Global existence and slow grow-up in a quasilinear Keller-Segel system with exponentially decaying diffusivity

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Abstract

The Neumann initial-boundary value problem for the chemotaxis system

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), \\ v_t = \Delta v - v + u, \end{cases} \qquad (\star)$$

is considered in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with smooth boundary. In compliance with refined modeling approaches, the diffusivity function D therein is allowed to decay considerably fast at large densities, where a particular focus will be on the mathematically delicate case when $D(s)$ decays exponentially as $s \to \infty$. In such situations, namely, straightforward Moser-type recursive arguments for the derivation of L^∞ estimates for u from corresponding L^p bounds seem to fail. Accordingly, results on global existence, and especially on quantitative upper bounds for solutions, so far mainly concentrate on cases when D decays at most algebraically, and hence are unavailable in the present context.

This work develops an alternative approach, at its core based on a Moser-type iteration for the quantity e^u, to establish global existence of classical solutions for all reasonably regular initial data, as well as a logarithmic upper estimate for the possible growth of $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ as $t \to \infty$, under the assumptions that with some $K_1 > 0$, $K_2 > 0$, $\beta^- > 0$ and $\beta^+ \in (-\infty, \beta^-]$ we have $K_1 e^{-\beta^- s} \leq D(s) \leq K_2 e^{-\beta^+ s}$ for all $s \geq 0$, and that the size of S relative to D can be estimated according to $\frac{S(s)}{D(s)} \leq K_3 e^{\gamma s}$ for all $s \geq 0$ with some $K_3 > 0$ and $\gamma \in [\frac{\beta^+ - \beta^-}{2}, \frac{\beta^+}{2})$.

Making use of the fact that this allows for certain superalgebraic growth of $\frac{S}{D}$, as a particular consequence of this and known results on nonexistence of global bounded solutions we shall see that in the prototypical case when $D(s) = e^{-\beta s}$ and $S(s) = s e^{-\alpha s}$ for all $s \geq 0$ and some positive α and β, the assumptions that $n \geq 2$ and that

$$\beta > 0 \qquad \text{and} \qquad \begin{cases} \alpha \in \left(\frac{\beta}{2}, \beta\right) & \text{if } n = 2, \\ \alpha \in \left(\frac{\beta}{2}, \beta\right] & \text{if } n \geq 3, \end{cases}$$

warrant the existence of classical solutions which are global but unbounded, and for which this infinite-time blow-up is slow in the sense that the corresponding grow-up rate is at most logarithmic.
To the best of our knowledge, this inter alia seems to constitute the first quantitative information on a blow-up rate in a parabolic Keller-Segel system of type $(\star)$ for widely arbitrary initial data, hence independent of a particular construction of possibly non-generic exploding solutions.

Key words: chemotaxis, degenerate diffusion, global existence, infinite-time blow-up, grow-up rate
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1 Introduction

This work is concerned with the parabolic initial-boundary value problem

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$, with smooth boundary, where D and S are prescribed functions on $[0, \infty)$ with values in $(0, \infty)$ and $[0, \infty)$, respectively. Systems of this type are used in mathematical biology to model the dynamics in populations of chemotactically moving individuals, represented through their density $u = u(x,t)$, that are attracted by a chemical stimulus, with concentration $v = v(x,t)$, which they produce themselves.

The theoretical study of such processes by means of cross-diffusive parabolic systems of the considered form was initiated by Keller and Segel in their seminal work ([14]), and numerous results on the apparently simplest reasonable version thereof, as obtained on letting $D \equiv 1$ and $S(s) := s$ for $s \geq 0$, indicate that (1.1) indeed is able to adequately describe the spontaneous emergence of structures, known to occur in many experimental frameworks, even in the mathematically extreme sense of singularity formation, that is, of finite-time blow-up with respect to the norm in $L^\infty(\Omega)$ in the first solution component. In this classical Keller-Segel system, such explosions have been rigorously detected for some radially symmetric solutions in the case $n = 2$ under the additional condition that $\int_\Omega u_0 > 8\pi$ ([10], [17]), and in the case $n \geq 3$ for arbitrary positive values of the total mass $\int_\Omega u_0$ ([25]), whereas it is known that if either $n = 1$, or $n = 2$ and $\int_\Omega u_0 < 4\pi$, or $n \geq 3$ and (u_0, v_0) is suitably small in $L^{\frac{n}{2}}(\Omega) \times W^{1,n}(\Omega)$, then under appropriate regularity assumptions on the initial data there always exist global bounded solutions also in nonradial settings ([19], [18], [3]).

As a refinement of this simple model more appropriate for the description in biological situations when large values of u seem inadequate, more elaborate modeling approaches suggest to choose D and S as more general functions of the cell density, preferably remaining significantly below the above prototypes at large densities and thereby reflecting so-called volume-filling effects, that is, limitations in the ability of cells to move which due to their nonzero volume naturally arise when they are densely packed (cf. [20] and also the surveys [11] and [1] for more references on the background of such modeling aspects).

Indeed, numerous analytical studies on such refined, in general quasilinear, chemotaxis systems have revealed that blow-up phenomena can entirely be ruled out when relative to the diffusion rate D, the chemotactic sensitivity function S is weakened to a suitably large extent at large values of the cell density, *provided that the diffusivity $D(s)$ does not decay too fast as $s \to \infty$.* More precisely, it is known that whenever $n \geq 2$ and

$$\frac{S(s)}{D(s)} \leq Cs^{\frac{2}{n}-\varepsilon} \qquad \text{for all } s > 1 \tag{1.2}$$

with some $C > 0$ and $\varepsilon > 0$, for any suitably regular initial data the problem (1.1) possesses a global bounded classical solution, provided that in addition there exist $p > 0$ and $c > 0$ fulfilling

$$D(s) \geq cs^{-p} \qquad \text{for all } s > 1 \tag{1.3}$$

(see [23] and also e.g [16], [15], [21] and [13] for some precedent partial results in this direction). On the other hand, in this respect the condition (1.2) cannot be relaxed substantially, as indicated by results on the occurrence of unbounded solutions in radial cases when instead it is assumed that $\frac{S(s)}{D(s)}$ grows substantially faster than $s^{\frac{2}{n}}$ as $s \to \infty$ in the sense that e.g.

$$\liminf_{s\to\infty} \frac{s\left(\frac{S}{D}\right)'(s)}{\left(\frac{S}{D}\right)(s)} > \frac{2}{n}, \tag{1.4}$$

without any further restriction on the behavior of $D(s)$ for large s ([24]); in certain regimes of the parameters $p \in \mathbb{R}$ and $q \in \mathbb{R}$ in the prototypical version of (1.1) obtained by choosing $D(s) = (s+1)^{-p}$ and $S(s) = s(s+1)^{q-1}$ for $s \geq 0$, this blow-up is even known to take place within finite time, whereas within certain further ranges of p and q blow-up occurs only in infinite time ([5], [7]; cf. also [8] and [9] for related and more complete results on an associated parabolic-elliptic simplification of (1.1)).

To the best of our knowledge, in cases when D fails to satisfy (1.3) the question of global solvability in (1.1) has remained widely unsolved so far in the literature. This may reflect the circumstance that then Moser-type recursive procedures, constituting a natural and frequently employed approach to derive L^∞ estimates for u from corresponding L^p bounds (cf. e.g. [23]), apparently fail to yield the desired conclusion when applied in a straightforward manner, and consequently their availability seems restricted to special cases ([2], [7]). As a conceivable alternative, approaches based on De-Giorgi-type iterations have up to now been found useful only in particular situations, and with an additional drawback of not providing quantitative information on the growth of possibly unbounded solutions ([4]).

Main results. It is the purpose of the present work to investigate the questions of global solvability as well as of basic quantitative information on the large time behavior of solutions to (1.1) in cases when the diffusion rate therein is allowed to decay exponentially at large densities. From a technical point of view, a particular goal will consist in developing a Moser-type approach for such situations, aiming at the derivation of L^∞ bounds for u from estimates for $\int_\Omega e^{\beta_k u}$ with appropriately chosen sequences of numbers β_k diverging to $+\infty$ as $k \to \infty$.

To make our overall hypotheses more precise, we shall assume that there exists $\iota > 0$ such that

$$\begin{cases} D \in C^{1+\iota}([0,\infty)) & \text{is positive and} \\ S \in C^{1+\iota}([0,\infty)) & \text{is nonnegative with } S(0) = 0, \end{cases} \tag{1.5}$$

and that the behavior of D at large values of its argument can be controlled from below and from above by exponential bounds in the sense that there exist $\beta^- \in \mathbb{R}$, $\beta^+ \leq \beta^-$ and positive constants K_1 and K_2 such that

$$D(s) \geq K_1 e^{-\beta^- s} \qquad \text{for all } s \geq 0 \tag{1.6}$$

and

$$D(s) \leq K_2 e^{-\beta^+ s} \qquad \text{for all } s \geq 0. \tag{1.7}$$

Moreover, we shall require that the growth of S relative to D can be estimated according to

$$\frac{S(s)}{D(s)} \leq K_3 e^{\gamma s} \qquad \text{for all } s \geq 0 \tag{1.8}$$

with some $\gamma \in \mathbb{R}$ and $K_3 > 0$.

Under the additional assumption that the initial data satisfy

$$\begin{cases} u_0 \in W^{1,\infty}(\Omega) \quad \text{with } u_0 \geq 0 \text{ in } \Omega \qquad \text{and} \\ v_0 \in W^{1,\infty}(\Omega) \quad \text{with } v_0 \geq 0 \text{ in } \Omega, \end{cases} \tag{1.9}$$

our main result then asserts global solvability, as well as a logarithmic bound on a possibly occurring asymptotic growth, in the following sense.

Theorem 1.1 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, that D and S comply with (1.5), and that there exist $\beta^- > 0, \beta^+ \in (-\infty, \beta^-]$ and*

$$\gamma \in \Big[\frac{\beta^+ - \beta^-}{2}, \frac{\beta^+}{2}\Big) \tag{1.10}$$

such that (1.6), (1.7) and (1.8) are valid with certain positive constants K_1, K_2 and K_3. Then for any u_0 and v_0 satisfying (1.9), there exists a pair (u, v) of nonnegative functions which for any $\vartheta > n$ is uniquely determined by the inclusions

$$\begin{cases} u \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)), \\ v \in C^0(\bar{\Omega} \times [0,\infty)) \cap C^{2,1}(\bar{\Omega} \times (0,\infty)) \cap L^\infty_{loc}([0,\infty); W^{1,\vartheta}(\Omega)), \end{cases} \tag{1.11}$$

and which solves (1.1) in the classical sense in $\Omega \times (0,\infty)$. Moreover, for any $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that this solution satisfies

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \Big(\frac{1}{\beta^+ - 2\gamma} + \varepsilon\Big) \cdot \ln(1+t) + C(\varepsilon) \qquad \textit{for all } t > 0. \tag{1.12}$$

Let us underline that in the case when β^+ is positive, the condition (1.10) is mild enough so as to include some positive values of γ, in accordance with (1.8) thus allowing for situations in which S even may grow exponentially relative to D, and in which thus the condition (1.2) is quite drastically violated. In order to illustrate this and further aspects of Theorem 1.1, let us draw some conclusions of the above for the prototypical situation obtained on choosing $D(s) := e^{-\beta s}$ and $S(s) := se^{-\alpha s}$ with $\beta > 0$ and $\alpha \in \mathbb{R}$ in (1.1). For the corresponding problem

$$\begin{cases} u_t = \nabla \cdot (e^{-\beta u} \nabla u) - \nabla \cdot (u e^{-\alpha u} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.13}$$

from Theorem 1.1 we then firstly infer the following.

Corollary 1.2 *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and suppose that*

$$\beta > 0 \qquad \textit{and} \qquad \alpha > \frac{\beta}{2}. \tag{1.14}$$

Then for any (u_0, v_0) satisfying (1.9), the problem (1.13) possesses a global classical solution (u, v) which is uniquely determined by (1.11) for arbitrary $\vartheta > n$. Moreover, for all $\varepsilon > 0$ one can find $C(\varepsilon) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Big(\frac{1}{\min\{\beta, 2\alpha-\beta\}} + \varepsilon\Big) \cdot \ln(1+t) + C(\varepsilon) \qquad \textit{for all } t > 0. \tag{1.15}$$

Secondly, however, by making use of the mentioned option to choose γ in (1.8) positive we shall derive as a further consequence of Theorem 1.1 when combined with the known unboundedness result from [24] that the somewhat rare phenomenon of infinite-time blow-up can also be detected in (1.13) whenever D decays exponentially but $\frac{S}{D}$ exhibits suitably slow exponential growth at large densities. According to the growth estimates achieved so far, we furthermore obtain that any such grow-up must occur at most at a logarithmic rate. Since to the best of our knowledge this is the first quantitative information on infinite-time blow-up in a Keller-Segel system in the literature, and since this moreover seems to constitute the first estimate on a blow-up rate in a parabolic Keller-Segel system of type (1.1) for widely arbitrary initial data, hence independent of a particular construction of possibly non-generic exploding solutions as e.g. in [10], for reasons of adequate emphasis let us repeat the corresponding estimate from the above corollary again in the following.

Theorem 1.3 *Let $n \ge 2$, $R > 0$ and $\Omega := B_R(0) \subset \mathbb{R}^n$, and suppose that*

$$\beta > 0 \qquad \textit{and} \qquad \begin{cases} \alpha \in \big(\frac{\beta}{2}, \beta\big) & \textit{if } n = 2, \\ \alpha \in \big(\frac{\beta}{2}, \beta\big] & \textit{if } n \ge 3. \end{cases} \tag{1.16}$$

Then for all $m > 0$ there exist radially symmetric initial data which are such that (1.9) holds as well as $\int_\Omega u_0 = m$, and which are such that the problem (1.13) possesses a unique global solution (u, v) fulfilling (1.11) which blows up in infinite time in that

$$\limsup_{t\to\infty} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.17}$$

Moreover, this infinite-time blow-up occurs at a rate no faster than logarithmic in the sense that for each $\varepsilon > 0$ one can find $C(\varepsilon) > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Big(\frac{1}{\min\{\beta, 2\alpha-\beta\}} + \varepsilon\Big) \cdot \ln(1+t) + C(\varepsilon) \qquad \textit{for all } t > 0. \tag{1.18}$$

We finally apply Theorem 1.1 to a particular case of the volume-filling chemotaxis model proposed in [20], hence following the suggestion therein to link D and S in (1.1) via the relations

$$D(s) = Q(s) - sQ'(s) \quad \text{and} \quad S(s) = sQ(s), \qquad s \ge 0,$$

on the basis of a supposedly known function Q for which $Q(u)$ represents the probability that a cell, when located at a point of current cell density u, finds space in some neighboring site. Let us recall from the introductory discussion and the literature that if Q decays algebraically in that $Q(s) = (s+1)^{-\lambda}$ for $s \ge 0$ with some $\lambda > 0$, and hence $\frac{S(s)}{D(s)} = \frac{s}{1+\lambda}$ for all $s \ge 0$, then it is known that unbounded solutions exist whenever $n \ge 3$ ([24]), and that these explosions occur only in infinite time when in addition

$\lambda > 2 - \frac{2}{n}$ ([7]; cf. also [6] for a discussion on a related two-dimensional situation). For the corresponding prototypical choice in the case of exponential decay, as determined by

$$Q(s) := e^{-\beta s}, \qquad s \ge 0,$$

for $\beta > 0$, (1.1) takes the form

$$\begin{cases} u_t = \nabla \cdot \Big((1+\beta u) e^{-\beta u} \nabla u \Big) - \nabla \cdot (u e^{-\beta u} \nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.19}$$

For this system, in view of the asymptotically constant behavior of $\frac{S(s)}{D(s)} = \frac{s}{1+\beta s}$ one might expect from the discussion around (1.2) and (1.4) that global solutions always exist and remain bounded. Beyond re-establishing the claim herein on global existence, as already proved in [4], our Theorem 1.1, albeit not asserting boundedness in this general setup, at least provides an upper bound on solutions in the flavor of (1.12). More precisely, we have the following.

Proposition 1.4 *Let $n \ge 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary, and let $\beta > 0$ and (u_0, v_0) be such that (1.9) holds. Then (1.19) possesses a unique global classical solution (u,v) fulfilling (1.11) which is such that for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ with the property that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Big(\frac{1}{\beta} + \varepsilon \Big) \cdot \ln(1+t) + C(\varepsilon) \qquad \textit{for all } t > 0. \tag{1.20}$$

The structure of this paper is as follows. In Section 2 we collect some preliminaries, in particular including a comparison result for the nonlocal ordinary differential inequality (2.2) and a result on independence of a constant in a Gagliardo-Nirenberg inequality within certain ranges of the exponents appearing therein (Lemma 2.5). In Section 3 we will then derive a fundamental a priori estimate for e^u in $L^\beta(\Omega)$ for some appropriately large $\beta > 0$, thereby making essential use of the right inequality in (1.10). After a preparatory selection of parameters and a sequence $(\beta_k)_{k\in\mathbb{N}}$ diverging to $+\infty$ (Lemma 4.1), based on an autonomous ODI for $e^{\beta_k u}$ (Lemma 4.2) we will then proceed to develop this into an L^∞ estimate for e^u, and hence also to the claimed global existence results, by means of an iterative argument of Moser type in Lemma 5.1. The applications to the particular systems (1.13) and (1.19) will finally be presented in Section 6.

2 Preliminaries

The following basic statement on local existence and extensibility of solutions can be obtained in a straightforward manner by adapting well-established arguments to the present context ([4], [13], [22], [26]).

Lemma 2.1 *Suppose that D and S satisfy (1.5) and that u_0 and v_0 fulfill (1.9), and let $\vartheta > n$. Then there exist $T_{max} \in (0,\infty]$ and a unique couple of nonnegative functions*

$$\begin{cases} u \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})), \\ v \in C^0(\bar\Omega \times [0,T_{max})) \cap C^{2,1}(\bar\Omega \times (0,T_{max})) \cap L^\infty_{loc}([0,T_{max}); W^{1,\vartheta}(\Omega)) \end{cases}$$

such that (u,v) is a classical solution of (1.1) in $\Omega\times(0,T_{max})$, and such that we have the alternative

$$\textit{either } T_{max}=\infty, \qquad \textit{or} \qquad \limsup_{t\nearrow T_{max}}\Big(\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\vartheta}(\Omega)}\Big)=\infty. \tag{2.1}$$

2.1 Two ODE comparison results

In the derivation of our basic quantitative growth estimate in Lemma 3.2, we shall employ the following comparison argument involving a nonlocal ordinary differential inequality.

Lemma 2.2 *Let $T>0$, and suppose that $y\in C^0([0,T))\cap C^1((0,T))$ is nonnegative and such that*

$$y'(t)\le a\cdot\Big\{\max_{s\in[0,t]}y(s)\Big\}^\lambda \qquad \textit{for all } t\in(0,T) \tag{2.2}$$

with certain constants $a>0$ and $\lambda\in(0,1)$. Then

$$y(t)\le\Big\{y^{1-\lambda}(0)+(1-\lambda)at\Big\}^{\frac{1}{1-\lambda}} \qquad \textit{for all } t\in(0,T). \tag{2.3}$$

Proof. For fixed $\varepsilon>0$, we let $\overline{y}_\varepsilon\in C^1([0,\infty))$ denote the solution of the initial-value problem

$$\begin{cases} \overline{y}_\varepsilon'(t)=a\cdot\Big\{\overline{y}_\varepsilon(t)+\varepsilon\Big\}^\lambda, & t>0,\\ \overline{y}_\varepsilon(0)=y(0)+\varepsilon, \end{cases}$$

that is, we define

$$\overline{y}_\varepsilon(t):=\Big\{(y(0)+2\varepsilon)^{1-\lambda}+(1-\lambda)at\Big\}^{\frac{1}{1-\lambda}}-\varepsilon, \qquad t\ge0. \tag{2.4}$$

Then using that $\overline{y}_\varepsilon(0)>y(0)$, we see that

$$M_\varepsilon:=\Big\{T_0\in[0,T)\ \Big|\ y(t)<\overline{y}_\varepsilon(t) \text{ for all } t\in[0,T_0)\Big\}$$

is not empty and hence $t_\star:=\sup M_\varepsilon$ a well-defined element of $(0,T]$. To see that actually $t_\star=T$, assuming this to be false we would obtain from the regularity properties of y and $\overline{y}_\varepsilon$ that $y(t)\le\overline{y}_\varepsilon(t)$ for all $t\in[0,t_\star]$ and that $y'(t_\star)\ge\overline{y}_\varepsilon'(t_\star)$. Now since $\overline{y}_\varepsilon$ is nondecreasing, the latter entails that for any $t\in[0,t_\star]$,

$$y'(t_\star)\ge a\cdot\Big\{\overline{y}_\varepsilon(t_\star)+\varepsilon\Big\}^\lambda\ge a\cdot\Big\{\overline{y}_\varepsilon(t)+\varepsilon\Big\}^\lambda\ge a\cdot\Big\{y(t)+\varepsilon\Big\}^\lambda$$

and thus

$$y'(t_\star)>a\cdot\Big\{\max_{t\in[0,t_\star]}y(t)\Big\}^\lambda.$$

This contradiction to (2.2) shows that actually $y(t)\le\overline{y}_\varepsilon(t)$ for all $t\in[0,T)$ and any $\varepsilon>0$, so that (2.3) results on observing that by (2.4) we have

$$\overline{y}_\varepsilon(t)\to\Big\{y^{1-\lambda}(0)+(1-\lambda)at\Big\}^{\frac{1}{1-\lambda}} \qquad \text{as } \varepsilon\to0$$

for any $t \ge 0$. $\Box$

For later reference, let us furthermore state the following result of a straightforward elementary comparison argument.

Lemma 2.3 *Let a, d, α and δ denote positive constants, and suppose that for some $T > 0$, $y \in C^0([0,T)) \cap C^1((0,T))$ is positive on $[0,T)$ and such that*

$$y'(t) + dy^\delta(t) \le a(1+t)^\alpha \qquad \text{for all } t \in (0,T). \tag{2.5}$$

Then

$$y(t) \le \max\left\{ y(0)\,,\, \left(\frac{a}{d}\right)^{\frac{1}{\delta}} (1+t)^{\frac{\alpha}{\delta}} \right\} \qquad \text{for all } t \in [0,T). \tag{2.6}$$

PROOF. Writing

$$\overline{y}_\varepsilon(t) := \max\left\{ y(0)\,,\, \left(\frac{a}{d}\right)^{\frac{1}{\delta}} (1+t)^{\frac{\alpha}{\delta}} \right\} + \varepsilon$$

for $t \in [0,T)$ and $\varepsilon > 0$, we see that for any fixed $\varepsilon > 0$ we have $y(0) < \overline{y}_\varepsilon(0)$, and since $\overline{y}_\varepsilon$ possesses a nonnegative left derivative $D^-\overline{y}_\varepsilon(t)$ at each $t \in (0,T)$, using that $\overline{y}_\varepsilon^\delta(t) > \frac{a}{d}(1+t)^\alpha$ for all $t \in [0,T)$ we can estimate

$$D^-\overline{y}_\varepsilon(t) + d\overline{y}_\varepsilon^\delta(t) - a(1+t)^\alpha > d \cdot \frac{a}{d}(1+t)^\alpha - a(1+t)^\alpha = 0 \qquad \text{for all } t \in (0,T).$$

Therefore, an elementary comparison argument shows that $y < \overline{y}_\varepsilon$ throughout $[0,T)$, which yields (2.6) upon letting $\varepsilon \searrow 0$. $\Box$

2.2 Independence of constants from exponents in some interpolation inequalities

In the course of our Moser-type iteration (cf. Lemma 4.2), it will be important to notice that the constants appearing in some Gagliardo-Nirenberg inequalities can be chosen so as to be independent from the respective summability powers, provided that the latter remain within certain subcritical ranges. In proving our statement in this direction, as specified in Lemma 2.5, we will rely on the following straightforward consequence of the compact embedding of $W^{1,2}(\Omega)$ into $L^2(\Omega)$, to be used again independently also in Lemma 4.2.

Lemma 2.4 *Let $p_\star > 0$. Then there exists $C > 0$ such that for any $p \ge p_\star$ we have*

$$\|\varphi\|_{W^{1,2}(\Omega)}^2 \le C \cdot \left\{ \|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\varphi\|_{L^p(\Omega)}^2 \right\} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{2.7}$$

PROOF. Due to the compactness of the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Omega)$, by means of an associated Ehrling-type lemma we can find $c_1 > 0$ such that

$$\|\varphi\|_{L^2(\Omega)}^2 \le \frac{1}{2}\|\varphi\|_{W^{1,2}(\Omega)}^2 + c_1\|\varphi\|_{L^{p_\star}(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega).$$

Since the Hölder inequality says that herein

$$\|\varphi\|_{L^{p_\star}(\Omega)}^2 \le |\Omega|^{\frac{2(p-p_\star)}{pp_\star}} \|\varphi\|_{L^p(\Omega)}^2 \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

and since $\frac{2(p-p_\star)}{pp_\star} \le \frac{2}{p_\star}$, we conclude that for all $\varphi \in W^{1,2}(\Omega)$,

$$\|\varphi\|^2_{W^{1,2}(\Omega)} = \|\nabla\varphi\|^2_{L^2(\Omega)} + \|\varphi\|^2_{L^2(\Omega)} \le \|\nabla\varphi\|^2_{L^2(\Omega)} + \frac{1}{2}\|\varphi\|^2_{W^{1,2}(\Omega)} + c_1 \cdot \max\Big\{1\,,\,|\Omega|^{\frac{2}{p_\star}}\Big\} \cdot \|\varphi\|^2_{L^p(\Omega)},$$

from which (2.7) readily follows. $\Box$

Now the announced version of the Gagliardo-Nirenberg inequality reads as follows.

Lemma 2.5 *Let $p_\star, p^\star, r_\star$ and $r^\star$ be positive numbers satisfying*

$$r_\star \le r^\star < p_\star \le p^\star < \frac{2n}{(n-2)_+}. \tag{2.8}$$

Then there exists $C > 0$ such that for any choice of $p \in [p_\star, p^\star]$ and $r \in [r_\star, r^\star]$ we have

$$\|\varphi\|_{L^p(\Omega)} \le C\|\nabla\varphi\|^a_{L^2(\Omega)}\|\varphi\|^{1-a}_{L^r(\Omega)} + C\|\varphi\|_{L^r(\Omega)} \qquad \textit{for all } \varphi \in W^{1,2}(\Omega) \tag{2.9}$$

with

$$a = \frac{\frac{n}{r} - \frac{n}{p}}{1 - \frac{n}{2} + \frac{n}{r}} \in [a_\star, a^\star], \tag{2.10}$$

where

$$a_\star := \frac{\frac{n}{r^\star} - \frac{n}{p_\star}}{1 - \frac{n}{2} + \frac{n}{r^\star}} > 0 \qquad \textit{and} \qquad a_\star := \frac{\frac{n}{r_\star} - \frac{n}{p^\star}}{1 - \frac{n}{2} + \frac{n}{r_\star}} < 1. \tag{2.11}$$

Proof. According to the Gagliardo-Nirenberg inequality, since $r^\star < p^\star < \frac{2n}{(n-2)_+}$ we can find $c_1 \ge 1$ such that

$$\|\varphi\|_{L^{p^\star}(\Omega)} \le c_1\|\varphi\|^b_{W^{1,2}(\Omega)}\|\varphi\|^{1-b}_{L^{r^\star}(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega) \tag{2.12}$$

with

$$b = \frac{\frac{n}{r^\star} - \frac{n}{p^\star}}{1 - \frac{n}{2} + \frac{n}{r^\star}} \in (0,1).$$

Since the Hölder inequality asserts that for any such φ we have

$$\|\varphi\|_{L^p(\Omega)} \le \|\varphi\|^c_{L^{p^\star}(\Omega)}\|\varphi\|^{1-c}_{L^r(\Omega)}$$

and

$$\|\varphi\|_{L^{r^\star}(\Omega)} \le \|\varphi\|^d_{L^p(\Omega)}\|\varphi\|^{1-d}_{L^r(\Omega)}$$

with

$$c = \frac{\frac{1}{r} - \frac{1}{p}}{\frac{1}{r} - \frac{1}{p^\star}} \in (0,1) \qquad \text{and} \qquad d = \frac{\frac{1}{r} - \frac{1}{r^\star}}{\frac{1}{r} - \frac{1}{p}} \in (0,1),$$

from (2.12) we thus obtain that

$$\|\varphi\|_{L^p(\Omega)} \le \Big\{c_1\|\varphi\|^b_{W^{1,2}(\Omega)} \cdot \Big(\|\varphi\|^d_{L^p(\Omega)}\|\varphi\|^{1-d}_{L^r(\Omega)}\Big)^{1-b}\Big\}^c \cdot \|\varphi\|^{1-c}_{L^r(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

that is,

$$\|\varphi\|_{L^p(\Omega)} \le c_1^{\frac{c}{1-(1-b)cd}} \|\varphi\|_{W^{1,2}(\Omega)}^{\frac{bc}{1-(1-b)cd}} \|\varphi\|_{L^r(\Omega)}^{\frac{1-c+(1-b)c(1-d)}{1-(1-b)cd}} \qquad \text{for all } \varphi \in W^{1,2}(\Omega). \tag{2.13}$$

Now a straightforward computation reveals that with a as in (2.10) we have

$$\frac{bc}{1-(1-b)cd} = a \qquad \text{and} \qquad \frac{1-c+(1-b)c(1-d)}{1-(1-b)cd} = 1-a,$$

so that since $p^\star < \frac{2n}{(n-2):+}$ warrants that indeed $0 < a_\star \le a \le a^\star < 1$ with $a_\star$ and $a^\star$ taken from (2.11), we see that moreover

$$\frac{c}{1-(1-b)cd} = \frac{a}{b} \le \frac{a^\star}{b}$$

and hence

$$c_1^{\frac{c}{1-(1-b)cd}} \le c_2 := c_1^{\frac{a^\star}{b}},$$

because $c_1 \ge 1$. As the inequalities $a_\star \le a \le a^\star$ along with Lemma 2.4 furthermore entail that with some $c_3 > 0$ we have

$$\|\varphi\|_{W^{1,2}(\Omega)}^a \le c_3 \cdot \Big\{ \|\nabla\varphi\|_{L^2(\Omega)}^a + \|\varphi\|_{L^r(\Omega)}^a \Big\},$$

from (2.13) we altogether infer that

$$\|\varphi\|_{L^p(\Omega)} \le c_2 \|\varphi\|_{W^{1,2}(\Omega)}^a \|\varphi\|_{L^r(\Omega)}^{1-a} \le c_2c_3 \|\nabla\varphi\|_{L^2(\Omega)}^a \|\varphi\|_{L^r(\Omega)}^{1-a} + c_2c_3\|\varphi\|_{L^r(\Omega)} \qquad \text{for all } \varphi \in W^{1,2}(\Omega),$$

as desired. $\square$

3 Bounds for e^u in $L^\beta(\Omega)$ for large $\beta > 0$

The goal of this section is to properly exploit (1.8) in deriving a quantitative estimate on the growth of u, formulated in terms of the norm of e^u in $L^\beta(\Omega)$ with appropriately large $\beta > 0$. This will be achieved in Lemma 3.2 which, as furthermore also Lemma 4.2 below, makes use of the following consequence of an assumed boundedness property of $\int_\Omega e^{\beta u}$ on regularity of the chemoattractive gradient, obtained through a standard argument based on well-known smoothing properties of the second equation in (1.1) when viewed as an inhomogeneous linear heat equation.

Lemma 3.1 *Under the assumptions of Lemma 2.1, for all $\beta > 0, q > 1$ and $\delta > 0$ there exists $C(\beta, q, \delta) > 0$ such that*

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le C(\beta,q,\delta) \cdot \Big\{ \max_{s\in[0,t]} \int_\Omega e^{\beta u(\cdot,s)} \Big\}^\delta \qquad \textit{for all } t \in (0, T_{max}). \tag{3.1}$$

PROOF. Given $q > 1$ and $\delta > 0$, we fix $p > 1$ large enough such that

$$p > \frac{nq}{n+q} \tag{3.2}$$

and

$$\frac{1}{p} < \delta. \tag{3.3}$$

Then since (3.2) means that $q < \frac{np}{n-p}$, a standard reasoning on the basis of well-known smoothing estimates for the Neumann heat semigroup ([13, Lemma 4.1]) provides $c_1 > 0$ such that

$$\|v(\cdot,t)\|_{W^{1,q}(\Omega)} \le c_1 \cdot \Big\{ 1 + \max_{s\in[0,t]} \|u(\cdot,s)\|_{L^p(\Omega)} \Big\} \qquad \text{for all } t \in (0,T_{max}). \tag{3.4}$$

Now picking $c_2 > 0$ such that $\xi^p \le c_2 \, e^{\beta\xi}$ for all $\xi \ge 0$, we can herein estimate

$$\|u(\cdot,s)\|_{L^p(\Omega)} = \Big\{ \int_\Omega u^p(\cdot,s) \Big\}^{\frac{1}{p}} \le c_2^{\frac{1}{p}} \cdot \Big\{ \int_\Omega e^{\beta u(\cdot,s)} \Big\}^{\frac{1}{p}} \qquad \text{for all } s \in [0,T_{max}),$$

so that since

$$\frac{1}{|\Omega|} \int_\Omega e^{\beta u(\cdot,s)} \ge 1 \qquad \text{for all } s \in [0,T_{max}), \tag{3.5}$$

it follows from (3.3) that

$$\|u(\cdot,s)\|_{L^p(\Omega)} \;\le\; c_2^{\frac{1}{p}} |\Omega|^{\frac{1}{p}} \cdot \Big\{ \frac{1}{|\Omega|} \int_\Omega e^{\beta u(\cdot,s)} \Big\}^{\delta} \qquad \text{for all } s \in [0,T_{max}).$$

Once more making use of (3.5), from (3.4) we can therefore readily derive that (3.1) holds if we let $C(\beta,q,\delta) := c_1 \cdot \big(|\Omega|^{-\delta} + c_2^{\frac{1}{p}} |\Omega|^{\frac{1}{p}-\delta} \big)$, for instance. $\square$

Now if the growth of S relative to D is limited according to (1.8) with some γ satisfying the upper inequality in (1.10), we can indeed find the following time-dependent estimate for $\int_\Omega e^{\beta u}$ for all suitably large β.

Lemma 3.2 *Let u_0 and v_0 be compatible with (1.9), and suppose that D and S satisfy (1.7) and (1.8) with some $K_2 > 0, K_3 > 0$, $\beta^+ \in \mathbb{R}$ and*

$$\gamma < \frac{\beta^+}{2}. \tag{3.6}$$

Then for all $\beta > \beta^+ - 2\gamma$ and each $\varepsilon > 0$ one can find $C(\beta,\varepsilon) > 0$ with the property that

$$\int_\Omega e^{\beta u(x,t)} dx \le C(\beta,\varepsilon) \cdot (1+t)^{\frac{\beta}{\beta^+ - 2\gamma} + \varepsilon} \qquad \textit{for all } t \in (0,T_{max}). \tag{3.7}$$

PROOF. Using (1.1) and integrating by parts, we compute

$$\begin{aligned} \frac{1}{\beta^2} \frac{d}{dt} \int_\Omega e^{\beta u} &= \frac{1}{\beta} \int_\Omega e^{\beta u} \nabla \cdot \Big(D(u)\nabla u - S(u)\nabla v \Big) \\ &= -\int_\Omega e^{\beta u} D(u)|\nabla u|^2 + \int_\Omega e^{\beta u} S(u) \nabla u \cdot \nabla v \qquad \text{for all } t \in (0,T_{max}), \end{aligned} \tag{3.8}$$

where by Young's inequality,

$$\int_\Omega e^{\beta u} S(u)\nabla u\cdot\nabla v \le \int_\Omega e^{\beta u} D(u)|\nabla u|^2 + \frac{1}{4}\int_\Omega e^{\beta u}\frac{S^2(u)}{D(u)}|\nabla v|^2 \qquad \text{for all } t\in(0,T_{max}). \tag{3.9}$$

Here we first invoke (1.7) and (1.8) to find that

$$\begin{aligned} \frac{1}{4}\int_\Omega e^{\beta u}\frac{S^2(u)}{D(u)}|\nabla v|^2 &= \frac{1}{4}\int_\Omega e^{\beta u} D(u)\cdot\Big(\frac{S(u)}{D(u)}\Big)^2\cdot|\nabla v|^2 \\ &\le \frac{K_2K_3^2}{4}\int_\Omega e^{(\beta-\beta^++2\gamma)u}|\nabla v|^2 \qquad \text{for all } t\in(0,T_{max}), \end{aligned} \tag{3.10}$$

and in order to estimate the latter integral we observe that (3.6) entails that $\beta-\beta^++2\gamma<\beta$, so that it is possible to fix $\theta>1$ sufficiently close to 1 such that still $\theta(\beta-\beta^++2\gamma)<\beta$. As $\beta-\beta^++2\gamma$ is positive, we may therefore twice apply the Hölder inequality to see that

$$\begin{aligned} \int_\Omega e^{(\beta-\beta^++2\gamma)u}|\nabla v|^2 &\le \Big\{\int_\Omega e^{\theta(\beta-\beta^++2\gamma)u}\Big\}^{\frac{1}{\theta}}\Big\{\int_\Omega |\nabla v|^{\frac{2\theta}{\theta-1}}\Big\}^{\frac{\theta-1}{\theta}} \\ &\le |\Omega|^{\frac{\beta-\theta(\beta-\beta^++2\gamma)}{\beta\theta}}\Big\{\int_\Omega e^{\beta u}\Big\}^{\frac{\beta-\beta^++2\gamma}{\beta}}\Big\{\int_\Omega |\nabla v|^{\frac{2\theta}{\theta-1}}\Big\}^{\frac{\theta-1}{\theta}} \end{aligned} \tag{3.11}$$

for all $t\in(0,T_{max})$. Again using (3.6), we nox fix $\delta>0$ small enough satisfying

$$\delta<\frac{\beta^+-2\gamma}{2\beta} \tag{3.12}$$

and

$$\frac{\beta}{\beta^+-2\gamma-2\beta\delta}\le\frac{\beta}{\beta^+-2\gamma}+\varepsilon \tag{3.13}$$

to infer from Lemma 3.1 that there exists $c_1>0$ fulfilling

$$\Big\{\int_\Omega |\nabla v(\cdot,t)|^{\frac{2\theta}{\theta-1}}\Big\}^{\frac{\theta-1}{\theta}} = \|\nabla v(\cdot,t)\|^2_{L^{\frac{2\theta}{\theta-1}}(\Omega)} \le c_1\cdot\Big\{\max_{s\in[0,t]}\int_\Omega e^{\beta u(\cdot,s)}\Big\}^{2\delta} \qquad \text{for all } t\in(0,T_{max}).$$

Combined with (3.8)-(3.11), this yields $c_2>0$ such that

$$\frac{d}{dt}\int_\Omega e^{\beta u}\le c_2\cdot\Big\{\int_\Omega e^{\beta u}\Big\}^{\frac{\beta-\beta^++2\gamma}{\beta}}\cdot\Big\{\max_{s\in[0,t]}\int_\Omega e^{\beta u(\cdot,s)}\Big\}^{2\delta} \qquad \text{for all } t\in(0,T_{max}),$$

which implies that $y(t):=\int_\Omega e^{\beta u(x,t)}dx,\ t\in[0,T_{max})$, has the property that

$$\begin{aligned} y'(t) &\le c_2 y^{\frac{\beta-\beta^++2\gamma}{\beta}}(t)\cdot\Big\{\max_{s\in[0,t]} y(s)\Big\}^{2\delta} \\ &\le c_2\cdot\Big\{\max_{s\in[0,t]} y(s)\Big\}^{\frac{\beta-\beta^++2\gamma}{\beta}+2\delta} \qquad \text{for all } t\in(0,T_{max}). \end{aligned}$$

As (3.12) asserts that $\lambda := \frac{\beta-\beta^++2\gamma}{\beta} + 2\delta < 1$, an application of Lemma 2.2 thus shows that

$$y(t) \le \Big\{ y^{1-\lambda}(0) + (1-\lambda)c_2 t \Big\}^{\frac{1}{1-\lambda}} \qquad \text{for all } t \in (0, T_{max}),$$

which immediately entails (3.7), because

$$\frac{1}{1-\lambda} = \frac{\beta}{\beta^+ - 2\gamma - 2\beta\delta} \le \frac{\beta}{\beta^+ - 2\gamma} + \varepsilon$$

according to (3.13). □

4 Preparations for a recursive argument

Let us next prepare a Moser-type iteration within which we will estimate the norm of e^u in $L^{\beta_k}(\Omega)$ for an appropriately chosen sequence $(\beta_k)_{k\in\mathbb{N}} \subset (0,\infty)$. A fundamental ordinary differential inequality for these norms, of recursive nature in containing a source term involving $\int_\Omega e^{\beta_{k-1}u}$, will be derived in Lemma 4.2 based on a selection of $(\beta_k)_{k\in\mathbb{N}}$ achieved in Lemma 4.1. For a further exploitation of this ODI in the next section, Lemma 4.3 will provide an elementary estimate for sequences satisfying certain recursive inequalities with asymptotically quadratic source terms.

4.1 Selection of parameters

For definiteness in our subsequent procedure, by now making full use of our assumptions in (1.10) on γ let us fix a sequence $(\beta_k)_{k\in\mathbb{N}} \subset (0,\infty)$ and list some basic properties thereof.

Lemma 4.1 *Let $\beta^- > 0, \beta^+ \le \beta^-$ and $\gamma \in [\frac{\beta^+-\beta^-}{2}, \frac{\beta^+}{2})$, and fix $p_\star \in (2, \frac{2n}{(n-2)_+})$. Then there exists $\theta > 1$ with the property that for all $\delta > 0$ one can find $\beta_0 > \max\{1, \beta^-\}$ such that with*

$$\beta_k := 2^k \beta_0, \qquad k \in \mathbb{N}, \tag{4.1}$$

we have

$$\beta_k - \beta^- > 0 \quad \textit{and} \quad \beta_k - \beta^+ + 2\gamma > 0 \qquad \textit{for all } k \in \mathbb{N}, \tag{4.2}$$

and that with

$$a_k := \frac{\frac{n(\beta_k-\beta^-)}{\beta_k} - \frac{n(\beta_k-\beta^-)}{2\theta(\beta_k-\beta^++2\gamma)}}{1 - \frac{n}{2} + \frac{n(\beta_k-\beta^-)}{\beta_k}}, \qquad k \in \mathbb{N}, \tag{4.3}$$

and

$$q_k := \frac{\beta_k - \beta^-}{(\beta_k - \beta^+ + 2\gamma)a_k}, \qquad k \in \mathbb{N}, \tag{4.4}$$

we have

$$\frac{1}{2} \le \frac{\beta_k}{\beta_k - \beta^-} \le \frac{3}{2} \quad \textit{and} \quad \frac{2\beta_k}{\beta_k - \beta^-} \le p^\star \qquad \textit{for all } k \in \mathbb{N} \tag{4.5}$$

as well as

$$a_\star \le a_k \le a^\star \quad \textit{and} \quad q_\star \le q_k \le q^\star \qquad \textit{for all } k \in \mathbb{N} \tag{4.6}$$

with

$$a_\star := \frac{n}{n+6}, \qquad a^\star := \frac{4n-\frac{n}{p^\star}}{3n+1} < 1 \tag{4.7}$$

and

$$q_\star := \frac{\beta_0-\beta^-}{(\beta_0-\beta^++2\gamma)a^\star} > 1, \qquad q^\star := \frac{n+6}{n}, \tag{4.8}$$

and such that moreover

$$2\theta \le \frac{2\theta(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-} \le p^\star \qquad \text{for all } k\in\mathbb{N} \tag{4.9}$$

and

$$\frac{\beta_k-\beta^++2\gamma}{\beta_k-\beta^-}\cdot(1-a_k)\cdot\frac{q_k}{q_k-1} \le 1+\delta\cdot 2^{-k} \qquad \text{for all } k\in\mathbb{N} \tag{4.10}$$

as well as

$$\frac{\beta_k}{\beta_k-\beta^-} \le 1+\delta\cdot 2^{-k} \qquad \text{for all } k\in\mathbb{N}. \tag{4.11}$$

PROOF. We let

$$\varphi(\beta) := \frac{\beta-\beta^++2\gamma}{\beta-\beta^-}, \qquad \beta>\beta^-,$$

and observe that since $-\beta^++2\gamma \ge -\beta^-$ according to our assumption that $\gamma \ge \frac{\beta^+-\beta^-}{2}$, the function φ is positive and nonincreasing on (β^-,∞) with $\varphi(\beta)\searrow 1$ as $\beta\to\infty$. Using the hypothesis that $p^\star>2$ and the easily checked fact that indeed $a^\star<1$, we can therefore pick $\beta_\star>1$ large enough fulfilling

$$\beta_\star \ge \max\left\{3\beta^-\,,\,\frac{p^\star\beta^-}{p^\star-2}\right\} \tag{4.12}$$

as well as

$$2\varphi(\beta_\star) < p^\star$$

and

$$\varphi(\beta_\star) < \frac{1}{a^\star}, \tag{4.13}$$

and thereupon choose $\theta>1$ sufficiently close to 1 such that still

$$2\theta\varphi(\beta_\star) \le p^\star. \tag{4.14}$$

Now given $\delta>0$, we finally fix $\beta_0\ge\beta_\star$ such that

$$\frac{1}{\beta_0}\cdot\frac{n(\beta^--\beta^++2\gamma)}{1-\frac{n}{2}+\frac{n}{2\theta}-\frac{n(\beta^--\beta^++2\gamma)}{\beta_0}} \le \delta \tag{4.15}$$

and

$$\beta_0 \ge \frac{(1+\delta)\beta^-}{\delta}, \tag{4.16}$$

and let $(\beta_k)_{k\in\mathbb{N}}, (a_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ be defined through (4.1), (4.3) and (4.4), respectively. Then since $\beta_k \ge \beta_0$ for all $k \in \mathbb{N}$, (4.2) is obvious from (4.12) and the fact that $\beta_\star > 1$, and (4.9) is immediate from (4.14) and the observation that

$$\frac{2\theta(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} = 2\theta\varphi(\beta_k) \in [2\theta, 2\theta\varphi(\beta_\star)] \qquad \text{for all } k \in \mathbb{N}$$

by monotonicity of φ. Moreover, the first property in (4.2) along with the positivity of β^- warrants that

$$\frac{\beta_k}{\beta_k - \beta^-} - \frac{1}{2} = \frac{\beta_k + \beta^-}{2(\beta_k - \beta^-)} \ge 0 \qquad \text{for all } k \in \mathbb{N},$$

and from the first condition contained in (4.12) we see that

$$\frac{\beta_k}{\beta_k - \beta^-} - \frac{3}{2} = \frac{-\beta_k + 3\beta^-}{2(\beta_k - \beta^-)} \le 0 \qquad \text{for all } k \in \mathbb{N},$$

whereas the second ensures that

$$\frac{2\beta_k}{\beta_k - \beta^-} - p^\star = \frac{-(p^\star - 2)\beta_k + p^\star\beta^-}{\beta_k - \beta^-} \le 0 \qquad \text{for all } k \in \mathbb{N}.$$

Having thus proved (4.5), combining the first two inequalities therein with (4.9) we observe that as a particular outcome of Lemma 2.5 when applied to $r := \frac{\beta_k}{\beta_k - \beta^-} \in [\frac{1}{2}, \frac{3}{2}]$ and $p := \frac{2\theta(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \in [2\theta, p^\star] \subset [2, p^\star]$ we obtain

$$\frac{\frac{n}{3/2} - \frac{n}{2}}{1 - \frac{n}{2} + \frac{n}{3/2}} \le a_k \le \frac{\frac{n}{1/2} - \frac{n}{p^\star}}{1 - \frac{n}{2} + \frac{n}{1/2}} < 1 \qquad \text{for all } k \in \mathbb{N},$$

which can readily be seen to imply (4.6)-(4.8). To verify (4.10), we first use (4.3) and (4.4) in computing

$$\frac{\beta_k - \beta^+ + 2\gamma}{\beta_k - \beta^-} \cdot (1 - a_k) \cdot \frac{q_k}{q_k - 1} = \frac{\beta_k - \beta^+ + 2\gamma}{\beta_k} \cdot \frac{1 - \frac{n}{2} + \frac{n}{2\theta\varphi(\beta_k)}}{1 - \frac{n}{2} + \frac{n}{2\theta} - \frac{n(\beta^- - \beta^+ + 2\gamma)}{\beta_k}} \qquad \text{for all } k \in \mathbb{N},$$

so that since $\gamma < \frac{\beta^+}{2}$ guarantees that $\frac{\beta_k - \beta^+ + 2\gamma}{\beta_k} \le 1$ for all $k \in \mathbb{N}$, and again since $\varphi(\beta_k) \ge 1$ for all $k \in \mathbb{N}$ by monotonicity of φ, we can estimate

$$\begin{aligned}
\frac{\beta_k - \beta^+ + 2\gamma}{\beta_k - \beta^-} \cdot (1 - a_k) \cdot \frac{q_k}{q_k - 1} &\le \frac{1 - \frac{n}{2} + \frac{n}{2\theta}}{1 - \frac{n}{2} + \frac{n}{2\theta} - \frac{n(\beta^- - \beta^+ + 2\gamma)}{\beta_k}} \\
&= 1 + \frac{1}{\beta_k} \cdot \frac{n(\beta^- - \beta^+ + 2\gamma)}{1 - \frac{n}{2} + \frac{n}{2\theta} - \frac{n(\beta^- - \beta^+ + 2\gamma)}{\beta_k}} \\
&\le 1 + \frac{2^{-k}}{\beta_0} \cdot \frac{n(\beta^- - \beta^+ + 2\gamma)}{1 - \frac{n}{2} + \frac{n}{2\theta} - \frac{n(\beta^- - \beta^+ + 2\gamma)}{\beta_0}} \qquad \text{for all } k \in \mathbb{N},
\end{aligned}$$

because $\beta_k = 2^k\beta_0 \ge \beta_0$ for any $k \in \mathbb{N}$. In view of (4.15), this indeed shows (4.10).
Finally, (4.16) entails that since $\beta^- > 0$ we have

$$\frac{1}{\delta \cdot 2^{-k}} \cdot \frac{\beta_k}{\beta_k - \beta^-} - 1 = \frac{\beta^-}{\delta\beta_0 - \delta \cdot 2^{-k}\beta^-} - 1 \le \frac{\beta^-}{\delta\beta_0 - \delta\beta^-} - 1 = \frac{-\delta\beta_0 + (1 + \delta)\beta^-}{\delta(\beta_0 - \beta^-)} \le 0$$

for all $k \in \mathbb{N}$, and that thus also (4.11) is valid. $\square$

4.2 A recursive integral inequality

With the above definition at hand, we can now derive an ODI for $\int_\Omega e^{\beta_k u}$ in which according to the parameter estimates provided by Lemma 4.1 the dependence on $k \in \mathbb{N}$ can be controlled in an essentially explicit manner.

Lemma 4.2 *Suppose that D and S satisfy (1.6), (1.7) and (1.8) with some $\beta^- \in \mathbb{R}, \beta^+ \le \beta^-$ and $\gamma \in [\frac{\beta^+-\beta^-}{2}, \frac{\beta^+}{2})$, and some $K_1 > 0, K_2 > 0$ and $K_3 > 0$. Then for all $\delta > 0$ there exist $\beta_0 > \max\{1, \beta^-\}$ and $C(\delta) > 0$ such that with $(\beta_k)_{k\in\mathbb{N}}, (a_k)_{k\in\mathbb{N}}$ and $(q_k)_{k\in\mathbb{N}}$ as defined in (4.1), (4.3) and (4.4), for any choice of $k \in \mathbb{N}$ we have*

$$\begin{aligned}
\frac{d}{dt}\int_\Omega e^{\beta_k u} + \frac{1}{C(\delta)} \cdot \Big\{ \int_\Omega e^{\beta_k u} \Big\}^{\frac{\beta_k-\beta^-}{\beta_k}} \quad \le \quad & C^k(\delta) \cdot (1+t)^\delta \cdot \Big\{ \int_\Omega e^{\beta_{k-1} u} \Big\}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k} \cdot (1-a_k) \cdot \frac{q_k}{q_k-1}} \\
& + C^k(\delta) \cdot (1+t)^\delta \cdot \Big\{ \int_\Omega e^{\beta_{k-1} u} \Big\}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k}} \\
& + C(\delta) \cdot \Big\{ \int_\Omega e^{\beta_{k-1} u} \Big\}^{\frac{2(\beta_k-\beta^-)}{\beta_k}} \qquad \text{for all } t \in (0, T_{max}). \qquad (4.17)
\end{aligned}$$

PROOF. We fix any $p^\star \in (2, \frac{2n}{(n-2)_+})$ and let $\theta > 1$ be as thereupon provided by Lemma 4.1, due to the latter implying that given $\delta > 0$ we can find $\beta_0 > 0$ with the properties listed there. Then again by straightforward computation and Young's inequality, from (1.1), (1.7) and (1.8) we obtain that for $k \in \mathbb{N}$,

$$\begin{aligned}
\frac{1}{\beta_k^2}\frac{d}{dt}\int_\Omega e^{\beta_k u} &= -\int_\Omega e^{\beta_k u} D(u)|\nabla u|^2 + \int_\Omega e^{\beta_k u} S(u)\nabla u \cdot \nabla v \\
&\le -\frac{1}{2}\int_\Omega e^{\beta_k u} D(u)|\nabla u|^2 + \frac{1}{2}\int_\Omega e^{\beta_k u}\frac{S^2(u)}{D(u)}|\nabla v|^2 \\
&\le -\frac{1}{2}\int_\Omega e^{\beta_k u} D(u)|\nabla u|^2 + \frac{K_2K_3^2}{2}\int_\Omega e^{(\beta_k-\beta^++2\gamma)u}|\nabla v|^2 \quad \text{for all } t \in (0, T_{max}), \quad (4.18)
\end{aligned}$$

where unlike in Lemma 3.2 we now additionally make use of (1.6) to estimate

$$\begin{aligned}
\frac{1}{2}\int_\Omega e^{\beta_k u} D(u)|\nabla u|^2 &\ge \frac{K_1}{2}\int_\Omega e^{(\beta_k-\beta^-)u}|\nabla u|^2 \\
&= \frac{2K_1}{(\beta_k-\beta^-)^2}\int_\Omega \Big|\nabla e^{\frac{\beta_k-\beta^-}{2}u}\Big|^2 \qquad \text{for all } t \in (0, T_{max}). \qquad (4.19)
\end{aligned}$$

To prepare an appropriate control the last summand in (4.18), we fix an arbitrary number $\beta > \beta^+ - 2\gamma$ and employ Lemma 3.2 to obtain $\kappa > 0$ and $c_1 > 0$ such that

$$\int_\Omega e^{\beta u} \le c_1(1+t)^\kappa \qquad \text{for all } t \in (0, T_{max}).$$

We may therefore invoke Lemma 3.1 to infer the existence of $c_2 > 0$, actually only depending on δ due to the fact that θ and β are fixed numbers, such that with $q_\star > 1$ as in (4.8) we have

$$\|\nabla v(\cdot,t)\|_{L^{\frac{2\theta}{\theta-1}}(\Omega)} \le c_2(1+t)^{\frac{(q_\star-1)\delta}{2q_\star}} \qquad \text{for all } t \in (0, T_{max}), \qquad (4.20)$$

whence using the Hölder inequality we obtain that

$$
\begin{aligned}
\frac{K_2K_3^2}{2}\int_\Omega e^{(\beta_k-\beta^++2\gamma)u}|\nabla v|^2 &\le \frac{K_2K_3^2}{2}\cdot\left\{\int_\Omega e^{\theta(\beta_k-\beta^++2\gamma)u}\right\}^{\frac{1}{\theta}}\cdot\left\{\int_\Omega |\nabla v|^{\frac{2\theta}{\theta-1}}\right\}^{\frac{\theta-1}{\theta}} \\
&\le c_3(1+t)^{\frac{(q_\star-1)\delta}{q_\star}}\cdot\left\{\int_\Omega e^{\theta(\beta_k-\beta^++2\gamma)u}\right\}^{\frac{1}{\theta}} \qquad (4.21)
\end{aligned}
$$

for all $t\in(0,T_{max})$ with $c_3:=\frac{K_2K_3^2c_2^2}{2}$. In order to estimate the rightmost factor by means of the Gagliardo-Nirenberg inequality from Lemma 2.5, we observe that according to (4.9) we have

$$
2<2\theta\le\frac{2\theta(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}\le p^\star<\frac{2n}{(n-2)_+}\qquad\text{for all } k\in\mathbb{N},
$$

whereas (4.5) asserts that

$$
\frac{1}{2}\le\frac{\beta_k}{\beta_k-\beta^-}\le\frac{3}{2}<2\qquad\text{for all } k\in\mathbb{N}. \qquad (4.22)
$$

Therefore, Lemma 2.5 provides $c_4>0$ such that for any $k\in\mathbb{N}$ we have

$$
\begin{aligned}
\left\{\int_\Omega e^{\theta(\beta_k-\beta^++2\gamma)u}\right\}^{\frac{1}{\theta}} &= \left\|e^{\frac{\beta_k-\beta^-}{2}u}\right\|_{L^{\frac{2\theta(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}} \\
&\le c_4\left\|\nabla e^{\frac{\beta_k-\beta^-}{2}u}\right\|_{L^2(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}\cdot a_k}\cdot\left\|e^{\frac{\beta_k-\beta^-}{2}u}\right\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}\cdot(1-a_k)} \\
&\quad +c_4\left\|e^{\frac{\beta_k-\beta^-}{2}u}\right\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}}\qquad\text{for all } t\in(0,T_{max}) \qquad (4.23)
\end{aligned}
$$

with $a_k\in(0,1)$ determined by the relation

$$
-\frac{n(\beta_k-\beta^-)}{2\theta(\beta_k-\beta^++2\gamma)}=\Big(1-\frac{n}{2}\Big)a_k-\frac{n(\beta_k-\beta^-)}{\beta_k}(1-a_k),
$$

that is, with a_k given by (4.3).
We now combine (4.23) with (4.21) and apply Young's inequality in the form

$$
AB\le\eta A^q+(q-1)q^{-\frac{q}{q-1}}\eta^{-\frac{1}{q-1}}B^{\frac{q}{q-1}},
$$

valid for all $A \geq 0, B \geq 0, q > 1$ and $\eta > 0$, to see that with $q_k = \frac{\beta_k - \beta^-}{(\beta_k + \beta^+ + 2\gamma)a_k}$ as in (4.4) we have

$$
\begin{aligned}
&\frac{K_2 K_3^2}{2} \int_\Omega e^{(\beta_k - \beta^+ + 2\gamma)u} |\nabla v|^2 \\
\leq \quad & c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| \nabla e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^2(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot a_k} \cdot \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot (1 - a_k)} \\
& + c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-}} \\
\leq \quad & \frac{K_1}{(\beta_k - \beta^-)^2} \left\| \nabla e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^2(\Omega)}^2 \\
& + (q_k - 1) q_k^{-\frac{q_k}{q_k - 1}} \left\{ \frac{K_1}{(\beta_k - \beta^-)^2} \right\}^{-\frac{1}{q_k - 1}} \cdot \left\{ c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot (1 - a_k)} \right\}^{\frac{q_k}{q_k - 1}} \\
& + c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-}}.
\end{aligned}
\tag{4.24}
$$

Here thanks to the preparatory observation that $1 < q_\star \leq q_k \leq q^\star$ for all $k \in \mathbb{N}$, as made in Lemma 4.1, we can find $c_5 > 0$ independent of $k \in \mathbb{N}$ such that

$$
\begin{aligned}
&(q_k - 1) q_k^{-\frac{q_k}{q_k - 1}} \left\{ \frac{K_1}{(\beta_k - \beta^-)^2} \right\}^{-\frac{1}{q_k - 1}} \cdot \left\{ c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot (1 - a_k)} \right\}^{\frac{q_k}{q_k - 1}} \\
&\qquad \leq \quad c_5 (\beta_k - \beta^-)^{\frac{2}{q_k - 1}} (1+t)^{\frac{(q_\star - 1)\delta}{q_\star} \cdot \frac{q_k}{q_k - 1}} \left\| e^{\frac{\beta_k - \beta^-}{u}} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot (1 - a_k) \cdot \frac{q_k}{q_k - 1}}
\end{aligned}
$$

for all $t \in (0, T_{max})$, and moreover recalling the definition of $(\beta_k)_{k \in \mathbb{N}}$ we conclude that there exists $c_6 > 0$ fulfilling

$$
\begin{aligned}
&(q_k - 1) q_k^{-\frac{q_k}{q_k - 1}} \left\{ \frac{K_1}{(\beta_k - \beta^-)^2} \right\}^{-\frac{1}{q_k - 1}} \cdot \left\{ c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot (1 - a_k)} \right\}^{\frac{q_k}{q_k - 1}} \\
&\qquad \leq \quad c_6 \cdot 2^{\frac{2k}{q_\star - 1}} (1+t)^\delta \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-} \cdot (1 - a_k) \cdot \frac{q_k}{q_k - 1}}
\end{aligned}
$$

for all $t \in (0, T_{max})$. As clearly

$$
c_3 c_4 (1+t)^{\frac{(q_\star - 1)\delta}{q_\star}} \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-}} \leq c_3 c_4 (1+t)^\delta \left\| e^{\frac{\beta_k - \beta^-}{2} u} \right\|_{L^{\frac{\beta_k}{\beta_k - \beta^-}}(\Omega)}^{\frac{2(\beta_k - \beta^+ + 2\gamma)}{\beta_k - \beta^-}} \qquad \text{for all } t \in (0, T_{max}),
$$

from (4.24) together with (4.18) and (4.19) we infer that

$$\begin{aligned}
\frac{1}{\beta_k^2}\frac{d}{dt}\int_\Omega e^{\beta_k u} + \frac{K_1}{(\beta_k-\beta^-)^2}\int_\Omega \Big|\nabla e^{\frac{\beta_k-\beta^-}{2}u}\Big|^2 \\
&\le c_6\cdot 2^{\frac{2k}{q_\star-1}}(1+t)^\delta \Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}\cdot(1-a_k)\cdot\frac{q_k}{q_k-1}} \\
&\quad + c_3c_4(1+t)^\delta \Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}} \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{4.25}$$

In order to turn the Dirichlet integral herein into a zero-order absorptive term, we recall that by (4.22) and (4.5) we have

$$1 \le \frac{2\beta_k}{\beta_k-\beta^-} \le p^\star \qquad \text{for all } k\in\mathbb{N},$$

whence an application of Lemma 2.4 provides $c_7>0$ such that for any choice of $k\ge 1$ we can estimate

$$\begin{aligned}
\int_\Omega e^{\beta_k u} &= \Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{2\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2\beta_k}{\beta_k-\beta^-}} \\
&\le \Big\{c_7\cdot\Big(\Big\|\nabla e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^2(\Omega)}^2 + \Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^2\Big)\Big\}^{\frac{\beta_k}{\beta_k-\beta^-}} \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

Therefore,

$$\frac{K_1}{(\beta_k-\beta^-)^2}\int_\Omega\Big|\nabla e^{\frac{\beta_k-\beta^-}{2}u}\Big|^2 \ge \frac{K_1}{c_7(\beta_k-\beta^-)^2}\cdot\Big\{\int_\Omega e^{\beta_k u}\Big\}^{\frac{\beta_k-\beta^-}{\beta_k}} - \frac{K_1}{(\beta_k-\beta^-)^2}\Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^2$$

for all $t\in(0,T_{max})$, whence (4.25) yields the inequality

$$\begin{aligned}
\frac{d}{dt}\int_\Omega e^{\beta_k u} + \frac{K_1\beta_k^2}{(\beta_k-\beta^-)^2}\cdot\Big\{\int_\Omega e^{\beta_k u}\Big\}^{\frac{\beta_k-\beta^-}{\beta_k}} \\
&\le c_6\cdot 2^{\frac{2k}{q_\star-1}}\beta_k^2(1+t)^\delta \Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}\cdot(1-a_k)\cdot\frac{q_k}{q_k-1}} \\
&\quad + c_3c_4\beta_k^2(1+t)^\delta \Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k-\beta^-}} \\
&\quad + \frac{K_1\beta_k^2}{(\beta_k-\beta^-)^2}\Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)}^2 \qquad \text{for all } t\in(0,T_{max}).
\end{aligned}$$

Since $\beta_k = 2\beta_{k-1}$ and hence

$$\Big\|e^{\frac{\beta_k-\beta^-}{2}u}\Big\|_{L^{\frac{\beta_k}{\beta_k-\beta^-}}(\Omega)} = \Big\{\int_\Omega e^{\beta_{k-1}u}\Big\}^{\frac{\beta_k-\beta^-}{\beta_k}} \qquad \text{for all } t\in(0,T_{max}),$$

once more recalling (4.22) we readily end up with (4.17) on choosing $C(\delta)>0$ suitably large. □

4.3 Bounds in recursions involving asymptotically quadratic nonlinearities

In general, the right-hand side of (4.17) may contain powers of $\int_\Omega e^{\beta_{k-1}u}$ which are subquadratic, but which thanks to the observations made in Lemma 4.1 will at least become quadratic asymptotically at a sufficiently fast rate. This will be essential to our next step, to be achieved in Lemma 5.1 below on the basis of the following elementary estimate which has implicitly been used in precedent Moser iterations for quasilinear parabolic equations such as e.g. in [23, Lemma A.1].

Lemma 4.3 *Let $(M_k)_{k\in\mathbb{N}_0} \subset [1,\infty)$ be such that*

$$M_k \le b^k M_{k-1}^{\theta_k} \qquad \text{for all } k \in \mathbb{N} \tag{4.26}$$

with some $b \ge 1$ and $(\theta_k)_{k\in\mathbb{N}} \subset (0,\infty)$ having the property that there exists $d > 0$ fulfilling

$$\theta_k \le 2(1 + d \cdot 2^{-k}) \qquad \text{for all } k \in \mathbb{N}. \tag{4.27}$$

Then

$$M_k \le b^{k+e^d\cdot 2^{k+1}} \cdot M_0^{e^d\cdot 2^k} \qquad \text{for all } k \in \mathbb{N}. \tag{4.28}$$

Proof. By straightforward induction, from (4.26) we first obtain that

$$M_k \le b^{k+\sum_{j=1}^k (j-1)\cdot\prod_{i=j}^k \theta_i} \cdot M_0^{\prod_{i=1}^k \theta_i} \qquad \text{for all } k \in \mathbb{N}. \tag{4.29}$$

Here using (4.27) and the fact that $\ln(1+\xi) \le \xi$ for all $\xi \ge 0$ we can estimate

$$\begin{aligned} \prod_{i=j}^k \theta_i &\le 2^{k+1-j} e^{\sum_{i=j}^k \ln(1+d\cdot 2^{-k})} \\ &\le 2^{k+1-j} e^{\sum_{i=j}^k d\cdot 2^{-k}} \\ &\le c_1 \cdot 2^{k+1-j} \qquad \text{for all } k \in \mathbb{N} \text{ and each } j \in \{1,...,k\} \end{aligned}$$

with $c_1 := \exp\Big\{\sum_{i=1}^\infty d\cdot 2^{-k}\Big\} = e^d$. Therefore, (4.29) along with the inequalities $b \ge 1$ and $M_0 \ge 1$ implies that

$$M_k \le b^{k+c_1\cdot 2^{k+1}\sum_{j=1}^k (j-1)\cdot 2^{-j}} \cdot M_0^{c_1\cdot 2^k} \qquad \text{for all } k \in \mathbb{N},$$

which directly yields (4.28), because $\sum_{j=1}^k (j-1)\cdot 2^{-j} \le \frac{1}{4}\sum_{l=1}^\infty l\cdot 2^{-(l-1)} = 1$ for all $k \in \mathbb{N}$. □

5 Bounds for e^u in $L^\infty(\Omega)$. Proof of Theorem 1.1

By appropriately applying the results of the previous section along with the outcome of Lemma 3.2, we can now accomplish the main step toward both the statement on global existence as well as the upper estimate claimed in Theorem 1.1.

Lemma 5.1 *Suppose that (1.5), (1.6), (1.7) and (1.8) hold with some $\beta^- \in \mathbb{R}, \beta^+ \le \beta^-, \gamma \in [\frac{\beta^+-\beta^-}{2}, \frac{\beta^+}{2})$ and positive K_1, K_2 and K_3, and that (u_0, v_0) satisfies (1.9). Then for all $\varepsilon > 0$ there exists $C(\varepsilon) > 0$ such that for the solution of (1.1) we have*

$$\|e^{u(\cdot,t)}\|_{L^\infty(\Omega)} \le C(\varepsilon)\cdot(1+t)^{\frac{1}{\beta^+-2\gamma}+\varepsilon} \qquad \text{for all } t \in (0, T_{max}). \tag{5.1}$$

PROOF. Given $\varepsilon > 0$, let us first fix

$$\mu_0 := \frac{1}{\beta^+ - 2\gamma} + \frac{\varepsilon}{2}, \tag{5.2}$$

whence it is possible to pick $\eta > 0$ sufficiently small such that

$$\mu_0 \, e^\eta \le \frac{1}{\beta^+ - 2\gamma} + \varepsilon. \tag{5.3}$$

We finally choose $\delta \in (0,1]$ fulfilling

$$\delta \le \frac{\mu_0}{3\mu_0 + 2} \cdot \eta \tag{5.4}$$

and thereafter let $\theta > 1$, $\beta_0 > \max\{1, \beta^-\}$ and $(\beta_k)_{k\in\mathbb{N}}$ be as obtained from an application of Lemma 4.1 to any fixed $p^\star \in (2, \frac{2n}{(n-2)_+})$. We then recursively define

$$\kappa_0 := \beta_0\mu_0 \qquad \text{and} \qquad \kappa_k := 2(1+\eta\cdot 2^{-k})\kappa_{k-1}, \qquad k \in \mathbb{N}, \tag{5.5}$$

that is, we let

$$\kappa_k := \left(\prod_{j=1}^k (1+\eta \cdot 2^{-j}) \right) \cdot 2^k \kappa_0 \qquad \text{for } k \in \mathbb{N}, \tag{5.6}$$

and for nonnegative integers k we moreover introduce the numbers

$$M_k := \max\left\{ 1 \, , \sup_{t\in(0,T_{max})} (1+t)^{-\kappa_k} \int_\Omega e^{\beta_k u(\cdot,t)} \right\}, \tag{5.7}$$

which are all finite thanks to Lemma 3.2, because

$$\kappa_k > 2^k \kappa_0 > \frac{\beta_k}{\beta^+ - 2\gamma} \qquad \text{for all } k \in \mathbb{N}$$

thanks to (5.6), (5.5), (4.1) and (5.2). In order to estimate M_k for $k \in \mathbb{N}$, we first apply Lemma 4.2 to gain constants $c_1 > 0$ and $c_2 \ge 1$ such that for any choice of $k \in \mathbb{N}$, the function y_k defined on $[0, T_{max})$ by letting $y_k(t) := \int_\Omega e^{\beta_k u(x,t)} dx$, $t \in [0, T_{max})$, satisfies

$$\begin{aligned} y_k'(t) + c_1 y_k^{\frac{\beta_k - \beta^-}{\beta_k}}(t) \quad \le \quad & c_2^k (1+t)^\delta \cdot \left\{ \int_\Omega e^{\beta_{k-1} u} \right\}^{\frac{2(\beta_k - \beta^- + 2\gamma)}{\beta_k} \cdot (1-a_k) \cdot \frac{q_k}{q_k - 1}} \\ & + c_2^k (1+t)^\delta \cdot \left\{ \int_\Omega e^{\beta_{k-1} u} \right\}^{\frac{2(\beta_k - \beta^+ + 2\gamma}{\beta_k}} \\ & + c_2 \left\{ \int_\Omega e^{\beta_{k-1} u} \right\}^{\frac{2(\beta_k - \beta^-)}{\beta_k}} \qquad \text{for all } t \in (0, T_{max}), \end{aligned}$$

where a_k and q_k are as defined in Lemma 4.1. Here since by (5.7) we have

$$\int_\Omega e^{\beta_{k-1} u} \le M_{k-1}(1+t)^{\kappa_{k-1}} \qquad \text{for all } t \in (0, T_{max}),$$

using (4.10) and that $\frac{2(\beta_k-\beta^-)}{\beta_k} \le 2$ and $\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k} \le 2$ by positivity of both β^- as well as $\beta^+-2\gamma$, from this we infer that

$$
\begin{aligned}
y_k'(t) + c_1 y_k^{\frac{\beta_k-\beta^-}{\beta_k}}(t) &\le c_2^k (1+t)^\delta \cdot \Big\{ M_{k-1}(1+t)^{\kappa_{k-1}} \Big\}^{\frac{2(\beta_k-\beta^-+2\gamma)}{\beta_k}\cdot(1-a_k)\cdot\frac{q_k}{q_k-1}} \\
&\quad + c_2^k (1+t)^\delta \cdot \Big\{ M_{k-1}(1+t)^{\kappa_{k-1}} \Big\}^{\frac{2(\beta_k-\beta^++2\gamma)}{\beta_k}} \\
&\quad + c_2 \Big\{ M_{k-1}(1+t)^{\kappa_{k-1}} \Big\}^{\frac{2(\beta_k-\beta^-)}{\beta_k}} \\
&\le c_2^k (1+t)^\delta \cdot \Big\{ M_{k-1}(1+t)^{\kappa_{k-1}} \Big\}^{2(1+\delta\cdot 2^{-k})} \\
&\quad + c_2^k (1+t)^\delta \cdot \Big\{ M_{k-1}(1+t)^{\kappa_{k-1}} \Big\}^{2} \\
&\quad + c_2 \Big\{ M_{k-1}(1+t)^{\kappa_{k-1}} \Big\}^{2} \\
&\le 3c_2^k M_{k-1}^{2(1+\delta\cdot 2^{-k})} (1+t)^{2(1+\delta\cdot 2^{-k})\kappa_{k-1}+\delta} \qquad \text{for all } t\in(0,T_{max}), \qquad (5.8)
\end{aligned}
$$

where in increasing the respective exponents we also rely on the inequality $M_{k-1} \ge 1$ guaranteed by (5.7).

Now Lemma 2.3 enables us to conclude from (5.8) that

$$
y_k(t) \le \max\left\{ y_k(0)\,,\, \left(\frac{3c_2^k}{c_1} \cdot M_{k-1}^{2(1+\delta\cdot 2^{-k})} \right)^{\frac{\beta_k}{\beta_k-\beta^-}} \cdot (1+t)^{\frac{\beta_k}{\beta_k-\beta^-}\cdot[2(1+\delta\cdot 2^{-k})\kappa_{k-1}+\delta]} \right\} \qquad \text{for all } t\in(0,T_{max}). \tag{5.9}
$$

Here since (4.11) warrants that

$$
\frac{\beta_k}{\beta_k-\beta^-} \le 1+\delta\cdot 2^{-k} \tag{5.10}
$$

and hence implies that

$$
\frac{\beta_k}{\beta_k-\beta^-} \le 2 \tag{5.11}
$$

as a particular consequence of our restriction that $\delta \le 1$, we see that

$$
\begin{aligned}
\left(\frac{3c_2^k}{c_1} \cdot M_{k-1}^{2(1+\delta\cdot 2^{-k})} \right)^{\frac{\beta_k}{\beta_k-\beta^-}} &\le \left(\frac{3c_2^k}{c_1}\right)^2 \cdot M_{k-1}^{2(1+\delta\cdot 2^{-k})^2} \\
&\le \left(\frac{3c_2^k}{c_1}\right)^2 \cdot M_{k-1}^{2(1+\eta\cdot 2^{-k})}, \qquad (5.12)
\end{aligned}
$$

for our assumption (5.4) implies that $\delta \le \frac{\eta}{3}$ and hence

$$
(1+\delta\cdot 2^{-k})^2 - (1+\eta\cdot 2^{-k}) = (2\delta+\delta^2\cdot 2^{-k}-\eta)\cdot 2^{-k} \le (3\delta-\eta)\cdot 2^{-k} \le 0,
$$

again because $\delta \le 1$.
To treat the time-dependent factor on the right of (5.9), we now make full use of (5.4), which along with (5.10), (5.11) and (5.5) guarantees that

$$\frac{\beta_k}{\beta_k - \beta^-} \cdot \Big[2(1+\delta\cdot 2^{-k})\kappa_{k-1} + \delta\Big] \le 2(1+\delta\cdot 2^{-k})^2\kappa_{k-1} + 2\delta \le 2(1+\eta\cdot 2^{-k})\kappa_{k-1} = \kappa_k,$$

because once more due to the fact that $\delta \le 1$, (5.4) can be used to estimate

$$\begin{aligned}
2(1+\delta\cdot 2^{-k})^2\kappa_{k-1} + 2\delta - 2(1+\eta\cdot 2^{-k})\kappa_{k-1} &= 2(2\delta + \delta^2\cdot 2^{-k} - \eta)\cdot 2^{-k}\kappa_{k-1} + 2\delta \\
&\le (3\delta - \eta)\cdot 2^{-k}\kappa_{k-1} + 2\delta \\
&\le \Big(3\cdot\frac{\mu_0}{3\mu_0+2}\cdot\eta - \eta\Big)\cdot 2^{-k}\kappa_{k-1} + \frac{2\mu_0}{3\mu_0+2}\cdot\eta \\
&= \frac{2\eta}{3\mu_0+2}\cdot\Big(-2^{1-k}\kappa_{k-1} + \mu_0\Big),
\end{aligned}$$

and because (5.6) and the restriction $\beta_0 > 1$ ensure that

$$-2^{1-k}\kappa_{k-1} + \mu_0 \le -2^{1-k}\cdot 2^{k-1}\kappa_0 + \mu_0 = -\beta_0\mu_0 + \mu_0 < 0.$$

Accordingly, (5.9) shows that

$$y_k(t) \le \max\Big\{y_k(0)\,,\,\Big(\frac{3c_2^k}{c_1}\Big)^2 M_{k-1}^{2(1+\eta\cdot 2^{-k})}(1+t)^{\kappa_k}\Big\} \qquad \text{for all } t\in(0,T_{max}),$$

which implies that

$$(1-t)^{-\kappa_k}\int_\Omega e^{\beta_k u(\cdot,t)} \le \max\Big\{\int_\Omega e^{\beta_k u_0}\,,\,\Big(\frac{3c_2^k}{c_1}\Big)^2 M_{k-1}^{2(1+\eta\cdot 2^{-k})}\Big\} \qquad \text{for all } t\in(0,T_{max}),$$

and that hence

$$M_k \le \max\Big\{1\,,\int_\Omega e^{\beta_k u_0}\,,\,\Big(\frac{3c_2^k}{c_1}\Big)^2 M_{k-1}^{2(1+\eta\cdot 2^{-k})}\Big\}. \tag{5.13}$$

Now if there exists $(k_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $k_j \to \infty$ as $j\to\infty$ and

$$M_{k_j} \le \max\Big\{1\,,\int_\Omega e^{\beta_{k_j} u_0}\Big\} \qquad \text{for all } j\in\mathbb{N},$$

it is evident from (5.7) that

$$\begin{aligned}
\big\|e^{u(\cdot,t)}\big\|_{L^\infty(\Omega)} &= \lim_{j\to\infty}\Big\{\int_\Omega e^{\beta_{k_j}u(\cdot,t)}\Big\}^{\frac{1}{\beta_{k_j}}} \\
&\le \limsup_{j\to\infty}\Big\{(1+t)^{\kappa_{k_j}}M_{k_j}\Big\}^{\frac{1}{\beta_{k_j}}} \\
&\le \limsup_{j\to\infty}(1+t)^{\frac{\kappa_{k_j}}{\beta_{k_j}}}\cdot\max\Big\{1\,,\,\limsup_{j\to\infty}\Big\{\int_\Omega e^{\beta_{k_j}u_0}\Big\}^{\frac{1}{\beta_{k_j}}}\Big\} \\
&= \max\Big\{1\,,\,\big\|e^{u_0}\big\|_{L^\infty(\Omega)}\Big\}\cdot\limsup_{j\to\infty}(1+t)^{\frac{\kappa_{k_j}}{\beta_{k_j}}} \qquad \text{for all } t\in(0,T_{max}).
\end{aligned} \tag{5.14}$$

Observing that by (4.1), (5.6) and (5.5) we have

$$\begin{aligned}
\frac{\kappa_k}{\beta_k} &= \frac{\kappa_0}{\beta_0}\prod_{j=1}^k (1+\eta\cdot 2^{-j}) \\
&= \mu_0\, e^{\sum_{j=1}^k \ln(1+\eta\cdot 2^{-j})} \\
&\le \mu_0\, e^{\sum_{j=1}^k \eta\cdot 2^{-j}} \\
&\le \mu_0\, e^{\eta} \\
&\le \frac{1}{\beta^+ - 2\gamma} + \varepsilon \qquad \text{for all } k\in\mathbb{N}
\end{aligned} \tag{5.15}$$

thanks to (5.3), (5.14) shows that in this case we have

$$\left\|e^{u(\cdot,t)}\right\|_{L^\infty(\Omega)} \le \max\left\{1\,,\,\left\|e^{u_0}\right\|_{L^\infty(\Omega)}\right\}\cdot(1+t)^{\frac{1}{\beta^+-2\gamma}+\varepsilon} \qquad \text{for all } t\in(0,T_{max}). \tag{5.16}$$

Conversely, if such a sequence does not exist, then (5.13) implies the existence of some suitably large $b\ge 1$ such that

$$M_k \le b^k M_{k-1}^{2(1+\eta\cdot 2^{-k})} \qquad \text{for all } k\in\mathbb{N},$$

whence Lemma 4.3 applies so as to ensure that

$$M_k \le b^{k+e^{\eta}\cdot 2^{k-1}}\cdot M_0^{e^{\eta}\cdot 2^k} \qquad \text{for all } k\in\mathbb{N}.$$

Consequently, by a reasoning similar to that in (5.14) we obtain

$$\begin{aligned}
\left\|e^{u(\cdot,t)}\right\|_{L^\infty(\Omega)} &= \lim_{k\to\infty}\left\{\int_\Omega e^{\beta_k u(\cdot,t)}\right\}^{\frac{1}{\beta_k}} \\
&\le \limsup_{k\to\infty}\left\{(1+t)^{\kappa_k}M_k\right\}^{\frac{1}{\beta_k}} \\
&\le \left\{\limsup_{k\to\infty}(1+t)^{\frac{\kappa_k}{\beta_k}}\right\}\cdot\left\{\limsup_{k\to\infty}M_k^{\frac{1}{\beta_k}}\right\} \qquad \text{for all } t\in(0,T_{max}),
\end{aligned}$$

where by (4.1) and the trivial inequality $k\le 2^k$ we have

$$\begin{aligned}
M_k^{\frac{1}{\beta_k}} &\le b^{\frac{k+e^{\eta}\cdot 2^{k+1}}{\beta_k}}\cdot M_0^{\frac{e^{\eta}\cdot 2^k}{\beta_k}} \\
&= b^{\frac{k+e^{\eta}\cdot 2^{k+1}}{2^k\beta_0}}\cdot M_0^{\frac{e^{\eta}\cdot 2^k}{2^k\beta_0}} \\
&\le b^{\frac{1+2e^{\eta}}{\beta_0}}\cdot M_0^{\frac{e^{\eta}}{\beta_0}} =: c_3 \qquad \text{for all } k\in\mathbb{N}.
\end{aligned}$$

In view of (5.15), we thus infer that in this case

$$\left\|e^{u(\cdot,t)}\right\|_{L^\infty(\Omega)} \le c_3(1+t)^{\frac{1}{\beta^+-2\gamma}+\varepsilon} \qquad \text{for all } t\in(0,T_{max}),$$

which combined with (5.16) completes the proof. $\Box$

Our main results thereby become obvious.

PROOF of Theorem 1.1. Due to the extensibility criterion (2.1), Lemma 5.1 together with Lemma 2.1 and Lemma 3.1 firstly entails global solvability in the indicated class, whereupon the estimate (1.12) directly results from (5.1). $\Box$

6 Applications. Proofs of Corollary 1.2, Theorem 1.3 and Proposition 1.4

We can next verify the claimed consequences of Theorem 1.1 on the particular systems (1.13) and (1.19). Let us first address (1.13), that is, the Neumann problem for

$$\begin{cases} u_t = \nabla\cdot(e^{-\beta u}\nabla u) - \nabla\cdot(ue^{-\alpha u}\nabla v), & x\in\Omega,\ t>0,\\ v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{6.1}$$

and make sure that indeed for any choice of $\beta>0$ and $\alpha>\frac{\beta}{2}$ a global solution exists which satisfies the estimate (1.15).

PROOF of Corollary 1.2. Observing that (1.14) ensures that $\min\{\beta,2\alpha-\beta\}\equiv\beta-2(\beta-\alpha)_+$ is positive, given $\varepsilon>0$ we can fix $\eta>0$ suitably small such that

$$\frac{1}{\beta-2(\beta-\alpha)_+-2\eta}\le\frac{1}{\beta-2(\beta-\alpha)_+}+\frac{\varepsilon}{2}, \tag{6.2}$$

and such that moreover

$$2\eta<\beta-2(\beta-\alpha)_+, \tag{6.3}$$

where by positivity of η and (6.3), the number $\gamma:=(\beta-\alpha)_++\eta$ then satisfies

$$\gamma>0 \qquad\text{and}\qquad \gamma<(\beta-\alpha)_++\frac12\Big(\beta-2(\beta-\alpha)_+\Big)=\frac{\beta}{2}. \tag{6.4}$$

As the positivity of η moreover asserts that $\kappa:=\gamma+\alpha-\beta$ is positive, using that therefore $se^{-\kappa s}\le\frac{1}{\kappa e}$ for all $s\ge0$ we obtain that

$$\frac{se^{-\alpha s}}{e^{-\beta s}}=se^{-\kappa s}\cdot e^{\gamma s}\le\frac{1}{\kappa e}e^{\gamma s}\qquad\text{for all } s\ge0,$$

and that hence (1.6), (1.7) and (1.8) hold for $D(s):=e^{-\beta s}$ and $S(s):=se^{-\alpha s}$, $s\ge0$, if we let $\beta^-:=\beta^+:=\beta$ as well as $K_1:=K_2:=1$ and $K_3:=\frac{1}{\kappa e}$. Since (6.4) in particular warrants the validity of (1.10), we may thus apply Theorem 1.1 to obtain a global classical solution having the indicated uniqueness and regularity properties and furthermore satisfying

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le\Big(\frac{1}{\beta-2\gamma}+\frac{\varepsilon}{2}\Big)\cdot\ln(1+t)+c_1\qquad\text{for all } t>0$$

with some $c_1 = c_1(\varepsilon) > 0$. Since herein by definition of γ and (6.2) we can estimate

$$\frac{1}{\beta - 2\gamma} + \frac{\varepsilon}{2} = \frac{1}{\beta - 2(\beta-\alpha)_+ - 2\eta} \le \frac{1}{\beta - 2(\beta-\alpha)_+} + \frac{\varepsilon}{2} = \frac{1}{\min\{\beta, 2\alpha-\beta\}} + \frac{\varepsilon}{2},$$

this implies (1.15) with $C(\varepsilon) := c_1$. $\square$

Now if $\alpha > \frac{\beta}{2}$ is not too large in the sense that $\alpha < \beta$ if $n = 2$ and $\alpha \le \beta$ if $n = 3$, as specified in (1.16), the ratio $\frac{e^{-\alpha s}}{e^{-\beta s}}$ of the chemotactic sensitivity and the cell diffusivity in (6.1) grows even exponentially and hence faster than any algrebraic function of s as $s \to \infty$. In light of a known result on nonexistence of global bounded solutions in such constellations, the conclusion that in this case there exist global solutions which blow up in infinite time at a slow rate controlled by (1.17) is thus straightforward:

Proof of Theorem 1.3. Once more writing $D(s) := s^{-\beta}$ and $S(s) := se^{-\alpha s}$ for $s \ge 0$, observing that then again $\frac{S(s)}{D(s)} = se^{(\alpha-\beta)s}$ for all $s \ge 0$, we conclude from (1.16) that in the case $n = 2$ when $\alpha < \beta$ we have

$$\frac{\frac{S(s)}{D(s)}}{s \ln s} = \frac{e^{(\beta-\alpha)s}}{\ln s} \to +\infty \qquad \text{as } s \to \infty,$$

whereas if $n \ge 3$, then the inequality $\alpha \le \beta$ ensures that

$$\frac{s\Big(\frac{S}{D}\Big)'(s)}{\frac{S}{D}(s)} = 1 + (\beta - \alpha)s \ge 1 \qquad \text{for all } s \ge 1.$$

Therefore, for any prescribed $m > 0$ we obtain from a known result ([24, Theorem 5.1, Corollary 5.2]) that there exist radial initial data (u_0, v_0) satisfying (1.9) as well as $\int_\Omega u_0 = m$, which are such that (1.13) does not possess any global classical solution for which u belongs to $L^\infty(\Omega \times (0,\infty))$. Since on the other hand Corollary 1.2 guarantees the existence of a global classical solution fulfilling (1.15), this solution must actually satisfy (1.17). $\square$

For (1.19), that is, the corresponding initial-boundary value problem for

$$\begin{cases} u_t = \nabla \cdot \Big((1+\beta u)e^{-\beta u}\nabla u\Big) - \nabla \cdot (ue^{-\beta u}\nabla v), & x \in \Omega,\ t > 0, \\ v_t = \Delta v - v + u, & x \in \Omega,\ t > 0, \end{cases}$$

Theorem 1.1 immediately yields global existence and the logarithmic upper bound claimed in Proposition 1.4:

Proof of Proposition 1.4. Given $\varepsilon > 0$, we fix $\beta^+ \in (0, \beta)$ sufficiently close to β such that

$$\frac{1}{\beta^+} < \frac{1}{\beta} + \frac{\varepsilon}{2}. \tag{6.5}$$

which implies that writing $D(s) := (1 + \lambda s)e^{-\beta s}$ and $S(s) := se^{-\beta s}$ for $s \ge 0$, we obtain that

$$e^{\beta^+ s}D(s) = (1+\beta s)e^{-(\beta-\beta^+)s} \le K_2 := 1 + \frac{\beta}{(\beta-\beta^+)e} \qquad \text{for all } s \ge 0,$$

and that hence (1.7) holds. Since clearly (1.6) is valid with $\beta^- := \beta$ and $K_1 := 1$, and since moreover

$$\frac{S(s)}{D(s)} = \frac{s}{1+\beta s} \le K_3 := \frac{1}{\beta} \qquad \text{for all } s \ge 0,$$

we may thus apply Theorem 1.1 to $\gamma := 0 \in (\frac{\beta^+-\beta}{2}, \frac{\beta^+}{2})$ to infer that the claimed existence and uniqueness statement holds, and that we can find $c_1 = c_1(\varepsilon) > 0$ fulfilling

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \Big(\frac{1}{\beta^+} + \frac{\varepsilon}{2}\Big) \cdot \ln(1+t) + c_1 \qquad \text{for all } t > 0.$$

In view of (6.5), this establishes (1.20) with $C(\varepsilon) := c_1$. $\square$

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