) > 0$, contradicting (4.18). Choosing $\xi = \xi_j$ in (4.20) and using that (4.19) and (4.10) entail that

$$\limsup_{j \to \infty} d_t(\xi_j, t_\star) \geq 0 \qquad \text{and} \qquad d_\xi(\xi_j, t_\star) \to 0 \qquad \text{as } j \to \infty,$$

we obtain on taking $j \to \infty$ that

$$0 \quad \leq \quad \underline{U}_\xi(\xi_\star, t_\star) \cdot \int_0^{t_\star} D(\xi_\star, t_\star, s) d(\xi_\star, s) ds + \underline{U}_\xi(\xi_\star, t_\star) \cdot \int_0^{t_\star} D(\xi_\star, t_\star, s) \cdot \varepsilon(s) ds - \varepsilon'(t_\star). \tag{4.21}$$

Here since $d(\xi_\star, s) < 0$ for all $s \in [0, t_\star)$ by definition of $t_\star$, we have

$$\underline{U}_\xi(\xi_\star, t_\star) \cdot \int_0^{t_\star} D(\xi_\star, t_\star, s) d(\xi_\star, s) ds \leq 0,$$

because $D \geq 0$ and $\underline{U}_\xi \geq 0$ by (4.10) and (4.12). Furthermore, (4.12) and our choice of c_1 ensure that

$$\underline{U}_\xi(\xi_\star, t_\star) \cdot \int_0^{t_\star} D(\xi_\star, t_\star, s) \cdot \varepsilon(s) ds \leq M c_1 \int_0^{t_\star} \varepsilon(s) ds,$$

so that recalling (4.17), from (4.21) we obtain

$$\begin{aligned} 0 \quad &\leq \quad M c_1 \int_0^{t_\star} \varepsilon_0 e^{\alpha s} ds - \alpha \varepsilon_0 e^{\alpha t_\star} \\ &= \quad \frac{M c_1 \varepsilon_0}{\alpha} (e^{\alpha t_\star} - 1) - \alpha \varepsilon_0 e^{\alpha t_\star} \\ &< \quad \Big(\frac{M c_1}{\alpha^2} - 1 \Big) \cdot \alpha \varepsilon_0 e^{\alpha t_\star}. \end{aligned}$$

Since $\frac{Mc_1}{\alpha^2} = 1$ according to our definition of α, this absurd conclusion shows that actually $t_\star = T$ and hence $\underline{U} \le \overline{U} + \varepsilon_0 e^{\alpha t}$ throughout $[0,1] \times [0,T]$. On taking $\varepsilon_0 \searrow 0$ we finally arrive at the desired inequality. □

5 Boundedness for $\int_\Omega u_0 < 8\pi\delta$. Proof of Theorem 1.2

In this section we shall make sure that small-mass solutions remain bounded in the sense of Theorem 1.2.

5.1 A pointwise upper bound for U

As a preliminary, let us prove the following elementary lemma.

Lemma 5.1 *Let $m > 0$ and $\varepsilon > 0$, and suppose that $\varphi \in W^{1,\infty}((0,1))$ is such that $\varphi(0) = 0$ and $\varphi(\xi) \le \frac{m}{2\pi}$ for all $\xi \in (0,1)$. Then there exists $b \in (0,1)$ such that*

$$\varphi(\xi) \le \frac{m}{2\pi} \cdot \frac{(b+1+\varepsilon)\xi}{b+\xi} \qquad \text{for all } \xi \in (0,1). \tag{5.1}$$

Proof. Since $\varphi(0) = 0$ and $\varphi_\xi \in L^\infty(\Omega)$, we can find $c_1 > 0$ such that $\varphi(\xi) \le c_1\xi$ for all $\xi \in (0,1)$, where we may assume that $c_1 > \frac{m}{2\pi}$. Therefore, our assumption warrants that

$$\varphi(\xi) \le \hat{\varphi}(\xi) := \min\left\{c_1\xi, \frac{m}{2\pi}\right\} \qquad \text{for all } \xi \in (0,1). \tag{5.2}$$

Now writing $\varphi_b(\xi) := \frac{m}{2\pi} \cdot \frac{(b+1+\varepsilon)\xi}{b+\xi}$ for $\xi \in [0,1]$ and $b \in (0,1)$, we see that the quotient $\frac{\hat{\varphi}}{\varphi_b}$ admits a continuous extension Q_b to all of $[0,1]$ such that

$$Q_b(\xi) = \begin{cases} \frac{2\pi}{m} \cdot \frac{c_1(b+\xi)}{b+1+\varepsilon} & \text{if } \xi \in [0,\xi_1], \\ \frac{b+\xi}{(b+1+\varepsilon)\xi} & \text{if } \xi \in (\xi_1,1], \end{cases}$$

where $\xi_1 := \frac{m}{2\pi c_1} \in (0,1)$ thanks to our choice of c_1. Since

$$\frac{\partial}{\partial b}\frac{b+\xi}{b+1+\varepsilon} = \frac{1+\varepsilon-\xi}{(b+1+\varepsilon)^2} > 0 \qquad \text{for all } \xi \in [0,1],$$

it follows that as $b \searrow 0$ we have

$$Q_b(\xi) \searrow Q(\xi) := \begin{cases} \frac{2\pi}{m} \cdot \frac{c_1\xi}{1+\varepsilon} & \text{if } \xi \in [0,\xi_1], \\ \frac{1}{1+\varepsilon} & \text{if } \xi \in (\xi_1,1]. \end{cases}$$

As Q is continuous in $[0,1]$, Dini's theorem asserts that the convergence $Q_b \to Q$ is actually uniform in $[0,1]$. Since $Q(\xi) \le \frac{1}{1+\varepsilon} < 1$ for all $\xi \in [0,1]$, we can therefore pick some sufficiently small $b \in (0,1)$ such that $Q_b(\xi) \le 1$ for all $\xi \in [0,1]$, which in view of (5.2) implies (5.1). □

By means of a comparison argument, we can now prove that under the assumption $\int_\Omega u_0 < 8\pi\delta$, it is possible to control the mass concentrating in small balls around the origin uniformly with respect to $t \in (0,\infty)$ in the following sense.

Lemma 5.2 *Let $\delta \geq 0$, and assume that u_0 has the property that*

$$m \equiv \int_\Omega u_0 < 8\pi\delta. \tag{5.3}$$

Then there exists $C > 0$ such that the function U defined in (4.2) satisfies

$$U(\xi,t) \leq C\xi \qquad \textit{for all } \xi \in (0,1) \textit{ and } t > 0. \tag{5.4}$$

Proof. Since $m < 8\pi\delta$, we can find $\varepsilon > 0$ such that

$$8 > \frac{m}{\pi\delta}(1+\varepsilon),$$

and thereupon choose $t_0 > 0$ large fulfilling

$$8 \geq \frac{m}{\pi\delta}(1+\varepsilon) + 4c_1 e^{-\frac{\delta}{\tau}t_0}, \tag{5.5}$$

where

$$c_1 := \frac{1}{2}\|w_0\|_{L^\infty(\Omega)}. \tag{5.6}$$

With these values of ε and t_0 fixed, using that for all $t \in [0,t_0]$ we have $U(0,t) = 0$ and $U(\xi,t) \leq U(1,t) = \frac{m}{2\pi}$ for all $\xi \in [0,1]$ by (4.3) and (4.4), we can apply Lemma 5.1 to find $b \in (0,1)$ satisfying

$$\max_{t\in[0,t_0]} U(\xi,t) \leq \frac{m}{2\pi} \cdot \frac{(b+1+\varepsilon)\xi}{b+\xi} \qquad \text{for all } \xi \in [0,1]. \tag{5.7}$$

This means that if we let

$$\overline{U}(\xi,t) := \frac{a\xi}{b+\xi} \qquad \text{for } \xi \in [0,1] \text{ and } t \geq 0,$$

with

$$a := \frac{m}{2\pi} \cdot (b+1+\varepsilon),$$

then $U(\xi,t) \leq \overline{U}(\xi,t)$ for all $\xi \in [0,1]$ and $t \in [0,t_0]$. Moreover, clearly $0 = U(0,t) \leq \overline{U}(0,t) = 0$ and $U(1,t) = \frac{m}{2\pi} < \frac{m}{2\pi} \cdot \frac{b+1+\varepsilon}{b+1} = \overline{U}(1,t)$ for all $t \geq t_0$. Computing

$$\overline{U}_t = 0, \quad \overline{U}_\xi = \frac{ab}{(b+\xi)^2} \quad \text{and} \quad \overline{U}_{\xi\xi} = -\frac{2ab}{(b+\xi)^3} \qquad \text{for } \xi \in (0,1) \text{ and } t > t_0,$$

we see that with $\mathcal{P}$ as defined in (4.6) we have

$$\begin{aligned}
\mathcal{P}\overline{U}(\xi,t) &= \frac{8ab\xi}{(b+\xi)^3} - \left\{ \frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \Big(\frac{a\xi}{b+\xi} - \frac{m}{2\pi}\xi\Big) ds \right\} \cdot \frac{ab}{(b+\xi)^2} \\
&\quad -2\Big(W_0(\xi) - K_0\xi\Big) \cdot e^{-\frac{\delta}{\tau}t} \cdot \frac{ab}{(b+\xi)^2} \\
&= \frac{ab\xi}{(b+\xi)^2} \cdot \left\{ \frac{8}{b+\xi} - \frac{2}{\delta}(1-e^{-\frac{\delta}{\tau}t}) \cdot \Big(\frac{a}{b+\xi} - \frac{m}{2\pi}\Big) - 2\Big(\frac{W_0(\xi)}{\xi} - K_0\Big) \cdot e^{-\frac{\delta}{\tau}t} \right\}
\end{aligned} \tag{5.8}$$

for all $\xi \in (0,1)$ and $t > t_0$. Here we use the definition of a and the nonnegativity of $e^{-\frac{\delta}{\tau}t}$ to estimate

$$\begin{aligned}
\frac{2}{\delta}(1-e^{-\frac{\delta}{\tau}t}) \cdot \Big(\frac{a}{b+\xi} - \frac{m}{2\pi}\Big) &= \frac{2}{\delta}(1-e^{-\frac{\delta}{\tau}t}) \cdot \frac{m}{2\pi} \cdot \Big(\frac{b+1+\varepsilon}{b+\xi} - 1\Big) \\
&= \frac{2}{\delta}(1-e^{-\frac{\delta}{\tau}t}) \cdot \frac{m}{2\pi} \cdot \frac{1+\varepsilon-\xi}{b+\xi} \\
&\le \frac{2}{\delta} \cdot \frac{m}{2\pi} \cdot \frac{1+\varepsilon}{b+\xi} \qquad \text{for all } \xi \in (0,1) \text{ and } t > t_0.
\end{aligned}$$

Since by (4.7) and (5.6) we have

$$W_0(\xi) \le \|w_0\|_{L^\infty(\Omega)} \cdot \frac{(\sqrt{\xi})^2}{2} = c_1 \xi \qquad \text{for all } \xi \in (0,1),$$

we moreover see that

$$2\Big(\frac{W_0(\xi)}{\xi} - K_0\Big) \cdot e^{-\frac{\delta}{\tau}t} \le 2c_1 e^{-\frac{\delta}{\tau}t} \le 2c_1 e^{-\frac{\delta}{\tau}t_0} \qquad \text{for all } \xi \in (0,1) \text{ and } t > t_0.$$

According to (5.5), the identity (5.8) thus shows that

$$\begin{aligned}
\mathcal{P}\overline{U}(\xi,t) &\ge \frac{ab\xi}{(b+\xi)^2} \cdot \Big\{\frac{8}{b+\xi} - \frac{2}{\delta} \cdot \frac{m}{2\pi} \cdot \frac{1+\varepsilon}{b+\xi} - 2c_1 e^{-\frac{\delta}{\tau}t_0}\Big\} \\
&= \frac{ab\xi}{(b+\xi)^3} \cdot \Big\{8 - \frac{m}{\pi\delta}(1+\varepsilon) - 2c_1(b+\xi)e^{-\frac{\delta}{\tau}t_0}\Big\} \\
&\ge \frac{ab\xi}{(b+\xi)^3} \cdot \Big\{8 - \frac{m}{\pi\delta}(1+\varepsilon) - 4c_1 e^{-\frac{\delta}{\tau}t_0}\Big\} \\
&\ge 0 \qquad \text{for all } \xi \in (0,1) \text{ and } t > t_0,
\end{aligned}$$

where we have used that $b+\xi \le b+1 \le 2$, because $b<1$. By comparison on the basis of Lemma 4.2, we thereby conclude that $\overline{U} \ge U$ in $(0,1)\times(0,\infty)$, which in particular shows that

$$U(\xi,t) \le \frac{m(b+1+\varepsilon)}{2\pi b} \cdot \xi \qquad \text{for all } \xi \in (0,1) \text{ and } t > 0,$$

and thereby completes the proof. □

5.2 Boundedness away from the origin

In the case $\delta > 0$ when the third equation in (1.1) contains an absorption term, radial solutions can become unbounded in their first component u only near the spatial origin. This is contained in the following lemma, the outcome of which will be an essential ingredient to our ε-regularity result in Section 5.3, and hence in establishing Theorem 1.2.

Lemma 5.3 *Let $\delta > 0$. Then for all $r_0 \in (0,1)$ there exists $C(r_0) > 0$ such that the solution of (1.1) satisfies*

$$u(x,t) \le C(r_0) \qquad \text{for all } x \in \Omega \setminus B_{r_0} \text{ and } t > 0. \tag{5.9}$$

Proof. We evidently only need to consider the case when $u_0 \not\equiv 0$, in which we proceed in six setps.
<u>Step 1.</u> We first claim that there exists $c_1 > 0$ such that

$$|v_r(r,t)| \le \frac{c_1}{r} \qquad \text{for all } r \in (0,1) \text{ and } t > 0. \tag{5.10}$$

To verify this, we write the second equation in (1.1) in the form

$$\frac{1}{r}(rv_r)_r = \mu(t) - w \qquad \text{for } r \in (0,1) \text{ and } t > 0,$$

multiply this by r and integrate using $v_r(0,t) = 0$ for all $t > 0$ to see that

$$rv_r(r,t) = \frac{\mu(t)r^2}{2} - \int_0^r \rho w(\rho,t)d\rho \qquad \text{for all } r \in (0,1) \text{ and } t > 0.$$

Since $w \ge 0$ and $\int_0^1 \rho w(\rho,t)d\rho = \frac{\mu(t)}{2}$ for all $t > 0$ by (1.2), from this we obtain

$$-\frac{\mu(t)}{2} \le rv_r(r,t) \le \frac{\mu(t)r^2}{2} \qquad \text{for all } r \in (0,1) \text{ and } t > 0.$$

This implies (5.10) if we choose $c_1 := \frac{1}{2}\|\mu\|_{L^\infty((0,\infty))}$ which is finite according to (2.5).
<u>Step 2.</u> We next assert that for all $p \in (0,1)$ and each $r_0 \in (0,1)$ we can find $c_2(p,r_0) > 0$ fulfilling

$$\int_t^{t+1} \int_{\Omega\setminus B_{r_0}} |\nabla u^{\frac{p}{2}}|^2 \le c_2(p,r_0) \qquad \text{for all } t > 0. \tag{5.11}$$

To this end, we fix a radially symmetric $\zeta \in C^\infty(\overline{\Omega})$ such that $0 \le \zeta \le 1$ in Ω, $\zeta \equiv 1$ in $\Omega \setminus B_{r_0}$ and $\zeta \equiv 0$ in $B_{\frac{r_0}{2}}$, and multiply the first equation in (1.1) by $\zeta^2 u^{p-1}$ to see upon integrating by parts that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega \zeta^2 u^p &= (1-p)\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 - 2\int_\Omega \zeta u^{p-1}\nabla u \cdot \nabla\zeta \\ &\quad -(1-p)\int_\Omega \zeta^2 u^{p-1}\nabla u \cdot \nabla v + 2\int_\Omega \zeta u^p \nabla v \cdot \nabla\zeta \qquad \text{for all } t > 0, \end{aligned} \tag{5.12}$$

where we note that u, and hence also u^{p-1} and u^{p-2}, is smooth and positive in $\overline{\Omega} \times (0,\infty)$ thanks to our assumption that $u_0 \not\equiv 0$ and the strong maximum principle. Now by Young's inequality, the Hölder inequality and (2.3) we have

$$\begin{aligned} \left| -2\int_\Omega \zeta u^{p-1}\nabla u \cdot \nabla\zeta \right| &\le \frac{1-p}{2}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 + \frac{1}{1-p}\int_\Omega u^p|\nabla\zeta|^2 \\ &\le \frac{1-p}{2}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 + \frac{m^p}{1-p}\Big(\int_\Omega |\nabla\zeta|^{\frac{2}{1-p}}\Big)^{1-p} \qquad \text{for all } t > 0. \end{aligned}$$

By the same token combined with (5.10),

$$\begin{aligned} \left| -(1-p)\int_\Omega \zeta^2 u^{p-1}\nabla u \cdot \nabla v \right| &\le \frac{1-p}{4}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 + (1-p)\int_\Omega \zeta^2 u^p|\nabla v|^2 \\ &\le \frac{1-p}{4}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 + (1-p)m^p\Big(\int_{\Omega\setminus B_{\frac{r_0}{2}}} |\nabla v|^{\frac{2}{1-p}}\Big)^{1-p} \\ &\le \frac{1-p}{4}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 + c_3(p,r_0) \qquad \text{for all } t > 0 \end{aligned}$$

with $c_3(p,r_0) := c_1^2 \cdot (1-p)m^p \cdot \left(2\pi \int_{\frac{r_0}{2}}^1 r^{1-\frac{2}{1-p}} dr\right)^{1-p}$. Similarly, we find $c_4(p,r_0) > 0$ fulfilling

$$\left|2\int_\Omega \zeta u^p \nabla v \cdot \nabla\zeta\right| \le 2m^p \left(\int_{\Omega\setminus B_{\frac{r_0}{2}}} |\nabla v|^{\frac{1}{1-p}} \cdot |\nabla\zeta|^{\frac{1}{1-p}}\right)^{1-p} \le c_4(p,r_0) \qquad \text{for all } t>0,$$

whence (5.12) altogether yields $c_5(p,r_0) > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega \zeta^2 u^p \ge \frac{1-p}{4}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 - c_5(p,r_0) \qquad \text{for all } t>0.$$

After a time integration and another application of the Hölder inequality and (2.3), we thus obtain

$$\begin{aligned}
\frac{1-p}{4}\int_t^{t+1}\int_\Omega \zeta^2 u^{p-2}|\nabla u|^2 &\le \frac{1}{p}\int_\Omega \zeta^2 u^p(\cdot,t+1) + c_5(p,r_0) \\
&\le \frac{m^p \cdot \pi^{1-p}}{p} + c_5(p,r_0) \qquad \text{for all } t>0,
\end{aligned}$$

which entails (5.11) in view of the fact that $\zeta \equiv 1$ in $\Omega \setminus B_{r_0}$.

Step 3. We now make sure that for all $r_0 \in (0,1)$ we can find $c_6(r_0) > 0$ satisfying

$$\int_t^{t+1} \|u(\cdot,s)\|_{L^\infty(\Omega\setminus B_{r_0})} ds \le c_6(r_0) \qquad \text{for all } t>0. \tag{5.13}$$

To this end, let us fix an arbitrary $p \in (0,1)$. Then again by radial symmetry we may combine the one-dimensional version of the Gagliardo-Nirenberg inequality with the outcome of Step 2 and (2.3) to fix positive constants $c_7(r_0)$, $c_8(r_0)$ and $c_9(r_0)$ such that

$$\begin{aligned}
\int_t^{t+1} \|u^{\frac{p}{2}}(\cdot,s)\|_{L^\infty((r_0,1))}^{\frac{2(p+1)}{p}} ds &\le c_7(r_0)\int_t^{t+1}\Big\{\|(u^{\frac{p}{2}})_r(\cdot,s)\|_{L^2((0,1))}^2 \cdot \|u^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}((r_0,1))}^{\frac{2}{p}} \\
&\qquad + \|u^{\frac{p}{2}}(\cdot,s)\|_{L^{\frac{2}{p}}((r_0,1))}^{\frac{2(p+1)}{p}}\Big\} ds \\
&\le c_8(r_0)\int_t^{t+1}\Big\{\|(u^{\frac{p}{2}})_r(\cdot,s)\|_{L^2((r_0,1))}^2 + 1\Big\} ds \\
&\le c_9(r_0) \qquad \text{for all } t>0.
\end{aligned}$$

Since $\|u^{\frac{p}{2}}(\cdot,s)\|_{L^\infty((r_0,1))}^{\frac{2(p+1)}{p}} = \|u(\cdot,s)\|_{L^\infty((r_0,1))}^{p+1}$, an application of the Hölder inequality thereupon yields (5.13).

Step 4. We proceed to show that for any $r_0 \in (0,1)$ there exists $c_{10}(r_0) > 0$ such that

$$\|w(\cdot,t)\|_{L^\infty(\Omega\setminus B_{r_0})} \le c_{10}(r_0) \qquad \text{for all } t>0. \tag{5.14}$$

Indeed, given $t>0$ we write $I_j := (t-j-1, t-j) \cap (0,\infty)$ for nonnegative integers j, and representing $w(\cdot,t)$ according to $w(\cdot,t) = e^{-\frac{\delta}{\tau}t}w_0 + \frac{1}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}u(\cdot,s)ds$ we can estimate

$$\|w(\cdot,t)\|_{L^\infty(\Omega\setminus B_{r_0})} \le e^{-\frac{\delta}{\tau}t}\|w_0\|_{L^\infty(\Omega)} + \frac{1}{\tau}\sum_{j=0}^\infty \int_{I_j} e^{-\frac{\delta}{\tau}(t-s)}\|u(\cdot,s)\|_{L^\infty(\Omega\setminus B_{r_0})} ds$$

$$
\begin{aligned}
&\leq \quad e^{-\frac{\delta}{\tau}t}\|w_0\|_{L^\infty(\Omega)} + \frac{1}{\tau}\sum_{j=0}^{\infty} e^{-\frac{\delta}{\tau}j}\int_{I_j}\|u(\cdot,s)\|_{L^\infty(\Omega\setminus B_{r_0})}ds \\
&\leq \quad e^{-\frac{\delta}{\tau}t}\|w_0\|_{L^\infty(\Omega)} + c_6(r_0)\sum_{j=0}^{\infty} e^{-\frac{\delta j}{\tau}}.
\end{aligned}
$$

Since the rightmost series converges thanks to our assumption $\delta > 0$, this proves (5.14).

Step 5. Next, we prove that for each $r_0 \in (0,1)$ we can fix $c_{11}(r_0) > 0$ such that

$$
\int_t^{t+1}\int_{\Omega\setminus B_{r_0}} |\nabla u|^2 \leq c_{11}(r_0) \qquad \text{for all } t \geq 1. \tag{5.15}
$$

Since $t \geq 1$, Step 3 allows us to pick $t_0 \in (t-1,t)$ such that

$$
\|u(\cdot,t_0)\|_{L^\infty(\Omega\setminus B_{\frac{r_0}{2}})} \leq c_6\Big(\frac{r_0}{2}\Big). \tag{5.16}
$$

Then using ζ as introduced in Step 2, by a straightforward testing procedure we infer that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \zeta u^2(\cdot,s) + \int_\Omega \zeta|\nabla u(\cdot,s)|^2 &= -\int_\Omega u\nabla u\cdot\nabla\zeta + \int_\Omega \zeta u\nabla u\cdot\nabla v + \int_\Omega u^2\nabla v\cdot\nabla\zeta \\
&= \frac{1}{2}\int_\Omega u^2\Delta\zeta - \frac{1}{2}\int_\Omega \zeta u^2\Delta v + \frac{1}{2}\int_\Omega u^2\nabla v\cdot\nabla\zeta \\
&\leq \frac{1}{2}\int_\Omega u^2\Big(\Delta\zeta + \zeta w + \nabla v\cdot\nabla\zeta\Big) \qquad \text{for all } s \in (t_0,t+1).
\end{aligned}
$$

Thus, by Step 4, Step 1 and (2.3), we can find $c_{12}(r_0) > 0$ satisfying

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \zeta u^2(\cdot,s) + \int_\Omega \zeta|\nabla u(\cdot,s)|^2 &\leq c_{12}(r_0)\int_{\Omega\setminus B_{\frac{r_0}{2}}} u^2 \\
&\leq c_{12}(r_0)\cdot m\cdot\|u(\cdot,s)\|_{L^\infty(\Omega\setminus B_{\frac{r_0}{2}})} \qquad \text{for all } s \in (t_0,t+1),
\end{aligned}
$$

whence integrating and using (5.16) and Step 3 shows that

$$
\begin{aligned}
\frac{1}{2}\int_\Omega \zeta u^2(\cdot,t+1) + \int_{t_0}^{t+1}\int_\Omega \zeta|\nabla u|^2 &\leq \frac{1}{2}\int_\Omega \zeta u^2(\cdot,t_0) + c_{12}(r_0)\cdot m\cdot\int_{t_0}^{t+1}\|u(\cdot,s)\|_{L^\infty(\Omega\setminus B_{\frac{r_0}{2}})}ds \\
&\leq \frac{\pi}{2}c_6^2\Big(\frac{r_0}{2}\Big) + c_{12}(r_0)\cdot m\cdot 2c_6\Big(\frac{r_0}{2}\Big).
\end{aligned}
$$

As $t_0 < t$, this implies (5.15).

Step 6. Conclusion. Again with ζ as in Step 2, we let $\widetilde{u}(r,t) := \zeta(r)u(r,t)$ for $r \in [0,1]$ and $t \geq 0$. Then

$$
\widetilde{u}_t = \widetilde{u}_{rr} + f(r,t) \qquad \text{for all } r \in (0,1) \text{ and } t > 0, \tag{5.17}
$$

with

$$
f(r,t) := \frac{1}{r}\zeta u_r - 2\zeta_r u_r - \zeta_{rr}u - \zeta u_r v_r - \mu(t)\zeta u + \zeta uw \qquad \text{for } r \in (0,1) \text{ and } t > 0.
$$

By the outcome of Step 1, Step 3, Step 4 and Step 5, for some $c_{13}(r_0) > 0$ we have

$$\int_{t_0}^{t_0+2} \|f(\cdot,s)\|^2_{L^2((0,1))} ds \le c_{13}(r_0) \qquad \text{for all } t_0 \ge 1. \tag{5.18}$$

Now given $t \ge 2$, once more by Step 3 we can fix $t_0 \in (t-1,t)$ fulfilling

$$\|\widetilde{u}(\cdot,t_0)\|_{L^\infty((0,1))} \le c_6\Big(\frac{r_0}{2}\Big). \tag{5.19}$$

Since $\widetilde{u}_r = 0$ on $\partial(0,1)$, the variation-of-constants representation of $\widetilde{u}$ in terms of the one-dimensional Neumann heat semigroup $(e^{\tau\Delta})_{\tau\ge 0}$ on the interval $(0,1)$ shows that

$$\widetilde{u}(\cdot,t) = e^{(t-t_0)\Delta}\widetilde{u}(\cdot,t_0) + \int_{t_0}^t e^{(t-s)\Delta} f(\cdot,s)ds.$$

Therefore, using standard smoothing estimates ([31]) along with (5.19), the Hölder inequality and (5.18) we can find $c_{14} > 0$ such that

$$\begin{aligned}
\|\widetilde{u}(\cdot,t)\|_{L^\infty((0,1))} &\le \|\widetilde{u}(\cdot,t_0)\|_{L^\infty((0,1))} + c_{14}\int_{t_0}^t (t-s)^{-\frac{1}{4}} \|f(\cdot,s)\|_{L^2((0,1))} ds \\
&\le c_6\Big(\frac{r_0}{2}\Big) + c_{14}\Big(\int_{t_0}^t (t-s)^{-\frac{1}{2}} ds\Big)^{\frac{1}{2}} \cdot \Big(\int_{t_0}^t \|f(\cdot,s)\|^2_{L^2((0,1))} ds\Big)^{\frac{1}{2}} \\
&\le c_6\Big(\frac{r_0}{2}\Big) + c_{14}\cdot(2\sqrt{2})^{\frac{1}{2}} \cdot (c_{13}(r_0))^{\frac{1}{2}}.
\end{aligned}$$

Since $\widetilde{u}(r,t) = u(r,t)$ for all $r > r_0$, this establishes (5.9). □

5.3 An ε-regularity result. Proof of Theorem 1.2

In deriving Theorem 1.2 from Lemma 5.2, we shall make use of a regularity statement which says that solutions already must remain bounded if only their mass concentrating in an arbitrarily small ball centered at the origin is sufficiently small. A first step toward this is achieved in the following lemma.

Lemma 5.4 *Let $\delta > 0$. Then for all $p > 1$ there exists $\varepsilon = \varepsilon(p) > 0$ such that if for some $r_0 \in (0,1)$, a radial solution of (1.1) satisfies*

$$\int_{B_{r_0}} u(x,t)dx < \varepsilon \qquad \textit{for all } t > 0, \tag{5.20}$$

then

$$\sup_{t>0} \int_\Omega u^p(x,t)dx < \infty. \tag{5.21}$$

Proof. Using Young's inequality, given $p > 1$ we can find $c_1 = c_1(p) > 0$ such that

$$\frac{p-1}{p} A^p B + AB^p \le \frac{\delta}{2} B^{p+1} + c_1 A^{p+1} \qquad \text{for all } A \ge 0 \text{ and } B \ge 0. \tag{5.22}$$

Moreover, the Gagliardo-Nirenberg inequality says that with some $c_2 = c_2(p) > 0$ we have

$$\|\varphi\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \le c_2\|\nabla\varphi\|_{L^2(\Omega)}^2 \|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + c_2\|\varphi\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{5.23}$$

We claim that if (5.20) holds with some $r_0 \in (0,1)$ and

$$\varepsilon := \frac{p-1}{c_1 c_2 p^2}, \tag{5.24}$$

then (5.21) must be valid. To see this, we apply Lemma 3.1 and estimate the terms on the right-hand side of (3.1) by means of (5.22) to obtain

$$\frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1}\right\} + \frac{4(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \delta\int_\Omega w^{p+1} \le \frac{\delta}{2}\int_\Omega w^{p+1} + c_1\int_\Omega u^{p+1} \tag{5.25}$$

for all $t > 0$. We now fix $\zeta \in C_0^\infty(\Omega)$ such that $0 \le \zeta \le 1$ in Ω, $\zeta \equiv 1$ in $B_{\frac{r_0}{2}}$ and $\operatorname{supp}\zeta \subset B_{r_0}$, and split

$$c_1\int_\Omega u^{p+1} = c_1\int_\Omega \zeta^{\frac{2(p+1)}{p}} u^{p+1} + c_1\int_\Omega (1-\zeta^{\frac{2(p+1)}{p}})u^{p+1}, \tag{5.26}$$

where according to Lemma 5.3 we can find $c_3 = c_3(p, r_0) > 0$ such that

$$c_1\int_\Omega (1-\zeta^{\frac{2(p+1)}{p}})u^{p+1} \le c_1\int_{\Omega\setminus B_{\frac{r_0}{2}}} u^{p+1} \le c_3 \qquad \text{for all } t > 0. \tag{5.27}$$

The first term on the right of (5.26) can be estimated using (5.23) according to

$$\begin{aligned} c_1\int_\Omega \zeta^{\frac{2(p+1)}{p}} u^{p+1} &= c_1\|\zeta u^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\ &\le c_1c_2\|\nabla(\zeta u^{\frac{p}{2}})\|_{L^2(\Omega)}^2 \|\zeta u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + c_1c_2\|\zeta u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}}. \end{aligned} \tag{5.28}$$

Here since $\nabla(\zeta u^{\frac{p}{2}}) = \zeta\nabla u^{\frac{p}{2}} + u^{\frac{p}{2}}\nabla\zeta$, again by Lemma 5.3 we have

$$\begin{aligned} \|\nabla(\zeta u^{\frac{p}{2}})\|_{L^2(\Omega)}^2 &\le 2\int_\Omega \zeta^2|\nabla u^{\frac{p}{2}}|^2 + 2\int_\Omega u^p|\nabla\zeta|^2 \\ &\le 2\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_4 \qquad \text{for all } t > 0 \end{aligned}$$

with some $c_4 = c_4(p, r_0) > 0$, because $\operatorname{supp}\nabla\zeta \subset \Omega \setminus B_{\frac{r_0}{2}}$. Moreover, our hypothesis (5.20) asserts that

$$\|\zeta u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} = \int_\Omega \zeta^{\frac{2}{p}} u \le \int_{B_{r_0}} u < \varepsilon \qquad \text{for all } t > 0,$$

whence (5.28) implies that

$$c_1\int_\Omega \zeta^{\frac{2(p+1)}{p}} u^{p+1} \le 2c_1c_2\varepsilon\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_1c_2c_4\varepsilon + c_1c_2\varepsilon^{p+1} \qquad \text{for all } t > 0. \tag{5.29}$$

Since $2c_1c_2\varepsilon = \frac{2(p-1)}{p^2}$ by (5.24), from (5.25)-(5.29) we thus obtain

$$\begin{aligned} \frac{d}{dt}\left\{\frac{1}{p}\int_\Omega u^p + \frac{\tau}{p+1}\int_\Omega w^{p+1}\right\} &+ \frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{\delta}{2}\int_\Omega w^{p+1} \\ &\le c_3 + c_1c_2c_4\varepsilon + c_1c_2\varepsilon^{p+1} \qquad \text{for all } t>0. \end{aligned} \tag{5.30}$$

Here we may invoke the Poincaré inequality to find $c_5 = c_5(p) > 0$ fulfilling

$$\int_\Omega \varphi^2 \le c_5 \int_\Omega |\nabla\varphi|^2 + c_5\left(\int_\Omega |\varphi|^{\frac{2}{p}}\right)^p \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

which according to (2.3) entails that

$$\frac{2(p-1)}{p^2}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \ge \frac{2(p-1)}{c_5p^2}\int_\Omega u^p - \frac{2(p-1)}{p^2}m^p \qquad \text{for all } t>0$$

with $m := \int_\Omega u_0$. Therefore, (5.30) shows that $y(t) := \frac{1}{p}\int_\Omega u^p(\cdot,t) + \frac{\tau}{p+1}\int_\Omega w^{p+1}$, $t \ge 0$, satisfies

$$y'(t) + c_6 y(t) \le c_7 \qquad \text{for all } t>0,$$

where

$$c_6 := \min\left\{\frac{2(p-1)}{c_5p}\, , \, \frac{(p+1)\delta}{2\tau}\right\}$$

and $c_7 := c_3 + c_1c_2c_4\varepsilon + c_1c_2\varepsilon^{p+1} + \frac{2(p-1)}{p^2}m^p$. An ODE comparison thus yields (5.21). □

By applying the above to suitably large p and using additional regularity arguments, we can next make sure that the above assumptions already imply boundedness of u with respect to the norm in $L^\infty(\Omega)$.

Lemma 5.5 *Let $\delta > 0$. Then there exists $\varepsilon > 0$ such that if for some $r_0 \in (0,1)$ and some radial solution of (1.1) we have*

$$\int_{B_{r_0}} u(x,t)dx < \varepsilon \qquad \textit{for all } t>0, \tag{5.31}$$

then

$$\sup_{t>0} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty. \tag{5.32}$$

Proof. We pick any $p > 2$ and apply Lemma 5.4 which says that under the assumption (5.31) we can find $c_1 > 0$ such that

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le c_1 \qquad \text{for all } t>0.$$

Therefore, (2.1) shows that

$$\begin{aligned} \|w(\cdot,t)\|_{L^p(\Omega)} &\le e^{-\frac{\delta}{\tau}t}\|w_0\|_{L^p(\Omega)} + \frac{1}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\|u(\cdot,s)\|_{L^p(\Omega)}ds \\ &\le e^{-\frac{\delta}{\tau}t}\|w_0\|_{L^p(\Omega)} + \frac{c_1}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}ds \qquad \text{for all } t>0. \end{aligned}$$

Since $\delta > 0$, we know that $\int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \le \frac{\tau}{\delta}$, so that from this we obtain $c_2 > 0$ fulfilling

$$\|w(\cdot,t)\|_{L^p(\Omega)} \le c_2 \qquad \text{for all } t > 0.$$

From this and standard elliptic regularity theory we obtain a bound for v in $L^\infty((0,\infty);W^{2,p}(\Omega))$, which by the validity of the embedding $W^{2,p}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ implies that

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \qquad \text{for all } t \in (0,T)$$

with some $c_3 > 0$. Combined with (5.21), upon a Moser-type iteration ([33, Lemma 4.1]) this yields (5.32). □

Combining Lemma 5.5 with Lemma 5.2 now immediately yields boundedness of solutions in the subcritical mass case.

Proof of Theorem 1.2. We let $\varepsilon > 0$ be as provided by Lemma 5.5 and only need to verify the validity of (5.31) for some $r_0 \in (0,1)$. In order to choose the latter appropriately, we apply Lemma 5.2 to find $c_1 > 0$ such that for arbitrary $r_0 \in (0,1)$ we have

$$\int_{B_{r_0}} u(x,t)dx = 2\pi \int_0^{r_0} r(u(r,t)dr = 2\pi U(r_0^2,t) \le c_1 r_0^2 \qquad \text{for all } t > 0.$$

This means that if we now fix $r_0 \in (0,1)$ in such a way that $r_0 < \sqrt{\frac{\varepsilon}{c_1}}$, then indeed

$$\int_{B_{r_0}} u(x,t)dx < \varepsilon \qquad \text{for all } t > 0.$$

Lemma 5.5 thus ensures that (5.32) holds, whereupon recalling (2.1) and applying elliptic regularity theory we see that the statement in Theorem 1.2 becomes an evident consequence thereof. □

6 Unbounded solutions with $\int_\Omega u_0 > 8\pi\delta$. Proof of Theorem 1.3

6.1 A class of comparison functions

We shall next prove that whenever $m > 8\pi\delta$, some solutions at the mass level m asymptotically aggregate in the spirit of Theorem 1.3. To this end, we shall consider comparison functions $\underline{U} : [0,1] \times [0,\infty) \to \mathbb{R}$ of the form

$$\underline{U}(\xi,t) := \begin{cases} \frac{a(t)\xi}{b(t)+\xi} & \text{if } \xi \in [0,\xi_0] \text{ and } t \ge 0, \\ \frac{a(t)b(t)\xi + a(t)\xi_0^2}{(b(t)+\xi_0)^2} & \text{if } \xi \in (\xi_0,1] \text{ and } t \ge 0, \end{cases} \tag{6.1}$$

where $\xi_0 \in (0,1)$ and a and b are a suitably chosen positive functions on $[0,\infty)$. Let us first collect some basic properties of such functions, especially with regard to their behavior under the action of the operator $\mathcal{P}$ defined in (4.6).

Lemma 6.1 *Let $\xi_0 \in (0,1)$, and assume that $a \in C^1([0,\infty))$ and $b \in C^1([0,\infty))$ are positive. Then the function $\underline{U}$ given by (6.1) satisfies*

$$\underline{U} \in C^1([0,1]\times[0,\infty)) \cap C^0([0,\infty); W^{2,\infty}((0,1))) \cap C^0([0,\infty); C^2_{loc}([0,1]\setminus\{\xi_0\})).$$

Moreover, with $\mathcal{P}$ as in (4.6) we have

$$\begin{aligned}\frac{(b(t)+\xi)^2}{a(t)b(t)\xi}\cdot \mathcal{P}\underline{U}(\xi,t) &= \frac{a'(t)(b(t)+\xi)}{a(t)b(t)} - \frac{b'(t)}{b(t)} + \frac{8}{b(t)+\xi} \\ &\quad -\frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\Big\{\frac{a(s)}{b(s)+\xi} - \frac{m}{2\pi}\Big\}ds \\ &\quad -2\Big(\frac{W_0(\xi)}{\xi} - K_0\Big)\cdot e^{-\frac{\delta}{\tau}t} \qquad \text{for all } \xi\in(0,\xi_0) \text{ and } t>0 \end{aligned} \tag{6.2}$$

and

$$\begin{aligned}\frac{(b(t)+\xi_0)^2}{a(t)b(t)}\cdot \mathcal{P}\underline{U}(\xi,t) &= \frac{a'(t)\xi}{a(t)} + \frac{b'(t)\xi}{b(t)} + \frac{a'(t)\xi_0^2}{a(t)b(t)} - 2\frac{b'(t)\xi + \frac{b'(t)}{b(t)}\xi_0^2}{b(t)+\xi_0} \\ &\quad -\frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\Big\{\frac{a(s)b(s)\xi + a(s)\xi_0^2}{(b(s)+\xi_0)^2} - \frac{m}{2\pi}\xi\Big\}ds \\ &\quad -2\Big(W_0(\xi) - K_0\xi\Big)\cdot e^{-\frac{\delta}{\tau}t} \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0, \end{aligned} \tag{6.3}$$

where W_0 and K_0 are as defined in (4.7).

Proof. The claimed regularity properties can immediately be verified using the explicit form of $\underline{U}$ which clearly allows for piecewise differentiation, resulting in

$$U_\xi(\xi,t) = \begin{cases} \frac{ab}{(b+\xi)^2} & \text{for } \xi\in(0,\xi_0) \text{ and } t>0, \\ \frac{ab}{(b+\xi_0)^2} & \text{for } \xi\in(\xi_0,1) \text{ and } t>0, \end{cases} \tag{6.4}$$

and

$$U_{\xi\xi}(\xi,t) = \begin{cases} -\frac{2ab}{(b+\xi)^3} & \text{for } \xi\in(0,\xi_0) \text{ and } t>0, \\ 0 & \text{for } \xi\in(\xi_0,1) \text{ and } t>0, \end{cases} \tag{6.5}$$

as well as

$$U_t(\xi,t) = \begin{cases} \frac{a'\xi}{b+\xi} - \frac{ab'\xi}{(b+\xi)^2} & \text{for } \xi\in(0,\xi_0) \text{ and } t>0, \\ \frac{a'b\xi + ab'\xi + a'\xi_0^2}{(b+\xi_0)^2} - 2\frac{abb'\xi + ab'\xi_0^2}{(b+\xi_0)^3} & \text{for } \xi\in(\xi_0,1) \text{ and } t>0. \end{cases} \tag{6.6}$$

Moreover, for $\xi < \xi_0$ we obtain from (6.4)-(6.6) that

$$\begin{aligned}\mathcal{P}\underline{U}(\xi,t) &= \frac{a'\xi}{b+\xi} - \frac{ab'\xi}{(b+\xi)^2} + \frac{8ab\xi}{(b+\xi)^3} \\ &\quad -\frac{2}{\tau}\Big\{\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\cdot\Big\{\frac{a(s)\xi}{b(s)+\xi} - \frac{m}{2\pi}\xi\Big\}ds\Big\}\cdot\frac{ab}{(b+\xi)^2} \\ &\quad -\Big\{2\Big(W_0(\xi) - K_0\xi\Big)\cdot e^{-\frac{\delta}{\tau}t}\Big\}\cdot\frac{ab}{(b+\xi)^2},\end{aligned}$$

which is equivalent to (6.2). Likewise, (6.3) easily follows upon the observation that for $\xi > \xi_0$, the identity

$$\begin{aligned}
\frac{(b+\xi_0)^2}{ab} \cdot \Big(\underline{U}_t - 4\xi \underline{U}_{\xi\xi} \Big) &= \frac{a'b\xi + ab'\xi + a'\xi_0^2}{ab} - 2\frac{abb'\xi + ab'\xi_0^2}{ab(b+\xi_0)} \\
&= \frac{a'\xi}{a} + \frac{b'\xi}{b} + \frac{a'\xi_0^2}{ab} - 2\frac{b'\xi + \frac{b'}{b}\xi_0^2}{b+\xi_0}
\end{aligned}$$

holds. □

To make the above choice as efficient as possible for our purpose, $a(t)$ will be adjusted in such a way that at $\xi = 1$, the function $\underline{U}$ attains the same boundary value as U introduced in (4.2). The corresponding condition $\underline{U}(1,t) = \frac{m}{2\pi}$ for all $t \geq 0$, with $m := \int_\Omega u_0$, thus amounts to requiring that $a(t)$ is linked to ξ_0 and $b(t)$, and accordingly we shall concentrate on the case when

$$a(t) := \frac{m}{2\pi} \cdot \frac{(b(t)+\xi_0)^2}{b(t)+\xi_0^2} \qquad \text{for } t \geq 0 \tag{6.7}$$

in the sequel. Then for later use we note that if in addition we assume that b is differentiable, so will be a with

$$\begin{aligned}
a'(t) &= \frac{m}{2\pi} \cdot \frac{2(b+\xi_0)(b+\xi_0^2) - (b+\xi_0)^2}{(b+\xi_0^2)^2} \cdot b' \\
&= \frac{m}{2\pi} \cdot \frac{b^2 + 2b\xi_0^2 - \xi_0^2 + 2\xi_0^3}{(b+\xi_0^2)^2} \cdot b' \qquad \text{for all } t > 0,
\end{aligned} \tag{6.8}$$

6.2 Subsolution in an annulus

We first analyze in more depth the behavior of $\underline{U}$ in the outer region where $\xi > \xi_0$. Here it will not be necessary to fix ξ_0, and keeping this freedom will be important for our procedure in the corresponding inner part where $\xi < \xi_0$, in which we shall adjust ξ_0 in dependence of $m > 8\pi\delta$.
To begin with, let us draw a first conclusion from Lemma 6.1 under the assumption (6.7).

Lemma 6.2 *Let $\delta \geq 0$ and $m > 0$, and suppose that $\xi_0 \in (0,1)$, that $b \in C^1([0,\infty))$ is positive and nonincreasing such that*

$$b(t) \leq \xi_0^2 \qquad \textit{for all } t \geq 0, \tag{6.9}$$

and that $a \in C^1([0,\infty))$ is given by (6.7). Then the function $\underline{U}$ defined in (6.1) satisfies

$$\frac{(b(t)+\xi_0)^2}{a(t)b(t)} \cdot \mathcal{P}\underline{U}(\xi,t) \leq (1-\xi) \cdot \Big\{ -\frac{b'(t)}{b(t)} - \frac{m}{2\pi\tau} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \Big\} - 2\Big(W_0(\xi) - K_0\xi\Big) \cdot e^{-\frac{\delta}{\tau}t}$$
$$\textit{for all } \xi \in (\xi_0, 1) \textit{ and } t > 0, \tag{6.10}$$

where $\mathcal{P}$ is as in (4.6).

PROOF. From (6.8) we compute

$$\frac{a'(t)}{a(t)} = \frac{b^2+2b\xi_0^2-\xi_0^2+2\xi_0^3}{(b+\xi_0^2)(b+\xi_0)^2}\cdot b' \qquad \text{for all } t>0.$$

Thus, on the right-hand side of (6.3) we have

$$\begin{aligned} J_1(\xi,t) &:= \frac{a'\xi}{a}+\frac{b'\xi}{b}+\frac{a'\xi_0^2}{ab}-2\frac{b'\xi+\frac{b'}{b}\xi_0^2}{b+\xi_0} \\ &= \frac{b'}{b}\cdot\left\{\frac{(b^2+2b\xi_0^2-\xi_0^2+2\xi_0^3)(b\xi+\xi_0^2)}{(b+\xi_0^2)(b+\xi_0)^2}+\xi-\frac{2b\xi+2\xi_0^2}{b+\xi_0}\right\} \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0, \end{aligned}$$

whereupon a lengthy but straightforward computation yields

$$J_1(\xi,t) = -\frac{b'}{b}\cdot\frac{\xi_0^2(1-\xi)}{b+\xi_0^2} \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0. \tag{6.11}$$

Next, in the integrand on the right of (6.3) we again use (6.7) to see that

$$\begin{aligned} \frac{ab\xi+a\xi_0^2}{(b+\xi_0)^2}-\frac{m}{2\pi}\xi &= \frac{m}{2\pi}\cdot\left(\frac{b\xi+\xi_0^2}{b+\xi_0^2}-\xi\right) \\ &= \frac{m}{2\pi}\cdot\frac{\xi_0^2(1-\xi)}{b+\xi_0^2} \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0, \end{aligned}$$

so that

$$\begin{aligned} J_2(\xi,t) &:= -\frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\cdot\left(\frac{a(s)b(s)\xi+a(s)\xi_0^2}{(b(s)+\xi_0)^2}-\frac{m}{2\pi}\xi\right)ds \\ &= -\frac{m}{\pi\tau}\cdot(1-\xi)\cdot\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\cdot\frac{\xi_0^2}{b(s)+\xi_0^2}ds \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0. \end{aligned} \tag{6.12}$$

Now in (6.11) we can use the nonnegativity of b to find that

$$\frac{\xi_0^2(1-\xi)}{b+\xi_0^2}\le 1-\xi \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0,$$

whereas in (6.12) we employ (6.9) to estimate

$$\frac{\xi_0^2}{b+\xi_0^2}\ge\frac{1}{2} \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0.$$

Therefore, by means of the nonpositivity of $\frac{b'}{b}$ we have

$$J_1(\xi,t)+J_2(\xi,t)\le -\frac{b'}{b}\cdot(1-\xi)-\frac{m}{2\pi\tau}\cdot(1-\xi)\cdot\int_0^t e^{-\frac{\delta}{\tau}(t-s)}ds \qquad \text{for all } \xi\in(\xi_0,1) \text{ and } t>0.$$

In view of (6.3), this proves (6.10). □

Now the right-hand side of (6.10) suggests to choose b in such a way that $\frac{b'}{b}$ is a negative constant. In that case, namely, it turns out that the unfavorable contribution of $-\frac{b'}{b}$ in (6.10) can be controlled for large times by the integral on the right of (6.10), whereas for small t it will be dominated by the expression containing W_0 and K_0, provided that w_0 satisfies some rather mild condition.

Lemma 6.3 *Let $\delta \ge 0$ and $m > 0$, and suppose that for some $\xi_0 \in (0,1)$ and $\eta_0 > 0$, with W_0 and K_0 as in (4.7) we have*

$$\frac{W_0(\xi) - K_0\xi}{1-\xi} \ge \eta_0 \qquad \text{for all } \xi \in (\xi_0, 1). \tag{6.13}$$

Then for all $\alpha_\star > 0$ there exists $\alpha \in (0, \alpha_\star)$ such that for any choice of $b_0 \in (0, \xi_0^2)$, with

$$b(t) := b_0 e^{-\alpha t} \qquad \text{for } t \ge 0$$

and $a \in C^1([0,\infty))$ as in (6.7), the function $\underline{U}$ in (6.1) satisfies

$$\mathcal{P}\underline{U}(\xi, t) \le 0 \qquad \text{for all } \xi \in (\xi_0, 1) \text{ and } t > 0, \tag{6.14}$$

the operator $\mathcal{P}$ being defined through (4.6).

PROOF. We claim that (6.14) holds whenever $b_0 \in (0, \xi_0^2)$ and

$$\alpha < \alpha_0 := \min\left\{ \frac{m}{2\pi\tau e^{\frac{\delta}{\tau}}}, \frac{2\eta_0}{e^{\frac{2\delta}{\tau}}}, \alpha_\star \right\}. \tag{6.15}$$

Indeed, since (6.13) in particular implies that $W_0(\xi) - K_0\xi \ge 0$ for all $\xi \in (\xi_0, 1)$, from Lemma 6.2 we obtain that

$$\begin{aligned}
\frac{(b(t)+\xi_0)^2}{a(t)b(t)} \cdot \mathcal{P}\underline{U}(\xi,t) &\le (1-\xi) \cdot \left\{ -\frac{b'(t)}{b(t)} - \frac{m}{2\pi\tau} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \right\} \\
&= (1-\xi) \cdot \left\{ \alpha - \frac{m}{2\pi\tau} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \right\} \qquad \text{for all } \xi \in (\xi_0, 1) \text{ and } t > 0.
\end{aligned}$$

Here for large t we can estimate

$$\int_0^t e^{-\frac{\delta}{\tau}(t-s)} ds \ge \int_{t-1}^t e^{-\frac{\delta}{\tau}(t-s)} ds \ge \int_0^1 e^{-\frac{\delta}{\tau}} ds = e^{-\frac{\delta}{\tau}} \qquad \text{for all } t \ge 2,$$

so that the first restriction implied by (6.15) warrants that

$$\begin{aligned}
\frac{(b(t)+\xi_0)^2}{a(t)b(t)} \cdot \mathcal{P}\underline{U}(\xi,t) &\le (1-\xi) \cdot \left\{ \alpha - \frac{m}{2\pi\tau} e^{-\frac{\delta}{\tau}} \right\} \\
&\le 0 \qquad \text{for all } \xi \in (\xi_0, 1) \text{ and } t \ge 2.
\end{aligned}$$

For small values of t, however, (6.10) and (6.13) yield

$$\begin{aligned}
\frac{(b(t)+\xi_0)^2}{a(t)b(t)} \cdot \mathcal{P}\underline{U}(\xi,t) &\le (1-\xi)\alpha - 2\eta_0(1-\xi)e^{-\frac{\delta}{\tau}t} \\
&\le (1-\xi)\alpha - 2\eta_0(1-\xi)e^{-2\frac{\delta}{\tau}} \\
&\le 0 \qquad \text{for all } \xi \in (\xi_0, 1) \text{ and } t < 2
\end{aligned}$$

because of the second limitation on α asserted by (6.15). □

6.3 Subsolution near the origin

Our argument in the associated inner region will be more subtle, and here we will in particular rely on the supercriticality assumption $m > 8\pi\delta$. Let us begin by estimating the first term on the right of (6.2) under the hypothesis (6.7).

Lemma 6.4 *Let $m > 0$, and suppose that $b \in C^1([0,\infty)$ is positive and nonincreasing, and let $\xi_0 \in (0,1)$. Then the function $a \in C^1([0,\infty))$ defined in (6.7) satisfies*

$$\frac{a'(t)(b(t)+\xi)}{a(t)b(t)} \le \frac{1}{\xi_0} \cdot \frac{-b'(t)}{b(t)} \qquad \textit{for all } \xi \in (0,\xi_0) \textit{ and } t > 0.$$

Proof. In (6.8) we can trivially estimate

$$b^2(t) + 2b(t)\xi_0^2 - \xi_0^2 + \xi_0^3 \ge -\xi_0^2 \qquad \text{for all } t > 0$$

to obtain

$$a'(t) \le -\frac{m}{2\pi} \cdot \frac{\xi_0^2}{(b(t)+\xi_0^2)^2} \cdot b'(t) \qquad \text{for all } t > 0.$$

Therefore

$$\begin{aligned}
\frac{a'(t)(b(t)+\xi)}{a(t)b(t)} &\le \frac{-\frac{\xi_0^2}{(b(t)+\xi_0^2)^2} \cdot b'(t)}{\frac{(b(t)+\xi_0)^2}{b(t)+\xi_0^2}} \cdot \frac{b(t)+\xi}{b(t)} \\
&= \frac{\xi_0^2(b(t)+\xi)}{(b(t)+\xi_0^2)(b(t)+\xi_0)^2} \cdot \frac{-b'(t)}{b(t)} \qquad \text{for all } \xi \in (0,1) \text{ and } t > 0,
\end{aligned}$$

so that since $\frac{b(t)+\xi}{b(t)+\xi_0} \le 1$ for all $\xi \in (0,\xi_0)$ and $t > 0$, we find that

$$\begin{aligned}
\frac{a'(t)(b(t)+\xi)}{a(t)b(t)} &\le \frac{\xi_0^2}{(b(t)+\xi_0^2)(b(t)+\xi_0)} \cdot \frac{-b'(t)}{b(t)} \\
&\le \frac{1}{\xi_0} \cdot \frac{-b'(t)}{b(t)} \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t > 0,
\end{aligned}$$

again because $b \ge 0$ and $b' \le 0$. □

The technical key toward our proof of infinite-time blow-up in the supercritical case is contained in the following lemma which says that in the supercritical mass case we can achieve that $\underline{U}$ is a subsolution in the inner region for suitably large times upon an appropriate choice of the parameters.

Lemma 6.5 *Let $\delta \ge 0$ and*

$$m > 8\pi\delta, \tag{6.16}$$

and suppose that taking W_0 and K_0 from (4.7), we have

$$W_0(\xi) - K_0\xi \ge 0 \qquad \textit{for all } \xi \in (0,1). \tag{6.17}$$

Then there exist $\xi_0 \in (0,1)$ *and* $\alpha_\star > 0$ *with the property that for all* $\alpha \in (0,\alpha_\star)$ *one can find* $b_0 \in (0,\xi_0^2)$ *and* $t_0 > 0$ *such that with*

$$b(t) := b_0 e^{-\alpha t} \qquad \text{for } t \ge 0 \tag{6.18}$$

and $a \in C^1([0,\infty))$ *as given by (6.7), the function* $\underline{U}$ *in (6.1) satisfies*

$$\mathcal{P}\underline{U}(\xi,t) \le 0 \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t \ge t_0, \tag{6.19}$$

where $\mathcal{P}$ *is given by (4.6).*

Proof. We detail the proof for the case when δ is positive, leaving the minor modifications necessary for the limit case $\delta = 0$ to the reader. Since $m > 8\pi\delta$, we can pick $\varepsilon \in (0,1)$ small enough such that

$$c_1 := \frac{(1-\varepsilon)^3 m}{(1+\varepsilon)\pi\delta} - 8 > 0, \tag{6.20}$$

and thereafter fix $\alpha_\star > 0$ and $\xi_0 \in (0,1)$ small fulfilling

$$\alpha_\star \le \frac{c_1}{4} \tag{6.21}$$

and

$$\alpha_\star \le \frac{\delta \ln \frac{1}{1-\varepsilon}}{\tau \ln \frac{1}{\varepsilon}} \tag{6.22}$$

as well as

$$\xi_0 \le \frac{\varepsilon}{2}. \tag{6.23}$$

Given $\alpha \in (0,\alpha_\star)$, we then choose $b_0 > 0$ suitably small and $t_0 > 0$ sufficiently large such that

$$b_0 \le \varepsilon \xi_0^2 \tag{6.24}$$

and

$$t_0 \ge \frac{1}{\alpha} \cdot \ln \frac{1}{1-\varepsilon}, \tag{6.25}$$

and thereupon let b, a and $\underline{U}$ be defined by (6.18), (6.7) and (6.1). Then by (6.17), Lemma 6.1 implies that

$$\frac{(b(t)+\xi)^2}{a(t)b(t)\xi} \cdot \mathcal{P}\underline{U}(\xi,t) \le J(\xi,t) := \frac{a'(b+\xi)}{ab} - \frac{b'}{b} + \frac{8}{b+\xi} - \frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \left\{ \frac{a(s)}{b(s)+\xi} - \frac{m}{2\pi} \right\} ds \tag{6.26}$$

for all $\xi \in (0,\xi_0)$ and $t > 0$. Here by Lemma 6.4, we can estimate

$$\begin{aligned} J_1(\xi,t) &:= \frac{a'(b+\xi)}{ab} - \frac{b'}{b} \\ &\le -\left(\frac{1}{\xi_0}+1\right) \cdot \frac{b'}{b} \\ &= \left(\frac{1}{\xi_0}+1\right) \cdot \alpha \\ &\le \frac{2}{\xi_0} \cdot \alpha \qquad \text{for all } t > 0. \end{aligned} \tag{6.27}$$

Next, to estimate the integral in (6.26) we first note that (6.23) guarantees that

$$\begin{aligned} \varepsilon \cdot \frac{a(t)}{b(t)+\xi} - \frac{m}{2\pi} &= \frac{m}{2\pi} \cdot \Big\{ \varepsilon \cdot \frac{b+\xi_0}{b+\xi} \cdot \frac{b+\xi_0}{b+\xi_0^2} - 1 \Big\} \\ &\ge \frac{m}{2\pi} \cdot \Big\{ \varepsilon \cdot 1 \cdot \frac{\xi_0}{2\xi_0^2} - 1 \Big\} \\ &\ge 0 \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t>0, \end{aligned}$$

and that

$$a(t) \ge \frac{m}{2\pi} \cdot \frac{\xi_0^2}{b+\xi_0^2} \ge \frac{m}{2\pi} \cdot \frac{\xi_0^2}{(1+\varepsilon)\xi_0^2} = \frac{m}{2(1+\varepsilon)\pi} \qquad \text{for all } t>0$$

by (6.24). Therefore,

$$\begin{aligned} J_2(\xi,t) &:= \frac{8}{b(t)+\xi} - \frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \Big\{ \frac{a(s)}{b(s)+\xi} - \frac{m}{2\pi} \Big\} ds \\ &\le \frac{8}{b(t)+\xi} - \frac{2(1-\varepsilon)}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \frac{a(s)}{b(s)+\xi} ds \\ &\le \frac{8}{b(t)+\xi} - \frac{(1-\varepsilon)m}{(1+\varepsilon)\pi\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \frac{1}{b(s)+\xi} ds \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t>0 \end{aligned} \tag{6.28}$$

Now whenever $t \ge s \ge t - \frac{1}{\alpha}\ln\frac{1}{1-\varepsilon} \ge 0$, we have

$$\frac{b(t)}{b(s)} = e^{-\alpha(t-s)} \ge e^{-\ln\frac{1}{1-\varepsilon}} = 1-\varepsilon,$$

which implies that

$$\frac{b(t)+\xi}{b(s)+\xi} \ge \frac{b(t)+(1-\varepsilon)\xi}{b(s)+\xi} \ge 1-\varepsilon$$

for all $\xi > 0$ and any such t and s. By means of (6.25), we can hence estimate

$$\begin{aligned} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \frac{1}{b(s)+\xi} ds &\ge \frac{1-\varepsilon}{b(t)+\xi} \cdot \int_{t-\frac{1}{\alpha}\ln\frac{1}{1-\varepsilon}}^t e^{-\frac{\delta}{\tau}(t-s)} ds \\ &= \frac{1-\varepsilon}{b(t)+\xi} \cdot \frac{\tau}{\delta}\Big(1 - e^{-\frac{\delta}{\alpha\tau}\ln\frac{1}{1-\varepsilon}}\Big) \\ &\ge \frac{(1-\varepsilon)^2\tau}{(b(t)+\xi)\cdot\delta} \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t \ge t_0, \end{aligned}$$

because (6.22) ensures that

$$e^{-\frac{\delta}{\alpha\tau}\ln\frac{1}{1-\varepsilon}} \le \exp\Big(-\frac{\delta}{\tau}\ln\frac{1}{1-\varepsilon} \cdot \frac{\tau\ln\frac{1}{\varepsilon}}{\delta\ln\frac{1}{1-\varepsilon}} \Big) = \varepsilon.$$

Accordingly, (6.28) and (6.20) yield

$$\begin{aligned} J_2(\xi,t) &\le \frac{1}{b(t)+\xi}\cdot\left\{8-\frac{(1-\varepsilon)m}{(1+\varepsilon)\pi\tau}\cdot\frac{(1-\varepsilon)^2\tau}{\delta}\right\} \\ &= -\frac{c_1}{b(t)+\xi} \qquad \text{for all } \xi\in(0,\xi_0) \text{ and } t\ge t_0, \end{aligned}$$

which since by (6.24) we have

$$b(t)+\xi \le b_0+\xi_0 \le \varepsilon\xi_0^2+\xi_0 \le 2\xi_0 \qquad \text{for all } \xi\in(0,\xi_0) \text{ and } t>0$$

guarantees that

$$J_2(\xi,t) \le -\frac{c_1}{2\xi_0} \qquad \text{for all } \xi\in(0,\xi_0) \text{ and } t\ge t_0.$$

In conjunction with (6.27), (6.26) and (6.21), this shows that

$$J(\xi,t) \le \frac{2}{\xi_0}\cdot\alpha-\frac{c_1}{2\xi_0} \le 0 \qquad \text{for all } \xi\in(0,\xi_0) \text{ and } t\ge t_0$$

and thereby proves (6.19). □

Now for small times, we can also achieve that $\mathcal{P}\underline{U}\le 0$ if we suppose w_0 to be sufficiently strongly concentrated near the origin.

Lemma 6.6 *Let $\delta\ge 0$ and $m>0$, and suppose that $\alpha>0, b_0>0$ and $\xi_0\in(0,1)$, that*

$$b(t)=b_0e^{-\alpha t} \qquad \textit{for } t\ge 0, \tag{6.29}$$

and that $a\in C^1([0,\infty))$ is as given by (6.7). Then for all $t_0>0$ there exists $\Gamma_0(\alpha,b_0,\xi_0,t_0)>0$ such that whenever W_0 and K_0 as introduced in (4.7) satisfy

$$\frac{W_0(\xi)}{\xi}-K_0 \ge \Gamma_0(\alpha,b_0,\xi_0,t_0) \qquad \textit{for all } \xi\in(0,\xi_0), \tag{6.30}$$

the function $\underline{U}$ defined in (6.1) has the property that with $\mathcal{P}$ as in (4.6) we have

$$\mathcal{P}\underline{U}(\xi,t)\le 0 \qquad \textit{for all } \xi\in(0,\xi_0) \textit{ and } t\in(0,t_0). \tag{6.31}$$

Proof. By (6.2), we have

$$\begin{aligned} \frac{(b+\xi)^2}{ab\xi}\cdot\mathcal{P}\underline{U}(\xi,t) &= \frac{a'(b+\xi)}{ab}-\frac{b'}{b}+\frac{8}{b+\xi}-\frac{2}{\tau}\int_0^t e^{-\frac{\delta}{\tau}(t-s)}\cdot\left\{\frac{a(s)}{b(s)+\xi}-\frac{m}{2\pi}\right\}ds \\ &\quad -2\left(\frac{W_0(\xi)}{\xi}-K_0\right)\cdot e^{-\frac{\delta}{\tau}t} \qquad \text{for all } \xi\in(0,\xi_0) \text{ and } t>0, \end{aligned} \tag{6.32}$$

where Lemma 6.4 ensures that

$$\frac{a'(b+\xi)}{ab}-\frac{b'}{b} \le \left(\frac{1}{\xi_0}+1\right)\cdot\frac{-b'}{b}=\left(\frac{1}{\xi_0}+1\right)\cdot\alpha, \tag{6.33}$$

and where

$$\frac{8}{b+\xi} \le \frac{8}{b_0} e^{\alpha t_0} \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t \in (0,t_0) \tag{6.34}$$

according to (6.29). Moreover, due to (6.7) we see that

$$\frac{a}{b+\xi} - \frac{m}{2\pi} = \frac{m}{2\pi} \cdot \frac{b+\xi_0}{b+\xi} \cdot \frac{b+\xi_0}{b+\xi_0^2} - \frac{m}{2\pi} \ge 0 \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t > 0, \tag{6.35}$$

because for any choice of $\xi < \xi_0$ we have $b + \xi_0 \ge b + \xi$ and also $b + \xi_0 \ge b + \xi_0^2$ due to the fact that $\xi_0 < 1$. Therefore,

$$-\frac{2}{\tau} \int_0^t e^{-\frac{\delta}{\tau}(t-s)} \cdot \left\{ \frac{a(s)}{b(s)+\xi} - \frac{m}{2\pi} \right\} ds \le 0 \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t > 0,$$

which combined with (6.32)-(6.35) shows that

$$\frac{ab\xi}{(b+\xi)^2} \cdot \mathcal{P}\underline{U}(\xi,t) \le \left(\frac{1}{\xi_0}+1\right) \cdot \alpha + \frac{8}{b_0} e^{\alpha t_0} - 2\left(\frac{W_0(\xi)}{\xi} - K_0\right) \cdot e^{-\frac{\delta}{\tau} t} \qquad \text{for all } \xi \in (0,\xi_0) \text{ and } t \in (0,t_0). \tag{6.36}$$

Thus, if

$$\frac{W_0(\xi)}{\xi} - K_0 \ge \Gamma_0(\alpha,b_0,\xi_0,t_0) := \frac{1}{2} \cdot \left\{ \left(\frac{1}{\xi_0}+1\right) \cdot \alpha + \frac{8}{b_0} e^{\alpha t_0} \right\} \cdot e^{\frac{\delta}{\tau} t_0} \qquad \text{for all } \xi \in (0,\xi_0),$$

then (6.31) results from (6.36). □

By a careful selection of the parameters in (6.1) we can finally combine Lemma 6.3, Lemma 6.5 and Lemma 6.6 to establish our main result on infinite-time blow-up of supercritical-mass solutions.

PROOF of Theorem 1.3. We first take $\xi_0 \in (0,1)$ and $\alpha_\star > 0$ as provided by Lemma 6.5 and let

$$R := \sqrt{\xi_0}. \tag{6.37}$$

Again invoking Lemma 6.5, we then find $b_0 \in (0,\xi_0^2)$ and $t_0 > 0$ with the properties listed there, and thereupon pick $\alpha \in (0,\alpha_\star)$ as given by Lemma 6.3 when applied to $\eta_0 := \frac{\eta}{2}$. Having thus fixed α, b_0, ξ_0 and t_0, we finally fix $\Gamma_0 := \Gamma_0(\alpha,b_0,\xi_0,t_0)$ as yielded by Lemma 6.6, and claim that thereupon the conclusion of Theorem 1.3 holds if we let

$$\Gamma_u(m,\eta) := \frac{m}{\pi} \cdot \frac{(b_0+\xi_0)^2}{b_0(b_0+\xi_0^2)} \tag{6.38}$$

and

$$\gamma(m,\eta) := \frac{m}{\pi} \cdot \frac{b_0}{b_0+\xi_0^2} \tag{6.39}$$

as well as

$$\Gamma_w(m,\eta) := 2\Gamma_0. \tag{6.40}$$

To verify this, given u_0 and w_0 with the assumed properties we let W_0, K_0 and U be defined by (4.7) and (4.2), respectively, and fix $\underline{U}$ as in (6.1), with $b(t) := b_0 e^{-\alpha t}$ and $a \in C^1([0,\infty))$ given by (6.7). Then (4.7) and (1.9) imply that

$$
\begin{aligned}
W_0(\xi) - K_0\xi &= \int_0^{\sqrt{\xi}} \rho w_0(\rho) d\rho - K_0\xi \\
&= \Big\{ K_0 - \int_{\sqrt{\xi}}^1 \rho w_0(\rho) d\rho \Big\} - K_0\xi \\
&= (1-\xi)K_0 - \frac{1-\xi}{2} \fint_{B_1\setminus B_{\sqrt{\xi}}} w_0 \\
&\ge (1-\xi)K_0 - \frac{1-\xi}{2} \cdot \Big\{ \fint_{B_1} w_0 - \eta \Big\} \\
&= (1-\xi) \cdot \Big\{ K_0 - \frac{1}{2} \fint_{B_1} w_0 + \frac{\eta}{2} \Big\} \\
&= (1-\xi) \cdot \frac{\eta}{2} \qquad \text{for all } \xi \in (\xi_0, 1),
\end{aligned}
$$

because $\xi_0 = R^2$ by (6.37). Therefore,

$$
\frac{W_0(\xi) - K_0\xi}{1-\xi} \ge \frac{\eta}{2} = \eta_0 \qquad \text{for all } \xi \in (\xi_0, 1), \tag{6.41}
$$

so that Lemma 6.3 applies to show that according to our choice of α and the fact that $b_0 \in (0, \xi_0^2)$, taking $\mathcal{P}$ as in (4.6) we have

$$
\mathcal{P}\underline{U}(\xi, t) \le 0 \qquad \text{for all } \xi \in (\xi_0, 1) \text{ and } t > 0. \tag{6.42}
$$

We next combine (4.7) with (1.8) and (6.40) to see that

$$
\begin{aligned}
\frac{W_0(\xi)}{\xi} - K_0 &= \frac{1}{\xi} \int_0^{\sqrt{\xi}} \rho w_0(\rho) d\rho - \int_0^1 \rho w_0(\rho) d\rho \\
&= \frac{1}{2} \Big\{ \fint_{B_{\sqrt{\xi}}} w_0 - \fint_{B_1} w_0 \Big\} \\
&\ge \frac{1}{2} \cdot \Gamma_w(m, \eta) \\
&= \Gamma_0 \qquad \text{for all } \xi \in (0, \xi_0),
\end{aligned} \tag{6.43}
$$

again because $\xi_0 = R^2$ by (6.37). Consequently, Lemma 6.6 asserts that

$$
\mathcal{P}\underline{U}(\xi, t) \le 0 \qquad \text{for all } \xi \in (0, \xi_0) \text{ and } t \in (0, t_0). \tag{6.44}
$$

Moreover, since (6.41) together with (6.43) clearly implies that $W_0(\xi) - K_0\xi \ge 0$ for all $\xi \in (0,1)$, thanks to our choice of b_0 and t_0 and the fact that $\alpha < \alpha_\star$ we may employ Lemma 6.5 to infer that

$$
\mathcal{P}\underline{U}(\xi, t) \le 0 \qquad \text{for all } \xi \in (0, \xi_0) \text{ and } t \ge t_0. \tag{6.45}
$$

Now from the definition of $\underline{U}$ and Lemma 4.1 it is clear that

$$\underline{U}(0,t) = U(0,t) = 0 \quad \text{and} \quad \underline{U}(1,t) = U(1,t) = \frac{m}{2\pi} \qquad \text{for all } t > 0. \tag{6.46}$$

In order to show that furthermore

$$\underline{U}(\xi,0) \le U(\xi,0) \qquad \text{for all } \xi \in (0,1), \tag{6.47}$$

we first consider small values of ξ, for which from (1.6) and (6.37) we gain the inequality

$$U(\xi,0) = \int_0^{\sqrt{\xi}} \rho u_0(\rho) d\rho = \frac{\xi}{2} \fint_{B_{\sqrt{\xi}}} u_0 \ge \frac{\xi}{2} \cdot \Gamma_u(m,\eta) \qquad \text{for all } \xi \in (0,\xi_0).$$

On the other hand, (6.1), (6.7) and (6.38) show that

$$\underline{U}(\xi,0) = \frac{a(0)\xi}{b_0 + \xi} \le \frac{a(0)\xi}{b_0} \le \frac{1}{2}\Gamma_u(m,\eta) \cdot \xi \qquad \text{for all } \xi \in (0,\xi_0),$$

and that hence (6.47) is valid for any such ξ. For large ξ, by (6.7) we have

$$\begin{aligned}
\underline{U}(\xi,0) &= \frac{a(0) \cdot (b_0\xi + \xi_0^2)}{(b_0 + \xi_0^2)^2} \\
&= \frac{m}{2\pi} \cdot \frac{b_0\xi + \xi_0^2}{b_0 + \xi_0^2} \\
&= \frac{m}{2\pi} \cdot \left\{1 - \frac{b_0}{b_0 + \xi_0^2} \cdot (1-\xi)\right\} \qquad \text{for all } \xi \in (\xi_0, 1),
\end{aligned}$$

whereas (1.7), (6.39) and again (6.37) yield

$$\begin{aligned}
U(\xi,0) &= \int_0^1 \rho u_0(\rho) d\rho - \int_{\sqrt{\xi}}^1 \rho u_0(\rho) d\rho \\
&= \frac{m}{2\pi} - \frac{1-\xi}{2} \cdot \fint_{B_1 \setminus B_{\sqrt{\xi}}} u_0 \\
&\ge \frac{m}{2\pi} - \frac{1-\xi}{2} \cdot \gamma(m,\eta) \\
&= \frac{m}{2\pi} - \frac{m}{2\pi} \cdot \frac{b_0}{b_0 + \xi_0^2} \cdot (1-\xi) \qquad \text{for all } \xi \in (\xi_0, 1).
\end{aligned}$$

We thereby conclude that the claimed ordering in (6.47) indeed holds, so that on the basis of (6.42), (6.44), (6.45) and (6.46) we may invoke the comparison principle in Lemma 4.2 to infer that $\underline{U}(\xi,t) \le U(\xi,t)$ for all $\xi \in [0,1]$ and $t \ge 0$. In particular, this entails that for each fixed $t > 0$ we must have

$$U_\xi(0,t) = \lim_{\xi \searrow 0} \frac{U(\xi,t)}{\xi} \ge \lim_{\xi \searrow 0} \frac{\underline{U}(\xi,t)}{\xi} = \lim_{\xi \searrow 0} \frac{a(t)}{b(t) + \xi} = \frac{a(t)}{b(t)} = \frac{m}{2\pi} \cdot \frac{(b(t) + \xi_0)^2}{b(t) + \xi_0^2} \cdot \frac{1}{b(t)}.$$

Since $b(t) = b_0 e^{-\alpha t} \leq b_0 < \xi_0^2$ for all $t > 0$, this ensures that

$$u(0,t) = 2U_\xi(0,t) \geq \frac{m}{\pi} \cdot \frac{\xi_0^2}{2\xi_0^2} \cdot \frac{1}{b(t)} = \frac{m}{2\pi b_0} e^{\alpha t} \qquad \text{for all } t > 0$$

and hence completes the proof. □

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