Effects of signal-dependent motilities in a Keller-Segel-type reaction-diffusion system

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Abstract

This work considers the Keller-Segel-type parabolic system

$$\begin{cases} u_t = \Delta(u\phi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \qquad (\star)$$

in a smoothly bounded convex domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, under no-flux boundary conditions, which has recently been proposed as a model for processes of stripe pattern formation via so-called "self-trapping" mechanisms.

In the two-dimensional case, in stark contrast to the classical Keller-Segel model in which large-data solutions may blow up in finite time, for all suitably regular initial data the associated initial value problem is seen to possess a globally defined bounded classical solution, provided that the motility function $\phi \in C^3([0,\infty)) \cap W^{1,\infty}((0,\infty))$ is uniformly positive. In the corresponding higher-dimensional setting, it is shown that certain weak solutions exist globally, where in the particular three-dimensional case this solution actually is bounded and classical if the initial data are suitably small in the norm of $L^2(\Omega) \times W^{1,4}(\Omega)$. Finally, if still $n = 3$ but merely the physically interpretable quantity $\|\phi'\|_{L^\infty((0,\infty))} \int_\Omega u_0$ is appropriately small, then the above weak solutions are proved to become eventually smooth and bounded.

Key words: chemotaxis, global existence, boundedness, eventual regularity
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1 Introduction

In the modeling of self-enhanced chemotactic migration processes at macroscopic levels, Keller-Segel-type cross-diffusive systems of the form

$$\begin{cases} u_t = \nabla \cdot (D(u,v)\nabla u - u\chi(u,v)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

play an outstanding role. With regard to the apparently most striking among the potential features of (1.1), namely the spontaneous formation of singularities known to occur in the classical (aka minimal) Keller-Segel system obtained on letting $D \equiv \chi \equiv 1$ in two- and higher-dimensional frameworks (cf. [9], [36] and also [20]), a large variety of results in the literature underlines the crucial role of the interplay between the chemotactic sensitivity S and the diffusion rate D therein.

For instance, quite a comprehensive understanding could be achieved in the case when $D = D(u) > 0$ and $\chi = \chi(u) \geq 0$ are suitably smooth, independent of the signal concentration $v = v(x,t)$ and exclusively depending on the population density $u = u(x,t)$: Then, namely, known results indicate that essentially the asymptotic behavior of the ratio $\frac{\chi(u)}{D(u)}$ at large values of u determines whether or not unbounded solutions exist, with the algebraic growth rate of $1 \leq u \mapsto u^{\frac{2-n}{n}}$ apparently determining a critical behavior of this quotient in the sense that such unboundedness phenomena may occur when $\frac{\chi}{D}$ grows substantially faster ([5], [6], [34]), whereas in cases of accordingly subcritical growth of $\frac{\chi}{D}$, appropriate technical assumptions ensure global existence and boundedness of solutions to corresponding no-flux initial-boundary value problems in smoothly bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$ ([28], [11], [24]).

The respective knowledge is much less complete in situations when the parameter functions explicitly depend on v. Even in the comparatively simple case $D \equiv 1$ and $\chi = \chi(v)$ the interaction of diffusion, taxis and signal production in (1.1) seems complex enough so as to allow for partial results identifying conditions sufficient for global boundedness up to now, with an exception formed by the particular choice $\chi(v) = \frac{\chi_0}{v}$ of a singular sensitivity consistent with the Weber-Fechner law for which besides various results on global solvability ([2], [35], [14], [8], [26]) also some complementing statements on the occurrence of exploding solutions at least in certain parabolic-elliptic simplifications of (1.1) is available ([21]). To the best of our knowledge, chemotaxis systems additionally involving signal-dependent diffusion rates have been studied only quite rudimentarily so far with regard to possible singularity formation phenomena ([15]).

A special Keller-Segel-type model with signal-dependent motility. The present work is devoted to an analytical study of a cross-diffusive parabolic system in which both the cell diffusion rate and the chemotactic sensitivity depend nonlinearly on the signal concentration, assuming a particular functional link between these parameter functions suggested by a recent modeling approach. More precisely, we shall be concerned with the system

$$\begin{cases} u_t = \Delta(u\phi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases} \tag{1.2}$$

which formally corresponds to (1.1) upon choosing $D(u,v) := \phi(v)$ and $\chi(u,v) := -\phi'(v)$, where ϕ is a sufficiently smooth given positive function on $[0,\infty)$. This system (1.2) has recently been proposed to describe processes of stripe pattern formation via so-called "self-trapping" mechanisms ([7]), which has been investigated experimentally using a synthetic biology approach ([17]). As observed in the experiment, namely, bacteria of the species *E. Coli* can secrete a small signaling molecule acyl-homoserine lactone (AHL) with the property that at low AHL levels, these bacteria are motile and can thus perform essentially unperturbed random movement via usual swim-and-tumble processes, whereas high AHL levels substantially enhance the tumbling mechanism and thus lead to essentially immotile collective behavior at the macroscale ([7]).

In order to complete the framework of our study in accordance with this modeling background, let us assume that the parameter function ϕ in (1.2) satisfies

$$\phi \in C^3([0,\infty)) \tag{1.3}$$

and

$$k_\phi \le \phi(s) \le K_\phi \qquad \text{for all } s \ge 0 \tag{1.4}$$

as well as

$$|\phi'(s)| \le K_{\phi'} \qquad \text{for all } s \ge 0 \tag{1.5}$$

with certain positive constants k_ϕ, K_ϕ and $K_{\phi'}$, and consider (1.2) under the boundary conditions

$$\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, \qquad x \in \partial\Omega,\ t > 0, \tag{1.6}$$

and the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \qquad x \in \Omega, \tag{1.7}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^n$, $n \ge 2$, with smooth boundary, where our standing assumptions on the initial data will be that

$$\begin{cases} u_0 \in C^0(\bar\Omega) & \text{is nonnegative, } u_0 \not\equiv 0, \text{ and} \\ v_0 \in W^{1,\infty}(\Omega) & \text{is nonnegative.} \end{cases} \tag{1.8}$$

Main results. The goal of this work is to explore basic qualitative dynamical properties of the particular diffusion-taxis interplay implicity contained in (1.2), with a special focus on aspects related to the question how far singular solution behavior may occur.

The first of our main results in this direction asserts global existence of bounded solutions in two-dimensional settings, in sharp contrast to the minimal Keller-Segel model thus ruling out any such type of singularity formation.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^2$ be a bounded convex domain with smooth boundary, and suppose that ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi > 0, K_\phi > 0$ and $K_{\phi'} > 0$. Then for all u_0 and v_0 fulfilling (1.8), the problem (1.2), (1.6), (1.7) possesses a global classical solution $(u,v) \in (C^0(\bar\Omega \times [0,\infty)) \cap C^{2,1}(\bar\Omega \times (0,\infty)))^2$ such that both u and v are nonnegative in $\Omega \times (0,\infty)$, and such that (u,v) is bounded in the sense that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t > 0 \tag{1.9}$$

with some $C > 0$.

In higher-dimensional cases, it is at least possible to construct certain global weak solutions having some additional regularity features which exclude any possibility of finite-time collapse into persistent Dirac-type singularities, as known to occur in some Keller-Segel-type models ([18], [30]).

Theorem 1.2 *Let $n \geq 3$, and assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. Moreover, suppose that ϕ satisfies (1.3) as well as (1.4) and (1.5) with positive constants k_ϕ, K_ϕ and $K_{\phi'}$. Then for all u_0 and v_0 fulfilling (1.8), the problem (1.2), (1.6), (1.7) possesses at least one global weak solution in the sense of Definition 5.1, and this solution can be gained as the limit a.e. in $\Omega \times (0, \infty)$ of solutions $(u_{\varepsilon_k}, v_{\varepsilon_k})$ to the regularized problems (5.5) below along a suitably chosen sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0, 1)$ such that $\varepsilon_k \searrow 0$ as $k \to \infty$.*
Beyond (5.1) and (5.2), this solution has the additional regularity properties

$$\begin{cases} u \in L^2_{loc}([0, \infty); L^2(\Omega)) \cap L^{\frac{n+2}{n+1}}_{loc}([0, \infty); W^{1,\frac{n+2}{n+1}}(\Omega)) \quad \textit{and} \\ v \in L^\infty([0, \infty); W^{1,2}(\Omega)) \cap L^2_{loc}([0, \infty); W^{2,2}(\Omega)), \end{cases} \tag{1.10}$$

and there exists $C > 0$ such that

$$\int_t^{t+1} \int_\Omega u^2 \leq C \quad \textit{and} \quad \int_t^{t+1} \int_\Omega |\Delta v|^2 \leq C \qquad \textit{for all } t > 0. \tag{1.11}$$

In the physically relevant three-dimensional case, a suitable smallness condition on the initial data ensures that even global bounded classical solutions exist.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^3$ be bounded and convex with smooth boundary. Then for all $\kappa > 0$ and $K > 0$ one can find $\delta(\kappa, K) > 0$ with the property that whenever ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi > \kappa, K_\phi \geq k_\phi$ and $K_{\phi'} \in (0, K]$, for each (u_0, v_0) fulfilling (1.8) and*

$$\int_\Omega u_0^2 + \int_\Omega |\nabla v_0|^4 \leq \delta(\kappa, K), \tag{1.12}$$

the problem (1.2), (1.6), (1.7) possesses a global classical solution $(u, v) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^2$ which is bounded in the sense that (1.9) holds.

Appropriate smallness assumptions on the initial data, involving the norms of u_0 and v_0 in $L^{\frac{n}{2}}(\Omega)$ and in $W^{1,n}(\Omega)$ and thus remaining essentially in line with (1.12), are known to enforce global boundedness also in the minimal Keller-Segel system when $n \geq 3$ ([4], [33]), whereas the total cell mass apparently loses its relevance in this respect, yet present in the case $n = 2$ ([22], [19]), in such high-dimensional settings in the sense that for arbitrary $m > 0$, in balls $\Omega \subset \mathbb{R}^n$ one can find smooth u_0 and v_0 such that $\int_\Omega u_0 = m$ but that the corresponding solution blows up in finite time ([36]). In sharp contrast to the latter, the next theorem indicates that in the context of the present model, the mass functional may play a significant role with regard to solution regularity at least beyond a certain relaxation time. More precisely, a smallness condition relating the physically relevant total mass $\int_\Omega u_0$ to the quantity $\|\phi'\|_{L^\infty((0,\infty))}$ ensures that the global weak solution constructed in Theorem 1.2 in fact becomes smooth and classical eventually.

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^3$ be bounded and convex with smooth boundary. Then for all $\kappa > 0$ and $K > 0$ there exists $\delta(\kappa, K) > 0$ such that if ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi \ge \kappa, K_\phi \in [k_\phi, K]$ and $K_{\phi'} > 0$, and if (u_0, v_0) is such that (1.8) holds as well as*

$$K_{\phi'} \cdot \overline{u}_0 \le \delta(\kappa, K), \tag{1.13}$$

then the global weak solution of (1.2), (1.6), (1.7) gained in Theorem 1.2 has the property that there exists $t_0 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C \qquad \text{for all } t > t_0 \tag{1.14}$$

with some $C > 0$, and such that (u, v) belongs to $(C^{2,1}(\bar{\Omega} \times (t_0, \infty)))^2$ and solves the boundary value problem in (1.2) classically in $\Omega \times (t_0, \infty)$.

We remark that the above condition (1.13) is satisfied for arbitrary fixed initial data and given lower and upper bounds for ϕ if $|\phi'|$ is appropriately small throughout $[0, \infty)$, thus partially confirming the intuitive idea that when ϕ is suitably close to a constant $\phi_\star$, solutions to (1.2) should exhibit a behavior which is somewhat related to that of the respective limiting equation $u_t = \phi_\star \Delta u$, at least with regard to regularity.

Main ideas. Our analytical approach is guided by the idea to appropriately respect the particular structure of (1.2) which differs from the general system (1.1) in that both diffusive and cross-diffusive movement are captured by one single action of the Laplacian. Indeed, unlike in the latter model this allows for a lifting procedure consisting in an application of A^{-1} to both sides of the identity $u_t = A(u\phi(v) - \overline{u\phi(v)})$ formally associated with (1.2), where A denotes the realization of $-\Delta$ under homogeneous Neumann boundary conditions in the subspace of $L^2(\Omega)$ orthogonal to constants. On testing the resulting equation by $u - \overline{u}_0$ this will enable us to derive estimates for $\int_t^{t+\tau} \int_\Omega (u - \overline{u}_0)^2$ with appropriately small $\tau \in (0, 1]$ (Lemma 3.1), and thereafter, by smoothing properties of the second equation in (1.2), also for $\int_\Omega |\nabla v|^2$ and for $\int_t^{t+\tau} \int_\Omega |\Delta v|^2$ (Lemma 3.2).

In the spatially two-dimensional case, this regularity information turns out to be sufficient as a starting point for a bootstrap argument, through estimates for $\int_\Omega u \ln u$ and then for $\int_\Omega u^p + \int_\Omega |\nabla v|^{2p}$ with arbitrary $p > 1$ finally yielding bounds for u in $L^\infty(\Omega)$ and thereby establishing Theorem 1.1. In the higher-dimensional case, we will make use of a certain structural stability of the a priori estimates from Lemma 3.1 and Lemma 3.2 with respect to the coefficient functions in (1.2) by firstly considering a suitably regularized variant thereof for which global solutions can easily be seen to exist, and for which the above basic regularity properties imply compactness properties allowing for the construction of a weak solution by an appropriate limit procedure (Section 5).

The proofs both of Theorem 1.3 and of Theorem 1.4 are based on the observation that in the case $n = 3$ and under the assumtion that $K_{\phi'}\overline{u}_0$ is suitably small, the functional

$$\mathcal{E}(u, v) := \int_\Omega (u - \overline{u}_0)^2 + K_{\phi'}^2 \int_\Omega |\nabla v|^4$$

possesses a certain energy-like property under the additional condition that $\mathcal{E}(u, v)$ be sufficiently small (Lemma 6.1). Again by means of a bootstrap procedure, for initial data fulfilling a smallness assumption as in Theorem 1.3 the accordingly implied boundedness features of (u, v) in $L^p(\Omega) \times W^{1,2p}(\Omega)$, with $p := 2$ satisfying $p > \frac{n}{2}$ due to the fact that $n = 3$, warrant global existence and boundedness of classical solutions (Section 6.2). If merely $K_{\phi'}\overline{u}_0$ is small, by making full use of the quantitative dependence of the basic estimates from Lemma 3.1 and Lemma 3.2 it is possible to make sure that such solutions at least eventually comply with the hypotheses from Lemma 3.2, whereupon a similar series of regularity arguments, when applied to the respective approximate versions of (1.2), yields bounds for the corresponding approximate solutions in spaces of smooth functions after an appropriate waiting time (Lemma 6.3), and thereby establishes Theorem 1.4.

2 Preliminaries

2.1 Local existence and basic solution properties

In order to derive some common features of the original system (1.2) and the regularized variants (5.5) thereof, instead of (1.2), (1.6), (1.7) let us consider the more general auxiliary problem

$$\begin{cases} u_t = \Delta(u\phi(v)), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + f(u), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{2.1}$$

with

$$f \in C^1([0,\infty)) \tag{2.2}$$

satisfying

$$f(0) = 0 \qquad \text{and} \qquad 0 \le f'(s) \le 1 \quad \text{for all } s \ge 0. \tag{2.3}$$

For any such problem, standard theory yields the following result on local existence and extensibility of smooth solutions.

Lemma 2.1 *Suppose that ϕ and f satisfy (1.3), (1.4), (2.2) and (2.3) with some $k_\phi > 0$ and $K_\phi \ge k_\phi$. Then for all u_0 and v_0 fulfilling (1.8) there exist $T_{max} \in (0,\infty]$ and a pair of nonnegative functions u and v, both belonging to $C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max}))$, such that (u,v) solves (2.1) in the classical sense in $\Omega \times (0,T_{max})$, and such that*

$$\text{if } T_{max} < \infty, \quad \text{then} \quad \limsup_{t \nearrow T_{max}} \Big\{ \|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,q}(\Omega)} \Big\} = \infty \quad \text{for all } q > n. \tag{2.4}$$

Proof. Following well-established fixed point arguments and invoking standard parabolic regularity theory (cf. e.g. [27, Lemma2.1] or [39], for instance), one can readily verify the existence of a local-in-time classical solution, nonnegative in both components by the maximum principle, and satisfying the extensiblity criterion (2.4). $\Box$

The following properties of the spatial L^1 norms are immediate.

Lemma 2.2 *Assume that (1.3), (1.4), (2.2) and (2.3) hold with some $k_\phi > 0$ and $K_\phi \ge k_\phi$, and that (1.8) is valid. Then the solution of (2.1) satisfies*

$$\int_\Omega u(x,t)dx = \int_\Omega u_0 \qquad \textit{for all } t \in (0, T_{max}) \tag{2.5}$$

as well as

$$\int_\Omega v(x,t)dx \le \Big(\int_\Omega v_0\Big)e^{-t} + \Big(\int_\Omega u_0\Big)(1-e^{-t}) \qquad \textit{for all } t \in (0, T_{max}). \tag{2.6}$$

Proof. Integrating the first equation in (1.2) with respect to $x \in \Omega$, we see that $\frac{d}{dt}\int_\Omega u \equiv 0$, and that

$$\frac{d}{dt}\int_\Omega v \;=\; -\int_\Omega v + \int_\Omega f(u) \le -\int_\Omega v + \int_\Omega u \qquad \text{for all } t \in (0, T_{max})$$

thanks to $0 \le f(u) \le u$ as a consequence of (2.3). This yields (2.5) and moreover shows (2.6). $\square$

Now combining two standard testing procedures yields the following basic inequelity that will be referred to several times throughout the sequel.

Lemma 2.3 *Assume that (1.3), (1.4), (1.5) as well as (2.2) and (2.3) are valid, and that (1.8) holds. Then for all $p > 1$ and any $a > 0$, the solution of (2.1) satisfies*

$$\begin{aligned} \frac{d}{dt}\Big\{\int_\Omega u^p + a\int_\Omega |\nabla v|^{2p}\Big\} + 2pa\int_\Omega |\nabla v|^{2p} + \frac{2(p-1)k_\phi}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \frac{4(p-1)a}{p}\int_\Omega \Big|\nabla|\nabla v|^p\Big|^2 \\ \le \frac{p(p-1)K_{\phi'}^2}{2k_\phi}\int_\Omega u^p|\nabla v|^2 + \Big(4p(p-1)^2 + np\Big)a\int_\Omega u^2|\nabla v|^{2p-2} \qquad \textit{for all } t \in (0, T_{max}). \end{aligned} \tag{2.7}$$

Proof. By straightforward computation using (1.2) and Young's inequality, we obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u^p &= -p(p-1)\int_\Omega u^{p-2}\nabla u\cdot\nabla(u\phi(v)) \\ &= -p(p-1)\int_\Omega \phi(v)u^{p-2}|\nabla u|^2 - p(p-1)\int_\Omega \phi'(v)u^{p-1}\nabla u\cdot\nabla v \\ &\le -\frac{p(p-1)}{2}\int_\Omega \phi(v)u^{p-2}|\nabla u|^2 + \frac{p(p-1)}{2}\int_\Omega \frac{\phi'^2(v)}{\phi(v)}u^p|\nabla v|^2 \quad \text{for all } t \in (0, T_{max}) \end{aligned} \tag{2.8}$$

where from (1.4) and (1.5) we know that

$$\frac{p(p-1)}{2}\int_\Omega \phi(v)u^{p-2}|\nabla u|^2 \ge \frac{p(p-1)k_\phi}{2}\int_\Omega u^{p-2}|\nabla u|^2 = \frac{2(p-1)k_\phi}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \qquad \text{for all } t \in (0, T_{max}) \tag{2.9}$$

and

$$\frac{p(p-1)}{2}\int_\Omega \frac{\phi'^2(v)}{\phi(v)}u^p|\nabla v|^2 \le \frac{p(p-1)K_{\phi'}^2}{2k_\phi}\int_\Omega u^p|\nabla v|^2 \qquad \text{for all } t \in (0, T_{max}). \tag{2.10}$$

Next, relying on the convexity of Ω in estimating $\frac{\partial|\nabla v|^2}{\partial\nu} \le 0$ on $\partial\Omega$ ([16]) and making use of the identities $2\nabla v\cdot\nabla\Delta v = \Delta|\nabla v|^2 - 2|D^2v|^2$ and $\nabla|\nabla v|^{2p-2} = (p-1)|\nabla v|^{2p-4}\nabla|\nabla v|^2$, by means of a standard testing procedure applied to the second equation in (2.1) ([28]) we see that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + \frac{4(p-1)}{p}\int_\Omega \Big|\nabla|\nabla v|^p\Big|^2 + 2p\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 + 2p\int_\Omega |\nabla v|^{2p} \\
&\quad\le\ 2p\int_\Omega |\nabla v|^{2p-2}\nabla f(u)\cdot\nabla v \qquad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{2.11}$$

where integrating by parts and using Young's inequality we obtain

$$\begin{aligned}
2p\int_\Omega |\nabla v|^{2p-2}\nabla f(u)\cdot\nabla v &= -4p(p-1)\int_\Omega f(u)|\nabla v|^{2p-4}\nabla v\cdot(D^2v\cdot\nabla v) - 2p\int_\Omega f(u)|\nabla v|^{2p-2}\Delta v \\
&\le p\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 + 4p(p-1)^2\int_\Omega f^2(u)|\nabla v|^{2p-2} \\
&\quad +\frac{p}{n}\int_\Omega |\nabla v|^{2p-2}|\Delta v|^2 + np\int_\Omega f^2(u)|\nabla v|^{2p-2} \\
&\le 2p\int_\Omega |\nabla v|^{2p-2}|D^2v|^2 + \Big(4p(p-1)^2 + np\Big)\int_\Omega u^2|\nabla v|^{2p-2}
\end{aligned}$$

for all $t\in(0,T_{max})$, because $|\Delta v|^2 \le n|D^2v|^2$ in $\Omega\times(0,T_{max})$ and $0\le f(u)\le u$ in $\Omega\times(0,T_{max})$ as a consequence of (2.3). Therefore, (2.7) immediately results from an evident linear combination of (2.11) with (2.8) upon taking into account (2.9) and (2.10). $\Box$

2.2 Two ODE comparison results

The following statement on subsolutions of an apsorptive linear ODE, generalizing a corresponding inequality obtained in [25], will be needed in Lemma 3.2 and also in Lemma 4.1.

Lemma 2.4 *Let $T\in(0,\infty]$, and suppose that $y\in C^0([0,T))\cap C^1((0,T))$ is nonnegative and satisfies*

$$y'(t) + \lambda y(t) \le h(t) \qquad \textit{for all } t\in(0,T) \tag{2.12}$$

with a nonnegative function $h\in C^0((0,T))\cap L^1_{loc}([0,T))$ fulfilling

$$\int_t^{t+\tau} h(s)ds \le a + be^{-\lambda' t} \qquad \textit{for all } t\in(0,T-\tau), \tag{2.13}$$

where a,b,λ,λ' and τ are positive constants such that $\lambda<\lambda'$ and $\tau<T$. Then

$$y(t) \le \frac{a}{1-e^{-\lambda\tau}} + \Big\{y(0) + \frac{be^{(2\lambda'-\lambda)\tau}}{e^{(\lambda'-\lambda)\tau}-1}\Big\}\cdot e^{-\lambda t} \qquad \textit{for all } t\in(0,T). \tag{2.14}$$

Proof. Given $t\in(0,T)$, we fix $k\in\mathbb{N}$ such that $(k-1)\tau < t \le k\tau$, so that extending h to $(-\infty,T)$ by letting $h(t):=0$ for $t\le 0$, from (2.12) and a comparison argument we infer that

$$y(t) \ \le\ y(0)e^{-\lambda t} + \int_0^t e^{-\lambda(t-s)}h(s)ds$$

$$
\begin{aligned}
&= y(0)e^{-\lambda t} + \sum_{j=0}^{k-1} \int_{t-(j+1)\tau}^{t-j\tau} e^{-\lambda(t-s)} h(s)ds \\
&\le y(0)e^{-\lambda t} + \sum_{j=0}^{k-1} e^{-j\lambda\tau} \int_{t-(j+1)\tau}^{t-j\tau} h(s)ds.
\end{aligned}
\tag{2.15}
$$

Here by (2.13) and our assumption that $\lambda' > \lambda$,

$$
\begin{aligned}
\sum_{j=0}^{k-1} e^{-j\lambda\tau} \int_{t-(j+1)\tau}^{t-j\tau} h(s)ds &\le a\sum_{j=0}^{k-1} e^{-j\lambda\tau} + b\sum_{j=0}^{k-1} e^{-j\lambda\tau} e^{-\lambda'(t-(j+1)\tau)} \\
&= a\sum_{j=0}^{k-1} e^{-j\lambda\tau} + be^{-\lambda'(t-\tau)} \cdot \sum_{j=0}^{k-1} e^{j(\lambda'-\lambda)\tau} \\
&= a \cdot \frac{1-e^{-k\lambda\tau}}{1-e^{-\lambda\tau}} + be^{-\lambda'(t-\tau)} \cdot \frac{e^{k(\lambda'-\lambda)\tau}-1}{e^{(\lambda'-\lambda)\tau}-1} \\
&\le \frac{a}{1-e^{-\lambda\tau}} + be^{-\lambda'(t-\tau)} \cdot \frac{e^{(\lambda'-\lambda)(t+\tau)}}{e^{(\lambda'-\lambda)\tau}-1},
\end{aligned}
$$

because $k\tau < t+\tau$. Along with (2.15), this readily establishes (2.14). $\Box$

For superlinearly dampened differential inequalities with possibly large initial data, we prepare the following statement to be used in Lemma 6.2.

Lemma 2.5 *Let $a > 0$ and $b > 0$ and $\alpha > 1$, and suppose that with some $t_0 \in \mathbb{R}$ and $T \in (t_0, \infty]$ we are given a nonnegative function $y \in C^0([t_0, T)) \cap C^1((t_0, T))$ satisfying*

$$
y'(t) + ay^\alpha(t) \le b \qquad \textit{for all } t \in (t_0, T). \tag{2.16}
$$

Then

$$
y(t) \le C \cdot (t-t_0)^{-\frac{1}{\alpha-1}} + C \qquad \textit{for all } t \in (t_0, T) \tag{2.17}
$$

with

$$
C := \max\left\{ \Big(\frac{2}{(\alpha-1)a}\Big)^{\frac{1}{\alpha-1}}, \Big(\frac{2b}{a}\Big)^{\frac{1}{\alpha}} \right\}. \tag{2.18}
$$

PROOF. Without loss of generality assuming that $t_0 = 0$, with C as defined in (2.18) we let $\overline{y}(t) := Ct^{-\frac{1}{\alpha-1}} + C$ for $t \in (0, T)$ and compute

$$
\overline{y}'(t) + a\overline{y}^\alpha(t) - b = -\frac{C}{\alpha-1} t^{-\frac{\alpha}{\alpha-1}} + aC^\alpha \cdot \Big\{ t^{-\frac{1}{\alpha-1}} + 1 \Big\}^\alpha - b \qquad \text{for all } t \in (0, T).
$$

As (2.18) guarantees that

$$
\frac{\frac{1}{2}aC^\alpha \cdot \Big\{ t^{-\frac{1}{\alpha-1}} + 1 \Big\}^\alpha}{\frac{C}{\alpha-1} t^{-\frac{\alpha}{\alpha-1}}} \ge \frac{\frac{1}{2}aC^\alpha \cdot t^{-\frac{\alpha}{\alpha-1}}}{\frac{C}{\alpha-1} t^{-\frac{\alpha}{\alpha-1}}} = \frac{(\alpha-1)a}{2} C^{\alpha-1} \ge 1 \qquad \text{for all } t \in (0, T)
$$

and

$$\frac{\frac{1}{2}aC^\alpha \cdot \left\{t^{-\frac{1}{\alpha-1}}+1\right\}^\alpha}{b} \ge \frac{\frac{1}{2}aC^\alpha}{b} \ge 1 \qquad \text{for all } t\in(0,T),$$

this implies that

$$\overline{y}'(t)+a\overline{y}^\alpha(t)-b\ge 0 \qquad \text{for all } t\in(0,T).$$

Since $\overline{y}(t)\to+\infty$ as $t\searrow 0$ and hence $\overline{y}(t_1)>y(t_1)$ for all $t_1\in(0,t_\star)$ with some sufficiently small $t_\star\in(0,T)$, an ODE comparison on (t_1,T) shows that $\overline{y}\ge y$ on (t_1,T) for any such t_1, which on taking $t_1\searrow 0$ shows that indeed (2.17) holds. □

3 Fundamental a priori estimates for (2.1)

Now a cornerstone for all our subsequence analysis is obtained by properly exploiting the special structure of the diffusive processes in both (1.2) and (2.1), thus constituting an essential difference between these and the more general system (1.1).

Lemma 3.1 *Let $\kappa>0$. Then there exist $L=L(\kappa)>0$ and $\lambda=\lambda(\kappa)>0$ with the property that if ϕ satisfies (1.3) and (1.4) with some $k_\phi\ge\kappa$ and $K_\phi\ge k_\phi$, and if f complies with (2.2) and (2.3), then for any (u_0,v_0) fulfilling (1.8) one can find $C>0$ such that for the solution of (2.1) we have*

$$\int_t^{t+\tau}\int_\Omega (u-\overline{u}_0)^2 \le LB_{\phi,u_0}^2+Ce^{-\lambda t} \qquad \textit{for all } t\in(0,T_{max}-\tau), \tag{3.1}$$

where

$$\tau:=\min\left\{1\,,\,\frac{1}{2}T_{max}\right\} \tag{3.2}$$

and

$$B_{\phi,u_0}:=V_\phi\overline{u}_0 \tag{3.3}$$

with

$$V_\phi:=K_\phi-k_\phi. \tag{3.4}$$

Proof. We let A denote the self-adjoint realization of $-\Delta$ under homogeneous Neumann boundary conditions in the Hilbert space $L^2_\perp(\Omega):=\{\psi\in L^2(\Omega)\ |\ \int_\Omega\psi=0\}$ with domain given by $D(A):=\{\psi\in W^{2,2}(\Omega)\cap L^2_\perp(\Omega)\ |\ \frac{\partial\psi}{\partial\nu}=0 \text{ on } \partial\Omega\}$. Then according to known results from elliptic theory ([38]), since the spectrum of A is a discrete subset of the positive real half-line $(0,\infty)$, A possesses bounded self-adjoint fractional powers $A^{-\alpha}$ for any $\alpha>0$. In particular, rewriting the first equation in (2.1) in the form $u_t=-A\Big(u\phi(v)-\overline{u\phi(v)}\Big)$, in view of (2.5) we may compute

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega \left|A^{-\frac{1}{2}}(u-\overline{u}_0)\right|^2 &= \int_\Omega A^{-\frac{1}{2}}(u-\overline{u}_0)\cdot A^{-\frac{1}{2}}(u-\overline{u}_0)_t \\
&= \int_\Omega A^{-1}(u-\overline{u}_0)\cdot(u-\overline{u}_0)_t \\
&= -\int_\Omega A^{-1}(u-\overline{u}_0)\cdot A\Big(u\phi(v)-\overline{u\phi(v)}\Big) \\
&= -\int_\Omega (u-\overline{u}_0)\cdot\Big(u\phi(v)-\overline{u\phi(v)}\Big) \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{3.5}$$

Here we decompose

$$\begin{aligned}\int_\Omega (u-\overline{u}_0)\cdot\Big(u\phi(v)-\overline{u\phi(v)}\Big) &= \int_\Omega (u-\overline{u}_0)\cdot\Big(u\phi(v)-\overline{u}_0\phi(v)\Big)+\int_\Omega (u-\overline{u}_0)\cdot\Big(\overline{u}_0\phi(v)-\overline{u\phi(v)}\Big)\\ &= \int_\Omega \phi(v)(u-\overline{u}_0)^2+\int_\Omega (u-\overline{u}_0)\cdot\Big(\overline{u}_0\phi(v)-\overline{u\phi(v)}\Big)\end{aligned}$$

for $t\in(0,T_{max})$, where due to (2.5) and the definitions (3.4) and (3.3) of V_ϕ and B we can estimate

$$\begin{aligned}\Big|\overline{u}_0\phi(v(x,t))-\overline{u(\cdot,t)\phi(v(\cdot,t))}\Big| &= \Big|\overline{u(\cdot,t)}\phi(v(x,t))-\overline{u(\cdot,t)\phi(v(\cdot,t))}\Big|\\ &= \Big|\frac{1}{|\Omega|}\int_\Omega u(y,t)\cdot\Big\{\phi(v(x,t))-\phi(v(y,t))\Big\}dy\Big|\\ &\le V_\phi\cdot\frac{1}{|\Omega|}\int_\Omega u(y,t)dy\\ &= B_{\phi,u_0} \qquad \text{for all } x\in\Omega \text{ and } t\in(0,T_{max}).\end{aligned}$$

Thus, by Young's inequality and (1.4) we see that

$$\begin{aligned}\Big|\int_\Omega (u-\overline{u}_0)\cdot\Big(\overline{u}_0\phi(v)-\overline{u\phi(v)}\Big)\Big| &\le \frac12\int_\Omega\phi(v)(u-\overline{u}_0)^2+\frac12\int_\Omega\frac{1}{\phi(v)}\cdot B^2_{\phi,u_0}\\ &\le \frac12\int_\Omega\phi(v)(u-\overline{u}_0)^2+\frac{|\Omega|}{2k_\phi}B^2_{\phi,u_0} \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

and that hence

$$\begin{aligned}\int_\Omega (u-\overline{u}_0)\cdot\Big(u\phi(v)-\overline{u\phi(v)}\Big) &\ge \frac12\int_\Omega\phi(v)(u-\overline{u}_0)^2-\frac{|\Omega|}{2k_\phi}B^2_{\phi,u_0}\\ &\ge \frac{k_\phi}{2}\int_\Omega (u-\overline{u}_0)^2-\frac{|\Omega|}{2k_\phi}B^2_{\phi,u_0} \qquad \text{for all } t\in(0,T_{max}),\end{aligned}$$

so that from (3.5) and our assumption that $k_\phi\ge\kappa$ we obtain the inequality

$$\frac{d}{dt}\int_\Omega\Big|A^{-\frac12}(u-\overline{u}_0)\Big|^2+\kappa\int_\Omega(u-\overline{u}_0)^2\le c_1B^2_{\phi,u_0} \qquad \text{for all } t\in(0,T_{max}) \tag{3.6}$$

with $c_1:=\frac{|\Omega|}{\kappa}$. Now using that $\int_\Omega A^{-\frac12}(u-\overline{u}_0)=0$, we may invoke the Poincaré inequality to gain $\lambda>0$ such that

$$\frac{\kappa}{2}\int_\Omega(u-\overline{u}_0)^2=\frac{\kappa}{2}\int_\Omega\Big|\nabla A^{-\frac12}(u-\overline{u}_0)\Big|^2\ge\lambda\int_\Omega\Big|A^{-\frac12}(u-\overline{u}_0)\Big|^2 \qquad \text{for all } t\in(0,T_{max}),$$

so that (3.6) implies that for $y(t):=\int_\Omega\Big|A^{-\frac12}(u(\cdot,t)-\overline{u}_0)\Big|^2$ and $g(t):=\frac{\kappa}{2}\int_\Omega(u(\cdot,t)-\overline{u}_0)^2$, $t\in[0,T_{max})$, we have

$$y'(t)+\lambda y(t)+g(t)\le c_1B^2_{\phi,u_0} \qquad \text{for all } t\in(0,T_{max}). \tag{3.7}$$

As g is nonnegative, this firstly entails that

$$\begin{aligned} y(t) &\le y(0)e^{-\lambda t} + c_1 B_{\phi,u_0}^2 \int_0^t e^{-\lambda(t-s)}ds \\ &= y(0)e^{-\lambda t} + \frac{c_1 B_{\phi,u_0}^2}{\lambda}(1-e^{-\lambda t}) \\ &\le y(0)e^{-\lambda t} + \frac{c_1 B_{\phi,u_0}^2}{\lambda} \qquad \text{for all } t \in (0,T_{max}), \end{aligned}$$

whereupon an integration of (3.7) shows that

$$\int_t^{t+\tau} g(s)ds \le y(t) + c_1 B_{\phi,u_0}^2 \le y(0)e^{-\lambda t} + \frac{c_1 B_{\phi,u_0}^2}{\lambda} + c_1 B_{\phi,u_0}^2 \qquad \text{for all } t \in (0, T_{max}-\tau)$$

and thereby proves the lemma. □

For the evolution of the standard first-order energy functional associated with the inhomogeneous linear heat equation for v in (2.1), Lemma 3.1 has an immediate consequence which implies the following.

Lemma 3.2 *For all $\kappa > 0$ there exist $M = M(\kappa) > 0$ and $\mu = \mu(\kappa) > 0$ with the following property: If ϕ and f satisfy (1.3), (1.4), (2.2) and (2.3) with some $k_\phi \ge \kappa$ and $K_\phi \ge k_\phi$, then for any choice of (u_0, v_0) fulfilling (1.8) it is possible to choose $C > 0$ such that the solution of (2.1) satisfies*

$$\int_\Omega |\nabla v(\cdot,t)|^2 \le M B_{\phi,u_0}^2 + Ce^{-\mu t} \qquad \textit{for all } t \in (0,T_{max}) \tag{3.8}$$

and

$$\int_t^{t+\tau} \int_\Omega |\Delta v|^2 \le M B_{\phi,u_0}^2 + Ce^{-\mu t} \qquad \textit{for all } t \in (0,T_{max}-\tau), \tag{3.9}$$

where τ and B_{ϕ,u_0} are as in (3.2) and (3.3).

Proof. Given $\kappa > 0$, according to Lemma 3.1 we can pick $L > 0$ and $\lambda > 0$ such that for each (u_0, v_0) satisfying (1.8) one can find $c_1 > 0$ such that if (u, v) denotes the maximally extended solution of (2.1) with some ϕ and f fulfilling (1.3), (1.4), (2.2) and (2.3), then

$$\int_t^{t+\tau} \int_\Omega \Big(u(\cdot,s) - \overline{u}_0\Big)^2 ds \le L B_{\phi,u_0}^2 + c_1 e^{-\lambda t} \qquad \text{for all } t \in (0, T_{max}-\tau). \tag{3.10}$$

Now for any such solution, we test the second equation in (2.1) by $-\Delta v$ to obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla v|^2 + \int_\Omega |\Delta v|^2 + \int_\Omega |\nabla v|^2 &= -\int_\Omega f(u)\Delta v \\ &= -\int_\Omega \Big(f(u) - \overline{f(u)}\Big)\Delta v \\ &\le \frac{1}{2}\int_\Omega |\Delta v|^2 + \frac{1}{2}\int_\Omega \Big(f(u) - \overline{f(u)}\Big)^2 \end{aligned} \tag{3.11}$$

for all $t \in (0, T_{max})$, because $\int_\Omega \Delta v = \int_{\partial\Omega} \frac{\partial v}{\partial \nu} = 0$ for any such t. Here since $|f'| \le 1$ on $[0, \infty)$ by (2.3), by the mean value theorem the integrand in the rightmost term can be estimated according to

$$\begin{aligned} \Big| f(u(x,t)) - \overline{f(u(\cdot,t))} \Big| &= \Big| \frac{1}{|\Omega|} \int_\Omega \Big\{ f(u(x,t)) - f(u(y,t)) \Big\} dy \Big| \\ &\le \frac{1}{|\Omega|} \int_\Omega \Big| u(x,t) - u(y,t) \Big| dy \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max}), \end{aligned}$$

so that thanks to the Cauchy-Schwarz inequality and the Minkowski inequality,

$$\begin{aligned} \frac{1}{2} \int_\Omega \Big(f(u(x,t)) - \overline{f(u(\cdot,t))} \Big)^2 dx &\le \frac{1}{2|\Omega|^2} \int_\Omega \Big\{ \int_\Omega \Big| u(x,t) - u(y,t) \Big| dy \Big\}^2 dx \\ &\le \frac{1}{2|\Omega|} \int_\Omega \int_\Omega \Big(u(x,t) - u(y,t) \Big)^2 dy dx \\ &\le \frac{1}{|\Omega|} \int_\Omega \int_\Omega \Big\{ \Big(u(x,t) - \overline{u}_0 \Big)^2 + \Big(\overline{u}_0 - u(y,t) \Big)^2 \Big\} dy dx \\ &= \int_\Omega \Big(u(x,t) - \overline{u}_0 \Big)^2 dx + \int_\Omega \Big(\overline{u}_0 - u(y,t) \Big)^2 dy \\ &= 2 \int_\Omega (u - \overline{u}_0)^2 \qquad \text{for all } t \in (0, T_{max}). \end{aligned}$$

Now writing $y(t) := \int_\Omega |\nabla v(\cdot,t)|^2$ for $t \in [0, T_{max})$ and $g(t) := \int_\Omega |\Delta v(\cdot,t)|^2$ as well as $h(t) := 4 \int_\Omega (u(\cdot,t) - \overline{u}_0)^2$ for $t \in (0, T_{max})$, we thus see that for any fixed $\mu \in (0, 2]$ such that $\mu < \lambda$, (3.11) implies that

$$y'(t) + \mu y(t) + g(t) \le h(t) \qquad \text{for all } t \in (0, T_{max}). \tag{3.12}$$

As g is nonnegative, Lemma 2.4 therefore applies to show that in view of (3.10) we have

$$y(t) \le M B_{\phi,u_0}^2 + c_2 e^{-\mu t} \qquad \text{for all } t \in (0, T_{max})$$

with $M := \frac{4L}{1 - e^{-\mu\tau}}$ and $c_2 := \int_\Omega |\nabla v_0|^2 + \frac{4c_1 e^{(2\lambda - \mu)\tau}}{e^{(\lambda-\mu)\tau} - 1}$, and thereupon an integration of (3.12), again thanks to (3.10), yields

$$\begin{aligned} \int_t^{t+\tau} g(s) ds &\le y(t) + \int_t^{t+\tau} h(s) ds \\ &\le (4L + M) B_{\phi,u_0}^2 + c_2 e^{-\mu t} + 4c_1 e^{-\lambda t} \\ &\le (4L + M) B_{\phi,u_0}^2 + (4c_1 + c_2) e^{-\mu t} \qquad \text{for all } t \in (0, T_{max} - \tau), \end{aligned}$$

because $\mu < \lambda$. □

Upon a straightforward interpolation, the two inequalities in Lemma 3.2 entail the following.

Lemma 3.3 *Let $\kappa > 0$. Then there exist $N = N(\kappa) > 0$ and $\gamma = \gamma(\kappa) > 0$ such that whenever ϕ and f satisfy (1.3), (1.4), (2.2) and (2.3) with some $k_\phi \ge \kappa$ and $K_\phi \ge k_\phi$, for all (u_0, v_0) fulfilling (1.8) one can fix $C > 0$ such that for the solution of (2.1) we have*

$$\int_t^{t+\tau} \int_\Omega |\nabla v|^{\frac{2(n+2)}{n}} \le N B_{\phi,u_0}^{\frac{2(n+2)}{n}} + C e^{-\gamma t} \qquad \textit{for all } t \in (0, T_{max}) \tag{3.13}$$

with τ and B as given by (3.2) and (3.3).

Proof. From Lemma 3.2 we know that for each $\kappa > 0$ we can choose positive constants M and μ such that under the above conditions on ϕ and f, whenever (u_0, v_0) is such that (1.8) holds one can find $c_1 > 0$ fulfilling

$$\int_\Omega |\nabla v(\cdot,t)|^2 \le MB^2_{\phi,u_0} + c_1 e^{-\mu t} \qquad \text{for all } t \in (0, T_{max}) \tag{3.14}$$

and

$$\int_t^{t+\tau} \int_\Omega |\Delta v|^2 \le MB^2_{\phi,u_0} + c_1 e^{-\mu t} \qquad \text{for all } t \in (0, T_{max} - \tau). \tag{3.15}$$

As the Gagliardo-Nirenberg inequality along with standard elliptic regularity theory provides $c_2 > 0$ such that

$$\|\nabla\psi\|_{L^{\frac{2(n+2)}{n}}(\Omega)}^{\frac{2(n+2)}{n}} \le c_2 \|\Delta\psi\|^2_{L^2(\Omega)} \|\nabla\psi\|^{\frac{4}{n}}_{L^2(\Omega)} \qquad \text{for all } \psi \in W^{2,2}(\Omega) \text{ with } \tfrac{\partial\psi}{\partial\nu} = 0 \text{ on } \partial\Omega,$$

combining (3.14) with (3.15) shows that

$$\int_t^{t+\tau} \|\nabla v(\cdot,s)\|_{L^{\frac{2(n+2)}{n}}(\Omega)}^{\frac{2(n+2)}{n}} ds \le c_2 \Big(MB^2_{\phi,u_0} + c_1 e^{-\mu t} \Big)^{\frac{n+2}{n}} \qquad \text{for all } t \in (0, T_{max} - \tau).$$

Since $(a+b)^{\frac{n+2}{n}} \le 2^{\frac{2}{n}} (a^{\frac{n+2}{n}} + b^{\frac{n+2}{n}})$ for all $a \ge 0$ and $b \ge 0$, this readily implies (3.13) with $N := 2^{\frac{2}{n}} c_2 M^{\frac{n+2}{n}}$, $\gamma := \frac{n+2}{n}\mu$ and $C := 2^{\frac{2}{n}} c_1^{\frac{n+2}{n}} c_2$, for instance. □

4 The two-dimensional case

In this section we directly address the problem (1.2) in the two-dimensional case, thus specifying $f(s) := s$ for $s \ge 0$ in (2.1). In this framework, namely, the integrability exponent in Lemma 3.3 is large enough so as to allow for appropriately estimating the cross-diffusive contributions arising in an ODE describing the time evolution of the logarithmic entropy $\int_\Omega u \ln u$.

Lemma 4.1 *Let $n = 2$, suppose that ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi > 0, K_\phi > 0$ and $K_{\phi'} > 0$, and that (1.8) holds. Then there exists $C > 0$ such that the solution of (1.2), (1.6), (1.7) satisfies*

$$\int_\Omega u(\cdot,t) \ln u(\cdot,t) \le C \qquad \textit{for all } t \in (0, T_{max}). \tag{4.1}$$

Proof. Since u is positive in $\bar\Omega \times (0, T_{max})$ according to the strong maximum principle, we may test the first equation in (1.2) by $\ln u$ and use (2.5) and Young's inequality to see that

$$\begin{aligned}
\frac{d}{dt}\int_\Omega u \ln u &= -\int_\Omega \frac{1}{u} \nabla u \cdot \nabla (u\phi(v)) \\
&= -\int_\Omega \phi(v) \frac{|\nabla u|^2}{u} - \int_\Omega \phi'(v) \nabla u \cdot \nabla v \\
&\le -\frac{1}{2} \int_\Omega \phi(v) \frac{|\nabla u|^2}{u} + \frac{1}{2} \int_\Omega \frac{\phi'^2(v)}{\phi(v)} u |\nabla v|^2 \\
&\le -\frac{1}{2} \int_\Omega \phi(v) \frac{|\nabla u|^2}{u} + \int_\Omega u^2 + \frac{1}{16} \int_\Omega \frac{\phi'^4(v)}{\phi^2(v)} |\nabla v|^4 \qquad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{4.2}$$

where (1.4) and (1.5) imply that

$$\frac{1}{2}\int_\Omega \phi(v)\frac{|\nabla u|^2}{u} \ge c_1 \int_\Omega \frac{|\nabla u|^2}{u} \qquad \text{for all } t \in (0, T_{max})$$

and

$$\frac{1}{16}\int_\Omega \frac{\phi'^4(v)}{\phi^2(v)}|\nabla v|^4 \le c_2 \int_\Omega |\nabla v|^4 \qquad \text{for all } t \in (0, T_{max})$$

with $c_1 := \frac{k_\phi}{2}$ and $c_2 := \frac{K_{\phi'}^4}{16k_\phi^2}$. Now since by an interpolation argument it can easily be seen that there exists $c_3 > 0$ such that

$$\int_\Omega u \ln u \le c_1 \int_\Omega \frac{|\nabla u|^2}{u} + c_3 \qquad \text{for all } t \in (0, T_{max}),$$

from (4.2) we infer that $y(t) := \int_\Omega u(\cdot,t)\ln u(\cdot,t)$, $t \in [0, T_{max})$, and $h(t) := \int_\Omega u^2(\cdot,t) + c_2 \int_\Omega |\nabla v(\cdot,t)|^4 + c_3$, $t \in (0, T_{max})$, satisfy

$$y'(t) + y(t) \le h(t) \qquad \text{for all } t \in (0, T_{max}).$$

As Lemma 3.1 and Lemma 3.3 along with (2.5) entail the existence of $c_4 > 0$ such that with $\tau = \min\{1, \frac{1}{2}T_{max}\}$ we have

$$\int_t^{t+\tau} h(s)ds \le c_4 \qquad \text{for all } t \in (0, T_{max} - \tau)$$

with $c_4 := LB_{\phi,u_0}^2 + c_2 NB_{\phi,u_0}^4 + \overline{u}_0^2|\Omega|\tau + c_3\tau + C(1 + c_2)$, where B_{ϕ,u_0}, L, N and C are constants of Lammata 3.1 and 3.2, Lemma 2.4 thus warrants that

$$y(t) \le \frac{c_4}{1 - e^{-\tau}} + y(0)e^{-t} \qquad \text{for all } t \in (0, T_{max})$$

and thereby clearly entails (4.1). $\square$

Similar to corresponding situations in the minimal two-dimensional Keller-Segel system ([22], [1]), through a variant of the Gagliardo-Nirenberg inequality due to [3] the above slight improvement of the L^1 information from (2.5) is sufficient to ensure higher regularity estimates.

Lemma 4.2 *Let $n = 2$, suppose that ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi > 0, K_\phi > 0$ and $K_{\phi'} > 0$, and that (1.8) holds. Then for all $p > 1$ one can find $C(p) > 0$ such that the solution of (1.2), (1.6), (1.7) has the properties that*

$$\int_\Omega u^p(\cdot,t) \le C(p) \qquad \textit{for all } t \in (0, T_{max}) \tag{4.3}$$

and

$$\int_\Omega |\nabla v(\cdot,t)|^{2p} \le C(p) \qquad \textit{for all } t \in (0, T_{max}). \tag{4.4}$$

PROOF. We first apply Lemma 2.3 to see that with $c_1 := \min\{\frac{2(p-1)k_\phi}{p}, \frac{4(p-1)}{p}\}$ and $c_2 := \max\{\frac{p(p-1)K^2_{\phi'}}{2k_\phi}, 4p(p-1)^2+2p\}$ we have

$$\begin{aligned}
&\frac{d}{dt}\Big\{\int_\Omega u^p + \int_\Omega |\nabla v|^{2p}\Big\} + \Big\{\int_\Omega u^p + \int_\Omega |\nabla v|^{2p}\Big\} \\
&\qquad +c_1\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_1\int_\Omega \Big|\nabla|\nabla v|^p\Big|^2 \\
&\quad \le c_2\int_\Omega u^p|\nabla v|^2 + c_2\int_\Omega u^2|\nabla v|^{2p-2} + \int_\Omega u^p \qquad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{4.5}$$

because $2p \ge 1$. In order to prepare an adequate estimation of the three rightmost summands herein, we invoke the Gagliardo-Nirenberg inequality (see [32] for a version suitable in the present case involving small summability powers) to find $c_3 > 0$ such that

$$\|\psi\|^{\frac{2(p+1)}{p}}_{L^{\frac{2(p+1)}{p}}(\Omega)} \le c_3\|\nabla\psi\|^2_{L^2(\Omega)}\|\psi\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} + c_3\|\psi\|^{\frac{2(p+1)}{p}}_{L^{\frac{2}{p}}(\Omega)} \qquad \text{for all } \psi\in W^{1,2}(\Omega), \tag{4.6}$$

and recall Lemma 3.2 to fix $c_4 > 0$ fulfilling

$$\int_\Omega |\nabla v|^2 \le c_4 \qquad \text{for all } t\in(0,T_{max}). \tag{4.7}$$

Then employing Young's inequality, we see that setting $\eta_1 := \frac{c_1}{c_3c_4}$ we can choose $c_5 > 0$ such that for all $t \in (0, T_{max})$,

$$\begin{aligned}
c_2\int_\Omega u^p|\nabla v|^2 + c_2\int_\Omega u^2|\nabla v|^{2p-2} + \int_\Omega u^p \;&\le\; \Big\{\frac{\eta_1}{2}\int_\Omega |\nabla v|^{2p+2} + c_5\int_\Omega u^{p+1}\Big\} \\
&\quad +\Big\{\frac{\eta_1}{2}\int_\Omega |\nabla v|^{2p+2} + c_5\int_\Omega u^{p+1}\Big\} \\
&\quad +\Big\{c_5\int_\Omega u^{p+1} + 1\Big\} \\
&= \eta_1\int_\Omega |\nabla v|^{2p+2} + 3c_5\int_\Omega u^{p+1} + 1.
\end{aligned} \tag{4.8}$$

Next, since $\xi\ln\xi \ge -\frac{1}{e}$ for all $\xi > 0$, from Lemma 4.1 we obtain $c_6 > 0$ satisfying

$$\int_\Omega |u\ln u| \le c_6 \qquad \text{for all } t\in(0,T_{max}), \tag{4.9}$$

and from an extended Gagliardo-Nirenberg inequality generalizing an observation originally made in [3] (see [29, Lemma A.5]) we infer that writing $\eta_2 := \frac{2c_1}{3pc_5c_6}$ we can find $c_7 > 0$ with the property that

$$\|\psi\|^{\frac{2(p+1)}{p}}_{L^{\frac{2(p+1)}{p}}(\Omega)} \le \eta_2\|\nabla\psi\|^2_{L^2(\Omega)}\Big\||\psi|\ln\psi|^{\frac{p}{2}}\Big\|^{\frac{2}{p}}_{L^{\frac{2}{p}}(\Omega)} + c_7\|\psi\|^{\frac{2(p+1)}{p}}_{L^{\frac{2}{p}}(\Omega)} + c_7 \qquad \text{for all nonnegative } \psi\in W^{1,2}(\Omega). \tag{4.10}$$

Now thanks to (4.6), (4.7) and our definition of η_1, on the right-hand side of (4.8) we can estimate

$$\begin{aligned}
\eta_1 \int_\Omega |\nabla v|^{2p+2} &= \eta_1 \Big\| |\nabla v|^p \Big\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le c_3\eta_1 \Big\| \nabla |\nabla v|^p \Big\|_{L^2(\Omega)}^2 \Big\| |\nabla v|^p \Big\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + c_3\eta_1 \Big\| |\nabla v|^p \Big\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le c_3c_4\eta_1 \Big\| \nabla |\nabla v|^p \Big\|_{L^2(\Omega)}^2 + c_3c_4^{p+1}\eta_1 \\
&= c_1 \int_\Omega \Big| \nabla |\nabla v|^p \Big|^2 + c_3c_4^{p+1}\eta_1 \qquad \text{for all } t \in (0, T_{max}), \qquad (4.11)
\end{aligned}$$

whereas combining (4.10) with (4.9) and (2.5) we see that by definition of η_2 we have

$$\begin{aligned}
3c_5 \int_\Omega u^{p+1} &= 3c_5 \|u^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\
&\le 3c_5\eta_2 \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \Big\| u^{\frac{p}{2}} |\ln u^{\frac{p}{2}}|^{\frac{p}{2}} \Big\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2}{p}} + 3c_5c_7 \|u^{\frac{p}{2}}\|_{L^{\frac{2}{p}}(\Omega)}^{\frac{2(p+1)}{p}} + 3c_5c_7 \\
&= \frac{3pc_5\eta_2}{2} \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \cdot \int_\Omega u|\ln u| + 3c_5c_7 \cdot \Big(\int_\Omega u_0 \Big)^{p+1} + 3c_5c_7 \\
&\le \frac{3pc_5c_6\eta_2}{2} \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + 3c_5c_7 \cdot \Big(\int_\Omega u_0 \Big)^{p+1} + 3c_5c_7 \\
&= c_1 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + 3c_5c_7 \cdot \Big(\int_\Omega u_0 \Big)^{p+1} + 3c_5c_7 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned}$$

Along with (4.11) and (4.8) this shows that (4.5) implies the existence of $c_8 > 0$ such that

$$\frac{d}{dt} \Big\{ \int_\Omega u^p + \int_\Omega |\nabla v|^{2p} \Big\} + \Big\{ \int_\Omega u^p + \int_\Omega |\nabla v|^{2p} \Big\} \le c_8 \qquad \text{for all } t \in (0, T_{max}),$$

from which both (4.3) and (4.4) result upon an ODE comparison. □

A Moser-type iteration results in the following.

Lemma 4.3 *Assume that $n = 2$, that ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi > 0, K_\phi > 0$ and $K_{\phi'} > 0$, and that (1.8) holds. Then there exists $C > 0$ such that for the solution of (1.2), (1.6), (1.7) we have*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le C \qquad \textit{for all } t \in (0, T_{max}).$$

PROOF. In view of Lemma 4.2, this can be seen by means of a Moser-type iteration (cf. [28, Lemma A.1] for a corresponding result precisely covering the present situation). □

We are now in a position to prove Theorem 1.1.

PROOF of Theorem 1.1. Since as a consequence of (2.6) we know that

$$\int_\Omega v \le \max \Big\{ \int_\Omega u_0 \,, \int_\Omega v_0 \Big\} \qquad \text{for all } t \in (0, T_{max}),$$

fixing any $q > 2$ we conclude from Lemma 4.2 and Lemma 4.3 that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^q(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max}) \tag{4.12}$$

with some $c_1 > 0$. Thanks to the extensibility criterion (2.4) in Lemma 2.1, this firstly ensures that (u,v) is global in time, whereupon the observation that $W^{1,q}(\Omega) \hookrightarrow L^\infty(\Omega)$ shows that (4.12) also entails (1.9). □

5 Global weak solutions in the case $n \ge 3$

In higher-dimensional domains, throughout this section assuming that ϕ satisfies (1.3), (1.4) and (1.5) and that (u_0, v_0) comply with (1.8), we shall seek for solutions in the following generalized framework.

Definition 5.1 *Let $\phi \in W^{1,\infty}((0,\infty))$ be nonnegative. Then by a* global weak solution *of (1.2), (1.6), (1.7) we mean a pair of nonnegative functions*

$$\begin{cases} u \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \quad \textit{and} \\ v \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)) \end{cases} \tag{5.1}$$

which are such that

$$u\nabla v \quad \textit{belongs to } L^1_{loc}(\bar\Omega \times [0,\infty)), \tag{5.2}$$

and which satisfy

$$\int_0^\infty \int_\Omega u\varphi_t + \int_\Omega u_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \phi(v)\nabla u \cdot \nabla\varphi + \int_0^\infty \int_\Omega \phi'(v) u \nabla v \cdot \nabla \varphi \tag{5.3}$$

as well as

$$\int_0^\infty \int_\Omega v\varphi_t + \int_\Omega v_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \nabla v \cdot \nabla\varphi + \int_0^\infty \int_\Omega v\varphi - \int_0^\infty \int_\Omega u\varphi \tag{5.4}$$

for all $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$.

In order to construct such a weak solution through an approximation procedure, let us consider the regularized problems

$$\begin{cases} u_{\varepsilon t} = \Delta(u_\varepsilon \phi(v_\varepsilon)), & x \in \Omega,\ t > 0, \\ v_{\varepsilon t} = \Delta v_\varepsilon - v_\varepsilon + f_\varepsilon(u_\varepsilon), & x \in \Omega,\ t > 0, \\ \frac{\partial u_\varepsilon}{\partial \nu} = \frac{\partial v_\varepsilon}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u_\varepsilon(x,0) = u_0(x), \quad v_\varepsilon(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{5.5}$$

for $\varepsilon \in (0,1)$, where

$$f_\varepsilon(s) := \frac{s}{1+\varepsilon s}, \qquad s \ge 0, \tag{5.6}$$

clearly satisfies (2.2) and (2.3) for any such ε.

Indeed, all these regularized problems are globally solvable in classical sense:

Lemma 5.1 *For each $\varepsilon \in (0,1)$, the problem (5.5) possesses a global classical solution $(u_\varepsilon, v_\varepsilon)$.*

PROOF. We only need to prove that for each fixed $\varepsilon \in (0,1)$ the corresponding maximal existence time from Lemma 2.1 satisfies $T_{max} = \infty$. To this end, combining the observation that

$$|f_\varepsilon(u_\varepsilon)| \le \frac{1}{\varepsilon} \qquad \text{in } \Omega \times (0, T_{max})$$

with well-known smoothing properties of the Neumann heat semigroup (see e.g. [10, Lemma 4.1]) yields the existence of a constant $c_1 = c_1(\varepsilon) > 0$ such that

$$\|v_\varepsilon\|_{W^{1,\infty}(\Omega)} \le c_1 \qquad \text{for all } t \in (0, T_{max}). \tag{5.7}$$

Therefore, in the identity $u_{\varepsilon t} = \nabla \cdot (\phi(v_\varepsilon)\nabla u_\varepsilon) + \nabla \cdot (u_\varepsilon \phi'(v_\varepsilon)\nabla v_\varepsilon)$, besides

$$k_\phi \le \phi(v_\varepsilon(x,t)) \le K_\phi \qquad \text{for all } x \in \Omega \text{ and } t \in (0, T_{max})$$

we have

$$\|\phi'(v_\varepsilon(\cdot,t))\nabla v_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_1 K_{\phi'} \qquad \text{for all } t \in (0, T_{max}).$$

Thus, by means a Moser-type iteration (cf. [29, Lemma 3.12]) applied to the first equation in (5.5) we obtain $c_2 = c_2(\varepsilon) > 0$ such that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_2(\varepsilon) \qquad \text{for all } t \in (0, T_{max}),$$

which in conjunction with (5.7) and Lemma 2.1 warrants that indeed $T_{max} = \infty$. $\Box$

Now in view of the fact that Definition 5.1 involves the spatial gradient of u not addressed so far, our net goal consists in deriving appropriate a priori information for the corresponding approximates. This will be achieved through a further testing procedure involving non-convex functionals of the first solution component.

Lemma 5.2 *For all $T > 0$ one can find $C(T) > 0$ with the property that for each $\varepsilon \in (0,1)$, the solution of (5.5) satisfies*

$$\int_0^T \int_\Omega u_\varepsilon^{-\frac{2n}{n+2}} |\nabla u_\varepsilon|^2 \le C(T). \tag{5.8}$$

PROOF. As u_ε is positive throughout $\bar\Omega \times (0,\infty)$, we may use $u_\varepsilon^{-\frac{n-2}{n+2}}$ as a test function in (5.5) to see, again thanks to (1.4), (1.5) and Young's inequality, that

$$\begin{aligned} \frac{n+2}{4}\frac{d}{dt}\int_\Omega u_\varepsilon^{\frac{4}{n+2}} &= \int_\Omega \nabla u_\varepsilon^{-\frac{n-2}{n+2}} \cdot \nabla(u_\varepsilon \phi(v_\varepsilon)) \\ &= \frac{n-2}{n+2}\int_\Omega \phi(v_\varepsilon) u_\varepsilon^{-\frac{2n}{n+2}} |\nabla u_\varepsilon|^2 + \frac{n-2}{n+2}\int_\Omega \phi'(v_\varepsilon) u_\varepsilon^{-\frac{n-2}{n+2}} \nabla u_\varepsilon \cdot \nabla v_\varepsilon \\ &\ge c_1 \int_\Omega u_\varepsilon^{-\frac{2n}{n+2}} |\nabla u_\varepsilon|^2 - c_2 \int_\Omega u_\varepsilon^{\frac{4}{n+2}} |\nabla v_\varepsilon|^2 \qquad \text{for all } t > 0 \end{aligned} \tag{5.9}$$

with $c_1 := \frac{(n-2)k_\phi}{2(n+2)} > 0$ and $c_2 := \frac{(n-2)K_{\phi'}^2}{2(n+2)k_\phi} > 0$. Here once more by Young's inequality,

$$\int_\Omega u_\varepsilon^{\frac{4}{n+2}} |\nabla v_\varepsilon|^2 \le \int_\Omega u_\varepsilon^2 + \int_\Omega |\nabla v_\varepsilon|^{\frac{2(n+2)}{n}} \qquad \text{for all } t>0,$$

so that an integration of (5.9) in time yields

$$\begin{aligned} c_1 \int_0^T \int_\Omega u_\varepsilon^{-\frac{2n}{n+2}} |\nabla u_\varepsilon|^2 + \frac{n+2}{4} \int_\Omega u_0^{\frac{4}{n+2}} \\ \le \quad \frac{n+2}{4} \int_\Omega u_\varepsilon^{\frac{4}{n+2}}(\cdot,T) + c_2 \int_0^T \int_\Omega u_\varepsilon^2 + c_2 \int_0^T \int_\Omega |\nabla v_\varepsilon|^{\frac{2(n+2)}{n}} \qquad \text{for all } T>0. \end{aligned} \tag{5.10}$$

Since

$$\int_\Omega u_\varepsilon^{\frac{4}{n+2}}(\cdot,T) \le |\Omega|^{\frac{n-2}{n+2}} \Big(\int_\Omega u_\varepsilon(\cdot,T) \Big)^{\frac{4}{n+2}} = |\Omega|^{\frac{n-2}{n+2}} \Big(\int_\Omega u_0 \Big)^{\frac{4}{n+2}} \qquad \text{for all } T>0$$

due to (2.5), invoking Lemma 3.1 and Lemma 3.3 we readily see that (5.10) implies (5.8). $\square$

By interpolation between the latter and the estimate from Lemma 3.1, we immediately obtain the following inequality which no longer involves weight functions.

Corollary 5.3 *For each $T>0$ one can find $C(T)>0$ such that*

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^{\frac{n+2}{n+1}} \le C(T) \tag{5.11}$$

for all $\varepsilon \in (0,1)$.

PROOF. By Young's inequality,

$$\begin{aligned} \int_0^T \int_\Omega |\nabla u_\varepsilon|^{\frac{n+2}{n+1}} &= \int_0^T \int_\Omega \Big\{ u_\varepsilon^{-\frac{2n}{n+2}} |\nabla u_\varepsilon|^2 \Big\}^{\frac{n+2)}{2(n+1)}} \cdot u_\varepsilon^{\frac{n}{n+1}} \\ &\le \int_0^T \int_\Omega u_\varepsilon^{-\frac{2n}{n+2}} |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^2 \qquad \text{for all } T>0, \end{aligned}$$

and hence (5.11) results from a combination of Lemma 5.2 with Lemma 3.1. $\square$

To prepare the derivation of some strong compactness properties of $(u_\varepsilon)_{\varepsilon\in(0,1)}$ by means of the Aubin-Lions lemma, we once more use Lemma 3.1 to obtain the following regularity property with respect to the time variable.

Lemma 5.4 *Given any $T>0$, one can find $C(T)>0$ fulfilling*

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^2_{(W_0^{2,2}(\Omega))^\star} dt \le C(T) \tag{5.12}$$

for all $\varepsilon \in (0,1)$.

PROOF. Multiplying the first equation in (5.5) by an arbitrary $\psi \in C_0^\infty(\Omega)$, on two integrations by parts we see that thanks to (1.4) we have

$$\begin{aligned}\Big| \int_\Omega u_{\varepsilon t}(\cdot,t)\psi \Big| &= \Big| \int_\Omega u_\varepsilon(\cdot,t)\phi(v_\varepsilon(\cdot,t))\Delta\psi \Big| \\ &\le K_\phi \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \|\Delta\psi\|_{L^2(\Omega)} \qquad \text{for all } t>0\end{aligned}$$

and hence

$$\|u_{\varepsilon t}(\cdot,t)\|_{(W_0^{2,2}(\Omega))^\star} \le K_\phi \|u_\varepsilon(\cdot,t)\|_{L^2(\Omega)} \qquad \text{for all } t>0.$$

Therefore,

$$\int_0^T \|u_{\varepsilon t}(\cdot,t)\|^2_{(W_0^{2,2}(\Omega))^\star} dt \le K_\phi^2 \int_0^T \int_\Omega u_\varepsilon^2 \qquad \text{for all } T>0,$$

and thus an application of Lemma 3.1 proves (5.12). □

Now a straightforward extraction procedure on the basis of the estimates gained above leads to our main result on global weak solvability in the higher-dimensional case.

PROOF of Theorem 1.2. From Lemma 3.1 and Lemma 3.2 we know that there exists $c_1 > 0$ such that for all $\varepsilon \in (0,1)$, the solution of (5.5) satisfies

$$\int_t^{t+1} \int_\Omega u_\varepsilon^2 \le c_1 \quad \text{and} \quad \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^2 \le c_1 \quad \text{as well as} \quad \int_t^{t+1} \int_\Omega |\Delta v_\varepsilon|^2 \le c_1 \qquad \text{for all } t>0. \tag{5.13}$$

Recalling (2.6), we see that this in particular ensures that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}([0,\infty); L^2(\Omega)) \tag{5.14}$$

and that

$$(v_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^\infty((0,\infty); W^{1,2}(\Omega)) \cap L^2_{loc}([0,\infty); W^{2,2}(\Omega)), \tag{5.15}$$

where in view of the second equation in (5.5) and (5.6), the latter immediately implies that also

$$(v_{\varepsilon t})_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}([0,\infty); L^2(\Omega)). \tag{5.16}$$

Since Corollary 5.3 and Lemma 5.4 furthermore assert that

$$(u_\varepsilon)_{\varepsilon\in(0,1)} \quad \text{is bounded in } L_{loc}^{\frac{n+2}{n+1}}([0,\infty); W^{1,\frac{n+2}{n+1}}(\Omega)) \tag{5.17}$$

and that

$$(u_{\varepsilon t})_{\varepsilon\in(0,1)} \quad \text{is bounded in } L^2_{loc}([0,\infty); (W_0^{2,2}(\Omega))^\star), \tag{5.18}$$

by means of a straightforward extraction procedure involving the Aubin-Lions lemma ([31, Ch. III, Theorem 2.3]) we infer the existence of a sequence $(\varepsilon_k)_{k\in\mathbb{N}} \subset (0,1)$ such that $\varepsilon_k \searrow 0$ as $k\to\infty$, that

$$u_\varepsilon \to u, \quad v_\varepsilon \to v \quad \text{and} \quad \nabla v_\varepsilon \to \nabla v \qquad \text{a.e. in } \Omega\times(0,\infty) \tag{5.19}$$

as $\varepsilon = \varepsilon_k \searrow 0$, and such that for all $T > 0$ we have

$$u_\varepsilon \rightharpoonup u \qquad \text{in } L^2(\Omega \times (0,T)) \tag{5.20}$$

and

$$\nabla u_\varepsilon \rightharpoonup \nabla u \qquad \text{in } L^{\frac{n+2}{n+1}}(\Omega \times (0,T)) \tag{5.21}$$

as well as

$$v_\varepsilon \to v \qquad \text{in } L^1(\Omega \times (0,T)) \tag{5.22}$$

and

$$\nabla v_\varepsilon \to \nabla v \qquad \text{in } L^2(\Omega \times (0,T)) \tag{5.23}$$

as $\varepsilon = \varepsilon_k \searrow 0$ with some nonnegative limit functions u and v which satisfy both (5.1) and (5.2) as well as (1.10) and (1.11) due to (5.13) and (5.17).
Now for each $T > 0$ we can use (1.5) to estimate

$$\begin{aligned}
\Big\|\phi'(v_\varepsilon)\nabla v_\varepsilon - \phi'(v)\nabla v\Big\|_{L^2(\Omega\times(0,T))} &\le \Big\|\phi'(v_\varepsilon)(\nabla v_\varepsilon - \nabla v)\Big\|_{L^2(\Omega\times(0,T))} + \Big\|(\phi'(v_\varepsilon) - \phi'(v))\nabla v\Big\|_{L^2(\Omega\times(0,T))} \\
&\le K_{\phi'}\|\nabla v_\varepsilon - \nabla v\|_{L^2(\Omega\times(0,T))} + \Big\|(\phi'(v_\varepsilon) - \phi'(v))\nabla v\Big\|_{L^2(\Omega\times(0,T))},
\end{aligned} \tag{5.24}$$

where due to (5.19) and the continuity of ϕ' we have $\phi'(v_\varepsilon) \to \phi'(v)$ a.e. in $\Omega \times (0,T)$ and hence, by the dominated convergence theorem,

$$\Big\|(\phi'(v_\varepsilon) - \phi'(v))\nabla v\Big\|^2_{L^2(\Omega\times(0,T))} = \int_0^T \int_\Omega |\phi'(v_\varepsilon) - \phi'(v)|^2 |\nabla v|^2 \to 0 \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0,$$

because ϕ' is bounded and $|\nabla v|^2$ belongs to $L^2(\Omega \times (0,T))$. Consequently, from (5.24) and (5.23) we infer that

$$\phi'(v_\varepsilon)\nabla v_\varepsilon \to \phi'(v)\nabla v \qquad \text{in } L^2(\Omega \times (0,T)) \tag{5.25}$$

as $\varepsilon = \varepsilon_k \searrow 0$.
Furthermore, upon another application of Lebesgue's theorem we obtain from (5.19) that

$$\phi(v_\varepsilon) \to \phi(v) \qquad \text{in } L^{n+2}(\Omega \times (0,T)) \tag{5.26}$$

as $\varepsilon = \varepsilon_k \searrow 0$, for ϕ is bounded and continuous on $[0,\infty)$.
Now in order to verify that (u,v) indeed satisfies the identities (5.3) and (5.4), given $\varphi \in C_0^\infty(\bar\Omega \times [0,\infty))$ we use (5.5) to see that

$$\int_0^\infty \int_\Omega u_\varepsilon \varphi_t + \int_\Omega u_0 \varphi(\cdot,0) = \int_0^\infty \int_\Omega \phi(v_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi + \int_0^\infty \int_\Omega u_\varepsilon \phi'(v_\varepsilon)\nabla v_\varepsilon \cdot \nabla\varphi \tag{5.27}$$

for all $\varepsilon \in (0,1)$. Here choosing $T > 0$ large such that $\varphi \equiv 0$ in $\Omega \times (T,\infty)$, we conclude from (5.20) that

$$\int_0^\infty \int_\Omega u_\varepsilon \varphi_t \to \int_0^\infty \int_\Omega u \varphi_t \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0,$$

whereas combining (5.21) with (5.26) warrants that

$$\int_0^\infty \int_\Omega \phi(v_\varepsilon)\nabla u_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega \phi(v)\nabla u \cdot \nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0$$

and (5.20) in conjunction with (5.25) implies that

$$\int_0^\infty \int_\Omega u_\varepsilon \phi'(v_\varepsilon)\nabla v_\varepsilon \cdot \nabla\varphi \to \int_0^\infty \int_\Omega u\phi'(v)\nabla v \cdot \nabla\varphi \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0.$$

Therefore, (5.27) entails (5.3), and the derivation of (5.4) can be accomplished in quite a similar manner, relying on (5.22) and (5.23) together with the observation that

$$\frac{u_\varepsilon}{1+\varepsilon u_\varepsilon} \rightharpoonup u \quad \text{in } L^1(\Omega\times(0,T)) \qquad \text{as } \varepsilon = \varepsilon_k \searrow 0$$

by (5.20), because $\frac{1}{1+\varepsilon u_\varepsilon} \to 1$ in $L^2(\Omega\times(0,T))$ as $\varepsilon = \varepsilon_k \searrow 0$ by (5.19) and the dominated convergence theorem. $\Box$

6 The three-dimensional case

6.1 Preservation of a smallness smallness property

The following observation on preservation of certain smallness properties, rooted in the existence of a constant barrier for an associated time-dependent functional (cf. (6.15), will be essential to our proofs of both Theorem 1.3 and Theorem 1.4.

Lemma 6.1 *Let $n=3$. Then for all $\kappa>0$ there exist $\eta(\kappa)>0$, $\sigma(\kappa)>0$ and $C(\kappa)>0$ with the following property: If ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi \ge \kappa, K_\phi \ge k_\phi$ and $K_{\phi'}>0$, if f and (u_0,v_0) fulfill (2.2), (2.3) and (1.8) as well as*

$$K_{\phi'}\overline{u}_0 \le \eta(\kappa), \tag{6.1}$$

and if for some $t_0 \in [0,T_{max})$, for the solution of (2.1) we have

$$\int_\Omega \Big(u(\cdot,t_0)-\overline{u}_0\Big)^2 + K_{\phi'}^2 \int_\Omega |\nabla v(\cdot,t_0)|^4 \le \frac{\sigma(\kappa)}{K_{\phi'}^2}, \tag{6.2}$$

then

$$\int_\Omega \Big(u(\cdot,t)-\overline{u}_0\Big)^2 \le \frac{C(\kappa)}{K_{\phi'}^2} \qquad \textit{and} \qquad \int_\Omega |\nabla v(\cdot,t)|^4 \le \frac{C(\kappa)}{K_{\phi'}^4} \qquad \textit{for all } t\in(t_0,T_{max}). \tag{6.3}$$

Proof. We first let $c_1 = c_1(\kappa) := \frac{1}{\kappa} + 14$ and employ the Gagliardo-Nirenberg inequality and the Poincaré inequality to find $c_2>0$ and $c_3>0$ such that

$$\|\psi\|_{L^3(\Omega)}^3 \le c_2 \|\psi\|_{W^{1,2}(\Omega)}^{\frac{3}{2}} \|\psi\|_{L^2(\Omega)}^{\frac{3}{2}} \qquad \text{for all } \psi\in W^{1,2}(\Omega) \tag{6.4}$$

and

$$\|\psi\|_{W^{1,2}(\Omega)} \le c_3 \|\nabla\psi\|_{L^2(\Omega)} \qquad \text{for all } \psi \in W^{1,2}(\Omega) \text{ with } \int_\Omega \psi = 0. \tag{6.5}$$

Then by Young's inequality, there exist $c_4 = c_4(\kappa) > 0$ and $c_5 = c_5(\kappa) > 0$ fulfilling

$$2c_1(\kappa)c_2c_3^{\frac{3}{2}}ab \le \frac{\kappa}{2}a^{\frac{4}{3}} + c_4 b^4 \qquad \text{for all } a \ge 0 \text{ and } b \ge 0 \tag{6.6}$$

and

$$4c_1(\kappa)c_2 ab \le 2a^{\frac{4}{3}} + c_5 b^4 \qquad \text{for all } a \ge 0 \text{ and } b \ge 0. \tag{6.7}$$

Abbreviating $c_6 = c_6(\kappa) := \min\{2, \frac{\kappa}{2c_3^2}\}$, $c_7 = c_7(\kappa) := c_4(\kappa) + c_5(\kappa)$ and $c_8 = c_8(\kappa) := 2c_1(\kappa)|\Omega|$, we next take $\eta = \eta(\kappa) > 0$ small such that

$$\eta(\kappa) < \sqrt[6]{\frac{4c_6(\kappa)^3}{27c_7(\kappa)c_8^2(\kappa)}} \tag{6.8}$$

and finally define

$$\sigma = \sigma(\kappa) := \sqrt{\frac{c_6(\kappa)}{3c_7(\kappa)}}. \tag{6.9}$$

To see that then the claimed implication holds, we note that since by (2.5) we have

$$\frac{1}{|\Omega|}\int_\Omega u(\cdot,t) = \overline{u}_0 = \frac{1}{|\Omega|}\int_\Omega u_0 \qquad \text{for all } t \in (0, T_{max}) \tag{6.10}$$

and hence

$$\frac{d}{dt}\int_\Omega (u-\overline{u}_0)^2 = \frac{d}{dt}\Big\{\int_\Omega u^2 - 2\overline{u}_0\int_\Omega u + \overline{u}_0^2|\Omega|\Big\} = \frac{d}{dt}\int_\Omega u^2 \qquad \text{for all } t \in (0, T_{max}),$$

Lemma 2.3 can be applied to $a = K_{\phi'}^2$ and $p = 2$ to show that in view of our choice of c_1 and the hypothesis $k_\phi \ge \kappa$ we have

$$\begin{aligned}
\frac{d}{dt}\Big\{\int_\Omega (u-\overline{u}_0)^2 + K_{\phi'}^2\int_\Omega |\nabla v|^4\Big\} &+ 4K_{\phi'}^2\int_\Omega |\nabla v|^4 + \kappa\int_\Omega |\nabla u|^2 + 2K_{\phi'}^2\int_\Omega \Big|\nabla|\nabla v|^2\Big|^2 \\
&\le \frac{K_{\phi'}^2}{\kappa}\int_\Omega u^2|\nabla v|^2 + 14K_{\phi'}^2\int_\Omega u^2|\nabla v|^2 \\
&= c_1 K_{\phi'}^2\int_\Omega u^2|\nabla v|^2 \qquad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{6.11}$$

Here we use Young's inequality to estimate

$$\int_\Omega u^2|\nabla v|^2 = \int_\Omega (u-\overline{u}_0+\overline{u}_0)^2|\nabla v|^2 \le 2\int_\Omega (u-\overline{u}_0)^2|\nabla v|^2 + 2\overline{u}_0^2\int_\Omega |\nabla v|^2 \qquad \text{for all } t \in (0, T_{max})$$

as well as

$$
\begin{aligned}
K_{\phi'}^2 \int_\Omega (u-\overline{u}_0)^2 |\nabla v|^2 &= \int_\Omega \Big\{ K_{\phi'}^{\frac{2}{3}} (u-\overline{u}_0)^2 \Big\} \cdot \Big\{ K_{\phi'}^{\frac{4}{3}} |\nabla v|^2 \Big\} \\
&\le K_{\phi'} \int_\Omega |u-\overline{u}_0|^3 + K_{\phi'}^4 \int_\Omega |\nabla v|^6 \qquad \text{for all } t \in (0,T_{max})
\end{aligned}
$$

and, similarly,

$$
K_{\phi'}^2 \overline{u}_0^2 \int_\Omega |\nabla v|^2 \le K_{\phi'} \overline{u}_0^3 |\Omega| + K_{\phi'}^4 \int_\Omega |\nabla v|^6 \qquad \text{for all } t \in (0,T_{max}),
$$

from which by definition of c_8 we infer that for all $t \in (0,T_{max})$ we have

$$
\begin{aligned}
c_1 K_{\phi'}^2 \int_\Omega u^2 |\nabla v|^2 &\le 2c_1 K_{\phi'} \int_\Omega |u-\overline{u}_0|^3 + 2c_1 K_{\phi'}^4 \int_\Omega |\nabla v|^6 + 2c_1 |\Omega| K_{\phi'} \overline{u}_0^3 + 2c_1 K_{\phi'}^4 \int_\Omega |\nabla v|^6 \\
&= 2c_1 K_{\phi'} \int_\Omega |u-\overline{u}_0|^3 + 4c_1 K_{\phi'}^4 \int_\Omega |\nabla v|^6 + c_8 K_{\phi'} \overline{u}_0^3. \qquad (6.12)
\end{aligned}
$$

Now by (6.4), (6.5), (6.10) and (6.6),

$$
\begin{aligned}
2c_1 K_{\phi'} \int_\Omega |u-\overline{u}_0|^3 &\le 2c_1 c_2 K_{\phi'} \|u-\overline{u}_0\|_{W^{1,2}(\Omega)}^{\frac{3}{2}} \|u-\overline{u}_0\|_{L^2(\Omega)}^{\frac{3}{2}} \\
&\le 2c_1 c_2 c_3^{\frac{3}{2}} K_{\phi'} \|\nabla u\|_{L^2(\Omega)}^{\frac{3}{2}} \|u-\overline{u}_0\|_{L^2(\Omega)}^{\frac{3}{2}} \\
&\le \frac{\kappa}{2} \|\nabla u\|_{L^2(\Omega)}^2 + c_4 K_{\phi'}^4 \|u-\overline{u}_0\|_{L^2(\Omega)}^6 \\
&= \frac{\kappa}{2} \int_\Omega |\nabla u|^2 + c_4 K_{\phi'}^4 \Big\{ \int_\Omega (u-\overline{u}_0)^2 \Big\}^3 \qquad \text{for all } t \in (0,T_{max}), \quad (6.13)
\end{aligned}
$$

whereas combining (6.4) with (6.7) shows that

$$
\begin{aligned}
4c_1 K_{\phi'}^4 \int_\Omega |\nabla v|^6 &= 4c_1 K_{\phi'}^4 \Big\| |\nabla v|^2 \Big\|_{L^3(\Omega)}^3 \\
&\le 4c_1 c_2 K_{\phi'}^4 \Big\| |\nabla v|^2 \Big\|_{W^{1,2}(\Omega)}^{\frac{3}{2}} \Big\| |\nabla v|^2 \Big\|_{L^2(\Omega)}^{\frac{3}{2}} \\
&\le 4c_1 c_2 \Big\{ K_{\phi'}^{\frac{3}{2}} \Big\| |\nabla v|^2 \Big\|_{W^{1,2}(\Omega)}^{\frac{3}{2}} \Big\} \cdot \Big\{ K_{\phi'}^{\frac{5}{2}} \Big\| |\nabla v|^2 \Big\|_{L^2(\Omega)}^{\frac{3}{2}} \Big\} \\
&\le 2K_{\phi'}^2 \Big\| |\nabla v|^2 \Big\|_{W^{1,2}(\Omega)}^2 + c_5 K_{\phi'}^{10} \Big\| |\nabla v|^2 \Big\|_{L^2(\Omega)}^6 \\
&= 2K_{\phi'}^2 \int_\Omega \Big| \nabla |\nabla v|^2 \Big|^2 + 2K_{\phi'}^2 \int_\Omega |\nabla v|^4 + c_5 K_{\phi'}^{10} \Big\{ \int_\Omega |\nabla v|^4 \Big\}^3 \qquad \text{for all } t \in (0,T_{max}).
\end{aligned}
$$

Together with (6.12), (6.13) and (6.11), this entails that

$$
\begin{aligned}
&\frac{d}{dt} \Big\{ \int_\Omega (u-\overline{u}_0)^2 + K_{\phi'}^2 \int_\Omega |\nabla v|^4 \Big\} + 2K_{\phi'}^2 \int_\Omega |\nabla v|^4 + \frac{\kappa}{2} \int_\Omega |\nabla u|^2 \\
&\qquad \le c_4 K_{\phi'}^4 \Big\{ \int_\Omega (u-\overline{u}_0)^2 \Big\}^3 + c_5 K_{\phi'}^{10} \Big\{ \int_\Omega |\nabla v|^4 \Big\}^3 + c_8 K_{\phi'} \overline{u}_0^3 \qquad (6.14)
\end{aligned}
$$

for all $t \in (0, T_{max})$. Since again (6.5) and (6.10) ensure that

$$\frac{\kappa}{2}\int_\Omega |\nabla u|^2 \ge \frac{\kappa}{2c_3^2}\|u-\overline{u}_0\|^2_{W^{1,2}(\Omega)} \ge \frac{\kappa}{2c_3^2}\int_\Omega (u-\overline{u}_0)^2 \qquad \text{for all } t \in (0, T_{max}),$$

and since writing

$$y(t) := \int_\Omega \Big(u(\cdot,t)-\overline{u}_0\Big)^2 + K_{\phi'}^2 \int_\Omega |\nabla v(\cdot,t)|^4, \qquad t \in [0, T_{max}),$$

we clearly have

$$c_4 K_{\phi'}^4 \bigg\{ \int_\Omega (u-\overline{u}_0)^2 \bigg\}^3 + c_5 K_{\phi'}^{10} \bigg\{ \int_\Omega |\nabla v|^4 \bigg\}^3 \le c_4 K_{\phi'}^4 y^3(t) + c_5 K_{\phi'}^4 y^3(t) \qquad \text{for all } t \in (0, T_{max}),$$

recalling the definitions of c_6 and c_7 we therefore conclude from (6.14) that

$$y'(t) + c_6 y(t) \le c_7 K_{\phi'}^4 y^3(t) + c_8 K_{\phi'} \overline{u}_0^3 \qquad \text{for all } t \in (0, T_{max}), \tag{6.15}$$

that is,

$$y'(t) \le g(y(t)) \qquad \text{for all } t \in (0, T_{max}) \tag{6.16}$$

with

$$g(s) := -c_6 s + c_7 K_{\phi'}^4 s^3 + c_8 K_{\phi'} \overline{u}_0^3, \qquad s \ge 0.$$

Here we note that g attains its minimum over $[0,\infty)$ at $s_0 := \sqrt{\frac{c_6}{3c_7 K_{\phi'}^4}} = \frac{\sigma}{K_{\phi'}^2}$, with corresponding minimal value

$$\begin{aligned} g(s_0) &= -c_6 \sqrt{\frac{c_6}{3c_7 K_{\phi'}^4}} + c_7 K_{\phi'}^4 \sqrt{\frac{c_6}{3c_7 K_{\phi'}^4}}^3 + c_8 K_{\phi'} \overline{u}_0^3 \\ &= -\sqrt{\frac{4c_6^3}{27 c_7 K_{\phi'}^4}} + c_8 K_{\phi'} \overline{u}_0^3 \end{aligned} \tag{6.17}$$

being negative, for from (6.1) and (6.8) we know that

$$c_8 K_{\phi'} \overline{u}_0^3 \le \frac{c_8}{K_{\phi'}^2}\eta^3 < \frac{c_8}{K_{\phi'}^2} \cdot \sqrt{\frac{4c_6^3}{27c_7 c_8^2}} = \sqrt{\frac{4c_6^3}{27 c_7 K_{\phi'}^4}}.$$

Therefore, since (6.2) asserts that

$$y(t_0) \le \frac{\sigma}{K_{\phi'}^2} = s_0,$$

we may invoke an ODE comparison argument to conclude that (6.16) implies the inequality

$$y(t) \le s_1 \qquad \text{for all } t \in [t_0, T_{max}),$$

where $s_1 := \min\{s > s_0 \mid g(s) = 0\}$. As with $C := \sqrt{\frac{c_6}{c_7}}$ we obtain from the definition of g that

$$g\Big(\frac{C}{K_{\phi'}^2}\Big) = -c_6 \cdot \frac{C}{K_{\phi'}^2} + c_7 K_{\phi'}^4 \cdot \frac{C^3}{K_{\phi'}^6} + c_8 K_{\phi'} \overline{u}_0^3 = c_8 K_{\phi'} \overline{u}_0^3 \geq 0,$$

and that hence $s_1 \leq \frac{C}{K_{\phi'}^2}$, this directly yields (6.3). $\Box$

In the presently considered three-dimensional setting, controlling the norms appearing in Lemma 6.1 is sufficient for boundedness in any of the spaces $L^p(\Omega) \times W^{1,2p}(\Omega)$ with finite $p > 1$, which parallels a corresponding property of the minimal Keller-Segel system in n-dimensional domains where the same conclusion holds whenever for some $\varepsilon > 0$, solutions are known to fulfill bounds in $L^{\frac{n}{2}+\varepsilon}(\Omega) \times W^{1,n+\varepsilon}(\Omega)$ (or merely the norm of the first solution component in $L^{\frac{n}{2}+\varepsilon}(\Omega)$, cf. [1]).

Lemma 6.2 *Let $n = 3$, and suppose that (u_0, v_0), f and ϕ satisfy (1.8), (2.2) and (2.3) as well as (1.3), (1.4) and (1.5) with some $k_\phi > 0, K_\phi \geq k_\phi$ and $K_{\phi'} > 0$. Then for all $p > 2$ and each $\Sigma > 0$ there exists $C(p,\Sigma) > 0$ with the property that if for some $t_0 \in [0, T_{max})$ the solution of (2.1) has the property that*

$$\int_\Omega u^2(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^4 \leq \Sigma \qquad \textit{for all } t \in ((t_0-1)_+, T_{max}), \tag{6.18}$$

then

$$\int_\Omega u^p(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^{2p} \leq C(p,\Sigma) \qquad \textit{for all } t \in [t_0, T_{max}). \tag{6.19}$$

Proof. From Lemma 2.3 we obtain $c_1 > 0$ and $c_2 > 0$, as all constants $c_3, c_4, \ldots$ below possibly depending on p and Σ, such that

$$y(t) := \int_\Omega u^p(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^{2p}, \qquad t \in [0, T_{max}),$$

satisfies

$$y'(t) + c_1 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_1 \int_\Omega \Big|\nabla |\nabla v|^p\Big|^2 \leq c_2 \int_\Omega u^p |\nabla v|^2 + c_2 \int_\Omega u^2 |\nabla v|^{2p-2} \quad \text{for all } t \in (0, T_{max}), \tag{6.20}$$

where by Young's inequality,

$$c_2 \int_\Omega u^p |\nabla v|^2 + c_2 \int_\Omega u^2 |\nabla v|^{2p-2} \leq 2c_2 \int_\Omega u^{p+1} + 2c_2 \int_\Omega |\nabla v|^{2p+2} \qquad \text{for all } t \in (0, T_{max}). \tag{6.21}$$

Here we invoke the Gagliardo-Nirenberg inequality to obtain $c_3 > 0$ such that

$$2c_2 \int_\Omega u^{p+1} = 2c_2 \|u^{\frac{p}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \leq c_3 \|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^{\frac{6p-6}{3p-2}} \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{8p-4}{p(3p-2)}} + c_3 \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \qquad \text{for all } t \in (0, T_{max}),$$

so that since $\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}} = \int_\Omega u^2 \leq \Sigma$ for all $t \geq (t_0-1)_+$ by (6.18), and since $\frac{3p-3}{3p-2} < 1$, we may apply Young's inequality to see that with some $c_4 > 0$ we have

$$2c_2 \int_\Omega u^{p+1} \leq \frac{c_1}{2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_4 \qquad \text{for all } t \in [(t_0-1)_+, T_{max}). \tag{6.22}$$

In quite a similar manner, using that $\Big\||\nabla v|^p\Big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{4}{p}} = \int_\Omega |\nabla v|^4 \le \Sigma$ for all $t \in [(t_0-1)_+, T_{max})$ by (6.18), we can find $c_5 > 0$ and $c_6 > 0$ such that

$$\begin{aligned} 2c_2 \int_\Omega |\nabla v|^{2p+2} &= 2c_2 \Big\||\nabla v|^p\Big\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\ &\le c_5 \Big\|\nabla|\nabla v|^p\Big\|_{L^2(\Omega)}^{\frac{6p-6}{3p-2}} \Big\||\nabla v|^p\Big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{8p-4}{p(3p-2)}} + c_5 \Big\||\nabla v|^p\Big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{2(p+1)}{p}} \\ &\le \frac{c_1}{2} \int_\Omega \Big|\nabla|\nabla v|^p\Big|^2 + c_6 \qquad \text{for all } t \in [(t_0-1)_+, T_{max}). \end{aligned} \tag{6.23}$$

In order to introduce a superlinear absorptive term in (6.20), we let $\alpha := \frac{3p-2}{3p-6} > 1$ and first observe that

$$y^\alpha(t) \le 2^{\alpha-1}\Big\{\int_\Omega u^p\Big\}^\alpha + 2^{\alpha-1}\Big\{\int_\Omega |\nabla v|^{2p}\Big\}^\alpha \qquad \text{for all } t \in [0, T_{max}), \tag{6.24}$$

where two more applications of the Gagliardo-Nirenberg inequality along with (6.18) provide positive constants c_7, c_8, c_9 and c_{10} fulfilling

$$\begin{aligned} 2^{\alpha-1}\Big\{\int_\Omega u^p\Big\}^\alpha &= 2^{\alpha-1}\|u^{\frac{p}{2}}\|_{L^2(\Omega)}^{2\alpha} \\ &\le c_7\|\nabla u^{\frac{p}{2}}\|_{L^2(\Omega)}^2 \|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{8}{3p-6}} + c_7\|u^{\frac{p}{2}}\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{6p-4}{3p-6}} \\ &\le c_8 \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_8 \qquad \text{for all } t \in [(t_0-1)_+, T_{max}) \end{aligned}$$

as well as

$$\begin{aligned} 2^{\alpha-1}\Big\{\int_\Omega |\nabla v|^{2p}\Big\} &= 2^{\alpha-1}\Big\||\nabla v|^p\Big\|_{L^2(\Omega)}^{2\alpha} \\ &\le c_9 \Big\|\nabla|\nabla v|^p\Big\|_{L^2(\Omega)}^2 \Big\||\nabla v|^p\Big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{8}{3p-6}} + c_9 \Big\||\nabla v|^p\Big\|_{L^{\frac{4}{p}}(\Omega)}^{\frac{6p-4}{3p-6}} \\ &\le c_{10} \int_\Omega \Big|\nabla|\nabla v|^p\Big|^2 + c_{10} \qquad \text{for all } t \in [(t_0-1)_+, T_{max}). \end{aligned}$$

Writing $c_{11} := c_8 + c_{10}$, from (6.24) we thus infer that

$$y^\alpha(t) \le c_{11} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + c_{11} \int_\Omega \Big|\nabla|\nabla v|^p\Big|^2 + c_{11} \qquad \text{for all } t \in [(t_0-1)_+, T_{max}),$$

so that collecting (6.20), (6.21), (6.22) and (6.23) shows that

$$y'(t) + c_{12} y^\alpha(t) \le c_{13} \qquad \text{for all } t \in [(t_0-1)_+, T_{max}) \tag{6.25}$$

with $c_{12} := \frac{c_1}{2c_{11}}$ and $c_{13} := c_4 + c_6 + \frac{c_1}{2}$.
Now in the case $t_0 \le 1$ when the inequality in (6.25) holds for all $t \in [0, T_{max})$, we may use a simple comparison argument to see that

$$y(t) \le \max\Big\{y(0)\,,\,\Big(\frac{c_{13}}{c_{12}}\Big)^{\frac{1}{\alpha}}\Big\} \qquad \text{for all } t \in [0, T_{max})$$

and hence in particular

$$\int_\Omega u^p(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^{2p} \le \max\left\{ \int_\Omega u_0^p + \int_\Omega |\nabla v_0|^{2p} \, , \, \Big(\frac{c_{13}}{c_{12}}\Big)^{\frac{1}{\alpha}} \right\} \qquad \text{for all } t\in[t_0,T_{max}). \tag{6.26}$$

If $t_0>1$, however, we infer from Lemma 2.5 that with $c_{14} := \max\left\{ \Big(\frac{2}{(\alpha-1)c_{12}}\Big)^{\frac{1}{\alpha-1}} \, , \, \Big(\frac{2c_{13}}{c_{12}}\Big)^{\frac{1}{\alpha}} \right\}$ we have

$$y(t) \le c_{14} \cdot \Big(t-(t_0-1)\Big)^{-\frac{1}{\alpha-1}} + c_{14} \qquad \text{for all } t\in(t_0-1,T_{max}),$$

implying that in this case,

$$\int_\Omega u^p(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^{2p} \le 2c_{14} \qquad \text{for all } t\in[t_0,T_{max}).$$

Together with (6.26), this establishes (6.19). $\Box$

6.2 Small-data classical solutions. Proof of Theorem 1.3

Combining Lemma 6.1 with Lemma 6.2 and another Moser-type boundedness argument now readily yields the following.

Lemma 6.3 *Let $n=3$. Then for all $\kappa>0$ and $K>0$ there exists $\delta=\delta(\kappa,K)>0$ such that if ϕ fulfills (1.3), (1.4) and (1.5) with some $k_\phi>0, K_\phi\ge k_\phi$ and $K_{\phi'}\in(0,K]$, and if (u_0,v_0) is such that (1.8) holds as well as*

$$\int_\Omega u_0^2 + \int_\Omega |\nabla v_0|^4 \le \delta, \tag{6.27}$$

then there exists $C>0$ such that the solution of (1.2), (1.6), (1.7) satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|\nabla v(\cdot,t)\|_{L^4(\Omega)} \le C \qquad \textit{for all } t\in(0,T_{max}). \tag{6.28}$$

PROOF. Given $\kappa>0$ and $K>0$, we fix $\eta=\eta(\kappa)>0$ and $\sigma=\sigma(\kappa)>0$ as provided by Lemma 6.1 and let $\delta=\delta(\kappa,K)>0$ be small enough such that both

$$\delta \le \frac{|\Omega|\eta^2}{K^2} \tag{6.29}$$

and

$$\delta \le \frac{\sigma}{(1+K^2)K^2} \tag{6.30}$$

hold. Then assuming (u_0,v_0) to comply with the above hypotheses, using that $\Big(\int_\Omega u_0\Big)^2 \le |\Omega| \int_\Omega u_0^2$ by the Cauchy-Schwarz inequality we infer from (6.27) that thanks to (6.29) we have

$$K_{\phi'}\overline{u}_0 \le K|\Omega|^{-\frac{1}{2}} \left\{ \int_\Omega u_0^2 \right\}^{\frac{1}{2}} \le K|\Omega|^{-\frac{1}{2}}\delta^{\frac{1}{2}} \le \eta.$$

Since

$$\int_\Omega (u_0-\overline{u}_0)^2 = \int_\Omega u_0^2 - |\Omega|\overline{u}_0^2 \le \int_\Omega u_0^2,$$

it moreover follows from (6.27) when combined with (6.30) that

$$\int_\Omega (u_0-\overline{u}_0)^2 + K_{\phi'}^2 \int_\Omega |\nabla v_0|^4 \le \int_\Omega u_0^2 + K^2 \int_\Omega |\nabla v_0|^4 \le \delta + K^2\delta \le \frac{\sigma}{K^2} \le \frac{\sigma}{K_{\phi'}^2},$$

whence we may employ Lemma 6.1 with the choice $f(s) := s,\ s \ge 0$ to see that there exists $c_1 > 0$ such that

$$\int_\Omega \Big(u(\cdot,t)-\overline{u}_0\Big)^2 + \int_\Omega |\nabla v(\cdot,t)|^4 \le c_1 \qquad \text{for all } t \in (0,T_{max}).$$

As

$$\int_\Omega u^2(\cdot,t) = \int_\Omega \Big(u(\cdot,t)-\overline{u}_0\Big)^2 + |\Omega|\overline{u}_0^2 \qquad \text{for all } t \in (0,T_{max}),$$

this implies that

$$\int_\Omega u^2(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^4 \le c_1 + |\Omega|\overline{u}_0^2 \qquad \text{for all } t \in (0,T_{max}), \tag{6.31}$$

whereupon Lemma 6.2, applied to $t_0 := 0$ and arbitrary $p > \frac{15}{2}$, yields $c_2 > 0$ fulfilling

$$\int_\Omega u^p(\cdot,t) + \int_\Omega |\nabla v(\cdot,t)|^{2p} \le c_2 \qquad \text{for all } t \in (0,T_{max}).$$

In consequence, $F(x,t) := u(x,t)\phi'(v(x,t))\nabla v(x,t)$, $(x,t) \in \Omega \times (0,T_{max})$, has the property that

$$\begin{aligned}
\int_\Omega |F(\cdot,t)|^{\frac{2p}{3}} &\le K_{\phi'}^{\frac{2p}{3}} \int_\Omega u^{\frac{2p}{3}}(\cdot,t)|\nabla v(\cdot,t)|^{\frac{2p}{3}} \\
&\le K_{\phi'}^{\frac{2p}{3}} \Big\{\int_\Omega u^p(\cdot,t)\Big\}^{\frac{2}{3}} \Big\{\int_\Omega |\nabla v(\cdot,t)|^{2p}\Big\}^{\frac{1}{3}} \\
&\le K_{\phi'}^{\frac{2p}{3}} \cdot c_2 \qquad \text{for all } t \in (0,T_{max}).
\end{aligned}$$

As $\frac{2p}{3} > 5$, according to Lemma A.1 in [28] this guarantees that a Moser-type iteration can be applied to the first equation in (1.2), rewritten in the form $u_t = \nabla\cdot(D(x,t)\nabla u) + \nabla\cdot F(x,t)$ with $D(x,t) := \phi(v(x,t)) \in [k_\phi, K_\phi]$ for all $x \in \Omega$ and $t \in (0,T_{max})$, to show that there exists $c_3 > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le c_3 \qquad \text{for all } t \in (0,T_{max}).$$

Combined with (6.31), this proves (6.28). $\Box$

We thereby immediately arrive at our main result on global classical solvability for small data in three-dimensional domains.

PROOF of Theorem 1.3. In view of (2.6), the claim directly results from Lemma 6.3 and the extensibility criterion (2.4). $\Box$

6.3 Eventual regularity for small values of $K_{\phi'}\overline{u}_0$. Proof of Theorem 1.4

Let us finally make use of the precise quantitative information provided by Lemma 3.1 and Lemma 3.2 to reveal that under a smallness condition only involving $\|\phi'\|_{L^\infty((0,\infty))}$ and the total cell mass $\int_\Omega u_0$, the exponentially decay of the rightmost summands in (3.1) and (3.9) warrants that the requirements of Lemma 6.1 are fulfilled at least at some suitably large time, hence implying the following.

Lemma 6.4 *Let $n=3$. Then for each $\kappa>0$ and $K>0$ there exists $\delta(\kappa,K)>0$ such that whenever ϕ satisfies (1.3), (1.4) and (1.5) with some $k_\phi\ge\kappa, K_\phi\in[k_\phi,K]$ and $K_{\phi'}>0$, for all (u_0,v_0) fulfilling (1.8) and*

$$K_{\phi'}\cdot\overline{u}_0\le\delta(\kappa,K), \tag{6.32}$$

one can find $t_0>0$, $\alpha\in(0,1)$ and $C>0$ with the property that for all $\varepsilon\in(0,1)$, the solution of (5.5) satisfies

$$\|u_\varepsilon\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])}+\|v_\varepsilon\|_{C^{2+\alpha,1+\frac{\alpha}{2}}(\bar\Omega\times[t,t+1])}\le C \qquad \text{for all } t>t_0. \tag{6.33}$$

Proof. For fixed $\kappa>0$ and $K>0$, we let $\eta=\eta(\kappa)>0$ and $\sigma=\sigma(\kappa)>0$ denote the constants from Lemma 6.1, and take $L=L(\kappa)>0$ and $M=M(\kappa)>0$ as well as $\lambda=\lambda(\kappa)>0$ and $\mu=\mu(\kappa)>0$ as given by Lemma 3.1 and Lemma 3.2. By using the Sobolev inequality and elliptic regularity theory, we furthermore pick $c_1>0$ such that

$$\|\nabla\psi\|^2_{L^4(\Omega)}\le c_1\|\Delta\psi\|^2_{L^2(\Omega)} \qquad \text{for all } \psi\in W^{2,2}(\Omega) \text{ with } \tfrac{\partial\psi}{\partial\nu}=0 \text{ on } \partial\Omega, \tag{6.34}$$

and thereupon choose $\delta=\delta(\kappa,K)>0$ small enough such that

$$\delta\le\eta \tag{6.35}$$

and

$$\delta\le\sqrt{\frac{\sigma}{16LK^2}} \tag{6.36}$$

as well as

$$\delta\le\sqrt{\frac{\sigma}{128c_1^2M^2K^4}}. \tag{6.37}$$

Now assuming that (u_0,v_0) and ϕ satisfy (1.8) as well as (1.3), (1.4) and (1.5) with some $k_\phi\ge\kappa, K_\phi\in[k_\phi,K]$ and $K_{\phi'}>0$ fulfilling (6.32), from Lemma 3.1 and Lemma 3.2 we infer the existence of $c_2>0$ and $c_3>0$ such that for all $\varepsilon\in(0,1)$, the solution of (5.5) satisfies

$$\int_t^{t+1}\int_\Omega(u_\varepsilon-\overline{u}_0)^2\le LK^2\overline{u}_0^2+c_2e^{-\lambda t} \qquad \text{for all } t>0 \tag{6.38}$$

and

$$\int_t^{t+1}\int_\Omega|\Delta v_\varepsilon|^2\le MK^2\overline{u}_0^2+c_3e^{-\mu t} \qquad \text{for all } t>0, \tag{6.39}$$

because the number $B_{\phi,u_0}=(K_\phi-k_\phi)\overline{u}_0$ introduced in Lemma 3.1 satisfies $B_{\phi,u_0}\le K\overline{u}_0$ according to our hypotheses.

We claim that these choices imply that if, in implicit dependence on (u_0, v_0), we fix $t_\star > 0$ large enough fulfilling

$$4c_2 K_{\phi'}^2 e^{-\lambda t_\star} \le \frac{\sigma}{4} \qquad \text{and} \qquad 32 c_1^2 c_3^2 K_{\phi'}^4 e^{-2\mu t_\star} \le \frac{\sigma}{4}, \tag{6.40}$$

then for all $\varepsilon \in (0,1)$ there exists $t_\varepsilon \in (t_\star, t_\star + 1)$ such that

$$\int_\Omega \Big(u_\varepsilon(\cdot, t_\varepsilon) - \overline{u}_0 \Big)^2 + K_{\phi'}^2 \int_\Omega |\nabla v_\varepsilon(\cdot, t_\varepsilon)|^4 \le \frac{\sigma}{K_{\phi'}^2}. \tag{6.41}$$

To verify this, we first use (6.38) along with the Chebyshev inequality to see that for any such ε the set

$$S_1(\varepsilon) := \left\{ t \in (t_\star, t_\star + 1) \ \middle| \ \int_\Omega \Big(u_\varepsilon(\cdot, t) - \overline{u}_0 \Big)^2 > 4(LK^2 \overline{u}_0^2 + c_2 e^{-\lambda t_\star}) \right\}$$

must satisfy $|S_1(\varepsilon)| \le \frac{1}{4}$, whereas combining (6.39) with (6.34) shows that

$$\int_{t_\star}^{t_\star + 1} \left\{ \int_\Omega |\nabla v_\varepsilon(\cdot, t)|^4 \right\}^{\frac{1}{2}} dt \le c_1 M K^2 \overline{u}_0^2 + c_1 c_3 e^{-\mu t_\star},$$

which implies that also for

$$S_2(\varepsilon) := \left\{ t \in (t_\star, t_\star + 1) \ \middle| \ \left\{ \int_\Omega |\nabla v_\varepsilon(\cdot, t)|^4 \right\}^{\frac{1}{2}} dt > 4(c_1 M K^2 \overline{u}_0^2 + c_1 c_3 e^{-\mu t_\star}) \right\}$$

we have $|S_2(\varepsilon)| \le \frac{1}{4}$. Since therefore $\Big| (t_\star, t_\star + 1) \setminus (S_1(\varepsilon) \cup S_2(\varepsilon)) \Big| \ge \frac{1}{2}$, we can pick $t_\varepsilon \in (t_\star, t_\star + 1)$ such that simultaneously

$$\int_\Omega \Big(u_\varepsilon(\cdot, t_\varepsilon) - \overline{u}_0 \Big)^2 \le 4(LK^2 \overline{u}_0^2 + c_2 e^{-\lambda t_\star})$$

and

$$\int_\Omega |\nabla v(\cdot, t_\varepsilon)|^4 \le \left\{ 4(c_1 M K^2 \overline{u}_0^2 + c_1 c_3 e^{-\mu t_\star} \right\}^2 \le 32 c_1^2 M^2 K^4 \overline{u}_0^4 + 32 c_1^2 c_3^2 e^{-2\mu t_\star}$$

hold, and that hence, by (6.32), (6.36), (6.37) and (6.40), indeed

$$\begin{aligned} K_{\phi'}^2 \int_\Omega \Big(u_\varepsilon(\cdot, t_\varepsilon) - \overline{u}_0 \Big)^2 + K_{\phi'}^4 \int_\Omega |\nabla v_\varepsilon(\cdot, t_\varepsilon)|^4 &\le 4LK^2 K_{\phi'}^2 \overline{u}_0^2 + 4c_2 K_{\phi'}^2 e^{-\lambda t_\star} \\ &\quad + 32 c_1^2 M^2 K^4 K_{\phi'}^4 \overline{u}_0^4 + 32 c_1^2 c_3^2 K_{\phi'}^4 e^{-2\mu t_\star} \\ &\le 4LK^2 \delta^2 + 4c_2 K_{\phi'}^2 e^{-\lambda t_\star} \\ &\quad + 32 c_1^2 M^2 K^4 \delta^4 + 32 c_1^2 c_3^2 K_{\phi'}^4 e^{-2\mu t_\star} \\ &\le \frac{\sigma}{4} + \frac{\sigma}{4} + \frac{\sigma}{4} + \frac{\sigma}{4} = \sigma, \end{aligned}$$

thus establishing (6.41). Now as a consequence thereof, in view of the fact that $K_{\phi'}\overline{u}_0 \le \eta$ thanks to (6.32) and (6.35), Lemma 6.1 applies so as to yield $c_4 > 0$ such that for all $\varepsilon \in (0,1)$ we have

$$\int_\Omega \Big(u_\varepsilon(\cdot,t) - \overline{u}_0\Big)^2 + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4 \le c_4 \qquad \text{for all } t > t_\star + 1$$

and hence

$$\int_\Omega u_\varepsilon^2(\cdot,t) + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^4 \le c_4 + |\Omega|\overline{u}_0^2 \qquad \text{for all } t > t_\star + 1,$$

whereupon Lemma 6.2 applies to show that if we fix any $p > \frac{15}{2}$, then we can find $c_5(p) > 0$ such that for all $\varepsilon \in (0,1)$,

$$\int_\Omega u_\varepsilon^p(\cdot,t) + \int_\Omega |\nabla v_\varepsilon(\cdot,t)|^{2p} \le c_5 \qquad \text{for all } t > t_\star + 2. \tag{6.42}$$

In order to turn this into an estimate for u_ε with respect to the norm in $L^\infty(\Omega \times (t_\star + 3, \infty))$ by means of another Moser iteration, let us pick a nondecreasing function $\zeta \in C^\infty(\mathbb{R})$ such that $\zeta \equiv 0$ in $(-\infty, t_\star + 2)$ and $\zeta \equiv 1$ in $(t_\star + 3, \infty)$. Then

$$w_\varepsilon(x,t) := \zeta(t) u_\varepsilon(x,t), \qquad x \in \bar\Omega, t \ge 0,$$

satisfies

$$\begin{aligned} w_{\varepsilon t} &= \zeta(t) u_{\varepsilon t} + \zeta'(t) u_\varepsilon \\ &= \nabla \cdot (D_\varepsilon(x,t)\nabla w_\varepsilon) + \nabla \cdot F_\varepsilon(x,t) + G_\varepsilon(x,t), \qquad x \in \Omega,\ t > 0, \end{aligned}$$

with

$$D_\varepsilon(x,t) := \phi(v_\varepsilon(x,t))$$

and

$$F_\varepsilon(x,t) := \zeta(t) u_\varepsilon(x,t) \phi'(v_\varepsilon(x,t)) \nabla v_\varepsilon(x,t)$$

as well as

$$G_\varepsilon(x,t) := \zeta'(t) u_\varepsilon(x,t)$$

for $x \in \Omega$ and $t > 0$. Here due to the cut-off properties of ζ, (6.42) along with the Hölder inequality shows that

$$(F_\varepsilon)_{\varepsilon \in (0,1)} \qquad \text{is bounded in } L^\infty((0,\infty; L^{\frac{2p}{3}}(\Omega))$$

and that

$$(G_\varepsilon)_{\varepsilon \in (0,1)} \qquad \text{is bounded in } L^\infty((0,\infty; L^p(\Omega)).$$

Since $\frac{2p}{3} > 5$ and $p > \frac{5}{2}$, and since $k_\phi \le D_\varepsilon \le K_\phi$ in $\Omega \times (0,\infty)$, making use of the evident fact that $w_\varepsilon(\cdot,0) \equiv 0$ we may once more employ Lemma A.1 in [28] to infer the existence of $c_6 > 0$ such that for each $\varepsilon \in (0,1)$ we have

$$\|w_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_6 \qquad \text{for all } t > 0,$$

in particular implying that

$$\|u_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \le c_6 \qquad \text{for all } t > t_\star + 3.$$

Using this as a starting point, by means of a straightforward bootstrap procedure based on standard results from parabolic regularity theory alternately applied to the second and the first equations in (5.5) ([13], [23]), we readily conclude that (6.33) holds with $t_0 := t_\star + 4$, for instance. $\Box$

Thereby eventual smoothness and boundedness of our weak solutions under the assumptions from Theorem 1.4 becomes evident.

Proof of Theorem 1.4. In view of the Arzelà-Ascoli theorem, the claim immediately results from Lemma 6.4 on extracting an appropriate subsequence $(\varepsilon_{k_j})_{j\in\mathbb{N}}$ of the sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ provided by Theorem 1.2, and taking $\varepsilon = \varepsilon_{k_j} \searrow 0$. $\Box$

7 Conclusion

Our analysis has revealed that in comparison to the classical Keller-Segel system, linking diffusion and cross-diffusion through the particular functional form described in (1.2) substantially reduces the ability of the system to spontaneously generate singularities, up to complete blow-up suppression in two-dimensional settings, and may therefore indeed be appropriate to describe the dynamics of stripe pattern formation at large time scales, as suggested by the modeling approach in [7].

The present study may thereby be viewed as the attempt to provide one further step toward a more comprehensive understanding of how chemotactic cross-diffusion influences the dynamics in models for collective behavior in cell populations, in accordance with current trends, as reflected e.g. in the recent collection described in [37], focusing on the intention to more and more incorporate refined aspects of modeling.

In the particular context of (1.2), a natural next step, potentially accompanied or also guided by numerical simulations, might consist in exploring the corresponding global dynamical features in more detail, possibly in the sense of stabilization toward equilibria or also in more general frameworks including attractors.

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