Emergence of large population densities despite logistic growth restrictions in fully parabolic chemotaxis systems

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Abstract

We consider the no-flux initial-boundary value problem for Keller-Segel-type chemotaxis growth systems of the form

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \rho u - \mu u^2, & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases}$$

in a ball $\Omega\subset\mathbb{R}^n$, $n\ge 3$, with parameters $\chi>0, \rho\ge 0$ and $\mu>0$.

By means of an argument based on a conditional quasi-energy inequality, it is firstly shown that if $\chi=1$ is fixed, then for any given $K>0$ and $T>0$ one can find radially symmetric initial data, possibly depending on K and T, such that for arbitrary $\mu\in(0,1)$ the corresponding local-in-time classical solution (u,v) satisfies

$$u(x,t) > \frac{K}{\mu}$$

with some $x\in\Omega$ and $t\in(0,T)$; in fact, this growth phenomenon is actually identified as being generic in the sense that the set of all initial data having this property is dense in the set of all suitably regular radial initial data in a certain topology.

Secondly, turning a focus on possible effects of large chemotactic sensitivities, on the basis of the above it is shown that when $\rho\ge 0$ and $\mu>0$ are fixed, then for all $L>0, T>0$ and $\chi>\mu$ one can fix radial initial data $(u_{0,\chi}, v_{0,\chi})$ which decay in $L^\infty(\Omega)\times W^{1,\infty}(\Omega)$ as $\chi\to\infty$, and which are such that for the respective solution (u_χ, v_χ) there exist $x\in\Omega$ and $t\in(0,T)$ fulfilling

$$u_\chi(x,t) > L.$$

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1 Introduction

Including logistic proliferation terms may substantially influence the dynamics in chemotaxis systems. This firstly concerns the ability of the respective system to spontaneously generate singularities, as known to constitute one of the most striking features of the classical Keller-Segel system

$$\begin{cases} u_t = d\Delta u - \chi\nabla \cdot (u\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \tag{1.1}$$

which has widely been accepted as the simplest reasonable macroscopic model for the collective behavior in cell populations, quantified through their density $u = u(x,t)$, in chemotactic response to a signal produced by themselves and represented by its concentration $v = v(x,t)$ ([10], [7]). Indeed, whereas the nonlinear cross-diffusion process in (1.1) is known to enforce finite-time blow-up of some solutions with respect to the norm in L^∞ of its first component in two- or higher-dimensional cases ([6], [26]; cf. also the surveys [8], [3]), in the correspondingly modified variant thereof given by

$$\begin{cases} u_t = d\Delta u - \chi\nabla \cdot (u\nabla v) + \rho u - \mu u^2, \\ v_t = \Delta v - v + u, \end{cases} \tag{1.2}$$

the additional dissipative effect of the quadratic zero-order death term is known to rule out any such collapse when either $n = 2$ and $\mu > 0$ is arbitrary ([17]), or $n \geq 3$ and μ is sufficiently large ([25]); if in the latter case $n \geq 3$ the number $\mu > 0$ is arbitrary, then at least certain global weak solutions can be constructed, and if moreover $n = 3$ and ρ is suitably small, then these solutions eventually become smooth and classical ([13]). In line with this, systems of type (1.2) appear as subsystems at the core of numerous more complex models for chemotactic cell migration at large time scales, especially in situations when infinite densities turn out to be unrealistic, and thus seem of particular relevance in the modeling of tumor invasion processes ([4], [22], [20]), also in the context of multiscale approaches ([14], [21]).

However, effects of logistic source terms in fact may go significantly beyond such aspects of global existence and boundedness theory, and thus the interplay of Fisher-type cell kinetics with diffusion *and* chemotactic cross-diffusion is considerably more colorful than with merely diffusion. This is, inter alia, indicated by numerical evidence revealing quite a multifaceted and possibly even chaotic solution behavior already in spatially one-dimensional versions of (1.2) ([19]), as well as rich structures of associated steady-state sets in two-dimensional cases, including the occurrence of hexagonal patterns ([11]).

Apparently, however, up to now only few aspects of the solution behavior in (1.2) have been captured by rigorous analysis. For instance, it is known that if $\mu > \mu_0$ with some $\mu_0 = \mu_0(d, \chi, \rho, \Omega)$, then the corresponding nontrivial spatially homogeneous equilibrium of (1.2) is globally asymptotically stable (see [27] for a proof in the prototypical case $d = \chi = \rho = 1$), where even an explicit bound for μ_0 can be obtained in an associated parabolic-elliptic simplification of (1.2) in which the signal evolution is governed by the elliptic equation $0 = \Delta v - v + u$ ([23]). In presence of small values of μ when no such proliferation-dominated behavior can be expected, only little seems known beyond results on existence and dimension of exponential attractors in two-dimensional frameworks ([17], [16], [2]); after all, large-time extinction phemonena, as numerically observed to occur in large spatial regions ([19])

and initially discussed in [1] from a rigorous perspective, have recently been shown to necessarily be of local nature in the sense that for each global solution, the associated total mass of cells always persists throughout evolution ([24]).

More subtle qualitative facets of chemotaxis-growth interaction could up to now be rigorously detected only in simplified parabolic-elliptic settings and under the essential additional assumption that cell diffusion is suitably weak: In the hyperbolic-elliptic limit case $d = 0$ of such sytsems, namely, it can be observed that some solutions blow up in finite time with respect to the spatial L^∞ norm of the component u, even in spatially one-dimensional intervals ([28]), but also in radial higher-dimensional situations ([12]). Based on a suitable perturbation analysis, it can be shown that in either of these cases, under an appropriate assumption on the initial data it is possible to find $T > 0$ with the property that for each $M > 0$ there exists $d_0 > 0$ such that whenever $d \in (0, d_0)$, one can find a point x_d in the spatial domain Ω and $t_d \in (0, T)$ for which the solution (u_d, v_d) of an associated Neumann initial-boundary value problem in $\Omega \times (0, T)$ satisfies

$$u_d(x_d, t_d) \ge M.$$

In particular, this means that even in situations when solutions are known to be global and bounded, the influence of chemotactic cross-diffusion may force some solutions to exceed any given threshold dynamically, at least on intermediate time scales, which is in sharp contrast to the solution behavior e.g. in the diffusive Fisher-KPP problem corresponding to the choice $\chi = 0$ in (1.2), where such phenomena are ruled out by the availability of a maximum principle.

Main results. To the best of our knowledge, however, no rigorous results on solution behavior far from equilibrium are available for the fully parabolic system (1.2), nor for any chemotaxis-growth system involving possibly large diffusion rates. The purpose of the present work consists in developing an approach which enables us to accomplish some first steps in this direction, and especially to show that the dynamical emergence of structures, extreme in the sense that arbitrarily large population densities are involved, need not necessarily be a small-diffusion phenomenon.

For this purpose, firstly focusing only on the parameters relevant to cell proliferation we will consider the initial-boundary value problem

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla v) + \rho u - \varepsilon u^2, & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

in a ball $\Omega \subset \mathbb{R}^n$, $n \ge 3$, where the numbers $\rho \ge 0$ and $\varepsilon > 0$ as well as the initial data u_0 and v_0 are given.

As for this problem, the first of our main results reveals an unboundeness phenomenon, possibly transient in time, which can even be viewed generic with respect to the choice of initial data within an appropriate topology, and which can moreover be quantified in terms of the parameter ε in (1.3).

Theorem 1.1 *Let $n \ge 3$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$, let $\rho \ge 0$, and suppose that $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ are radially symmetric and positive in $\overline{\Omega}$. Then for all $K > 0$ and each*

$T \in (0,1)$ there exist sequences $(u_{0k})_{k\in\mathbb{N}} \subset C^0(\overline{\Omega})$ and $(v_{0k})_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ of radially symmetric positive functions u_{0k} and v_{0k} on $\overline{\Omega}$ such that

$$\int_\Omega u_{0k} = \int_\Omega u_0 \qquad \text{for all } k \in \mathbb{N}, \tag{1.4}$$

that

$$u_{0k} \to u_0 \quad \text{in } L^p(\Omega) \text{ for all } p \in \Big[1, \frac{2n}{n+2}\Big) \qquad \text{and} \qquad v_{0k} \to v_0 \quad \text{in } W^{1,2}(\Omega) \tag{1.5}$$

as $k \to \infty$, and that for all $k \in \mathbb{N}$ and $\varepsilon \in (0,1)$ one can find $t_{\varepsilon,k} \in (0,T)$ with the property that (1.3) possesses a classical solution $(u_{\varepsilon,k}, v_{\varepsilon,k}) \in (C^0(\overline{\Omega} \times [0, t_{\varepsilon_k}]) \cap C^{2,1}(\overline{\Omega} \times (0, t_{\varepsilon,k})))^2$ which is such that

$$u_{\varepsilon,k}(x_{\varepsilon,k}, t_{\varepsilon,k}) > \frac{K}{\varepsilon} \qquad \text{for some } x_{\varepsilon,k} \in \Omega. \tag{1.6}$$

In particular, this implies the following quantitative result on dynamical growth in (1.3) for a fixed pair of initial data.

Corollary 1.2 *Let $n \geq 3$ and $\Omega = B_R(0) \subset \mathbb{R}^n$ with some $R > 0$, and let $\rho \geq 0$. Then for all $K > 0$ and any $T > 0$ there exist radially symmetric positive functions $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ with the property that for each $\varepsilon \in (0,1)$ one can find $t_\varepsilon \in (0,T)$ with the property that (1.3) possesses a classical solution $(u_\varepsilon, v_\varepsilon) \in (C^0(\overline{\Omega} \times [0, t_\varepsilon]) \cap C^{2,1}(\overline{\Omega} \times (0, t_\varepsilon)))^2$ which is such that*

$$u_\varepsilon(x_\varepsilon, t_\varepsilon) > \frac{K}{\varepsilon} \qquad \text{for some } x_\varepsilon \in \Omega. \tag{1.7}$$

As a second by-product of Theorem 1.1, the particular quantitative information (1.6) contained therein will enable us to study possible effects of large chemotactic sensitivities in presence of a fixed logistic source. Specifically, for the version of (1.2) given by

$$\begin{cases} w_t = \Delta w - \chi \nabla \cdot (w \nabla z) + \rho w - \mu w^2, & x \in \Omega,\ t > 0, \\ z_t = \Delta z - z + w, & x \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = \frac{\partial z}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ w(x,0) = w_0(x), \quad z(x,0) = z_0(x), & x \in \Omega, \end{cases} \tag{1.8}$$

we shall obtain the following.

Theorem 1.3 *Let $n \geq 3, R > 0$ and $\Omega := B_R(0) \subset \mathbb{R}^n$, and let $\rho \geq 0$ and $\mu > 0$. Then for any choice of $L > 0, T \in (0,1)$ and $\chi > \mu$ one can find radially symmetric positive functions $w_{0\chi} \in C^0(\overline{\Omega})$ and $z_{0\chi} \in W^{1,\infty}(\Omega)$ such that*

$$\|w_{0\chi}\|_{L^\infty(\Omega)} \to 0 \quad \text{and} \quad \|z_{0\chi}\|_{W^{1,\infty}(\Omega)} \to 0 \qquad \text{as } \chi \to \infty, \tag{1.9}$$

and that for any $\chi > \mu$ there exists $t_\chi \in (0,T)$ such that (1.8) with $(w_0, z_0) := (w_{0\chi}, z_{0\chi})$ admits a positive classical solution $(w_\chi, z_\chi) \in (C^0(\overline{\Omega} \times [0, t_\chi]) \cap C^{2,1}(\overline{\Omega} \times (0, t_\chi)))^2$ satisfying

$$w_\chi(x_\chi, t_\chi) > L \qquad \text{for some } x_\chi \in \Omega. \tag{1.10}$$

The main idea: Exploiting a conditional quasi-energy inequality. Our approach is rooted in a contradictory argument based on an analysis of the quantity

$$\mathcal{F}(u,v) := \frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2 - \int_\Omega uv + \int_\Omega u\ln u, \tag{1.11}$$

which is well-known to play the role of a genuine Lyapunov functional for the unforced normalized Keller-Segel system obtained on letting $d = \chi = 1$ in (1.1), in the sense that

$$\frac{d}{dt}\mathcal{F}(u(\cdot,t),v(\cdot,t)) = -\mathcal{D}(u(\cdot,t),v(\cdot,t)) \tag{1.12}$$

holds along the respective trajectories, with the nonnegative dissipation rate given by

$$\mathcal{D}(u,v) := \int_\Omega \Big|\Delta v - v + u\Big|^2 + \int_\Omega \Big|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big|^2 \tag{1.13}$$

([15]). Whereas this subtle structure is apparently destroyed in presence of the kinetic terms in (1.2), it will turn out that at least a certain quasi-energy inequality can be derived under an appropriately mild boundedness hypothesis on the solution component u. Relying on a functional inequality relating $\mathcal{F}(u,v)$ to the associated dissipation rate, as obtained in [26] by making essential use of the fact that $n \ge 3$ (Lemma 3.2), under the assumption that within a suitably small time interval the solution of (1.2) satisfies $u \le \frac{K}{\varepsilon}$ with some $K > 0$, this will enable us to establish an autonomous ordinary differential inequality for $F(u,v)$ (Lemma 3.1, Lemma 3.6 and Lemma 3.11) which cannot hold throughout this time interval (Lemma 3.12). Exploiting this will yield the statements from Theorem 1.1 and Corollary 1.2 in Section 3, whereupon a stratightforward variable transformation will lead to a proof of Theorem 1.3 in Section 4.

2 Preliminaries

For definiteness in our subsequent arguments, let us first recall from [26] that any given pair of suitably regular positive radial functions on Ω can conveniently be approximated by low-energy data.

Lemma 2.1 *Let $u_0 \in C^0(\overline{\Omega})$ and $v_0 \in W^{1,\infty}(\Omega)$ be radially symmetric and positive in $\overline{\Omega}$. Then there exist $(u_{0j})_{j\in\mathbb{N}} \subset C^0(\overline{\Omega})$ and $(v_{0j})_{j\in\mathbb{N}} \subset W^{1,\infty}(\Omega)$ such that for all $j \in \mathbb{N}$, u_{0j} and v_{0j} are radially symmetric and positive in $\overline{\Omega}$ with*

$$\int_\Omega u_{0j} = \int_\Omega u_0,$$

that

$$u_{0j} \to u_0 \quad \text{in } L^p(\Omega) \text{ for all } p \in \Big[1, \frac{2n}{n+2}\Big) \qquad \text{and} \qquad v_{0j} \to v_0 \quad \text{in } W^{1,2}(\Omega)$$

as $j \to \infty$, and that with $\mathcal{F}$ as in (1.11) we have

$$\mathcal{F}(u_{0j}, v_{0j}) \to -\infty \qquad \text{as } j \to \infty.$$

PROOF. This immediately results from the statement in [26, Lemma 6.1]. $\square$

When employed as initial data in (1.3), all these functions give rise to corresponding local-in-time classical solutions.

Lemma 2.2 *For all $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$, there exists $T_{\varepsilon,j} \in (0,\infty]$ such that the problem (1.3) with $u_0 := u_{0j}$ and $v_0 := v_{0j}$ possesses a positive classical solution $(u,v) \equiv (u_{\varepsilon,j}, v_{\varepsilon,j}) \in (C^0(\overline{\Omega} \times [0,T_{\varepsilon,j})) \cap C^{2,1}(\overline{\Omega} \times (0,T_{\varepsilon,j})))^2$ which is such that $v_{\varepsilon,j} \in C^0([0,T_{\varepsilon,j}); W^{1,2}(\Omega))$ and that*

$$\text{if } T_{\varepsilon,j} < \infty, \text{ then } \quad \limsup_{t \nearrow T_{\varepsilon,j}} \|u_{\varepsilon,j}(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

PROOF. It is well-known ([25]) that the problem in question is solvable in the indicated class, with some $T_{\varepsilon,j} \in (0,\infty]$ which is such that

$$\text{if } T_{\varepsilon,j} < \infty, \text{ then } \ \|u_{\varepsilon,j}(\cdot,t)\|_{L^\infty(\Omega)} + \|v_{\varepsilon,j}(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty \qquad \text{as } t \nearrow T_{\varepsilon,j}. \tag{2.2}$$

To see that actually (2.1) holds, assuming on the contrary that $T_{\varepsilon,j}$ be finite, but that $u_{\varepsilon,j}$ be bounded in $\Omega \times (0,T_{\varepsilon,j})$, by applying standard arguments from parabolic regularity theory to the second equation in (1.3) ([9]) we could find $c_1 > 0$ such that

$$\|v_{\varepsilon,j}(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_1 \qquad \text{for all } t \in \Big(\frac{1}{2}T_{\varepsilon,j}, T_{\varepsilon,j}\Big).$$

This contradicts (2.2) and thereby verifies (2.1). $\square$

3 A conditional quasi-energy inequality for (1.3)

The following generalization of the energy identity (1.12) to the chemotaxis-growth system (1.3) is straightforward but fundamental to our approach.

Lemma 3.1 *Suppose that $(u,v) \in (C^{2,1}(\overline{\Omega} \times (0,T)))^2$ is a positive classical solution of the boundary value problem in (1.3) for some $\varepsilon \in (0,1)$, $\rho \ge 0$ and $T > 0$. Then with $\mathcal{F}$ and $\mathcal{D}$ taken from (1.11) and (1.13) we have*

$$\begin{aligned} \frac{d}{dt}\mathcal{F}\Big(u(\cdot,t),v(\cdot,t)\Big) &= -\mathcal{D}\Big(u(\cdot,t),v(\cdot,t)\Big) + \varepsilon\int_\Omega u^2 v + \rho\int_\Omega u\ln u + \rho\int_\Omega u \\ &\quad -\varepsilon\int_\Omega u^2\ln u - \rho\int_\Omega uv - \varepsilon\int_\Omega u^2 \qquad \textit{for all } t \in (0,T). \end{aligned} \tag{3.1}$$

PROOF. This can be seen by straightforward computation: Indeed, from (1.3) we obtain on integrating by parts that

$$\begin{aligned} \frac{d}{dt}\Big\{\frac{1}{2}\int_\Omega |\nabla v|^2 + \frac{1}{2}\int_\Omega v^2\Big\} &= \int_\Omega (-\Delta v + v)\cdot v_t \\ &= -\int_\Omega v_t^2 + \int_\Omega u v_t \end{aligned} \tag{3.2}$$

and

$$-\frac{d}{dt}\int_\Omega uv = -\int_\Omega \Big\{\Delta u - \nabla\cdot(u\nabla v) + \rho u - \varepsilon u^2\Big\}\cdot v - \int_\Omega uv_t$$
$$= \int_\Omega \nabla u\cdot\nabla v - \int_\Omega u|\nabla v|^2 - \rho\int_\Omega uv + \varepsilon\int_\Omega u^2 v - \int_\Omega uv_t \tag{3.3}$$

as well as

$$\frac{d}{dt}\int_\Omega u\ln u = \int_\Omega \Big\{\Delta u - \nabla\cdot(u\nabla v) + \rho u - \varepsilon u^2\Big\}\cdot\Big\{\ln u + 1\Big\}$$
$$= -\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega \nabla u\cdot\nabla v + \rho\int_\Omega u\ln u - \varepsilon\int_\Omega u^2\ln u + \rho\int_\Omega u - \varepsilon\int_\Omega u^2 \tag{3.4}$$

for all $t\in(0,T)$. Recalling that

$$\Big\{\int_\Omega \nabla u\cdot\nabla v - \int_\Omega u|\nabla v|^2\Big\} + \Big\{-\int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega \nabla u\cdot\nabla v\Big\} = -\int_\Omega\Big|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big|^2 \qquad \text{for all } t\in(0,T),$$

on adding (3.3) and (3.4) to (3.2) we readily arrive at (3.1). □

In order to draw appropriate conclusions from (3.1), we recall from [26] that in the case $n\ge 3$ considered here, the expression $\int_\Omega uv$ can essentially be controlled by a sublinear power of the dissipation rate $\mathcal{D}(u,v)$ from (1.13) in the sense of the following functional inequality that is actually valid for a large class of radially symmetric functions on Ω.

Lemma 3.2 *Let $m_0>0, M>0, B>0$ and $\kappa>n-2$. Then there exists $A(m_0,M,B,\kappa)>0$ such that for any choice of $m\in(0,m_0]$, the inequality*

$$\int_\Omega uv \le A(m_0,M,B,\kappa)\cdot\Big\{\Big\|\Delta v - v + u\Big\|_{L^2(\Omega)}^{2\theta} + \Big\|\frac{\nabla u}{\sqrt{u}} - \sqrt{u}\nabla v\Big\|_{L^2(\Omega)} + 1\Big\} \tag{3.5}$$

holds for all

$$(u,v)\in\mathcal{S}(m,M,B,\kappa) := \left\{(\tilde u,\tilde v)\in C^1(\overline\Omega)\times C^2(\overline\Omega)\ \middle|\ \begin{array}{l}\tilde u \text{ and } \tilde v \text{ are positive and radially symmetric}\\ \text{with } \frac{\partial\tilde v}{\partial\nu}=0 \text{ on } \partial\Omega \text{ and such that}\\ \int_\Omega \tilde u = m,\ \int_\Omega v\le M \text{ and}\\ \tilde v(x)\le B|x|^{-\kappa} \text{ for all } x\in\Omega\end{array}\right\},$$

where $\theta := \frac{1}{1+\frac{n}{2(n+2)\kappa}}$.

Proof. For $m := m_0$, (3.5) has precisely been formulated in [26, Lemma 4.1]. As the reasoning in the corresponding proof in [26, Section 4] shows, however, the respective constant on the right-hand side of (3.5) depends on u only through an upper bound for $\int_\Omega u$, whence we conclude that (3.5) is valid for all $(u,v) \in \mathcal{S}(m,M,B,\kappa)$ and actually any $m \le m_0$. $\Box$

Let us next make sure that within a suitably small time interval, all the solutions under consideration indeed remain in the set $\mathcal{S}(m,M,B,\kappa)$ for appropriate m, M, B and κ. To this end, we firstly note the following basic observation on the mass evolution in the first component of the solution obtained in Lemma 2.2.

Lemma 3.3 *For any $\varepsilon \in (0,1)$ and each $j \in \mathbb{N}$ we have*

$$\int_\Omega u_{\varepsilon,j}(\cdot,t) \le e^\rho \int_\Omega u_0 \qquad \textit{for all } t \in \Big(0, \min\{1, T_{\varepsilon,j}\}\Big). \tag{3.6}$$

PROOF. Since

$$\frac{d}{dt}\int_\Omega u_{\varepsilon,j} = \rho \int_\Omega u_{\varepsilon,j} - \varepsilon \int_\Omega u_{\varepsilon,j}^2 \le \rho \int_\Omega u_{\varepsilon,j} \qquad \text{for all } t \in (0, T_{\varepsilon,j})$$

by (1.3), on integration we infer that

$$\int_\Omega u_{\varepsilon,j} \le \left\{ \int_\Omega u_{0j} \right\} \cdot e^{\rho t} \qquad \text{for all } t \in (0, T_{\varepsilon,j}),$$

which implies (3.6) due to the fact that $\int_\Omega u_{0j} = \int_\Omega u_0$. $\Box$

Secondly, based on Lemma 3.3 and features of parabolic regularization, also the second solution component can be seen to comply with the requirements contained in Lemma 3.2.

Lemma 3.4 *Let $\kappa > n-2$. Then there exists $B(\kappa) > 0$ such that for all $\varepsilon \in (0,1)$ and any $j \in \mathbb{N}$ we have*

$$v_{\varepsilon,j}(x,t) \le B(\kappa)|x|^{-\kappa} \qquad \textit{for all } x \in \Omega \textit{ and } t \in \Big(0, \min\{1, T_{\varepsilon,j}\}\Big). \tag{3.7}$$

PROOF. Thanks to Lemma 3.3 and the assumed radial symmetry, this can be seen by straightforward modification of the reasoning in [26, Section 3]; for completeness, let us briefly outline a proof: Without loss of generality assuming that $\kappa \le n-1$ and then writing $p := \frac{n}{\kappa+1} > 1$, we have $p < \frac{n}{n-1}$, so that a standard result on regularization in the inhomogeneous linear heat equation $v_t = \Delta v - v + u$ ([9]) applies so as to provide $c_1 > 0$ such that

$$\|v_{\varepsilon,j}(\cdot,t)\|_{W^{1,p}(\Omega)} \le c_1 \cdot \left\{ \|v_{0j}\|_{W^{1,2}(\Omega)} + \sup_{s\in(0,t)} \|u_{\varepsilon,j}(\cdot,s)\|_{L^1(\Omega)} \right\} \qquad \text{for all } t \in (0, T_{\varepsilon,j}),$$

whence by Lemma 3.3 and the boundedness of $(v_{0j})_{j\in\mathbb{N}}$ in $W^{1,2}(\Omega)$, as asserted by Lemma 2.1, we can find $c_2 > 0$ such that

$$\|\nabla v_{\varepsilon,j}(\cdot,t)\|_{L^p(\Omega)} + \|v_{\varepsilon,j}(\cdot,t)\|_{L^1(\Omega)} \le c_2 \qquad \text{for all } t \in (0, \widehat{T}), \tag{3.8}$$

where $\widehat{T} := \min\{1, T_{\varepsilon,j}\}$. For each fixed $t \in (0, \widehat{T})$, we can therefore find $r_0(t) \in (\frac{R}{2}, R)$ such that letting $v(r,t) := v_{\varepsilon,j}(x,t)$ for $x \in \partial B_r(0)$ and $t \in (0, \widehat{T})$ we have

$$v(r_0(t),t) \le \frac{c_2}{|B_R(0) \setminus B_{\frac{R}{2}}(0)|}$$

and hence

$$
\begin{aligned}
v(r,t) &= v(r_0(t),t) + \int_{r_0(t)}^r v_r(\sigma,t)d\sigma \\
&\le \frac{c_2}{|B_R(0)\setminus B_{\frac{R}{2}}(0)|} + \left\{ \int_{r_0(t)}^r \sigma^{n-1} \Big|v_r(\sigma,t)\Big|^p d\sigma \right\}^{\frac{1}{p}} \cdot \left| \int_{r_0(t)}^r \sigma^{-\frac{n-1}{p-1}} d\sigma \right|^{\frac{p-1}{p}} \qquad (3.9)
\end{aligned}
$$

for all $r \in (0,R)$. As can be verified by explicit evaluation, herein we have

$$
\begin{aligned}
\left| \int_{r_0(t)}^r \sigma^{-\frac{n-1}{p-1}} d\sigma \right|^{\frac{p-1}{p}} &\le 2^{\frac{n-p}{p}} \cdot \Big(\frac{p-1}{n-p}\Big)^{\frac{p-1}{p}} \cdot r^{-\frac{n-p}{p}} \\
&= 2^{\frac{n-p}{p}} \cdot \Big(\frac{p-1}{n-p}\Big)^{\frac{p-1}{p}} \cdot r^{-\kappa} \qquad \text{for all } r \in (0,R),
\end{aligned}
$$

whence on using (3.8) we can readily derive (3.7) from (3.9). □

Therefore, Lemma 3.2 indeed becomes applicable for the solutions from Lemma 2.2 at least for suitably small times:

Lemma 3.5 *There exist $\theta \in (0,1)$ and $C_0 > 0$ with the property that for all $\varepsilon \in (0,1)$ and any $j \in \mathbb{N}$, the solution gained in Lemma 2.2 satisfies*

$$
\int_\Omega u_{\varepsilon,j}(\cdot,t)v_{\varepsilon,j}(\cdot,t) \le C_0 \cdot \left\{ \mathcal{D}^\theta\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) + 1 \right\} \qquad \text{for all } t \in \Big(0, \min\{1, T_{\varepsilon,j}\}\Big), \qquad (3.10)
$$

where $\mathcal{D}$ is taken from (1.13).

Proof. We fix any $\theta \in [\frac{1}{2}, 1)$ such that

$$
\theta > \frac{1}{1 + \frac{n}{2(n+2)(n-2)}},
$$

so that it is possible to pick $\kappa > n-2$ such that still

$$
\theta \ge \theta_0 := \frac{1}{1 + \frac{n}{2(n+2)\kappa}}.
$$

Thereupon, Lemma 3.4 applies so as to yield $B > 0$ fulfilling

$$
v_{\varepsilon,j}(x,t) \le B|x|^{-\kappa} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,\widehat{T}),
$$

where again $\widehat{T} := \min\{1, T_{\varepsilon,j}\}$. In particular, this entails that

$$
\int_\Omega v_{\varepsilon,j}(\cdot,t) \le M := B \int_\Omega |x|^{-\kappa} dx \qquad \text{for all } t \in (0,\widehat{T}),
$$

with M being finite due to the fact that $\kappa < n$. Along with Lemma 3.3, this enables us to conclude from Lemma 3.2 that with A as introduced there we have

$$
\int_\Omega u_{\varepsilon,j}v_{\varepsilon,j} \le A\Big(e^\rho \int_\Omega u_0, M, B, \kappa\Big) \cdot \left\{ \Big\|\Delta v_{\varepsilon,j} - v_{\varepsilon,j} + u_{\varepsilon,j}\Big\|_{L^2(\Omega)}^{2\theta_0} + \Big\|\frac{\nabla u_{\varepsilon,j}}{\sqrt{u_{\varepsilon,j}}} - \sqrt{u_{\varepsilon,j}}\nabla v_{\varepsilon,j}\Big\|_{L^2(\Omega)} + 1 \right\} \qquad (3.11)
$$

for all $t \in (0, \widehat{T})$. Here using that $\theta_0 \le \theta$ and that $\theta \ge \frac{1}{2}$, by definition of $\mathcal{D}$ and Young's inequality we can estimate

$$\begin{aligned}
\Big\| \Delta v_{\varepsilon,j} - v_{\varepsilon,j} + u_{\varepsilon,j} \Big\|_{L^2(\Omega)}^{2\theta_0} + \Big\| \frac{\nabla u_{\varepsilon,j}}{\sqrt{u_{\varepsilon,j}}} - \sqrt{u_{\varepsilon,j}} \nabla v_{\varepsilon,j} \Big\|_{L^2(\Omega)} + 1 \\
\le \ \Big\| \Delta v_{\varepsilon,j} - v_{\varepsilon,j} + u_{\varepsilon,j} \Big\|_{L^2(\Omega)}^{2\theta} + \Big\| \frac{\nabla u_{\varepsilon,j}}{\sqrt{u_{\varepsilon,j}}} - \sqrt{u_{\varepsilon,j}} \nabla v_{\varepsilon,j} \Big\|_{L^2(\Omega)}^{2\theta} + 3 \\
\le \ 2\mathcal{D}^\theta(u_{\varepsilon,j}, v_{\varepsilon,j}) + 3 \qquad \text{for all } t \in (0, \widehat{T}),
\end{aligned}$$

so that (3.10) results from (3.11). $\Box$

As a first important application of the latter, we can use (3.10) to adequately control the crucial ill-signed summand $\varepsilon \int_\Omega u^2 v$ on the right of (3.1) whenever εu satisfies an upper estimate which we finally plan to disprove. We can thereby turn the identity (3.1) into an inequality exclusively containing $\mathcal{F}$ and $\mathcal{D}$ as follows.

Lemma 3.6 *Let $\theta \in (0,1)$ and C_0 be as in Lemma 3.5, let $K > 0$ and $T_\star \in (0,1)$, and suppose that for some $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$ we have $T_{\varepsilon,j} \ge T_\star$ and*

$$u_{\varepsilon,j}(x,t) \le \frac{K}{\varepsilon} \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0, T_\star). \tag{3.12}$$

Then with $\mathcal{F}$ and $\mathcal{D}$ as in (1.11) and (1.13),

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t) v_{\varepsilon,j}(\cdot,t)\Big) \ \le \ & -\mathcal{D}\Big(u_{\varepsilon,j}(\cdot,t) v_{\varepsilon,j}(\cdot,t)\Big) + 2\rho \mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t) v_{\varepsilon,j}(\cdot,t)\Big) \\
& + (K + 2\rho) C_0 D^\theta \Big(u_{\varepsilon,j}(\cdot,t) v_{\varepsilon,j}(\cdot,t)\Big) \\
& + (K + 2\rho) C_0 + |\Omega| \cdot \Big(\rho e + \frac{\rho}{e} + \frac{1}{2e}\Big) \qquad \textit{for all } t \in (0, T_\star).
\end{aligned} \tag{3.13}$$

PROOF. According to Lemma 3.1, $(u,v) := (u_{\varepsilon,j}, v_{\varepsilon,j})$ satisfies

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(u,v) \ = \ & -\mathcal{D}(u,v) + \varepsilon \int_\Omega u^2 v + \rho \int_\Omega u \ln u + \rho \int_\Omega u \\
& -\varepsilon \int_\Omega u^2 \ln u - \rho \int_\Omega uv - \varepsilon \int_\Omega u^2 \qquad \text{for all } t \in (0, T_{\varepsilon,j}),
\end{aligned} \tag{3.14}$$

where clearly

$$-\rho \int_\Omega uv - \varepsilon \int_\Omega u^2 \le 0 \qquad \text{for all } t \in (0, T_{\varepsilon,j}), \tag{3.15}$$

and where since $\varepsilon \in (0,1)$ we have

$$-\varepsilon \int_\Omega u^2 \ln u \le \varepsilon \cdot \frac{|\Omega|}{2e} \le \frac{|\Omega|}{2e} \qquad \text{for all } t \in (0, T_{\varepsilon,j}) \tag{3.16}$$

due to the validity of the inequality $\xi^2 \ln \xi \ge -\frac{1}{2e}$ for all $\xi > 0$. Moreover, again using that $\xi \ln \xi \ge -\frac{1}{e}$ for all positive ξ we can estimate

$$\begin{aligned}
\rho \int_\Omega u \ln u + \rho \int_\Omega u &= \rho \int_\Omega u \ln u + \rho \int_{\{u \ge e\}} u + \rho \int_{\{u<e\}} u \\
&\le \rho \int_\Omega u \ln u + \rho \int_{\{u \ge e\}} u \ln u + \rho \int_{\{u<e\}} u \\
&= 2\rho \int_\Omega u \ln u - \rho \int_{\{u<e\}} u \ln u + \rho \int_{\{u<e\}} u \\
&\le 2\rho \int_\Omega u \ln u + \frac{\rho|\Omega|}{e} + \rho e |\Omega| \\
&= 2\rho \cdot \Big\{ \mathcal{F}(u,v) - \frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{1}{2} \int_\Omega v^2 + \int_\Omega uv \Big\} + \frac{\rho|\Omega|}{e} + \rho e |\Omega| \\
&\le 2\rho \mathcal{F}(u,v) + 2\rho \int_\Omega uv + \frac{\rho|\Omega|}{e} + \rho e|\Omega| \qquad \text{for all } t \in (0, T_{\varepsilon,j}). \qquad (3.17)
\end{aligned}$$

Since finally our assumption (3.12) ensures that

$$\varepsilon \int_\Omega u^2 v \le \varepsilon \|u\|_{L^\infty(\Omega)} \int_\Omega uv \le K \int_\Omega uv \qquad \text{for all } t \in (0, T_\star),$$

and since from Lemma 3.5 we know that

$$(K + 2\rho) \cdot \int_\Omega uv \le (K + 2\rho) C_0 \cdot \Big\{ \mathcal{D}^\theta(u,v) + 1 \Big\} \qquad \text{for all } t \in (0, T_\star),$$

on using (3.15)-(3.17) we infer from (3.14) that indeed (3.13) is valid. □

In order to relate the summands in (3.13) containing $\mathcal{D}$ to certain expressions only involving $\mathcal{F}$, we once more apply Lemma 3.5 to achieve the following estimate on $\mathcal{D}$ from below in terms of $\mathcal{F}$.

Lemma 3.7 *Let $\theta \in (0,1)$ and $C_0 > 0$ be as in Lemma 3.5. Then for any choice of $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$,*

$$\mathcal{D}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \ge \left\{ \frac{-\mathcal{F}(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t))}{C_0} - 1 - \frac{|\Omega|}{C_0 e} \right\}_+^{\frac{1}{\theta}} \qquad \textit{for all } t \in \Big(0, \min\{1, T_{\varepsilon,j}\}\Big), \quad (3.18)$$

where $\mathcal{D}$ and $\mathcal{F}$ are as in (1.13) and (1.11).

Proof. Writing $(u,v) := (u_{\varepsilon,j}, v_{\varepsilon,j})$ and $\widehat{T} := \min\{1, T_{\varepsilon,j}\}$, from Lemma 3.5 we know that

$$\int_\Omega uv \le C_0 \cdot \Big\{ \mathcal{D}^\theta(u,v) + 1 \Big\} \qquad \text{for all } t \in (0, \widehat{T})$$

and hence

$$\begin{aligned}
-\mathcal{F}(u,v) &= -\frac{1}{2} \int_\Omega |\nabla v|^2 - \frac{1}{2} \int_\Omega v^2 + \int_\Omega uv - \int_\Omega u \ln u \\
&\le \int_\Omega uv - \int_\Omega u \ln u \\
&\le C_0 \mathcal{D}^\theta(u,v) + C_0 + \frac{|\Omega|}{e} \qquad \text{for all } t \in (0, \widehat{T}),
\end{aligned}$$

because $\xi \ln \xi \geq -\frac{1}{e}$ for all $\xi > 0$. By nonnegativity of $\mathcal{D}$, this immediately yields (3.18). $\Box$

As long as $\mathcal{F}$ attains suitably large negative numbers, this implies that up to a multiplicative constant, D even dominates a superlinear power of $\mathcal{F}$ itself.

Corollary 3.8 *Suppose that for some $T_\star \in (0,1)$, $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$ we have $T_{\varepsilon,j} \geq T_\star$ and*

$$\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \leq -2C_0 - \frac{2|\Omega|}{e} \qquad \text{for all } t \in (0,T_\star), \tag{3.19}$$

with C_0 and $\mathcal{F}$ taken from Lemma 3.5 and (1.11), respectively. Then the quantity $\mathcal{D}$ defined in (1.13) satisfies

$$\mathcal{D}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \geq \left\{\frac{-\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big)}{2C_0}\right\}^{\frac{1}{\theta}} \qquad \text{for all } t \in (0,T_\star). \tag{3.20}$$

Proof. Once more with $(u,v) := (u_{\varepsilon,j}, v_{\varepsilon,j})$, (3.19) says that

$$\frac{-\mathcal{F}(u,v)}{C_0} - 1 - \frac{|\Omega|}{e} \geq \frac{-\mathcal{F}(u,v)}{C_0} + \frac{\mathcal{F}(u,v)}{2C_0} = \frac{-\mathcal{F}(u,v)}{2C_0} \qquad \text{for all } t \in (0,T_\star).$$

In view of Lemma 3.7, the latter being applicable since $T_\star < 1$ and $T_\star \leq T_{\varepsilon,j}$, this directly yields (3.20). $\Box$

We next intend to make sure that as long as εu is conveniently small and $-\mathcal{F}$ is suitably large, $\mathcal{D}$ also substantially exceeds the last three summands in (3.13), the first among which is considered in the following.

Lemma 3.9 *Let $K > 0$ and $T_\star \in (0,1)$, and suppose $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$ are such that $T_{\varepsilon,j} \geq T_\star$ and*

$$u_{\varepsilon,j}(x,t) \leq \frac{K}{\varepsilon} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T_\star), \tag{3.21}$$

and that (3.19) holds as well as

$$\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \leq -2C_0 \cdot \Big\{4(K+2\rho)C_0\Big\}^{\frac{\theta}{1-\theta}} \qquad \text{for all } t \in (0,T_\star), \tag{3.22}$$

where $\theta \in (0,1)$ and $C_0 > 0$ are as provided by Lemma 3.5 and $\mathcal{F}$ is as in (1.11). Then with $\mathcal{D}$ taken from (1.13) we have

$$(K+2\rho)C_0 \cdot \mathcal{D}^\theta\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \leq \frac{1}{4}\mathcal{D}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \qquad \text{for all } t \in (0,T_\star). \tag{3.23}$$

Proof. Due to (3.19), Corollary 3.8 may be applied so as to guarantee that

$$\mathcal{D}(u,v) \geq \left\{\frac{-\mathcal{F}(u,v)}{2C_0}\right\}^{\frac{1}{\theta}} \qquad \text{for all } t \in (0,T_\star)$$

with $(u,v) := (u_{\varepsilon,j}, v_{\varepsilon,j})$. Therefore, using (3.22) and the fact that $\theta < 1$ we can estimate

$$\begin{aligned}
\frac{\frac{1}{4}\mathcal{D}(u,v)}{(K+2\rho)C_0 \cdot \mathcal{D}^\theta(u,v)} &= \frac{1}{4(K+2\rho)C_0} \cdot \mathcal{D}^{1-\theta}(u,v) \\
&\geq \frac{1}{4(K+2\rho)C_0} \cdot \left\{ \frac{-\mathcal{F}(u,v)}{2C_0} \right\}^{\frac{1-\theta}{\theta}} \\
&\geq 1 \qquad \text{for all } t \in (0, T_\star),
\end{aligned}$$

which is equivalent to (3.23). $\square$

The last two summands in (3.13) can be dealt with similarly.

Lemma 3.10 *Let $K > 0$ and $T_\star \in (0,1)$, and assume that $\varepsilon \in (0,1)$ and $j \in \mathbb{N}$ have the properties that $T_{\varepsilon,j} \geq T_\star$ and*

$$u_{\varepsilon,j}(x,t) \leq \frac{K}{\varepsilon} \qquad \textit{for all } x \in \Omega \textit{ and } t \in (0,T_\star), \tag{3.24}$$

that (3.19) is valid, and such that with $\theta \in (0,1)$ and $C_0 > 0$ from Lemma 3.5 we have

$$-\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \geq -2C_0 \cdot 4^\theta \cdot \left\{ (K+2\rho)C_0 + |\Omega| \Big(\rho e + \frac{\rho}{e} - \frac{1}{2e}\Big) \right\}^\theta \qquad \textit{for all } t \in (0,T_\star), \tag{3.25}$$

where $\mathcal{F}$ is as in (1.11). Then the functional $\mathcal{D}$ from (1.13) satisfies

$$(K+2\rho)C_0 + |\Omega| \cdot \Big(\rho e + \frac{\rho}{e} + \frac{1}{2e}\Big) \leq \frac{1}{4}\mathcal{D}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big) \qquad \textit{for all } t \in (0,T_\star). \tag{3.26}$$

Proof. We again use that thanks to Corollary 3.8 our assumption that (3.19) holds ensures that for $(u,v) := (u_{\varepsilon,j}, v_{\varepsilon,j})$ we have

$$\mathcal{D}(u,v) \geq \left\{ \frac{-\mathcal{F}(u,v)}{2C_0} \right\}^{\frac{1}{\theta}} \qquad \text{for all } t \in (0,T_\star).$$

Therefore, namely, from (3.25) we immediately obtain that

$$\begin{aligned}
\frac{1}{4}\mathcal{D}(u,v) &\geq \frac{1}{4} \cdot \left\{ 4^\theta \cdot \left\{ (K+2\rho)C_0 + |\Omega| \Big(\rho e + \frac{\rho}{e} + \frac{1}{2e}\Big) \right\}^\theta \right\}^{\frac{1}{\theta}} \\
&= (K+2\rho)C_0 + |\Omega| \Big(\rho e + \frac{\rho}{e} + \frac{1}{2e}\Big) \qquad \text{for all } t \in (0,T_\star),
\end{aligned}$$

as claimed. $\square$

In conclusion, if all of the above hypotheses are met, $\mathcal{F}$ will satisfy a superlinear autonomous ordinary differential inequality.

Lemma 3.11 *Let $\mathcal{F}$ be as in (1.11), let $K>0$ and $T_\star<1$, and suppose that $\varepsilon\in(0,1)$ and $j\in\mathbb{N}$ are such that $T_{\varepsilon,j}\ge T_\star$, and that (3.21), (3.19), (3.22) and (3.25) are valid with $\theta\in(0,1)$ and $C_0>0$ taken from Lemma 3.5. Then*

$$\frac{d}{dt}\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t),v_{\varepsilon,j}(\cdot,t)\Big)\le-\frac{1}{2}\cdot\left\{\frac{-\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t),v_{\varepsilon,j}(\cdot,t)\Big)}{2C_0}\right\}^{\frac{1}{\theta}}\qquad \textit{for all } t\in(0,T_\star). \tag{3.27}$$

Proof. According to the assumed inequality in (3.19), we particularly know that $(u,v):=(u_{\varepsilon,j},v_{\varepsilon,j})$ satisfies

$$\mathcal{F}(u,v)\le 0\qquad \text{for all } t\in(0,T_\star),$$

whereas the hypotheses that (3.21), (3.22) and (3.25) be valid guarantee that

$$(K+2\rho)C_0\cdot\mathcal{D}^\theta(u,v)\le\frac{1}{4}\mathcal{D}(u,v)\qquad\text{for all } t\in(0,T_\star)$$

as well as

$$(K+2\rho)C_0+|\Omega|\Big(\rho e+\frac{\rho}{e}+\frac{1}{2e}\Big)\le\frac{1}{4}\mathcal{D}^\theta(u,v)\qquad\text{for all } t\in(0,T_\star)$$

due to Lemma 3.9 and Lemma 3.10. Therefore, from Lemma 3.6 we obtain that

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}(u,v) &\le -\mathcal{D}(u,v)+2\rho\mathcal{F}(u,v)+(K+2\rho)C_0\cdot\mathcal{D}^\theta(u,v)\\
&\quad +(K+2\rho)C_0+|\Omega|\Big(\rho e+\frac{\rho}{e}+\frac{1}{2e}\Big)\\
&\le -\mathcal{D}(u,v)+\frac{1}{4}\mathcal{D}(u,v)+\frac{1}{4}\mathcal{D}(u,v)\\
&= -\frac{1}{2}\mathcal{D}(u,v)\qquad\text{for all } t\in(0,T_\star),
\end{aligned}$$

so that another application of Corollary 3.8 establishes (3.27). □

The latter inequality, however, cannot hold throughout the considered time interval if the energy functional attains suitably large negative values initially. The contradiction thereby obtained leads to the following conlcusion.

Lemma 3.12 *Let $K>0$ and $T\in(0,1)$. Then there exists $j_0(K,T)\in\mathbb{N}$ with the property that for all $j\ge j_0(K,T)$ and each $\varepsilon\in(0,1)$ one can find $x_{\varepsilon,j}\in\Omega$ and $t_{\varepsilon,j}\in(0,\min\{T,T_{\varepsilon,j}\})$ such that*

$$u_{\varepsilon,j}(x_{\varepsilon,j},t_{\varepsilon,j})>\frac{K}{\varepsilon}. \tag{3.28}$$

Proof. Given $K>0$ and $T\in(0,1)$, we abbreviate

$$c_1:=2C_0+\frac{2|\Omega|}{e}$$

and

$$c_2 := 2C_0 \cdot \Big\{4(K+2\rho)C_0\Big\}^{\frac{\theta}{1-\theta}}$$

as well as

$$c_3 := 2C_0 \cdot 4^\theta \cdot \Big\{(K+2\rho)C_0 + |\Omega|\Big(\rho e + \frac{\rho}{e} + \frac{1}{2e}\Big)\Big\}^\theta$$

and

$$c_4 := \bigg\{\frac{4(2C_0)^{\frac{1}{\theta}}\theta}{(1-\theta)T}\bigg\}^{\frac{\theta}{1-\theta}},$$

and then obtain from Lemma 2.1 that there exists $J_0 = j_0(K,T) \in \mathbb{N}$ fulfilling

$$\mathcal{F}(u_{0j}, v_{0j}) < -\max\{c_1, c_2, c_3, c_4\} \qquad \text{for all } j \geq j_0. \tag{3.29}$$

Then in order to verify that j_0 has the claimed property, assuming this to be false we could find $j > j_0$ and $\varepsilon \in (0,1)$ such that in view of (2.1) we would have

$$T_{\varepsilon,j} \geq T \tag{3.30}$$

and

$$u_{\varepsilon,j}(x,t) \leq \frac{K}{\varepsilon} \qquad \text{for all } x \in \Omega \text{ and } t \in (0,T). \tag{3.31}$$

For these fixed values of ε and j, we would thus obtain that

$$y(t) := -\mathcal{F}\Big(u_{\varepsilon,j}(\cdot,t), v_{\varepsilon,j}(\cdot,t)\Big), \qquad t \in [0,T),$$

is well-defined with its initial value satisfying

$$y(0) > y_0 := \max\{c_1, c_2, c_3, c_4\} \tag{3.32}$$

according to (3.29). Therefore, by continuity of y,

$$S := \Big\{T_\star \in (0,T) \ \Big| \ y(t) > y_0 \text{ for all } t \in [0,T_\star)\Big\}$$

would be nonempty and hence also

$$T_\star := \sup S$$

well-defined. To see that we actually must have $T_\star = T$, we observe that (3.32) especially entails that $y \geq c_1$ and $y \geq c_2$ as well as $y \geq c_3$ on $(0,T_\star)$, which along with (3.30) and (3.31) asserts the hypotheses of Lemma 3.11. An application of the latter thus shows that

$$y'(t) \geq \frac{1}{2} \cdot \Big(\frac{y(t)}{2C_0}\Big)^{\frac{1}{\theta}} \qquad \text{for all } t \in (0,T_\star), \tag{3.33}$$

so that, in particular, $y' \ge 0$ on $(0,T_\star)$ and hence $y \ge y(0) > y_0$ on $(0,T_\star)$. This would clearly be incompatible with the assumption that $T_\star < T$, meaning that indeed $T_\star = T$ and that hence the inequality in (3.33) is valid for all $t \in (0,T)$. On integration, however, this would entail that

$$\frac{y^{1-\frac{1}{\theta}}(t) - y^{1-\frac{1}{\theta}}(0)}{1-\frac{1}{\theta}} \ge \frac{1}{2(2C_0)^{\frac{1}{\theta}}} \cdot t \qquad \text{for all } t \in (0,T)$$

and hence

$$\begin{aligned} y^{1-\frac{1}{\theta}}(t) &\le y^{1-\frac{1}{\theta}}(0) - \frac{1-\theta}{2(2C_0)^{\frac{1}{\theta}}\theta} \cdot t \\ &< y_0^{1-\frac{1}{\theta}} - \frac{1-\theta}{2(2C_0)^{\frac{1}{\theta}}\theta} \cdot t \qquad \text{for all } t \in (0,T) \end{aligned}$$

by (3.32). Since $y_0 \ge c_4$ and thus

$$y_0^{1-\frac{1}{\theta}} \le c_4^{1-\frac{1}{\theta}} = \frac{(1-\theta)T}{4(2C_0)^{\frac{1}{\theta}}\theta},$$

namely, this would lead to the absurd conclusion that

$$y^{1-\frac{1}{\theta}}\Big(\frac{T}{2}\Big) < \frac{(1-\theta)T}{4(2C_0)^{\frac{1}{\theta}}\theta} - \frac{1-\theta}{2(2C_0)^{\frac{1}{\theta}}\theta} \cdot \frac{T}{2} = 0.$$

This contradiction shows that actually no such $j > j_0$ and $\varepsilon \in (0,1)$ can exist. $\square$

We thereby immediately arrive at our main results on (1.3).

Proof of Theorem 1.1. With $j_0(K,T) \in \mathbb{N}$ taken from Lemma 3.12, we only need to relabel the solution sequence from Lemma 2.2 by substituting $k := j - j_0(K,T)$ for $j > j_0(K,T)$, and thereupon apply Lemma 3.12. $\square$

Also, our statement on a corresponding growth phenomenon in (1.3) for fixed initial data thus becomes evident.

Proof of Corollary 1.2. The statement follows on applying Theorem 1.1 to any fixed initial data and thereafter choosing $(u_0,v_0) := (u_{01},v_{01})$, for instance. $\square$

4 Large densities enforced by large sensitivities

Thanks to the particular quantitative information on the minimal size of solutions achieved in Theorem 1.1, a simple variable transformation enables us to apply the latter, without further preparations, so as to obtain our main result on enforcement of large population densities by large chemotactic sensitivities in presence of otherwise fixed parameter values.

Proof of Theorem 1.3. An application of Theorem 1.1 to $K := \mu L$ and arbitrary fixed nontrivial smooth radial initial data particularly shows that there exists a pair $(u_0,v_0) \in C^0(\overline{\Omega}) \times W^{1,\infty}(\Omega)$ of radially symmetric positive functions u_0 and v_0 such that for each $\varepsilon \in (0,1)$, under the initial condition

$(u_\varepsilon(\cdot,0), v_\varepsilon(\cdot,0)) = (u_0, v_0)$, with some $\widetilde{t}_\varepsilon \in (0,T)$ the boundary value problem in (1.3) possesses a solution $(u_\varepsilon, v_\varepsilon) \in (C^0(\overline{\Omega} \times [0,\widetilde{t}_\varepsilon]) \cap C^{2,1}(\overline{\Omega} \times (0,\widetilde{t}_\varepsilon)))^2$ fulfilling

$$u_\varepsilon(\widetilde{x}_\varepsilon, \widetilde{t}_\varepsilon) > \frac{\mu L}{\varepsilon} \qquad \text{for some } \widetilde{x}_\varepsilon \in \Omega. \tag{4.1}$$

Given $\mu > \chi$, we now only need to observe that

$$w \equiv w_\chi := \frac{u_\varepsilon}{\chi} \quad \text{and} \quad z \equiv z_\chi := \frac{v_\varepsilon}{\chi} \qquad \text{with } \varepsilon := \frac{\mu}{\chi} \in (0,1)$$

satisfy the boundary-value problem in (1.8) with

$$w_\chi(\cdot,0) = w_{0\chi} := \frac{u_0}{\chi} \quad \text{and} \quad z_\chi(\cdot,0) = z_{0\chi} := \frac{v_0}{\chi} \qquad \text{in } \Omega. \tag{4.2}$$

Then, namely, (1.9) is obvious from (4.2), whereas (4.1) ensures that if we let $x_\chi := \widetilde{x}_{\mu/\chi}$ and $t_\chi := \widetilde{t}_{\mu/\chi}$ for $\chi > \mu$, then again with $\varepsilon = \frac{\mu}{\chi}$ we have

$$w_\chi(x_\chi, t_\chi) = \frac{u_\varepsilon(\widetilde{x}_\varepsilon, \widetilde{t}_\varepsilon)}{\chi} > \frac{\frac{\mu L}{\varepsilon}}{\chi} = L,$$

whence also (1.10) holds. $\Box$

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